Liquidity story again: a closed door? (16/4/03)

1) If we propose to relate the change ΔX in log price to hedging demand ΔH by

(1) $$\varphi(\Delta X) = \xi + \Delta H$$

then we want to make all round trips costly. Assuming we can neglect the market's demands ξ, using ψ for the inverse function to φ, which we assume is increasing, $\varphi(0)=0$, we find that if we propose a demand h, then the cost of filling that is

(2) $$S_0 \int_0^h e^{\psi(v)}dv \equiv S_0 \int_0^h F(v)\,dv$$

So to make a round trip costly, if we propose demands $h_1, \dots, h_N$, and if we write

$$V(\underline{h}) \equiv V(h_1, \dots, h_N)$$

(3) $$= \sum_{j=1}^{N} \exp\Big\{ \sum_{r<j} \psi(h_r) \Big\} \int_0^{h_j} e^{\psi(v)}\,dv$$

for the net cost of implementing those demands, then $V(\underline{h}) \geq 0$ if $\sum h_j = 0$.
Note that

(4) $$\boxed{V([h;x]) = \int_0^h F(v)\,dv + F(h)\,V(x)}$$

for all vectors x (using Scilab notation $[h;x]$). The costly round trip condition requires for all vectors x that

(5) $$\boxed{V(x) \geq -\int_0^h F(v)\,dv \Big/ F(h) \qquad \text{where } h = -x\cdot 1}$$

2) Suppose that for a particular ψ we wanted to prove this by induction on the length of the vector x. From (4), we would get the lower bound

$$V([h;x]) \geq \int_0^h F(v)\,dv - F(h)\int_0^t F(v)\,dv \Big/ F(t) \qquad (t \equiv -x\cdot 1)$$

and to have any hope we would need that this was at least as big as the lower bound

$$-\int_0^{t-h} F(v)\,dv \Big/ F(t-h)$$

which we'd try to establish to extend (5) to these longer vectors. Writing

$$G(y) = \int_0^y F(v)\,dv \Big/ F(y),$$

we will have to have

$$G(h) - G(t) \geq -\frac{1}{F(h)} G(t-h)$$

for all $t, h \in \mathbb{R}$, or again

$$(6) \qquad \boxed{\frac{G(y)}{F(h)} \geq G(h+y) - G(h)}$$

If we assume ψ is differentiable, and holding h fixed, by dividing (6) by $y>0$ and letting $y \downarrow 0$ we get

$$\frac{G'(0)}{F(h)} \geq G'(h)$$

and dividing by $y<0$ and letting $y \uparrow 0$ we get

$$\frac{G'(0)}{F(h)} \leq G'(h)$$

Obviously $G'(0)=1$, so we have deduced that

$$F(h)G'(h) = 1 \qquad \forall h$$

Now $\quad (FG)' = F = FG' + GF' = 1 + GF'$

so

$$\frac{F^2}{F'} = \frac{F}{F'} + \int_0^x F(v)\,dv$$

and differentiating gives

$$(7) \qquad 2F - \frac{F^2 F''}{(F')^2} = 1 - \frac{FF''}{(F')^2} + F$$

$$\Rightarrow \quad 1-F = \frac{FF''}{(F')^2}(1-F) \quad \Rightarrow \quad \frac{FF''}{(F')^2} = 1 \quad \Rightarrow \quad F' = \text{const.}\, F \quad \Rightarrow \quad \boxed{F(x) = e^{\gamma x}}$$

<u>which is very restrictive...</u>

3) On the other hand, in order to be able to ignore the random effect, we would need that the hedging demands were large enough. If we were to insist that the costly round trip condition holds under the restriction that $|h_j| \geq \varepsilon$ for all j, then the analysis rolls as before, except that (7) only holds outside $[-\varepsilon, \varepsilon]$, and this allows for some freedom near zero. We would have to have $\quad F(x) = \begin{cases} A_1 e^{\gamma_1 x} & x \geq \varepsilon \\ A_2 e^{\gamma_2 x} & x \leq \varepsilon \end{cases}$ + something else in between.

4) Let's look at what would happen in the special case of $\psi(x) = \gamma x$, and market orders are IID $N(\mu, \sigma^2)$. If our large agent buys h, then sells h, the cost will be

$$h \quad \frac{e^{\gamma(\xi+h)}-1}{\gamma(\xi+h)} \quad - \; h \quad \frac{e^{\gamma(\xi'-h)}-1}{\gamma(\xi'-h)} \quad e^{\gamma(\xi+h)}$$

where ξ, ξ' are independent $N(\mu, \sigma^2)$. Thus if we define

$$f(h) \equiv E \; \frac{e^{\gamma(\xi+h)}-1}{\gamma(\xi+h)} = E \; \frac{1}{\gamma}\int_0^{\gamma} e^{t(\xi+h)}\,dt = \frac{1}{\gamma}\int_0^{\gamma} \exp\{t(\mu+h) + \tfrac{1}{2}\sigma^2 t^2\}\,dt$$

then the cost of this simple round trip will be on average

$$h\; f(h) \; - \; h \; \exp\left(\gamma(\mu+h) + \tfrac{1}{2}\gamma^2\sigma^2\right) \; f(-h)$$

If the volatility were zero, and we took $h = \mu$, then the round-trip cost is $(\gamma=1)$

$$\mu \left\{ \frac{e^{2\mu}-1}{2\mu} - e^{2\mu} \right\} = -\tfrac{1}{2} - e^{2\mu}\left(\mu - \tfrac{1}{2}\right)$$

which gets big negative as μ gets big positive... this is certainly not what we want to see.

Could it be that we have a chance provided the vol is large relative to the drift? Taking $h\to 0$, the mean cost of the simple round trip will be positive if $\mu + \frac{1}{2}\gamma\sigma^2 < 0$, which is to say, μ must be sufficiently negative!! <u>This makes no sense as a model.</u>

If ψ has a Taylor expansion, $\psi(t) = \sum_{i\geq 1} \psi_i t^i$, then near zero

$$A(0,t) = 1 + \tfrac{1}{2}\psi_1 t + \tfrac{1}{6}(\psi_1^2 + 2\psi_2)t^2 + \tfrac{1}{24}(\psi_1^3 + 6\psi_1\psi_2 + 6\psi_3)t^3 + \cdots$$

Liquidity story again: modifying the approach (24/4/03)

1) The existing modelling approach appears unable to do sensible things with round trips. So let's modify the model, and suppose that each period there will be market demands* of ξ_+ buys, ξ_- sells, where the pair (ξ_+, ξ_-) may not be independent, but we'll assume independence from period to period. The hedger demands h; what then happens?

(i) If $h > 0$, there is buy demand $h + \xi_+$, so the hedger has to pay

$$\frac{h}{h+\xi_+} \int_0^{h+\xi_+} e^{\Psi(v)}\,dv \qquad \equiv h\ A(0, h+\xi_+)$$

and the price shifts to

$$\frac{1}{\xi_+ + \xi_- + h} \int_{-\xi_-}^{\xi_+ + h} e^{\Psi(v)}\,dv \qquad \equiv A(-\xi_-, \xi_+ + h)$$

which is the average price paid during the period

(ii) If $h = -x < 0$, then the hedger receives

$$\frac{x}{x+\xi_-} \int_{-(x+\xi_-)}^{0} e^{\Psi(v)}\,dv \qquad \equiv x\ A(-x-\xi_-, 0)$$

and the price moves to

$$\frac{1}{\xi_+ + \xi_- + |h|} \int_{-\xi_- + h}^{\xi_+} e^{\Psi(v)}\,dv \qquad \equiv A(-\xi_- + h, \xi_+)$$

Where we are using the notation

$$\boxed{A(a, b) = \frac{1}{b-a} \int_a^b e^{\Psi(v)}\,dv}$$

for the average of $\exp(\Psi)$ over $[a, b]$. In these terms then, the cost to the hedger is

$$h^+ A(0, h^+ + \xi_+) - h^- A(-\xi_- - h^-, 0)$$

and the price shifts by a factor

$$A(-\xi_- - h^-, \xi_+ + h^+).$$

How does this do with round trips? If we consider firstly what happens if the hedger buys $h > 0$ and then next period sells h, the cost to him will be

* Some market orders may get filled with other market orders; we think of $\xi_\pm$ as the orders which have to be filled from the limit order book

$$h\, A(0, h+\xi_+) - h\, A(-\xi_-' - h, 0).\, A(-\xi_-, \xi_+ + h)$$

$$\geq h\, A(-\xi_-, \xi_+ + h)\left\{1 - A(-\xi_-' - h, 0)\right\}$$

$$> 0$$

Likewise, selling $h > 0$ then immediately buying back $h > 0$ will cost

$$-h\, A(-\xi_- - h, 0) + h\, A(0, \xi_+' + h).\, A(-\xi_- - h, \xi_+)$$

$$> h\, A(-\xi_- - h, 0)\left\{-1 + A(0, \xi_+' + h)\right\}$$

$$> 0$$

in view of our standing assumptions that ψ is increasing continuous, $\psi(0) = 0$.

Notes: even if h is very small relative to $\xi_\pm$, there is still a non-vanishing difference between buy + sell prices — we are in effect looking at a model with transaction costs.

If the hedge amounts were zero, the changes in log price are IID, so we can expect a Black-Scholes limit.

2) Another possible modification is to suppose as before that the hedger pays

$$\frac{h}{\xi + h} \int_0^{\xi + h} e^{\psi(v)}\, dv$$

for his shares, and that the new share price is

$$\frac{1}{\xi + h} \int_0^{\xi + h} e^{\psi(v)}\, dv$$

instead of $\exp\{\psi(\xi + h)\}$.

But this looks to be bad when we examine the round trip of buying h, then selling h. The cost would be

$$h\, A(0, \xi + h) - A(0, \xi + h)\, h\, A(0, \xi' - h)$$

$$= h\, A(0, \xi + h)\left[1 - A(0, \xi' - h)\right]$$

and this would require $\left.\begin{array}{ll} \text{for } h > 0, & E\, A(0, \xi' - h) < 1 \\ \text{for } h < 0, & E\, A(0, \xi' - h) > 1 \end{array}\right\}$ so $E\, A(0, \xi') = 1$??

Not really sustainable: so it looks like the story where the order book gets eaten away on both sides is how it must be.

3) <u>How does this all look in the limit as $\Delta t \downarrow 0$?!</u>

Suppose that we aim to follow a hedging strategy in discrete time, changing portfolio at the times $t_j \equiv j\Delta t$, using the rule that the hedging demand in period j will be

$$h_j \equiv H(t_{j-1}, X_{t_{j-1}}) - H(t_{j-2}, X_{t_{j-2}})$$

where H is some suitably smooth bounded function. Let G_j denote the amount of cash just before time t_j, $S \equiv e^X$, H_j the number of shares held just before t_j. Then

$$\begin{cases} H_j = H(t_{j-1}, X_{t_{j-1}}), & (\#\text{ of shares held in } (t_{j-1}, t_j)) \\ G_j - G_{j-1} = S_{t_{j-1}}\{-h_j^+ A(0, h_j + \xi_{j+}) & (\text{change in cash at time } t_{j-1}) \\ \qquad + h_j^- A(h_j - \xi_{j-}, 0)\} & \\ \Delta X_j \equiv X_{t_j} - X_{t_{j-1}} & \\ \qquad = \log A(-\xi_{j-} - h_j^-, \xi_{j+} + h_j^+) & \end{cases}$$

describes the evolution of the system. Assuming that the $\xi_{j\pm}$ and the h_j are all going to be $O(\sqrt{\Delta t})$, we can do some series expansions. If we suppose we can write

$$\psi(v) = \sum_{j\geq 1} \psi_j v^j,$$

then

$$\Delta X_j = \tfrac{1}{2}\psi_1(\xi_{j+} - \xi_{j-} + h_j) + \tfrac{1}{24}\psi_1^2(\xi_{j+} + \xi_{j-} + |h_j|)^2$$
$$+ \tfrac{1}{3}\psi_2\{(\xi_{j+} - \xi_{j-} + h_j)^2 + (\xi_{j+} + h_j^+)(\xi_{j-} + h_j^-)\} + o((\Delta t)^{3/2})$$

Now let's suppose that

$$E\,\xi_\pm = \mu\sqrt{\Delta t} + \theta_\pm \Delta t$$

$$\operatorname{cov}\begin{pmatrix}\xi_+ \\ \xi_-\end{pmatrix} = \Delta t\begin{pmatrix}\sigma_{++} & \sigma_{+-} \\ \sigma_{-+} & \sigma_{--}\end{pmatrix}$$

so that conditional on $\mathcal{F}(t_{j-1})$,

$$E\left[\Delta X_j \mid \mathcal{F}(t_{j-1})\right] = \tfrac{1}{2}\psi_1\{(\theta_+ - \theta_-)\Delta t + h_j\} + \frac{\psi_1^2}{24}\Big[h_j^2 + 4\mu^2\Delta t + \Delta t(\sigma_{++} + 2\sigma_{+-} + \sigma_{--})$$
$$+ 4|h_j|\mu\sqrt{\Delta t}\Big] +$$

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$$+ 4|h_j|\mu\sqrt{\Delta t}\Big] +$$

$$+\ \tfrac{1}{3}\psi_2\Big[\ h_j^2 + \Delta t(\sigma_{++} - 2\sigma_{+-} + \sigma_{--}) + |h_j|\,\mu\sqrt{\Delta t} + \mu^2\Delta t + \sigma_{+-}\Delta t\Big] + O(\Delta t^{3/2})$$

so that

$$\Big\{ E(\Delta X_j \mid \mathcal{F}_{t_{j-1}}) - \tfrac{1}{2}\psi_1 h_j \Big\}/\Delta t$$

$$= \tfrac{1}{2}\psi_1(\theta_+ - \theta_-) + \frac{\psi_1^2}{24}\Big[\sigma_{++} + 2\sigma_{+-} + \sigma_{--} + 4\mu^2 + 4\frac{|h_j|}{\sqrt{\Delta t}}\mu + \frac{h_j^2}{\Delta t}\Big]$$

$$+ \frac{\psi_2}{3}\Big[\sigma_{++} - \sigma_{+-} + \sigma_{--} + \mu^2 + \frac{|h_j|}{\sqrt{\Delta t}}\mu + \frac{h_j^2}{\Delta t}\Big] + O(\Delta t^{1/2}) \equiv \alpha_j + O(\Delta t^{1/2})$$

and we can get

$$\mathrm{var}(\Delta X_j \mid \mathcal{F}_{t_{j-1}}) = \tfrac{1}{4}\psi_1^2\,\Delta t\,(\sigma_{++} - 2\sigma_{+-} + \sigma_{--}) + o(\Delta t)$$

similarly.

Thus

$$X_{t_j} - \sum_{r=1}^{j} \tfrac{1}{2}\psi_1 h_r - \sum_{r=1}^{j} \alpha_j \Delta t$$

is close to a BM with variance $\frac{1}{4}\psi_1^2(\sigma_{++} - 2\sigma_{+-} + \sigma_{--})$, where the α_j as above are $O(1)$. More compactly,

$$X_{t_j} - \tfrac{1}{2}\psi_1\, H(t_{j-1}, X_{t_{j-1}}) - \sum_{r=1}^{j} \alpha_j \Delta t$$

is nearly a BM with the appropriate variance, $\sigma^2 = \frac{1}{4}\psi_1^2(\sigma_{++} - 2\sigma_{+-} + \sigma_{--})$.

Thus if X were to be the solution of an SDE,

$$dX_t = a(t, X_t)\,dW_t + b(t, X_t)\,dt,$$

we would expect

$$\boxed{d\Big\{X_t - \tfrac{1}{2}\psi_1 H(t, X_t)\Big\} = \sigma\, dW_t + \Big\{\tfrac{1}{2}\psi_1(\theta_+ - \theta_-) + \frac{\psi_1^2}{24}\big(\sigma_{++} + 2\sigma_{+-} + \sigma_{--} + (2\mu + c|aH_x|)^2\big) + \frac{\psi_2}{3}\big(\sigma_{++} - \sigma_{+-} + \sigma_{--} + \mu^2 + \mu c|aH_x| + (aH_x)^2\big)\Big\}\,dt}$$

where $c = E|W_1| = \sqrt{2/\pi}$.

If $\delta_t = \sigma W_t + \mu t$, $Q(t) = \frac{1}{2}\sigma^2 t^2 + \mu t - \rho$, then if $Q(t) = \frac{1}{2}\sigma^2 (t-\beta)(t+\alpha)$ we find that the optimal level at which to shut down is given by

$$b = -\frac{1}{\gamma} \log\left(1 + \frac{\gamma}{\beta}\right)$$

if the asset is held in unit or zero amount.

Gradual abandonment of an asset (28/4/03)

1) We consider the situation where the amount $A_t \geq 0$ of an asset held at time t is non-increasing. The asset delivers a dividend stream

$$\delta_t = W_t + \mu t$$

and the objective of the agent is to select A so as to obtain

$$\min \; E \int_0^\infty \exp\{-\rho t - \gamma A_t \delta_t\}\, dt$$

(to maximise the CARA felicity of consumption.) How would this be done? The obvious thing is to let $\underline{\delta}_t \equiv \inf\{\delta_s : s \leq t\}$, and suppose that the optimal form of A is

$$A_t = \varphi(\underline{\delta}_t)$$

for some increasing function φ to be determined.

2) <u>Lemma</u> For $x > y$, $a > 0$,

$$E^x\left[\int_0^{\tau_y} e^{-\rho t - \gamma a \delta_t}\, dt\right] = \frac{e^{-\gamma a y}}{Q(-\gamma a)}\left[e^{-\alpha(x-y)} - e^{-\gamma a(x-y)}\right],$$

where

$$Q(t) = \tfrac{1}{2}t^2 + \mu t - \rho = \tfrac{1}{2}(t-\beta)(t+\alpha)$$

with $\alpha, \beta > 0$.

Proof. If V denotes the value, then we have as usual

$$\tfrac{1}{2}V'' + \mu V' - \rho + e^{-\gamma a x} = 0$$

solved by
$$V(x) = -\frac{e^{-\gamma a(x-y)} - e^{-\alpha(x-y)}}{Q(-\gamma a)}\, e^{-\gamma a y}$$

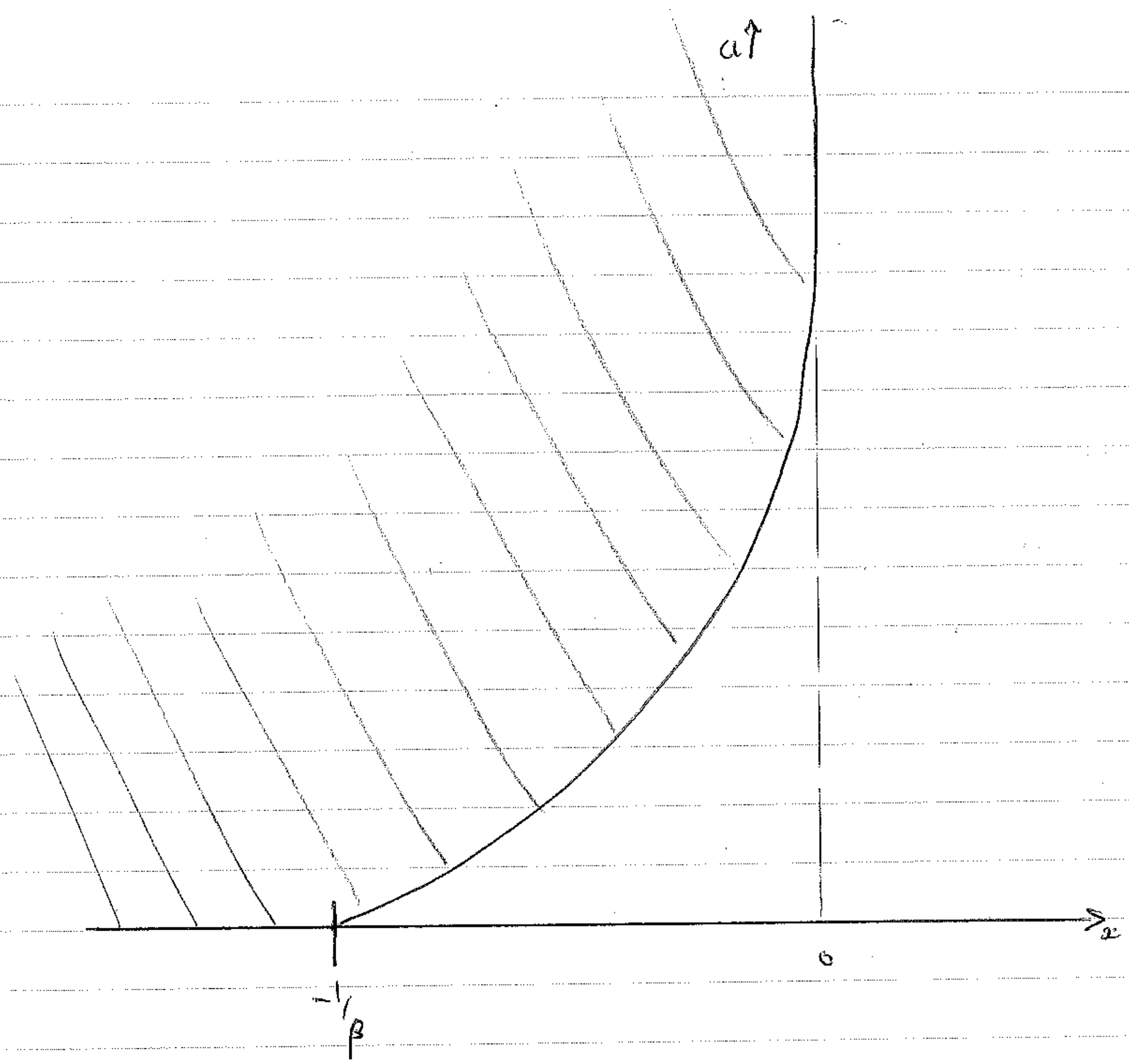
if we want to match the boundary conditions, 0 at y, bounded at infinity.

3) Suppose we postulate a form $A_t = \varphi(\underline{\delta}_t)$; what is the value of using this policy? We compute

$$E^x\left[\int_0^\infty \exp\{-\rho t - \gamma\, \varphi(\underline{\delta}_t)\, \delta_t\}\, dt\right]$$

$$= E^x\left[\sum_{y<x} \int_{\tau_{y-}}^{\tau_y} e^{-\rho t - \gamma \varphi(\underline{\delta}_t)\delta_t}\, dt\right]$$

$$= \int_{-\infty}^{x} dy\; e^{-\alpha(x-y)}\; n\left(\int_0^{\zeta} e^{-\rho u - \gamma\varphi(y)\{\xi_u + y\}}\, du\right) dy$$



Value if we weren't allowed to dump stuff would be $-e^{-\gamma a x}/Q(-\gamma a)$

so we need to compute

$$n\left(\int_0^\zeta \exp\{-\rho u - \gamma a\, \xi_u\}\, du\right)$$

$$= \lim \frac{1}{\varepsilon} E^\varepsilon \int_0^{\tau_0} \exp(-\rho u - \gamma a\, \xi_u)\, du$$

$$= \lim_x \frac{1}{x} \; \frac{e^{-\alpha x} - e^{-\gamma a x}}{Q(-\gamma a)}$$

$$= \frac{\gamma a - \alpha}{Q(-\gamma a)} = \frac{2(\gamma a - \alpha)}{(-\gamma a - \beta)(-\gamma a + \alpha)} = \frac{2}{\gamma a + \beta}$$

Thus the payoff is

$$\int_{-\infty}^{x} \exp\{-\alpha(x-y) - \gamma y\, \varphi(y)\} \; \frac{2\, dy}{\beta + \gamma \varphi(y)}$$

How do we choose monotone φ to maximize this? By calculus, if we do unrestricted optimisation over $\varphi(y)$, we get

$$\boxed{\begin{aligned} \varphi(y) &= +\infty && (y \geq 0) \\ &= \tfrac{1}{\gamma}\left\{ -\tfrac{1}{y} - \beta \right\}^+ && (y < 0) \end{aligned}}$$

This is plainly increasing, zero for $y \leq -1/\beta$, and $+\infty$ for $y \geq 0$.

What is the value of using this policy? If we start at (x, a), where $\varphi(x) > a$, and we set $\eta(a) \equiv \varphi^{-1}(a) = -1/(\beta + \gamma a)$, then the value is

$$\boxed{\begin{aligned} &e^{-\gamma a \eta} \; \frac{e^{-\alpha(x-\eta)} - e^{-\gamma a(x-\eta)}}{Q(-\gamma a)} + \frac{2e^{1-\alpha x}}{\lambda^2} \left\{ (\lambda y^* - 1) e^{\lambda y^*} - (\lambda \eta - 1) e^{\lambda \eta} \right\} \\ &\qquad + e^{-\alpha(x - y^*)} \; \frac{1}{\rho} \end{aligned}}$$

$(y^* \equiv -1/\beta, \; \lambda \equiv \alpha + \beta)$.

From this, it should be possible to confirm the solution by HJB; the value function $F = F(x, a)$ must satisfy

$$\tfrac{1}{2} F_{xx} + \mu F_x - \rho F + e^{-\gamma a x} \geq 0, \text{ equal in continuation region}, \quad F_a \leq 0, \text{ equal at exercise.}$$

I've checked this with Maple and it's OK!

4) <u>What is the price process here?</u> We have a state-price density process

$$\zeta_t = \exp\left[-\rho t - \gamma \delta_t \{ A_0 \wedge \varphi(\underline{\delta}_t) \} \right]$$

and our job is to value the dividend stream under this. From the earlier lemma, for $x > y$ we get

$$E^x \int_0^{\tau_y} \exp\{-\rho t - \gamma_a \delta_t\}\, \delta_t \, dt = \frac{e^{-\gamma_a x}}{Q^2}\{Q' - xQ\} + \frac{e^{-\gamma_a y - \alpha(x-y)}}{Q^2}(yQ - Q')$$

$\left(Q = Q(-\gamma_a) \text{ here} \right)$

Similarly,

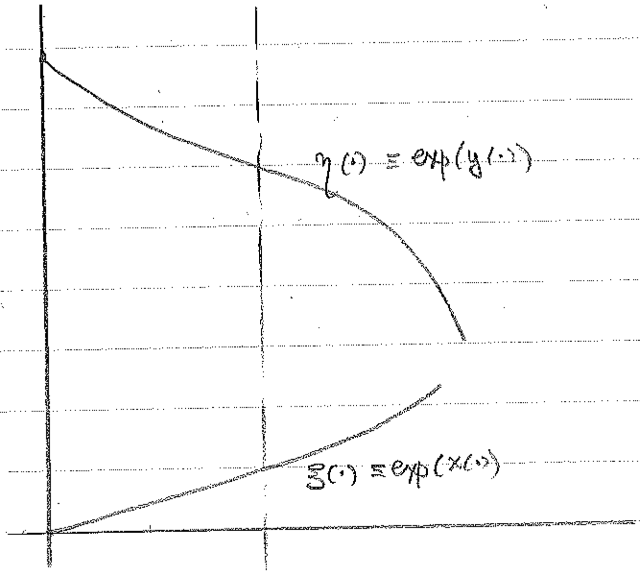
$$n\left(\int_0^{\zeta} \exp(-\rho u - \gamma_a \xi_u)(\xi_u + b)\, du \right)$$

$$= \frac{2b}{\gamma_a + \beta} + \frac{2}{(\gamma_a+\beta)^2} = \frac{2}{(\gamma_a+\beta)^2}\left\{ b(\gamma_a+\beta) + 1 \right\}$$

So dumping happens when the value of an upward excursion is exactly zero! This is entirely as you would expect - the value of the share is derived entirely from the dividends consumed prior to the first dumping.

Thus the price of the share when $\delta = x > \varphi^{-1}(a) \equiv y$ will be

$$\boxed{\frac{Q'(-\gamma_a) - x\,Q(-\gamma_a)}{Q(-\gamma_a)^2} + \frac{\{y\,Q(-\gamma_a) - Q'(-\gamma_a)\}\exp[(\gamma_a - \alpha)(x-y)]}{Q(-\gamma_a)^2}}$$



η(·) ≡ exp(y(·))
ξ(·) ≡ exp(x(·))

Convertible bonds: all-at-once expression using excursions (6/5/03)

1) Write $X_t \equiv \log V_t$, $\bar{X}_t \equiv \sup_{s\le t} X_s$, and suppose we run the process in the usual way until the smaller of

$$\zeta_B \equiv \inf\{t : X_t < x(m_t)\}, \quad \zeta_0 \equiv \inf\{t : X_t > y(0)\}$$

where $m_t = y^{-1}(\bar{X}_t) \equiv \tilde{m}(\bar{X}_t)$. There is a cashflow $\Phi(m_t, X_t)$ up til $\zeta \equiv \zeta_B \wedge \zeta_0$, and a payment $g(m_t)$ at bankruptcy time if bankruptcy comes first, else a payment of G at ζ_0 if ζ_0 comes first. Let the initial value of X be x_0. We therefore need to calculate

$$E^{x_0}\left[\int_0^\zeta e^{-rs}\Phi(m_s, X_s)\,ds + e^{-r\zeta}\left\{g(m_\zeta)\, I_{\{\zeta=\zeta_B\}} + G\, I_{\{\zeta=\zeta_0\}}\right\}\right]$$

2) (i) Downward excursions of X will be killed at rate r, or when they hit a barrier $b<0$. Rate of killed excursions

$$= \lim \frac{1}{\varepsilon}\left(1 - E^{-\varepsilon}\left[e^{-rH_0} : H_0 < H_b\right]\right)$$

$$= \frac{\alpha e^{\beta b} + \beta e^{-\alpha b}}{e^{-\alpha b} - e^{\beta b}} \equiv R(b), \text{ say.}$$

[We need to build $Ae^{-\alpha x} + Be^{\beta x}$ to vanish at b, equal 1 at zero, and this is done with the f^n $\left(e^{\beta b - \alpha x} - e^{-\alpha b + \beta x}\right)/\left(e^{\beta b} - e^{-\alpha b}\right)$, whose derivative at zero is $R(b)$]

(ii) Rate of excursions killed at the barrier, not r-killed,

$$= \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} E^{-\varepsilon}\left[e^{-rH_b} : H_b < H_0\right]$$

$$= \frac{\alpha+\beta}{e^{-\alpha b} - e^{\beta b}} \equiv R_B(b)$$

by similar reasoning.

(iii) The other thing we need is to compute the mean in the excursion measure of the integral component, so we need to compute

$$E^x\left[\int_0^{H_0\wedge H_b} \Phi(X_s)\, e^{-rs}\, ds\right]$$

where we'll suppose that $\Phi(x) = c_0 e^{qx}$. This requires us to find f to solve

$$\tfrac12\sigma^2 f'' + \mu f' - rf + c_0 e^{qx} = 0 \qquad (\mu = r - \delta - \tfrac12\sigma^2)$$

with the boundary conditions $f(0) = f(b) = 0$. Writing

$$Q(z) \equiv \tfrac{1}{2}\sigma^2 z^2 + \mu z - r$$

we find

$$f(x) = \frac{c_0}{Q(q)(e^{-\alpha b} - e^{\beta b})}\Big[(e^{qb} - e^{\beta b})e^{-\alpha x} + (e^{-\alpha b} - e^{qb})e^{\beta x} - (e^{-\alpha b} - e^{\beta b})\, e^{qx} \Big]$$

Taking the derivative of this at zero, we get

$$\boxed{\begin{aligned} & n\Big(\int_0^{H_b \wedge H_0} \Phi(X_s) e^{-rs}\, ds \Big) \\ &= \frac{c_0}{Q(q)(e^{-\alpha b} - e^{\beta b})}\Big[\alpha(e^{qb} - e^{\beta b}) + \beta(e^{qb} - e^{-\alpha b}) + q\,(e^{-\alpha b} - e^{\beta b}) \Big] \\ &\equiv K(c_0, q, b) \end{aligned}}$$

3) If we now assume $\Phi(m,x) = c_0(m)\exp\{q(m)x\}$, we have the excursion expression

$$\boxed{\begin{aligned} & E^{x_0}\Big[\int_0^{\zeta} e^{-rs}\Phi(m_s, X_s)\,ds + e^{-r\zeta}\big\{ g(m_\zeta) I_{\{\zeta = \zeta_B\}} + G\, I_{\{\zeta = \zeta_0\}} \big\} \Big] \\ &= \int_{x_0}^{y_0} \exp\Big(-\int_{x_0}^{z} R(b(s))\,ds \Big). \\ &\qquad \Big[K(\tilde{c}_0(z), \tilde{q}(z), b(z)) + R_B(b(z))\, g(\tilde{m}(z)) \Big]\, dz \\ &\qquad + \exp\Big(-\int_{x_0}^{y_0} R(b(s))\,ds \Big) G \end{aligned} \qquad \left[\begin{aligned} \tilde{x} &\equiv x \circ \tilde{m} \\ \tilde{g} &\equiv g \circ \tilde{m} \\ b(z) &\equiv \tilde{x}(z) - z \end{aligned}\right]}$$

For the cashflows of interest for our applications, we have $q = 0$, or $q = 1$. But then

$$\boxed{K(c_0, 0, b) = \frac{c_0\, \varphi_0(b)}{r(e^{-\alpha b} - e^{\beta b})}, \qquad K(c_0, 1, b) = \frac{c_0\, \varphi_1(b)}{\delta(e^{-(\alpha+1)b} - e^{(\beta-1)b})}}$$

in our earlier notation $\left(\varphi_0(x) \equiv \int_0^x \alpha\beta(e^{\beta t} - e^{-\alpha t})\,dt,\ \varphi_1(x) \equiv \int_0^x (\alpha+1)(\beta-1)e^{-t}(e^{\beta t} - e^{-\alpha t})\,dt \right)$.

Intriguing ......

We therefore define

$$\kappa_0(b) \equiv \frac{\varphi_0(b)}{r(e^{-\alpha b} - e^{\beta b})}, \qquad \kappa_1(b) \equiv \frac{\varphi_1(b)\, e^{b}}{\delta(e^{-\alpha b} - e^{\beta b})}, \quad \text{so that}$$

$$S(\tilde m(y), e^{y}) = \int_y^{y_0} \exp\left(-\int_y^{z} R(b(s))\,ds\right)\left[\frac{\delta e^{z}}{n-\tilde m(z)}\,\kappa_1(b(z)) - \frac{\tilde m(z)\rho'}{n-\tilde m(z)}\,\kappa_0(b(z))\right]dz$$

$$+\frac{\eta_0}{n}\exp\left[-\int_y^{y_0} R(b(s))\,ds\right]$$

Substituting $z = \log\eta(t)$, writing $\tilde m(y) \equiv m$, we get

$$S(m,\eta(m)) = \int_0^m \exp\left\{-\int_t^m R(b(\log\eta(w)))\,\frac{|\eta'(w)|\,dw}{\eta(w)}\right\}\left[\frac{\delta\,\eta(t)}{n-t}\kappa_1(b(\log\eta(t))) - \frac{t\rho'}{n-t}\kappa_0(b(\log\eta(t)))\right].$$

$$\frac{|\eta'(t)|\,dt}{\eta(t)} + \frac{\eta_0}{n}\exp\left(-\int_0^m R(b(\log\eta(w)))\,\frac{|\eta'(w)|\,dw}{\eta(w)}\right)$$

Now $b(\log\eta(w)) = \log\Theta(w)$, which brings things that bit closer to our former way of dealing with the problem.

The corresponding expression for the bond is

$$B(m,\eta(m)) = \int_0^m \exp\left\{-\int_t^m R(b(\log\eta(w)))\,\frac{|\eta'(w)|}{\eta(w)}\,dw\right\}\left[\rho\kappa_0(b(\log\eta(t))) + \frac{p\,\xi(t)}{t}R_B(b(\log\eta(t)))\right].$$

$$\frac{|\eta'(t)|}{\eta(t)}\,dt + \frac{\eta_0}{n}\exp\left\{-\int_0^m R(b(\log\eta(w)))\,\frac{|\eta'(w)|\,dw}{\eta(w)}\right\}.$$

4) Taking the above expression for $S(m,\eta(m))$, multiplying throughout by $\exp\left\{\int_0^m R(\log\Theta_w)\,|\eta'(w)|\,dw/\eta(w)\right\}$ and differentiating gives us

$$\boxed{\begin{aligned} &-R(\log\Theta_m)\frac{\eta'(m)}{\eta(m)}S(m,\eta(m)) + \frac{\partial S}{\partial m}(m,\eta(m)) + \eta'(m)\frac{\partial S}{\partial V}(m,\eta(m)) \\ &= \left\{\frac{\delta\eta(m)}{n-m}\kappa_1(\log\Theta_m) - \frac{m\rho'}{n-m}\kappa_0(\log\Theta_m)\right\}\left\{\frac{-\eta'(m)}{\eta(m)}\right\}\end{aligned}}$$

If $\frac{\partial S}{\partial m} = 0$ at the upper boundary, can we learn anything new from this?? Working this in with known forms of S gives $0=0$... and using the fact that $B = S$ at η gives the existing expression (3) on p 35 of WN XV for ξ.

Liquidity modelling: how does it look in the limit? (13/5/03)

Returning to the situation of pp 6–7, what do things look like for dC in the limit as $\Delta t \downarrow 0$? We have

$$\Delta X_j = \tfrac{1}{2}\psi_1 h_j + \Delta M_j + \tfrac{1}{2}\psi_1(\theta_+ - \theta_-)\Delta t + \frac{\psi_1^2}{24}\Big[\sigma_{++} + 2\sigma_{+-} + \sigma_{--} + 4\mu^2 + 4\frac{|h_j|}{\sqrt{\Delta t}}\mu + \frac{h_j^2}{\Delta t}\Big]\Delta t$$

$$+ \frac{\psi_2}{3}\Big[\sigma_{++} - \sigma_{+-} + \sigma_{--} + \mu^2 + \frac{|h_j|}{\sqrt{\Delta t}}\mu + \frac{|h_j|^2}{\Delta t}\Big]\Delta t + O(\Delta t^{3/2})$$

where

$$\Delta M_j \equiv \tfrac{1}{2}\psi_1\left\{\xi_{j+} - E\xi_{j+} - (\xi_{j-} - E\xi_{j-})\right\}$$

By a similar argument, doing a Taylor expansion of $A(0,t)$, we have

$$\Delta C_j \equiv S_{t_{j-1}}\left\{-h_j^+ A(0, h_j + \xi_{j+}) + h_j^- A(0, h_j - \xi_{j-})\right\}$$

$$= S_{t_{j-1}}\left\{-h_j^+\left(1 + \tfrac{\psi_1}{2}(h_j + \xi_{j+})\right) + h_j^-\left(1 + \tfrac{\psi_1}{2}(h_j - \xi_{j-})\right)\right\} + O(\Delta t^{3/2})$$

$$= -S_{t_{j-1}} h_j + \Delta N_j - \left(\tfrac{\psi_1}{2}\mu\sqrt{\Delta t}\,|h_j| + \tfrac{\psi_1}{2}h_j^2\right)S_{t_{j-1}} + O(\Delta t^{3/2})$$

where $\Delta N_j \equiv S_{t_{j-1}}\frac{\psi_1}{2}\left\{h_j^+(E\xi_{j+} - \xi_{j+}) + h_j^-(E\xi_{j-} - \xi_{j-})\right\}$. We thus have the covariation terms

$$E\,\Delta M_j^2 = \tfrac{1}{4}\psi_1^2(\sigma_{++} - 2\sigma_{+-} + \sigma_{--})\Delta t, \qquad E\,\Delta N_j^2 = \tfrac{1}{4}\psi_1^2 S_{t_{j-1}}^2\left\{(h_j^+)^2\sigma_{++} + (h_j^-)^2\sigma_{--}\right\}\Delta t$$

$$E(\Delta M_j\,\Delta N_j) = \tfrac{1}{4}\psi_1^2 S_{t_{j-1}}\left(h_j^+(\sigma_{+-} - \sigma_{++}) - h_j^-(\sigma_{+-} - \sigma_{--})\right)\Delta t$$

$$= \tfrac{1}{4}\psi_1^2 S_{t_{j-1}}\left(h_j\sigma_{+-} - \sigma_{++}h_j^+ + \sigma_{--}h_j^-\right)\Delta t.$$

Thus we expect in the limit that $d[N] = 0$, and

$$\boxed{\begin{aligned} dC &= -S\,dH - \tfrac{1}{2}\psi_1 S\left((aH_x)^2 + c\mu|aH_x|\right)dt \\ d(X - \tfrac{1}{2}\psi_1 H) &= \sigma\,dW + \Gamma\,dt \end{aligned}}$$

with

$$\Gamma \equiv \tfrac{1}{2}\psi_1(\theta_+ - \theta_-) + \frac{\psi_1^2}{24}\left(\sigma_{++} + 2\sigma_{+-} + \sigma_{--} + (2\mu + c|aH_x|)^2\right)$$

$$+ \frac{\psi_2}{3}\left(\sigma_{++} - \sigma_{+-} + \sigma_{--} + \mu^2 + \mu c\,|aH_x| + (aH_x)^2\right)$$

This gives us

$$\boxed{a = \frac{\sigma}{1 - \frac{1}{2}\psi_1 H_x}, \quad b = \frac{\Gamma + \frac{1}{2}\psi_1 (\dot{H} + \frac{1}{2}a^2 H_{xx})}{1 - \frac{1}{2}\psi_1 H_x}}$$

and from these

$$\boxed{\begin{aligned} dC + SdH + dSdH &\equiv d(HS + C) - HdS \\ &= S\left(\sigma a H_x - \tfrac{1}{2}\psi_1 c \mu |aH_x|\right)dt \end{aligned}}$$

This shows by how much the gains-from-trade expression fails.

A useful little lemma (14/5/03)

1) Suppose that $Z \sim N(0, I)$ is n-dimensional Gaussian vector. Then we have the following

Lemma The expectation

$$E \exp\{-(Z^T a)(b^T Z) - c^T Z\}$$

is finite if and only if $1 + a^T b > |a|\,|b|$, and if this holds then the value of the expectation is

$$\frac{\exp(\frac{1}{2} c^T \Sigma c)}{\sqrt{\det \Sigma^{-1}}} = \frac{\exp(\frac{1}{2} c^T \Sigma c)}{\{(1+a^T b)^2 - |a|^2 |b|^2\}^{1/2}} \qquad \left[\Sigma \equiv (I + ab^T + ba^T)^{-1}\right]$$

Proof We have

$$(2\pi)^{n/2} E \exp(-Z^T a b^T Z - c^T Z) = \int \exp\{-\tfrac{1}{2} z^T z - z^T a b^T z - c^T z\}\, dz$$

$$= \int \exp\{-\tfrac{1}{2} z^T \Sigma^{-1} z - c^T z\}\, dz \qquad (\Sigma^{-1} \equiv I + ab^T + ba^T)$$

$$= \int \exp\left[-\tfrac{1}{2}(z - \Sigma c)^T \Sigma^{-1} (z - \Sigma c) + \tfrac{1}{2} c^T \Sigma c\right] dz$$

$$= (2\pi)^{n/2} \sqrt{\det \Sigma}\, \exp(\tfrac{1}{2} c^T \Sigma c)$$

provided Σ is positive-definite. Now Σ^{-1} has two eigenvectors

$$|b|\, a \pm |a|\, b$$

with eigenvalues $1 + b^T a \pm |a|\,|b|$; these are both positive if the stated condition holds, and all other evalues are 1. Thus

$$\det \Sigma^{-1} = (1 + a^T b)^2 - |a|^2 |b|^2$$

2) It is easy to extend the result to the case $Z \sim N(0, V)$; we get

$$\boxed{E \exp(-(Z^T a)(Z^T b) - Z^T c) = \frac{\exp\{+\tfrac{1}{2} c^T (V^{-1} + ab^T + ba^T)^{-1} c\}}{\{(I + a^T V b)^2 - a^T V a \,.\, b^T V b\}^{1/2}}}$$

$$k = -\frac{1}{2}\begin{pmatrix}\xi \\ \gamma\tilde{a}\end{pmatrix} \cdot \Sigma\Sigma^T \begin{pmatrix}\xi \\ \gamma\tilde{a}\end{pmatrix} + \frac{1}{2}\xi^2\sigma_j^2 + \rho + \gamma\tilde{a}\cdot\tilde{\mu} + \frac{1}{2}(\theta^T\tilde{a})^2\gamma^2$$

$$\Sigma = \begin{pmatrix}\sigma_j & 0 \\ \theta & \tilde{\sigma}\end{pmatrix} \qquad \text{if } \xi = \mu_j/\sigma_j^2$$

$$\Sigma\Sigma^T = \begin{pmatrix}\sigma_j^2 & \sigma_j\theta^T \\ \sigma_j\theta & \tilde{\sigma}\tilde{\sigma}^T + \theta\theta^T\end{pmatrix}$$

Abandoning assets: a meiopic policy (15/5/03)

1) We return to a multivariate version of the problem considered earlier, with n-vector

$$\delta_t = \sigma W_t + \mu t + \delta_0$$

of dividend processes, and n-vector A_t of amounts at time t. We expect that when it comes to abandoning assets, we should do better by acting now than by waiting even a little longer. Let's consider the abandonment of one of the assets, asset j, say, and let's write $\tilde{\delta}$ for the assets without the j^{th}. Let's re-express

$$\begin{pmatrix} d\delta^j \\ d\tilde{\delta} \end{pmatrix} = \begin{pmatrix} \sigma_j & 0 \\ \theta & \tilde{\sigma} \end{pmatrix} \begin{pmatrix} dB \\ d\tilde{W} \end{pmatrix} + \begin{pmatrix} \mu^j \\ \tilde{\mu} \end{pmatrix} dt$$

to present the correlation of the assets in a helpful way. Suppose we're at some value δ_0 where it's just critical whether or not to drop (some of) asset j. So let's consider how the objective would change if we were to wait until $\tau_\varepsilon \equiv \inf\{t : \delta^j_t < \delta^j_0 - \varepsilon\}$. We need to be able to compute things like

$$E \int_0^{\tau_\varepsilon} \exp\{-\rho t - \gamma \tilde{a}\cdot\tilde{\delta}_t - \lambda \delta^j_t\}\, dt$$

$$= E \int_0^{\tau_\varepsilon} \exp\{-\rho t - \gamma\tilde{a}\cdot(\tilde{\delta}_0 + \theta B_t + \tilde{\sigma}\tilde{W}_t + \tilde{\mu}t) - \lambda(\delta^j_0 + \sigma_j B_t + \mu^j t)\}\, dt$$

$$= E \int_0^{\tau_\varepsilon} \exp\{-(\rho + \gamma\tilde{a}\cdot\tilde{\mu} - \tfrac{1}{2}\gamma^2 \tilde{a}\cdot\tilde{\sigma}\tilde{\sigma}^T\tilde{a})t - \gamma\tilde{a}\cdot\tilde{\delta}_0 - \gamma\tilde{a}\cdot\theta(\delta^j_t - \delta^j_0 - \mu^j t)/\sigma_j - \lambda\delta^j_t\}\, dt$$

$$= E \int_0^{\tau_\varepsilon} \exp\{-\kappa t - \gamma\tilde{a}\cdot\tilde{\delta}_0 - \varphi\delta^j_t + \gamma\tilde{a}\cdot\theta\,\delta^j_0/\sigma_j\}\, dt$$

where

$$\kappa = \rho + \gamma\tilde{a}\cdot\tilde{\mu} - \tfrac{1}{2}\gamma^2\tilde{a}\cdot\tilde{\sigma}\tilde{\sigma}^T\tilde{a} - \gamma\tilde{a}\cdot\theta\,\mu^j/\sigma_j \quad ; \quad \varphi = \lambda + \gamma\tilde{a}\cdot\theta/\sigma_j .$$

Thus we want to compute

$$E \int_0^{\tau_\varepsilon} \exp(-\kappa t - \varphi\delta^j_t)\, dt$$

and as we found on p 8, the value of this is

$$\frac{\exp\{-\varphi(\delta^j_0 - \varepsilon)\}}{Q_j(-\varphi)}\left[e^{-\alpha_j\varepsilon} - e^{-\varphi\varepsilon}\right] \sim \frac{2e^{-\varphi\delta^j_0}}{\sigma_j^2(\varphi+\beta_j)}\,\varepsilon$$

where $Q_j(z) = \frac{1}{2}\sigma_j^2 z^2 + \mu^j z - \kappa$, and $-\alpha_j$ is the negative root of Q_j, β_j the positive one.

2) What would this meiopic policy give us in the case of 0-1 holdings of the asset? So we here are considering what happens if we hold all the assets up to some default time,

and then shut the thing down. We therefore compare the case $\lambda = 0$ with the case $\lambda = \gamma$. The critical value of δ_0^j now satisfies

$$\boxed{\exp(\gamma \delta_0^j) = \frac{\beta_j + \gamma \tilde{a}\cdot\theta/\sigma_j}{\gamma + \beta_j + \gamma \tilde{a}\cdot\theta/\sigma_j}}$$

(This depends on amounts of other assets held, but not on levels of their dividend processes)

In the case of just one asset, this agrees with what we got on p 24, WN <u>XXI</u>.

3) <u>What would this meiopic policy say if there was gradual dumping of assets?</u>

The infinitesimal value of delaying is

$$\frac{2\exp(-\varphi\delta_0^j)\,\varepsilon}{\sigma_j^2(\varphi + \beta_j)} \qquad , \; \varphi = \gamma q_j + \tilde{a}\cdot\theta/\sigma_j)$$

and for a given value of $\delta_0^j < 0$ we would seek the value of q_j (the amount to hold) at which this is minimised; we find

$$\boxed{q_j = -\frac{1}{\gamma}\left(\frac{1}{\delta_0^j} + \beta_j + \frac{\gamma\tilde{a}\cdot\theta}{\sigma_j}\right)} \quad \text{or} \quad \boxed{\delta_0^j = \frac{-1}{\gamma q_j + \gamma\tilde{a}\cdot\theta/\sigma_j + \beta_j}}$$

Again, in the case of a single asset, this agrees with the result of p 9. Once again, the critical level is affected by amounts held of other assets, but not by the values of other δ^i.

4) <u>How close could this be to optimal?</u>

The only assumption used in the above analysis is that while asset j falls from δ_0^j to $\delta_0^j - \varepsilon$, the holdings of other assets did not change. If we are in a region where only one asset is remotely likely to be knocked down, then the optimal behaviour has to be very close to this. However, we can actually deduce an inequality, because if we are at a level δ_0^j where (according to the above meiopic analysis) we would do better to wait, then we would certainly do better to wait if there was a possibility that we followed a better rule for the other assets! <u>So the optimal intervention levels will always be below those given by the meiopic policy.</u>

5) In the case of two assets $\sigma\sigma^T = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}$, $v_{12} = \tilde{\rho}\sqrt{v_{11}v_{22}}$, if β_j^0 denotes the positive root of $\frac{1}{2}v_{jj}z^2 + \mu_j z - \rho$, β_j denotes the positive root of $\frac{1}{2}v_{jj}z^2 + \mu_j z - \kappa_j$, then the critical level for asset 1 on its own exceeds the critical level with both iff

$$\beta_1^0 > \beta_1 + \gamma\tilde{\rho}\sqrt{v_{22}/v_{11}} \iff \sqrt{\mu_1^2 + 2\rho v_{11}} - \sqrt{\mu_1^2 + 2\kappa_1 v_{11}} > \gamma v_{12}$$

In the case of uncorrelated assets, this is the condition $\gamma v_{22} > 2\mu_2$.

$X = \alpha \cdot 1 + \bar{\beta} V + \beta \cdot \varepsilon + \psi \cdot \nabla$ $\qquad \bar{\beta} \equiv \beta \cdot 1$

Extending the single-period Kyle model (16/5/03)

1) There are a number of places where the Kyle model can be criticised: When does a noise trader become an informed trader? Should the informed traders be allowed to add noise to their demand? Why are quantities demanded insensitive to price? With risk-neutral informed traders, what should one do about the multiplicity of solutions? Is the modelling of the market-maker over-simplified?

Here is some variant of the Kyle model, developed in a single period, which may answer some of the criticisms.

2) There are J agents. Agent j has CARA utility with coef of ARA equal to γ_j. He sees the true value V of the single asset corrupted by noise ε_j, so he sees $V+\varepsilon_j$. He plans to offer a demand function

$$\alpha_j + \beta_j(V+\varepsilon_j) + \psi_j \nu_j - \lambda_j p$$

as a function of price, where the ν_j are IID $N(0,1)$ randomisations of demand, and the constants $\alpha_j, \beta_j, \psi_j, \lambda_j$ are to be chosen. The total demand is thus

$$D = \sum_j (\alpha_j + \beta_j(V+\varepsilon_j) + \psi_j \nu_j) - \sum_j \lambda_j p \equiv X - \Lambda p$$

The market-maker begins with initial endowment θ_0 of the asset, and he sees some signal $V+\varepsilon_M$, as well as some noisy observation $X+\eta$ of the random component of demand. Once the true value is revealed, he will have

$$Dp + (\theta_0 - D)V = (X-\Lambda p)p + (\theta_0 - X + \Lambda p)V$$

and his objective is to pick p in such a way as to

$$\min \; E \; -\exp\left(-\gamma_M \{ Dp + (\theta_0 - D)V \} \right)$$

3) Let's work out what the market-maker should do. Given his information,

$$\begin{pmatrix} X \\ V \end{pmatrix} \sim N\left(\begin{pmatrix} \hat{X} \\ \hat{V} \end{pmatrix}, S^2 \right)$$

where S is pds, thus $\begin{pmatrix} X \\ V \end{pmatrix} = \begin{pmatrix} \hat{X} \\ \hat{V} \end{pmatrix} + S\begin{pmatrix} \xi \\ \eta \end{pmatrix}$ where ξ, η are $N(0,1)$ indept. Write $\zeta = (\xi,\eta)^T$, $a = S^T e_1$, $b = S^T e_2$; the quantity to be minimised is

$$e^{\gamma_M \Lambda p^2} E \exp\left(-\gamma_M \{ p(\hat{X}+a\cdot\zeta) + (\theta_0 + \Lambda p)(\hat{V}+b\cdot\zeta) - (\hat{X}+a\cdot\zeta)(\hat{V}+b\cdot\zeta) \} \right)$$

$$= e^{\gamma_M \Lambda p^2} E \exp\left(-\gamma_M \left(p\hat{X} + (\theta_0+\Lambda p)\hat{V} - \hat{X}\hat{V} + (pa + (\theta_0+\Lambda p)b - \hat{X}b - \hat{V}a)\cdot\zeta - (a\cdot\zeta)(b\cdot\zeta) \right) \right)$$

If $\tilde{\Sigma} = S\Sigma S = \left(S^{-2} - \gamma_M (e_1 e_2^T + e_2 e_1^T)\right)^{-1}$ then

$$p = \frac{\gamma_M (e_1 + \Lambda e_2) \cdot \tilde{\Sigma} \left(\hat{V} e_1 + (\hat{X} - \theta_0) e_2 \right) - \hat{X} - \Lambda \hat{V}}{\gamma_M (e_1 + \Lambda e_2) \cdot \tilde{\Sigma} (e_1 + \Lambda e_2) + 2\Lambda}$$

$$\Sigma_j = \left(I + \gamma_j A_j^T \varphi_j w^T + \gamma_j w \varphi_j^T A_j \right)^{-1}$$

Now we can use the Lemma on p 16, with

$$c = \gamma_M \left(p a + (\theta_0 + \Lambda p) b - \hat{V} a - \hat{X} b \right)$$

$$\Sigma = \left(I - \gamma_M \, (a b^T + b a^T) \right)^{-1}$$

and thereby find the equivalent form of the problem, namely to minimise over p the expression

$$\gamma_M \Lambda p^2 + \tfrac{1}{2} \, c \cdot \Sigma c - \gamma_M \, p (\hat{X} + \Lambda \hat{V}) .$$

We find

$$\boxed{p = \frac{\gamma_M \, (a + \Lambda b) \cdot Z (\hat{V} a + \hat{X} b - \theta_0 b) - \hat{X} - \Lambda \hat{V}}{\gamma_M \, (a + \Lambda b) \cdot Z (a + \Lambda b) + 2\Lambda}}$$

which is linear in estimated random demand components, estimated true value, and initial position.

4) Let's take a slightly more abstract approach, and write

$$Z = \begin{pmatrix} V \\ \varepsilon \\ \nu \\ \theta_0 \\ \eta \end{pmatrix} \equiv \Sigma_Z^{1/2} \, \tilde{Z} \qquad \tilde{Z} \sim N(0, I)$$

for the zero-mean (wlog) Gaussian vector which encapsulates all the noise. We know that the market maker will set his price as a constant plus some linear combination of θ_0, $V + \varepsilon_M$, $X + \eta$, so

$$p = \alpha_M + \varphi_M^T \, A_M \tilde{Z} \qquad \text{[In fact, the above analysis shows that } \alpha_M = 0]$$

where $A_M \tilde{Z} = \begin{pmatrix} \theta_0 \\ X + \eta \\ V + \varepsilon_M \end{pmatrix}$. Similarly, agent j's demand is a linear combination of $(V + \varepsilon_j, \nu_j, p)$, so that agent j's demand is

$$\alpha_j + \varphi_j^T A_j \tilde{Z}$$

where $A_j \tilde{Z} = (V + \varepsilon_j, \nu_j, p)^T$. Let's set $V - p \equiv w^T \tilde{Z}$. Once the exact form of the price has been determined, agent j's problem is to

$$\min_{\alpha_j, \varphi_j} E \exp\left\{ - \gamma_j \, (\alpha_j + \varphi_j^T A_j \tilde{Z}) \, w^T \tilde{Z} \right\}$$

$$= \min_{\alpha_j, \varphi_j} \frac{\exp\left\{ \tfrac{1}{2} \gamma_j^2 \alpha_j^2 \, w^T \Sigma_j \, w \right\}}{\left\{ (1 + \gamma_j \varphi_j^T A_j w)^2 - |w|^2 \cdot \gamma_j^2 |A_j^T \varphi_j|^2 \right\}^{1/2}}$$

Note $|w|^2 - w^T A_j^T (A_j A_j^T)^{-1} A_j w = \operatorname{var}(V - p \mid V + \xi_j, \nu_j, p)$, so we get

$$\varphi_j^T A_j \tilde{Z} = \frac{E(V - p \mid V + \xi_j, \nu_j, p)}{\gamma_j \operatorname{var}(V - p \mid V + \xi_j, \nu_j, p)}$$, obviously!

$$= \frac{V + \xi_j - p - E(\xi_j \mid V + \xi_j, \nu_j, p)}{\gamma_j \operatorname{var}(\xi_j \mid V + \xi_j, \nu_j, p)}$$

$$K^{-1} = \frac{1}{k \sigma_V^2} \begin{pmatrix} \sigma_V^2 & -\bar{\beta} \sigma_V^2 \\ -\bar{\beta} \sigma_V^2 & \bar{\beta}^2 \sigma_V^2 + k \end{pmatrix} \qquad \left(k = |\psi|^2 + \sum \beta_j^2 \sigma_j^2 \right)$$

$$S \Sigma S = \left(\Delta^{-1} + K^{-1} - \gamma_M (e_1 e_2^T + e_2 e_1^T) \right)^{-1} = \begin{pmatrix} \sigma_\eta^{-2} + 1/k & -\gamma_M - \bar{\beta}/k \\ -\gamma_M - \bar{\beta}/k & \sigma_M^{-2} + \bar{\beta}^2/k + \frac{1}{\sigma_V^2} \end{pmatrix}^{-1}$$

So from this we immediately get

$$\boxed{\alpha_j = 0}$$

and after some further calculations

$$\boxed{\varphi_j = \frac{(A_j A_j^T)^{-1} A_j w}{\gamma_j \{ |w|^2 - w^T A_j^T (A_j A_j^T)^{-1} A_j w \}}}$$

Note: $A_j w = E \begin{pmatrix} V+\varepsilon_j \\ \nu_j \\ p \end{pmatrix} (V-p)$

5) If we try to flesh this out a bit, assuming the ε, ν, η noises are indep of V, we have the covariance of

$$\begin{pmatrix} X \\ V \\ X+\eta \\ V+\varepsilon_M \end{pmatrix} \text{ is } \begin{pmatrix} K & K \\ K & K+\Delta \end{pmatrix}$$

$$K \equiv \begin{pmatrix} \bar\beta^2 \sigma_V^2 + \Sigma \beta_j^2 \sigma_j^2 + \Sigma \psi_j^2, & \bar\beta \sigma_V^2 \\ \bar\beta \sigma_V^2 & \sigma_V^2 \end{pmatrix}$$

$$\Delta = \begin{pmatrix} \sigma_\eta^2 & \cdot \\ \cdot & \sigma_M^2 \end{pmatrix}$$

$$\bar\beta \equiv \Sigma \beta_j$$

so that the conditional means of X, V given what MM sees will be

$$K (K+\Delta)^{-1} \begin{pmatrix} X+\eta \\ V+\varepsilon_M \end{pmatrix} = (I + \Delta K^{-1})^{-1} \begin{pmatrix} X+\eta \\ V+\varepsilon_M \end{pmatrix} = \begin{pmatrix} \hat X \\ \hat V \end{pmatrix}$$

and the conditional covariance is

$$K - K(K+\Delta)^{-1} K = \Delta (I + K^{-1}\Delta)^{-1} = \Delta^{\frac12} (I + \Delta^{\frac12} K^{-1} \Delta^{\frac12})^{-1} \Delta^{\frac12}$$

6). Suppose that the Market maker chooses to make a price

$$p = a_1 (X+\eta) + a_2 (V+\varepsilon_M) + a_3 \theta_0$$

$$\equiv \tilde a_1 \hat X + \tilde a_2 \hat V + a_3 \theta_0$$

We then have

$$p = a_1 (\bar\beta V + \beta\cdot\varepsilon + \psi\cdot\nu + \eta) + a_2 (V+\varepsilon_M) + a_3 \theta_0$$

$$= (a_1 \bar\beta + a_2) V + a_1 \beta\cdot\varepsilon + a_1 \psi\cdot\nu + a_1 \eta + a_2 \varepsilon_M + a_3 \theta_0$$

so

$$\operatorname{var}(p) \equiv \sigma_p^2 = (a_1\bar\beta + a_2)^2 \sigma_V^2 + a_1^2 k + a_1^2 \sigma_\eta^2 + a_2^2 \sigma_M^2 + a_3 \sigma_\theta^2 + 2a_3 (a_1\bar\beta + a_2) \sigma_{\theta V}$$

$$\left(k \equiv |\psi|^2 + \Sigma \beta_j^2 \sigma_j^2 \right)$$

From the viewpoint of agent j, the thing that matters is the covariance matrix of $(\xi_j, V+\xi_j, \nu_j, p)$, which is

$$J = \begin{pmatrix} \sigma_j^2 & \sigma_j^2 & 0 & a_1\beta_j\sigma_j^2 \\ \sigma_j^2 & \sigma_V^2+\sigma_j^2 & 0 & (a_1\bar\beta+a_2)\sigma_V^2 + a_3\sigma_{V\theta}^2 + a_1\beta_j\sigma_j^2 \\ 0 & 0 & 1 & a_1\psi_j \\ a_1\beta_j\sigma_j^2 & \cdots & a_1\psi_j & \sigma_p^2 \end{pmatrix} \equiv \begin{pmatrix} \sigma_j^2 & v_0 \\ v_0^T & J_0 \end{pmatrix}$$

Agent j's demand is

$$\frac{E(V-p \mid V+\xi_j, \nu_j, p)}{\gamma_j \operatorname{var}(V-p \mid V+\xi_j, \nu_j, p)} = \frac{V+\varepsilon_j - p - E(\xi_j \mid V+\xi_j, \nu_j, p)}{\gamma_j \operatorname{var}(\xi_j \mid V+\xi_j, \nu_j, p)} = \varphi \cdot \begin{pmatrix} V+\xi_j \\ \nu_j \\ p \end{pmatrix}$$

which (when worked through) gives us agent j's demand $\beta_j(V+\xi_j) + \psi_j\nu_j - \lambda_j p$. If we solve $\varphi_2 = \psi_j$ for ψ_j, it leads to the conclusion that

$$\varphi_1 = \beta_j = (1+\beta_j\gamma_j\sigma_j^2)/\gamma_j\sigma_j^2$$

which is absurd; so, in fact, $\boxed{\psi_j = 0 \quad \forall j}$

Liquidity modelling: some general observations (29/5/03)

A common structure to various discrete-time models of liquidity/feedback effects is that if S is the current price, and h is the amount required by the hedger, then next period the price has moved to

$$S \,.\, \tilde{F}(h, \xi_n) \qquad\qquad (\tilde{F} > 0)$$

and the cost to the hedger of making this trade is

$$S \,.\, \tilde{C}(h, \xi_n)$$

for functions $\tilde{C}, \tilde{F}$ which are quite general (assume $\tilde{C}(0, \cdot) = 0$) and inputs ξ_n which are IID from one period to the next. Suppose we make the requirement

$$\boxed{\begin{array}{l} \text{A (deterministic) sequence } h_1, \dots, h_N \text{ of demands such that } \sum h_j \geq 0 \\ \text{should on average cost a non-negative amount} \end{array}}$$

where does this lead?

If $C(h) \equiv E\,\tilde{C}(h, \xi_1)$, $F(h) \equiv E\,\tilde{F}(h, \xi_1)$, then we shall have the mean cost of the sequence $(h_1, \dots, h_N) = h$ is

$$V_N(h) = \sum_{j=1}^{N} C(h_j) \prod_{r<j} F(h_r)$$

Notice that (Scilab notation!)

$$\boxed{V_N([h; x]) = C(h) + F(h)\, V_{N-1}(x)}$$

so that if we set

$$v_n(t) \equiv \inf \{ V_n(x) : x \cdot 1 = t \}$$

we shall have

$$\boxed{v_N(t) = \inf_h \{ C(h) + F(h)\, v_{N-1}(t-h) \}}$$

<u>Assume</u> $\tilde{C}(\cdot, \xi)$ <u>is increasing</u>; then $v_1 = C$ is increasing, and so by induction is every v_N. Notice also that (taking $h=0$)

$$\boxed{v_N(t) \leq F(0)\, v_{N-1}(t)}$$

Our requirement is that $v_N(0) \geq 0$ for all N; together with $v_1(0) = C(0) = 0$, this tells us that

$$v_N(0) = 0 \qquad \text{for all } N.$$

Now we exploit the fact that $V_2((h, -h)) = C(h) + F(h)\, C(-h) \geq 0$

to see that for $h>0$

$$\frac{C(h)}{-C(-h)} \geq F(h) \quad , \text{and} \quad \frac{-C(-h)}{C(h)} \leq F(-h)$$

Thus

$$\frac{C(h)}{-C(-h)} \geq \max\left\{F(h), \frac{1}{F(-h)}\right\}$$

and letting $h \downarrow 0$, <u>assuming that C possesses left + right derivatives at 0</u>

$$\boxed{\frac{C'(0+)}{C'(0-)} \geq \max\left\{F(0), \frac{1}{F(0)}\right\}}$$

One simple consequence of this is

$$C'(0+) \geq C'(0-).$$

Another is

<u>If</u> C <u>is differentiable at 0</u>, <u>then</u> $F(0) = 1$

Extending this analysis to $\underline{h} = (h, h, \ldots, h, -nh)$, we see that $(F(0) \neq 1)$

$$C(h)\frac{F(h)^n - 1}{F(h) - 1} + C(-nh)F(h)^n \geq 0$$

so that
$$\frac{nC(h)}{-C(-nh)} \geq n\frac{F(h)^n(F(h)-1)}{F(h)^n - 1}$$

Let $h \downarrow 0$ to see that

$$\frac{C'(0+)}{C'(0-)} \geq n\frac{F(0)^n(F(0)-1)}{F(0)^n - 1}$$

If $n \to \infty$, we see that we <u>cannot</u> have $F(0) > 1$. Similar considerations on the sequence $(-nh, h, \ldots, h)$ shows that we cannot have $F(0) < 1$. This would force

$$\boxed{F(0) = 1.}$$

Even in the classical binomial model, where F doesn't depend on h, we don't need to have $F(0) = 1$!! So it looks like the 'requirement' is rather silly.

Liquidity: another try (3/6/03)

1) Suppose we try some discrete-time approximation to a model of liquidity effects, with time-step Δt. Let p_n denote the log-price of the asset in period n, and suppose that in any period $\lambda_a \Delta t$ shareholders consider whether to sell their shares, and $\lambda_b \Delta t$ agents consider whether to buy shares. There is a hedger who comes to the market wishing to buy ΔH_n shares, and the log-price $\tilde{p}_n$ at which he will buy is determined by

$$\lambda_a \Delta t \; A(\tilde{p}_n - p_{n-1} - \xi_a) = \lambda_b \Delta t \; B(\tilde{p}_n - p_{n-1} - \xi_b) + \Delta H_n$$

where (ξ_a, ξ_b) are random shocks to the perceived ask/bid prices, and the functions A and $-B$ are non-negative strictly increasing and C^1. The impact on the log-price is determined by

$$\Lambda_a A(p_n - p_{n-1} - \xi_a) = \Lambda_b B(p_n - p_{n-1} - \xi_b) + \Delta H_n$$

The interpretation of this is that the impact of ΔH_n on the market price p_n is relatively limited, because if we considered the whole of the issued shares, ΔH_n is a tiny fraction of that, but ΔH_n is a substantial fraction of the shares which were available to trade in the n^{th} time period.

[Aside: it may be a bit objectionable to separate out the effect of ΔH_n from all other shocks to supply in that same period. We could therefore add a term ΔX_n to the RHS of both equations to deal with that; we would have to suppose that ΔX_n was $O(\Delta t)$, otherwise the effect of ΔH_n would be completely swamped. This will also show that we have to use some $O(\sqrt{\Delta t})$ random effect in shifting the perceived level of price, otherwise we can't expect any diffusion-type behaviour at the end. It seems reasonable that ΔX_n, the demand in a period of length Δt, should be $O(\Delta t)$, rather than $O(\sqrt{\Delta t})$!]

2) Notice that (Λ_a, Λ_b) might be different from (λ_a, λ_b), and could in principle be allowed to vary randomly with time. For now, let's suppose that (Λ_a, Λ_b) are fixed and that

$$\Lambda_a A(0) = \Lambda_b B(0)$$

so as to rule out large jumps in p_n.

The wealth equation for the hedging agent is

$$\Delta(HS+C)_n = H_n S_n - H_{n-1} S_{n-1} + \Delta C_n = H_{n-1} \Delta S_n + \Delta H_n (S_n - e^{\tilde{p}_n})$$

Let's just for now suppose $\xi_a = \xi_b$ for simplicity of exposition, and write $\varphi = \lambda_a A - \lambda_b B$, $\Phi = \Lambda_a A - \Lambda_b B$, with inverses ψ, Ψ respectively. Then

$$\tilde{p}_n - p_{n-1}\xi = \psi\left(\frac{\Delta H_n}{\Delta t} + \frac{\Delta X_n}{\Delta t}\right)$$

$$p_n - p_{n-1} - \xi = \Psi\left(\Delta H_n + \Delta X_n\right)$$

so that in the limit we expect the wealth equation for the hedger to read

$$\boxed{dW_t = H_t\, dS_t - \dot{H}_t S_t f(\dot{H}_t)\, dt}$$

where $f(R) = E\,\psi\left(R + \frac{\Delta X}{\Delta t}\right)$. This is like imposing a 'transaction' cost which is a function of $\dot{H}$

3) If we took $\varphi(x) = \sinh(\mu x)$, then $\psi(y) = \frac{1}{\mu}\log\left(\sqrt{1+y^2} + y\right)$.

Another example:

$$\varphi(x) = \frac{\lambda_a e^{\mu x} - \lambda_b}{1 + e^{\mu x}} \qquad \left(\text{so } A(x) = \frac{e^{\mu x}}{1+e^{\mu x}},\quad B(x) = 1 - A(x)\right)$$

$$\text{then } \psi(y) = \frac{1}{\mu}\log\left(\frac{\lambda_b + y}{\lambda_a - y}\right)$$

If $\varphi(x) = \int_{-\infty}^{x} \frac{\kappa\, dy}{\cosh y} = 2\kappa \tan^{-1}(e^x)$, then $\psi(y) = \log \tan(y/2\kappa)$.

4) Suppose we now consider the Merton problem

$$\max E \int_0^\infty e^{-\rho t}\, U(C_t)\, dt$$

with

$$\begin{cases} dW_t = rW_t\, dt + H_t(dS_t - rS_t\, dt) - C_t\, dt - \varepsilon \dot{H}_t S_t f(\dot{H}_t)\, dt \\ dS_t = S_t(\sigma\, dW_t + \mu\, dt) \end{cases}$$

and $U'(x) = x^{-R}$ as usual. As $\varepsilon \downarrow 0$, we expect to see something looking more and more like the Merton solution, where we get

$$H_t S_t = \pi_M W_t \qquad \left(\pi_M = \frac{\mu - r}{\sigma^2 R}\right)$$

$$V(w) = \gamma_*^{-R}\, U(w) \qquad \left(\gamma_* = \frac{\rho + (R-1)(r + \frac{1}{2}\sigma^2 \pi_M^2 R)}{R}\right)$$

$$C_t = \gamma_* W_t$$

If we write $v(z,\pi) = f(z, H)$ then the optimality conditions are

$$c^* = f_z^{-1/R}, \quad h^* = f_h / 2\varepsilon f_z$$

To prevent bankruptcy, we shall always need to have $H \geq 0$, and $Y \equiv z - H > 0$; we have

$$dY = Y(-\sigma d\tilde{W} - (\lambda - \sigma^2 R)dt) - (h + \varepsilon h f(h) + c)\, dt$$

HJB would say

$$\sup_{c,h}\left[U(c) - \tilde{\rho}\, v + \tfrac{1}{2}\sigma^2 Y^2 v_{YY} - \left(h + \varepsilon h f(h) + c + (\lambda - \sigma^2 R)Y\right) v_Y + h v_H \right] = 0$$

Check with Maple! $\longrightarrow$

By scaling, $V(w,H,S) = S^{1-R}\, V(w/S, H, 1) \equiv S^{1-R}\, v(z,\pi)$, where we define the new variables

$$z \equiv \frac{w}{S}, \quad \pi = HS/w = H/z$$

This reduces the problem somewhat. We obtain $(h \equiv \dot H,\ c \equiv C/S)$

$$\boxed{\begin{aligned} dz &= \sigma(H-z)\,dW - \{(\sigma^2-\lambda)(H-z) + \varepsilon h f(h) + c\}\,dt \\ d\pi &= \sigma\pi(1-\pi)\,dW - \pi(1-\pi)(\sigma^2\pi-\lambda)\,dt + \left(\pi c + h + \varepsilon\pi h f(h)\right) dt/z \end{aligned}}$$ $(\lambda = \mu - r)$

5) We can also make the reduction at the stochastic level: the objective is

$$E\int_0^\infty \exp(-\rho t)\, U(C_t)\,dt$$

$$= E\int_0^\infty \exp(-\rho t)\, S_t^{1-R}\, U(c_t)\,dt$$

$$= \tilde E\int_0^\infty \exp\{-\tilde\rho t\}\, U(c_t)\,dt \cdot S_0^{1-R}$$

where $\tilde\rho \equiv \rho - (1-R)(\mu - \tfrac12\sigma^2 R)$, and under $\tilde P$, $W_t = \tilde W_t + \sigma(1-R)t$, so

$$\begin{cases} dz = (H-z)\{\sigma\,d\tilde W + (\lambda - \sigma^2 R)\,dt\} - \varepsilon h f(h)\,dt - c\,dt \\ dH = h\,dt \end{cases}$$

If we have

$$\tilde v(z,H) = \sup \tilde E\left[\int_0^\infty e^{-\tilde\rho t}\, U(c_t)\,dt \,\Big|\, z_0 = z,\ H_0 = H\right]$$

under the dynamics above for z, H, then in the Merton situation we would expect to find

$$S_0^{1-R}\,\tilde v(z,H) = \gamma_*^{-R}\, U(w_0), \quad \text{so} \quad \tilde v(z,H) = \gamma_*^{-R}\, U(z)$$

6) If we express $V(w,H,S) = S^{1-R}\, V(w/S, H, 1) \equiv S^{1-R} F(z,H)$, then we get for the HJB

$$0 = \sup_{c,h}\left[\frac{c^{1-R}}{1-R} - \rho F + \tfrac12\sigma^2(H-z)^2 F_{zz} - \left((\sigma^2 R - \lambda)(H-z) + c + \varepsilon h f(h)\right)F_z + (1-R)(\mu - \tfrac12\sigma^2 R)F + hF_H\right]$$

so that

$$\boxed{\begin{aligned} c^* &= F_z^{-1/R} \\ \varepsilon F_z\left[f(h^*) + h^* f'(h^*)\right] &= F_H \end{aligned}}$$

(Since we shall assume f is positive increasing, there may be an issue about existence/uniqueness of h^*)

Assuming $f(k) = h$

simplifies things somewhat : we get

$$\boxed{h^* = F_H / 2\varepsilon F_z}$$

and then

$$\boxed{-\frac{(F_z)^{1-1/R}}{1-1/R} - \rho F + \tfrac{1}{2}\sigma^2 (H-z)^2 F_{zz} + (H-z)(\lambda - \sigma^2 R) F_z + \frac{F_H^2}{4\varepsilon F_z} + (1-R)(\mu - \tfrac{1}{2}\sigma^2 R) F = 0}$$

We know that as $\varepsilon \to 0$ we get $F(z, H) \to \gamma_*^{-R} \, z^{1-R}/(1-R)$, so we may propose a solution of the form

$$F(z, H) = F_0(z) \exp\{ \sum_{m \geq 0} \delta^m G_m(z, y) \} \qquad \left[F_0(z) \equiv \gamma_*^{-R} \frac{z^{1-R}}{1-R}\right]$$

where $y \equiv \delta (H - \pi_M z)$, $\boxed{\delta = \varepsilon^{1/4}}$, it appears from heuristic argument.

Possibly useful: If we write $V(w, H, S; \varepsilon)$ to emphasise dependence on the small parameter ε, we have a further scaling relationship; for all $\lambda > 0$

$$\boxed{V(\lambda w, \lambda H, S; \lambda^{-1}\varepsilon) = \lambda^{1-R} V(w, H, S; \varepsilon)}$$

so

$$\boxed{F(\lambda z, \lambda H; \lambda^{-1}\varepsilon) = \lambda^{1-R} F(z, H; \varepsilon)}$$

If we now suppose that we have a solution where $\varepsilon = \delta^\nu$, and we have an analytic expansion

$$F(z, H; \varepsilon) = \sum_{k \geq 0} \delta^k G_k(z, H),$$

then this leads us to the scaling relations

$$\boxed{G_k(\lambda z, \lambda H) = \lambda^{k/\nu + 1 - R} G(z, H)}$$

Thus if $g_k(x) = G_k(1, x)$, we shall have

$$\boxed{G_k(z, H) = z^{1-R+k/\nu} g_k(H/z)}$$

This can then be worked through nicely with Maple! (/WORK/SURB/LIQUID/lqd2.mws)

What we find is $\boxed{\nu = 2}$

Another variant is where r_t is an OU diffusion. This can also be handled in the same fashion.

Markov-modulated asset returns (23/6/03)

At the AMS Summer Conference, there seems to be a lot of interest in various Markov-modulated asset processes, where vol, rate of return and even the interest rate depends on some finite state Markov chain X

1) Wendell Fleming reviewed some results where we suppose wealth eq^n is

$$dw_t = r_t w_t\,dt + \theta_t\{\sigma_t\,dW_t + (\mu_t - r_t)\,dt\} - c_t\,dt$$

where $r_t = r(X_t)$, $\theta_t = \theta(X_t)w_t$, etc, and $C_t = w_t\,c(X_t)$. If we take the standard Merton objective

$$\max E\int_0^\infty e^{-\rho t}\,U(C_t)\,dt$$

where U is CRRA, then we get the usual scaling argument to show the form that optimal consumption + investment must take, namely, proportional to wealth. The objective is

$$E\int_0^\infty e^{-\rho t}\,w_t^{1-R}\,\frac{c_t^{1-R}}{1-R}\,dt$$

and if $z_t \equiv \log w_t$ we have the nice story

$$dz_t = \theta(X_t)\cdot\{\sigma(X_t)\,dW_t + (\mu-r)(X_t)\,dt\} + \{r(X_t) - c(X_t) - \tfrac{1}{2}|\sigma^T\theta(X_t)|^2\}\,dt$$

so we can write

$$w_t^{1-R} = e^{(1-R)z_t} = Z_t\,\exp\left[-\int_0^t a(X_s)\,ds\right]$$

where Z is the change-of-measure martingale

$$dZ_t = Z_t\,\theta(X_t)\cdot\sigma(X_t)\,dW_t$$

and $-a(x) = \{\theta(x)\cdot(\mu-r)(x) + r(x) - c(x) - \frac{R}{2}|\sigma^T\theta(x)|^2\}(1-R)$. Under the new measure, we still have the Markov chain with the same jumps, so our objective is to maximise

$$\boxed{\tilde{E}\int_0^\infty e^{-\rho t - A_t}\,\frac{c(X_t)^{1-R}}{1-R}\,dt = (\rho + a - Q)^{-1} f}$$

where $f(x) = \dfrac{c(x)^{1-R}}{1-R}$

$A_t = \int_0^t a(X_s)\,ds$

The optimisation is over θ and c. Looking at the LHS, it is clear that we always want

$$\boxed{\theta^*(x) = \sigma(x)^{-1}(\mu(x) - r(x))/R}$$

Optimising over c is also possible. If we set

$$b(x) \equiv \rho(x) + \left(\frac{R-1}{2R}\right)(\mu-r)(x)\cdot(\sigma\sigma^T)^{-1}(x)\,(\mu-r)(x)$$

then the objective to maximise is

$$\left(b + (1-R)c - Q \right)^{-1} f$$

where $f(x) = c(x)^{1-R}/(1-R)$. Differentiating with respect to c and doing some algebra, we obtain the equation

$$\boxed{R\ c^{1-R} = (b-Q)\, c^{-R}}$$

2) Kurt Helmes studies the same dynamics, but without consumption, and asks (among other things) to have for a given start value the best interval (a,b) for stopping on exit; so (one-dimensional here) with z denoting log price,

$$dz_t = \sigma(X_t)\, dW_t + \mu(X)\, dt \quad , \ z_0 = 0$$

we want to let $\tau = \inf\{ t : z_t \notin (\alpha,\beta) \}$ and then choose α, β to maximise

$$E\left[\exp\left(-\int_0^\tau r(X_s)\,ds\right)\ e^{z_\tau} \right]$$

But this is back to the good old noisy Wiener-Hopf story; for f to give

$$\exp\left(\lambda z_t - \int_0^t r(X_s)\,ds \right) f(X_t) \quad \text{is a martingale}$$

we need

$$(Q - r) f + \lambda \mu f + \tfrac{1}{2}\lambda^2 \sigma^2 f = 0$$

So if we write $g = \lambda f$, we obtain the matrix equation

$$\begin{pmatrix} 0 & I \\ -2\sigma^{-2}(Q-r) & -2\sigma^{-2}\mu \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = \lambda \begin{pmatrix} f \\ g \end{pmatrix}$$

So this system has $2n$ eigenvector / eigenvalue pairs, and we just want to make some linear combination of the basic martingales to be equal to e^{z_τ} at exit. This allows us to compute for each α, β what the payoff is, then (numerical) max over α, β.

WH for phase-type compound Poisson processes from WH for Matrices (24/6/03)

This is something which has always appeared possible in principle, but I've never worked out the details before. But it seems to be quite OK. Taking determinants is the key trick!

1) Take $E = E_- \cup \{0\} \cup E_+$ as the statespace of the Markov chain, $V(x) = 1$ $(x \in E_+)$; $= -1$ $(x \in E_-)$ and $V(0) = 0$. We partition the Q-matrix as

$$Q = \begin{array}{c} \left(\begin{array}{c|c|c} A & b & 0 \\ \hline u & -q & w \\ \hline 0 & c & D \end{array}\right) \end{array} \begin{array}{c} E_+ \\ 0 \\ E_- \end{array}$$

Assume $Q1 = -\eta \, I_{\{0\}}$, so that there is killing only in the zero state. We write $\boxed{q_0 = q - \eta}$

As usual, define $\tau_t^\pm = \inf\{s : \pm \varphi_s > t\}$, $Y_t^\pm = X(\tau_t^\pm)$, $\Pi_+(i,j) = P^i(Y_0^+ = j)$ for $i \in E_-$, $j \in E_+$, etc.

The additive functional $\varphi_t = \int_0^t V(X_s)\,ds$ is (very close to) a phase-type compound Poisson process - if we only view it when $X=0$, then it is exactly such. The characteristic exponent is

$$\psi(i\theta) = u\left((-i\theta - A)^{-1}b - 1\right) + w\left((i\theta - D)^{-1}c - 1\right)$$

$$= u(-i\theta - A)^{-1}b + w(i\theta - D)^{-1}c - q_0$$

The generators of $Y^\pm$ can also be expressed in terms of Π_+; but let's derive the expression for the WH factors of the CPP. If T denotes the $\exp(\eta)$ time, then

$$E\left[e^{-\lambda \bar{Z}(T)}\right] = \sum_{n \ge 0} \left\{ \frac{(u + w\Pi_+)}{q} (\lambda - A)^{-1} b \right\}^n \left(1 - \frac{u + w\Pi_+}{q} 1\right)$$

$$= \frac{q - u1 - w\Pi_+ 1}{q - (u + w\Pi_+)(\lambda - A)^{-1} b}$$

So the WH factors in these terms are

$$\boxed{\begin{aligned} \psi_+(i\theta) &\equiv E\, e^{i\theta \bar{Z}(T)} = \frac{\eta + w(I - \Pi_+)1}{\eta + q_0 - (u + w\Pi_+)(-i\theta - A)^{-1} b} \\ \psi_-(i\theta) &\equiv E\, e^{i\theta \underline{Z}(T)} = \frac{\eta + u(I - \Pi_-)1}{\eta + q_0 - (w + u\Pi_-)(i\theta - D^{-1})c} \end{aligned}}$$

The generators of $Y^\pm$ can be expressed

$$\boxed{G_+ = A + b q^{-1}(u + w\Pi_+), \quad G_- = D + c q^{-1}(w + u\Pi_-)}$$

2) The fact that $V(0)=0$ makes it hard to run the usual WH story, so let's look at the chain only when it's in $E_+ \cup E_-$, where it has Q-matrix

$$\begin{pmatrix} A + bq^{-1}u & bq^{-1}w \\ cq^{-1}u & D + cq^{-1}w \end{pmatrix} \equiv \bar{Q}$$

and we have

$$V^{-1}\bar{Q} = S\begin{pmatrix} G_+ & \cdot \\ \cdot & -G_- \end{pmatrix} S^{-1} \qquad \text{where } S = \begin{pmatrix} I & \Pi_- \\ \Pi_+ & I \end{pmatrix}$$

so that

$$V^{-1}\bar{Q} + i\theta = S\begin{pmatrix} G_+ + i\theta & \cdot \\ \cdot & -G_- + i\theta \end{pmatrix} S^{-1}$$

and hence

$$\det(V^{-1}\bar{Q} + i\theta) = \det(G_+ + i\theta)\det(-G_- + i\theta)$$

It will turn out that this (or something similar) is the WH factorisation. Recall that if x is a column vector, y a row vector $yx \neq -1$, then

$$(I + xy)^{-1} = I - \frac{xy}{1+yx} \quad , \det(I + xy) = 1 + yx$$

We have

$$V^{-1}\bar{Q} + i\theta = \begin{pmatrix} A + i\theta & \cdot \\ \cdot & i\theta - D \end{pmatrix} + \begin{pmatrix} b \\ -c \end{pmatrix}\begin{pmatrix} q^{-1}u & q^{-1}w \end{pmatrix}$$

$$= \begin{pmatrix} (A+i\theta) & \cdot \\ \cdot & (i\theta - D) \end{pmatrix}\left\{ I + xy \right\} \qquad x = \begin{pmatrix} (A+i\theta)^{-1}b \\ -(i\theta - D)^{-1}c \end{pmatrix}, y = q^{-1}(u, w)$$

so

$$\det(V^{-1}\bar{Q} + i\theta) = \det(A+i\theta)\det(i\theta - D)\left(1 + q^{-1}u(A+i\theta)^{-1}b - q^{-1}w(i\theta - D)^{-1}c\right)$$

Similarly,

$$\det(G_+ + i\theta) = \det\left(A + i\theta + bq^{-1}(u + w\Pi_+)\right)$$

$$= \det(A+i\theta)\det I + (A+i\theta)^{-1}bq^{-1}(u + w\Pi_+)$$

$$= \det(A+i\theta)\left(1 + q^{-1}(u + w\Pi_+)(A+i\theta)^{-1}b\right)$$

Assembling,

$$q\left(q - u(-i\theta - A)^{-1}b - w(i\theta - D)^{-1}c\right) = \left\{q - (u + w\Pi_+)(-i\theta - A)^{-1}b\right\}\left\{q - (w + u\Pi_-)(i\theta - D)^{-1}c\right\}$$

which we can rearrange to

$$\psi_+(i\theta)\,\psi_-(i\theta) \Big/ (\eta + u(I-\Pi_-)1)(\eta + w(I-\Pi_+)1)$$

$$= \frac{1}{q}\;\frac{1}{\eta - \psi(i\theta)}$$

So we conclude

$$\boxed{\psi_+(i\theta)\psi_-(i\theta) = \frac{\eta}{\eta - \psi(i\theta)} \cdot \frac{(\eta + u(I-\Pi_-)1)(\eta + w(I-\Pi_+)1)}{\eta(q_0+\eta)} = \frac{\eta}{\eta-\psi(i\theta)} \cdot \kappa(\eta)}$$

Setting $\theta = 0$ on both sides, we learn that $\kappa(\eta) = 1$, and we have the WH factorisation.

3) And on towards the 2-sided exit problem? If we want to consider exit from $[-x, y] \ni 0$, then we seek $f(\varphi, \cdot) = \exp(-\varphi V^{-1}Q)\, g$ for some g such that

$$(I \;\; \Pi_-)\begin{pmatrix} e^{-yG_+} & \cdot \\ \cdot & e^{yG_-} \end{pmatrix} S^{-1} g = e^{i\theta y}(-i\theta - A)^{-1} b \qquad \left[S = \begin{pmatrix} I & \Pi_- \\ \Pi_+ & I \end{pmatrix} \right]$$

$$(\Pi_+ \;\; I)\begin{pmatrix} e^{xG_+} & \cdot \\ \cdot & e^{-xG_-} \end{pmatrix} S^{-1} g = e^{-i\theta x}(i\theta - D)^{-1} c$$

So we have to solve

$$\begin{pmatrix} e^{-y(i\theta + G_+)} & \Pi_- e^{y(G_- - i\theta)} \\ \Pi_+ e^{x(G_+ + i\theta)} & e^{-x(G_- - i\theta)} \end{pmatrix} S^{-1} g = \begin{pmatrix} (-i\theta - A)^{-1} b \\ (i\theta - D)^{-1} c \end{pmatrix}$$

So

$$g = \begin{pmatrix} (I - \Pi_- K_-)(I - K_+K_-)^{-1} & (\Pi_- - K_+)(I - K_-K_+)^{-1} \\ (\Pi_+ - K_-)(I - K_+K_-)^{-1} & (I - \Pi_+K_+)(I - K_-K_+)^{-1} \end{pmatrix} \begin{pmatrix} e^{y(G_+ + i\theta)}(-i\theta - A)^{-1} b \\ e^{x(G_- - i\theta)}(i\theta - D)^{-1} c \end{pmatrix}$$

where $K_+ \equiv e^{y(G_+ + i\theta)}\,\Pi_-\, e^{y(G_- - i\theta)}$, $K_- \equiv e^{x(G_- - i\theta)}\,\Pi_+\, e^{x(G_+ + i\theta)}$. This has some sort of interpretation in terms of crossings of the interval $[-x, y]$.

Where now? LT in (x, y)? Could try this, but it doesn't look so easy

Some thoughts on a model of Cadenillas, Cvitanic + Zapatero (27/6/03)

1) There was a nice talk at Utah where Abel Cadenillas proposed a model for a stock

$$dS = (\mu S + \delta u)\,dt + v(\alpha S dt + S dW)$$

where δ, μ, α are fixed constants, $\alpha, \delta \geq 0$. The processes u and v are at the disposal of a manager, who is to be rewarded at time T with a package $f(S_T)$, so the objective of the manager is

$$\max \; E\left[f(S_T) - \int_0^T L(u_s)\,ds \right]$$

where L is some loss function. The interpretation is that the manager chooses a level of effort u, and a riskiness (v) of the firm's activities. The manager performs optimally, and the firm aims to set f so as to

$$\max E[\; N S_T - f(S_T)]$$

where N is the number of shares (or should this be risk-neutral expectation??) CCZ chose $L(u) = \frac{1}{2}u^2$, but I didn't find the modelling assumptions quite right...

2) Seems better that the manager's actions should not depend on the level of $\log S$, so as dynamics

$$dS = S\left[\{\mu + \delta u + \alpha v\}\,dt + v\,dW \right]$$

seems more convincing. In this case, with $L(u) = \frac{1}{2}u^2$ we have a danger of ill-posedness, so instead let's try

$$L(u) = \frac{u^2}{1-u}$$

(u is thought of as the proportion of the day the manager devotes to the firm; this doesn't exceed 1, and by starting off like u^2 we don't get a threshold at the start of the manager's effort). So let's introduce Lagrangian process Y

$$dY = Y(a\,dW + b\,dt)$$

and cast the problem in Lagrangian form:

$$\Lambda \equiv \max E\left[f(S_T) + \int_0^T \left\{ SY(\mu + \delta u + \alpha v) - \frac{u^2}{1-u} + SYb + SvaY \right\} dt - \left[YS \right]_0^T \right]$$

$$= \max \; E\left[\tilde{f}(Y_T) + Y_0 S_0 + \int_0^T \left\{ SY(\mu + (\alpha + a)v + b) + \delta SYu - \frac{u^2}{1-u} \right\} dt \right]$$

Maxing over v tells us $\boxed{\alpha + a = 0}$ and then we max over u (writing $\delta SY = \xi$ for short)

$$\xi - \frac{2u}{1-u} - \left(\frac{u}{1-u}\right)^2 = 0$$

so if we set $t \equiv \frac{u}{1-u}$, we are solving $t^2 + 2t - \xi = 0$, so

$$\boxed{t = \frac{u}{1-u} = \sqrt{1+\xi} - 1}$$

and the maximised value after some calculations is

$$\left(\sqrt{1+\xi}-1\right)^2$$

and the maximisation within the integral turns out to be of

$$\left(\sqrt{1+\xi}-1\right)^2 + c\xi \qquad c \equiv (\mu+b)/\delta$$

Now this is a convex function, so for a finite maximum we need $c+1 \leq 0$, and the best choice is $\xi = 0$; the maximised objective is thus

$$E\left[\tilde{f}(Y_T) + Y_0 S_0\right] \qquad dY = Y(-\alpha dW + b\,dt)$$

and the familiar argument shows that to minimise this we take $\boxed{b = -\mu-\delta}$

The dual problem is therefore

$$\boxed{\min_{Y_0} E\left[\tilde{f}(Y_T) + Y_0 S_0\right]}$$

where we take $\boxed{dY = Y(-\alpha dW - (\mu+\delta)\,dt)}$

Optimal hedging under Γ-constraints (11/7/03)

1) This is a question that Mete asked me at the Pascal conference, following on from some of the earlier discussion with Nizar. If we have a standard share

$$dS = \sigma S\, dW \qquad (\sigma \text{ constant}) \quad [dY = Y(a\,dW + b\,dt)]$$

and we make a portfolio X, where

$$dX_t = \theta\, dS_t \qquad [d\eta = \alpha\, dW + \beta\, dt]$$

but subject to the constraint

$$d\theta = \mu\, dt + \Gamma\, dW, \qquad [d\lambda = a_\theta\, dW + b_\theta\, dt]$$

where Γ is bounded in some way, how would we proceed in order to

$$\max E\left[U(X_T - g(S_T)) \right] \; ?$$

2) If we tried the standard Lagrangian approach, with the multiplier processes as given above in [·], we have the Lagrangian form

$$\max_{S,\theta,X} E\Big[U(X_T - g(S_T)) - [YS]_0^T - [X\eta]_0^T - [\lambda\theta]_0^T + \int_0^T \{ SYb + SYa\sigma + X\beta + \alpha\theta S\sigma + \lambda\mu + \theta b_\theta + \Gamma a_\theta \}\, dt \Big]$$

$$= \max E\Big[U(X_T - g(S_T)) - Y_T S_T - X_T\eta_T - \lambda_T\theta_T + Y_0S_0 + X_0\eta_0 + \lambda_0\theta_0 + \int_0^T (\lambda\mu + \Gamma a_\theta)\, dt \Big]$$

with the conditions $\boxed{\beta = 0, \ \alpha = 0 \ (\text{since } \theta S \text{ is quadratic}), \ b + a\sigma \le 0, \ b_\theta = 0.}$ This says that λ is a local martingale, and η is a constant, and we have the optimisation

$$\max_{S\ge 0, X} U(X - g(S)) - YS - X\eta = \max_{S\ge 0, \xi} U(\xi) - YS - \eta(\xi + g(S))$$

$$= \tilde{U}(\eta) - \min_S \{ \eta g(S) + YS \}$$

For finite values, we need $\eta > 0$, and we get

$$= \tilde{U}(\eta) - \eta\, \tilde{g}(Y/\eta),$$

where $\tilde{g}$ is the (convex) dual function of g; second dual $\tilde{\tilde{g}} \le g$. The Lagrangian becomes

$$\max E\Big[\tilde{U}(\eta_0) - \eta_0\, \tilde{g}(Y_T/\eta_0) - \lambda_T\theta_T + Y_0S_0 + \eta_0 X_0 + \lambda_0\theta_0 + \int_0^T (\lambda\mu + \Gamma a_\theta)\, dt \Big]$$

Maximising over θ_T gives a finite value only when $\lambda_T = 0$, so $\lambda \equiv 0$ (since λ is a martingale), and hence $a_\theta \equiv 0$... but this then means that the dual problem doesn't see the effect of a bound on Γ...

Of course, we could let ρ depend on $\xi \cdot \omega$, and even γ, though this might be less natural.

A simple model of production and investment (14/7/03)

1) Suppose we have a Markov chain ξ with finitely many states, controlling a simple linear economy: there is capital (K) and cash (B) evolving as

$$\begin{cases} \dot{B} = r(\xi)B - c + \beta(\xi)K - g(I) \\ \dot{K} = I \end{cases}$$

where $g(x) = x \vee (\gamma x)$, with $0 \leq \gamma \leq 1$ fixed. The idea of this is that we may invest into capital, but disinvestment realises only a fraction of the nominal value of the capital. As usual, we aim to obtain

$$\max E\left[\int_0^\infty e^{-\rho t}U(c_t)\,dt \;\middle|\; B_0 = B,\ K_0 = K,\ \xi_0 = \xi\right] \equiv V(\xi, B, K).$$

Assume U is CRRA. We will also insist on the solvency conditions

$$B + K \geq 0, \qquad B + \gamma K \geq 0$$

at all times. If we look at the HJB equation

$$\sup_{c,I}\left[U(c) - \rho V + QV + (rB + \beta K - c - g(I))V_B + I V_K\right] = 0$$

we can exploit the scaling properties to get us to

$$V(\xi, B, K) = (B+K)^{1-R}\, v(\xi, x) \qquad x \equiv B/(B+K)$$

where $v(\xi, x) = V(\xi, x, 1-x)$. Since $V_B = (B+K)^{-R}\left[(1-R)v + (1-x)v'\right]$, $V_K = (B+K)^{-R}\left[(1-R)v - xv'\right]$, we can redo the HJB equation

$$\sup_{c,i}\Big[U(c) - \rho v + Qv + \{rx + \beta(1-x) - c - g(i)\}\{(1-R)v + (1-x)v'\}$$
$$+ i\left((1-R)v - xv'\right)\Big] = 0$$

$$= (Q-\rho)v + \tilde{U}\left((1-R)v + (1-x)v'\right) + \left(rx + \beta(1-x)\right)\left((1-R)v + (1-x)v'\right)$$
$$+ \sup_i\left\{i\left((1-R)v - xv'\right) - g(i)\left((1-R)v + (1-x)v'\right)\right\}.$$

2) Here is a conjecture: immediately there's a change of state, the investor adjusts his portfolio to make

$$\frac{B}{B+K} = \alpha(\xi)$$

where $\alpha(\xi) \in \left[\frac{-\gamma}{1-\gamma}, \infty\right]$, and then maintains (B, K) in those proportions while consuming at rate $\lambda(\xi)(B+K)$.

If this is true, then we have a skew-product expression $(\alpha_t, w_t) \equiv \left(\frac{B_t}{B_t+K_t}, B_t + K_t\right)$

$$\dot{K} = I = (1-\alpha)\dot{w} = (1-\alpha)\kappa w$$

$$\dot{B} = \{r\alpha + \beta(1-\alpha) - \lambda - g((1-\alpha)\kappa)\} w = \alpha \dot{w}$$

To find κ, note that

$K \longmapsto \alpha\kappa + g((1-\alpha)\kappa) = \max\{\kappa, (\gamma+\alpha-\gamma\alpha)\kappa\}$

is a piecewise-linear f^n with inverse $y \longmapsto \min\{y, \frac{1}{\gamma+\alpha-\gamma\alpha} y\}$.

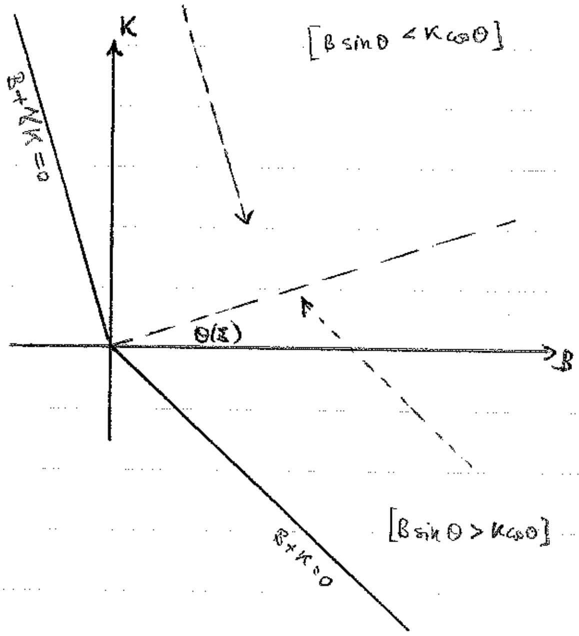
$$\tan\theta = \frac{1}{\alpha} - 1$$

and in between jumps we shall find

$$dw_t = \kappa(\xi_t)\, w_t$$

where κ, α, λ are related via

$$\boxed{\alpha\kappa + g((1-\alpha)\kappa) = r\alpha + \rho(1-\alpha) - \lambda}$$



What happens when there's a jump??

If we're at (B, K) when a jump of ξ occurs, and we now have to shift to $\theta = \theta(\xi)$ there are two situations:

(i) $B \sin\theta > K\cos\theta$

Then we take $y > 0$ and shift to

$$(B-y, K+y) = (B+K)\left(\frac{1}{1+\tan\theta}, \frac{\tan\theta}{1+\tan\theta}\right)$$
$$= (B+K)(\alpha, 1-\alpha).$$

$(y = (B\tan\theta - K)/(1+\tan\theta))$

(ii) $B\sin\theta < K\cos\theta$ (i.e. $\frac{B}{B+K} < \alpha$)

This time take $y > 0$ so that

$$(B+\gamma y, K-y) = (B+\gamma K)\left(\frac{1}{1+\gamma\tan\theta}, \frac{\tan\theta}{1+\gamma\tan\theta}\right) = (B+\gamma K)\left(\frac{\alpha}{\gamma+\alpha(1-\gamma)}, \frac{1-\alpha}{\gamma+\alpha(1-\gamma)}\right)$$

$[y = (K - B\tan\theta)/(1+\gamma\tan\theta)]$

In the first situation, there is no change in $w \equiv B+K$, and in the second, the change is

$$-\frac{(1-\gamma)(K - B\tan\theta)}{1+\gamma\tan\theta} = -\frac{(1-\gamma)\{\alpha K - B(1-\alpha)\}}{\gamma+\alpha(1-\gamma)}$$

We conclude that if we move from state i to state j, where $\alpha_i < \alpha_j$, there is a shrinkage factor

$$\frac{\gamma + \alpha_i - \gamma\alpha_i}{\gamma + \alpha_j - \gamma\alpha_j} \equiv \Delta_{ij}$$

in the wealth. Set $\Delta_{ij} = 1$ if $\alpha_i \geq \alpha_j$.

3) If this is what's going on, if we define

$$f_i \equiv E\left[\int_0^\infty e^{-\rho t} U(c_t)\,dt \;\middle|\; \xi_0 = i,\; B_0 + K_0 = 1,\; B_0 = \alpha_i(B_0+K_0)\right]$$

then

$$f_i = E\left[\int_0^\tau e^{-\rho t}\, U(\lambda_i e^{\kappa_i t})\, dt + e^{-\rho\tau + \kappa_i(1-R)\tau} \sum_{j\neq i} \frac{q_{ij}}{q_i} \Delta_{ij}^{1-R} f_j \right]$$

$$= \frac{U(\lambda_i)}{\rho + q_i - (1-R)\kappa_i} + \sum_{j\neq i} \frac{q_{ij}}{\rho + q_i - (1-R)\kappa_i} \Delta_{ij}^{1-R} f_j$$

Thus once we have chosen the α_i and the λ_i we can compute the κ's, and the Δ's, and obtain f by solving a linear equation.

Taking $\theta = \xi/\eta$ as the independent variable, we get $\tilde{b} = \log\theta$,
$z = \log\eta = \log(\xi/\theta)$ so $\dfrac{b'_\theta}{z'_\theta} = \dfrac{\xi}{\theta\xi' - \xi}$

Convertible bonds: the variational problem (16/7/03)

1) From the excursion analysis on p. 13, we have an expression for the share given in the form

$$S(\tilde{m}(y), e^{y}) = \int_y^{y_0} e^{-\int_y^z R(b_u)du} F(z, b_z)\,dz + \frac{\eta_0}{n} e^{-\int_y^{y_0} R(b_s)ds}$$

where

$$F(z,b) \equiv \frac{\delta e^{z} K_1(b) - \rho' \tilde{m}(z) K_0(b)}{n - \tilde{m}(z)} = \rho K_0(b) + p \frac{e^{z+b}}{\tilde{m}(z)} R_B(b)$$

If we imagine that $\tilde{m}$ has been chosen, we can think of the problem as being that of choosing b to achieve an extremal value for S. Now if we do the variational thing on S, changing b to $b+\varepsilon$, the first-order change is

$$0 = \int_y^{y_0} e^{-\int_y^z R(b_u)du} \Big[F_2(z,b_z)\,\varepsilon_z - \int_y^z R'(b_s)\varepsilon_s\,ds\, F(z,b_z) \Big] dz - \frac{\eta_0}{n} \int_y^{y_0} R'(b_s)\,\varepsilon_s\,ds\; e^{-\int_y^{y_0} R(b_u)du}$$

$$= \int_y^{y_0} \varepsilon_s \Big[-R'(b_s) \int_s^{y_0} F(z,b_z)\, e^{-\int_y^z R(b_u)du} dz + F_2(s,b_s)\, e^{-\int_y^s R(b_u)du} - \frac{\eta_0}{n} R'(b_s)\, e^{-\int_y^{y_0} R(b_u)du} \Big] ds$$

Since ε is arbitrary, we have $[\cdot]=0$, so this tells us that

$$0 = -R'(b_s) \int_s^{y_0} F(z,b_z)\, e^{\int_z^{y_0} R(b_u)du} dz + F_2(s,b_s)\, e^{\int_s^{y_0} R(b_u)du} - \frac{\eta_0}{n} R'(b_s)$$

Dividing by $R'(b_s)$ and differentiating gives us

$$0 = F(s,b_s) - R(b_s)\frac{F_2(s,b_s)}{R'(b_s)} + \frac{F_{12}(s,b_s)}{R'(b_s)} + b_s' \,\frac{R'(b_s) F_{22}(s,b_s) - R''(b_s) F_2(s,b_s)}{R'(b_s)^2}$$

So

$$\boxed{b_s' = \frac{R'(b_s)\big[R(b_s)F_2(s,b_s) - F_{12}(s,b_s) - R'(b_s)F(s,b_s) \big]}{R'(b_s)F_{22}(s,b_s) - R''(b_s)F_2(s,b_s)}}$$

2) All of this is working in terms of $z = \log \eta$ as the independent variable, but if instead we chose to work with θ as the independent variable, $\tilde{b}(\theta) \equiv b(z) = b(z(\theta))$, we get from a similar variational analysis

$$\boxed{\frac{\tilde{b}_\theta'}{z_\theta'} \left\{ \frac{R' F_{22} - R'' F_2}{R'^2} \right\} = \frac{R F_2 - F_{12} - R' F}{R'}}$$

(θ could be any independent variable, not necessarily our ξ/η choice ...)

3) So what actually happens if we take the form of F which appears in the expression for the share, and develop it a bit?

Grinding through Maple (GUNTHER/CB/MAPLE/varm2.mws), we derive an expression from the variational ODE, and the result is that the expression for $d\xi/dm$ that comes out is EXACTLY the same as at (4) - so this confirms what we knew, but adds nothing new.

4) Can we use the form (1) of S to get value out of the excursion expression? We have

$$S(\tilde{m}(y), e^y) = \frac{\tilde{m}(y)\rho' \varphi_0(-b_y) - re^y \varphi_1(-b_y)}{r(\alpha+\beta)\ (n-\tilde{m}(y))}$$

$$= \int_y^{y_0} e^{-\int_y^z R(b_s)ds} F(z, b_z)\, dz + \frac{y_0}{n} e^{-\int_y^{y_0} R(b_s)ds}$$

Multiplying by $\exp(\int_y^{y_0} R(b_s)ds)$ and then differentiating wrto y gives us

$$\frac{d}{dy} S(\tilde{m}(y), e^y) - R(b_y) S(\tilde{m}(y), e^y) = -F(y, b_y)$$

If we re-express $S(\tilde{m}(y), e^y)$ as a function of θ,

$$S = \frac{m(\theta)\rho' \varphi_0(-\log\theta) - r(\xi/\theta)\varphi_1(-\log\theta)}{r(\alpha+\beta)\ (n-m(\theta))} \equiv \bar{S}(\theta)$$

We get

$$\frac{d}{dy} S(\tilde{m}(y), e^y) = \frac{d\bar{S}}{d\theta}(\theta) \Big/ \frac{dy}{d\theta} = \frac{d\bar{S}}{d\theta}(\theta) \Big/ \left\{ -\frac{1}{\theta} + \frac{1}{\xi}\frac{d\xi}{d\theta} \right\},$$

As we find the equality

$$\boxed{\frac{d\bar{S}}{d\theta} \Big/ \left\{ -\frac{1}{\theta} + \frac{1}{\xi}\frac{d\xi}{d\theta} \right\} = R\bar{S} - F}$$

Again, if we now see where this leads, the conclusion (.../xcn3.mws) is the same as we got above; expression (4) for $d\xi/dm$ is what you get ...

Interestingly, getting correct BCs seems very critical for good answers — maybe worth doing the BCs to higher order?

If we have function values at $(x_0, x_1, x_2, x_3) = (x_0, x_0+\delta_1, x_0+\delta_2, x_0+\delta_3)$, we use weights

$$\left[-\frac{\delta_1\delta_2+\delta_1\delta_3+\delta_2\delta_3}{\delta_1\delta_2\delta_3}, \frac{\delta_2\delta_3}{\delta_1(\delta_1-\delta_2)(\delta_1-\delta_3)}, \frac{\delta_1\delta_3}{\delta_2(\delta_2-\delta_3)(\delta_2-\delta_1)}, \frac{\delta_1\delta_2}{\delta_3(\delta_3-\delta_1)(\delta_3-\delta_2)}\right]$$

for 1st derivative,

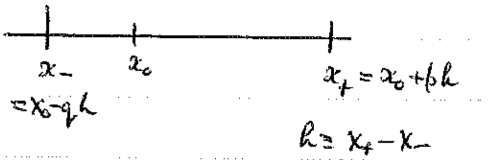
$$2\left[\frac{\delta_1+\delta_2+\delta_3}{\delta_1\delta_2\delta_3}, -\frac{\delta_2+\delta_3}{\delta_1(\delta_1-\delta_2)(\delta_1-\delta_3)}, -\frac{\delta_1+\delta_3}{\delta_2(\delta_2-\delta_1)(\delta_2-\delta_3)}, -\frac{\delta_1+\delta_2}{\delta_3(\delta_3-\delta_1)(\delta_3-\delta_2)}\right]$$

for second derivative

Unequally-spaced finite differences (23/7/03)

1) I'm interested to compute finite-difference approximations to derivatives up to order 2, when we don't assume equal spacing. In the middle of the (1-dimensional) region, we take

$$a\, f(x_0+ph) + b\, f(x_0) + c\, f(x_0 - qh)$$



$$\begin{aligned} &= (a+b+c)\, f(x_0) \\ &\quad + (ap - cq)\, h f'(x_0) \\ &\quad\quad + (ap^2 + cq^2)\, \tfrac{1}{2} h^2 f''(0) + O(h^3) \end{aligned}$$

so for first derivative we pick

$$\boxed{a = \frac{q}{ph}, \quad b = \frac{p-q}{qph}, \quad c = -\frac{p}{qh}}$$

and for the second we pick

$$\boxed{a = \frac{2}{ph^2}, \quad b = \frac{-2}{pq\,h^2}, \quad c = \frac{2}{qh^2}}$$

More generally, if we take the three points $(x_1, x_2, x_3) \equiv (x_1, x_1+\delta_1, x_1+\delta_2)$, we expand round x_1 and see

$$\begin{aligned} &a f(x_1) + b f(x_2) + c f(x_3) \\ &= (a+b+c)\, f(x_1) + f'(x_1)\,(b\delta_1 + c\delta_2) + \tfrac{1}{2} f''(x_1)\left(b\delta_1^2 + c\delta_2^2\right) \end{aligned}$$

so we shall pick

$$\boxed{a = -\frac{\delta_1+\delta_2}{\delta_1\delta_2}, \quad b = \frac{\delta_2}{\delta_1(\delta_2-\delta_1)}, \quad c = -\frac{\delta_1}{\delta_2(\delta_2-\delta_1)}}$$

for the first derivative,

$$\boxed{a = \frac{2}{\delta_1\delta_2}, \quad b = \frac{2}{\delta_1(\delta_1-\delta_2)}, \quad c = -\frac{2}{\delta_2(\delta_1-\delta_2)}}$$

for the second.

2) What about a 2-dimensional finite-difference scheme? This is actually the same for the derivatives $\frac{\partial}{\partial x_1}$, $\frac{\partial}{\partial x_2}$, etc — only the cross derivatives need any thought. But then we can make $\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}$ by successively applying the (FD approximations to) the derivatives one after the other.

Back to convertible bonds (27/8/03)

1) Taking the various expressions we already have, and plugging them into Maple (/WORK/GUNTHER/CB/MAPLE/cb1.mws) we find the differential equation for m in the form

$$\frac{dm}{d\theta} = m \, \frac{P(m,\theta)}{Q(m,\theta)}$$

where both P and Q are polynomials of degree 4 in m:

$$P(m,\theta) = \sum_{r=0}^{4} m^r \, p_r(\theta,s,u) \quad , \quad Q(m,\theta) = \sum_{r=0}^{4} m^r \, q_r(\theta,s,u)$$

where $s \equiv \theta^\beta$, $u \equiv \theta^\alpha$, and the functions p_r, q_r are multinomial in the three arguments; the highest degree appearing in each p_r is 9, the highest degree appearing in each q_r is 10. But more interestingly, the lowest degree is not always zero; in fact,

$$\boxed{\begin{aligned}
q_0(\theta,s,u) &= \theta^3 u \, \tilde{q}_0(\theta,s,u) \\
q_1(\theta,s,u) &= \theta^2 u \, \tilde{q}_1(\theta,s,u) \\
q_j(\theta,s,u) &= \theta \, \tilde{q}_j(\theta,s,u) \qquad (j = 2,3,4) \\
p_0(\theta,s,u) &= u\theta^2 \, \tilde{p}_0(\theta,s,u) \\
p_1(\theta,s,u) &= \theta \, \tilde{p}_1(\theta,s,u) \\
p_j(\theta,s,u) &= \tilde{p}_j(\theta,s,u) \qquad (j = 2,3,4)
\end{aligned}}$$

where the tilded expressions are multinomials with non-zero constant terms:

$$\boxed{\begin{aligned}
\tilde{q}_0(0,0,0) &= n^4 p(\alpha+\beta)\beta^2(\beta-1)(\alpha+1) & \tilde{p}_0(0,0,0) &= n^4 p\,\alpha\beta^2(\alpha+\beta)(\beta-1)(\alpha+1) \\
\tilde{q}_1(0,0,0) &= -n^3 p(\alpha+\beta)\beta\alpha(\beta-1)(\beta-2)(1-\kappa) & \tilde{p}_1(0,0,0) &= n^3\beta^2(\beta-1)^2(\alpha+1) \\
\tilde{q}_2(0,0,0) &= -n^2\beta\alpha(\beta-1)^2(1-\kappa) & \tilde{p}_2(0,0,0) &= -n^2\beta^2\alpha(\beta-1)^2(1-\kappa) \\
\tilde{q}_3(0,0,0) &= -n\kappa\,\alpha\beta(\beta-1)^3(1-\kappa) & \tilde{p}_3(0,0,0) &= n\,\alpha\beta^2(\beta-1)^2(1-\kappa)(1+\kappa) \\
\tilde{q}_4(0,0,0) &= \kappa(1-\kappa)\,\alpha\beta^2(\beta-1)^2 & \tilde{p}_4(0,0,0) &= -\kappa\,\alpha\beta^2(\beta-1)^2(1-\kappa)
\end{aligned}}$$

2) The idea is to seek a power-series solution of the form

$$m = \theta \sum_{i,j,k\geq 0} \theta^i s^j u^k \, a_{ijk} \, ,$$

to be approximated by

$$m_n = \theta \sum_{i,j,k\geq 0,\ i+j+k\leq n} \theta^i s^j u^k \, a_{ijk}$$

in some recursive solution. We shall write

$$Y_n = Q(m_n, \theta) \frac{dm_n}{d\theta} - m_n P(m_n, \theta)$$

and shall compute m_n by some recursive procedure. Let's see what we get for m_0, which is θa_{000}. The lowest order terms in the multinomial Y_0 will be coming from q_2 and from p_1 and p_2; equating those to zero gives us

$$0 = a_{000} \left\{ a_{000}^2 \theta^2 . \theta \tilde{q}_2(0,0,0) \right\} - a_{000}\theta \left\{ \theta^2 a_{000} \tilde{p}_1(0,0,0) + \theta^2 a_{000}^2 \tilde{p}_2(0,0,0) \right\}$$

and thus

$$a_{000} \left(\tilde{q}_2(0,0,0) - \tilde{p}_2(0,0,0) \right) - \tilde{p}_1(0,0,0) = 0$$

leading to $a_{000} = \dfrac{n\beta(\alpha+1)}{\alpha(\beta-1)(1-\tau)}$, as we get also from Maple.

<u>The recursive solution</u> <u>computes the terms in $z_n \equiv m_n - m_{n-1}$ (of degree $n+1$). by</u> <u>considering the terms in Y_n of degree</u> $n+3$, <u>and setting them to zero</u>. Let's define

$$\begin{cases} \Delta Q \equiv Q(m_{n+1},\theta) - Q(m_n,\theta) = \displaystyle\sum_{r=1}^{4} q_r(\theta,s,u) \sum_{j=1}^{r} \binom{r}{j} z^j m_n^{r-j} \\ \Delta P \equiv P(m_{n+1},\theta) - P(m_n,\theta) = \displaystyle\sum_{r=1}^{4} p_r(\theta,s,u) \sum_{j=1}^{r} \binom{r}{j} z^j m_n^{r-j} \end{cases}$$

where we abbreviate z_{n+1} to z. Then when we consider

$$Y_{n+1} - Y_n = \left(Q(m_n,\theta) + \Delta Q \right)\left(\frac{dm_n}{d\theta} + \frac{dz}{d\theta} \right) - (m_n + z)\left(P(m_n,\theta) + \Delta P \right)$$
$$- \left\{ Q(m_n,\theta) \frac{dm_n}{d\theta} - m_n P(m_n,\theta) \right\}$$

$$= Q(m_n,\theta) \frac{dz}{d\theta} + \Delta Q . \left\{ \frac{dm_n}{d\theta} + \frac{dz}{d\theta} \right\} - m_n \Delta P - z\left(P(m_n,\theta) + \Delta P \right)$$

we can look for the lowest degree terms in this expression. The lowest-degree terms in $P(m_n,\theta)$ will be degree 2, the lowest-degree terms in $Q(m_n,\theta)$ will be degree 3, whatever n. The lowest-degree terms in ΔQ is $n+4$; the lowest-degree terms in ΔP will be of degree $n+3$. <u>Thus the lowest-degree terms in $Y_{n+1} - Y_n$ will be of degree</u> $n+4$. This means in particular that all terms of degree $\leq n+3$ in Y_{n+1} will vanish, and we only need to work out the new terms in $z \equiv z_{n+1}$.

The degree-2 terms in $P(m_{n+1}, \theta)$ will be

$$\theta^2 a_{000} \left\{ \tilde{p}_1(0,0,0) + a_{000} \tilde{p}_2(0,0,0) \right\}$$

$$= -\theta^2 \frac{n^4\beta^3(1+\alpha)^2}{\alpha(1-\pi)}$$

and the degree-3 terms in Q will be

$$-\theta^3 \frac{n^4\beta^3(1+\alpha)^2}{\alpha(1-\pi)} = a_{000}^2\,\theta^2 .\ \theta\,\tilde{q}_2(0,0,0)$$

Exactly the same coefficient! The degree-(n+4) terms in ΔQ will be

$$2\theta^2 z .\ a_{000}\,\tilde{q}_2(0,0,0) = -2\theta^2 z\, n^3\beta^2(\beta-1)(\alpha+1)$$

and the degree-(n+3) terms in ΔP will be

$$-\theta z\, n^3\beta^2(\beta^2-1)(\alpha+1)$$

Collecting all terms of order (n+4) in $Y_{n+1} - Y_n$, we shall have

$$-\frac{n^4\beta^3(1+\alpha)^2\,\theta^2}{\alpha(1-\pi)}\left\{\theta\frac{dz}{d\theta} - (\beta-2)z\right\}$$

which we require must be equal to the terms of order (n+4) in $-Y_n$, and this allows us to solve for all the coefficients a_{ijk}, $i+j+k=n+1$, which appear in z_{n+1}.

We find terms of the form

$$-a_{ijk}\{1+i+j\beta+k\alpha-(\beta-2)\}\,\theta^{1+i}\beta^j u^k .\ \frac{n^4\beta^3(1+\alpha)^2}{\alpha(1-\pi)}\,\theta^2$$

3) As we compute further coefficients using Maple, it looks increasingly as if $a_{00k}=0$ for $k\geq 2$. To study this, we need to have more on

$$\tilde{q}_0(0,0,u) = n^4\,\beta(\alpha+\beta)(\beta-1)\left\{(\alpha+1)\beta^2 + u(\alpha+\beta)\beta\left(\alpha\beta(1-\pi)-2\alpha-2\right) + u^2(\alpha+\beta)^2\left(\alpha+1-\alpha\beta(1-\pi)\right)\right\}$$

$$\tilde{q}_1(0,0,u) = n^3\beta(\alpha+\beta)(\beta-1)(\beta-2)(1-\pi)\left(-\alpha\beta+\alpha(\alpha+\beta)u\right)$$

$$\tilde{q}_2(0,0,u) = n^2(\beta-1)^2(1-\pi)\,\alpha\left(-\beta+(\alpha+\beta)u\right)$$

$$\tilde{q}_3(0,0,u) = n(\beta-1)^3\alpha\pi(1-\pi)\left(-\beta+(\alpha+\beta)u\right)$$

$$\tilde{q}_4(0,0,u) = \alpha\beta\pi(1-\pi)(\beta-1)^2\left(\beta-(\alpha+\beta)u\right)$$

and

$$\tilde{p}_0(0,0,u) = \alpha\beta n^4 p(\alpha+\beta)\left\{ \beta(\beta-1)(\alpha+1) - u(\alpha+\beta)\left((\beta-1)(\alpha+1) - \alpha\beta p(1-\pi)\right) \right\}$$

$$\tilde{p}_1(0,0,u) = n^3(\beta-1)\Big\{ \beta^2(\beta-1)(\alpha+1) - \beta u(\alpha+\beta)\big((\alpha+1)(\alpha+2)(\beta-1) + \alpha\beta p(\alpha-1)(1-\pi)\big)$$
$$+ (\alpha+1)(\alpha+\beta)^2 u^2 \left[(\beta-1)(\alpha+1) - (1-\pi)\beta p\right] \Big\}$$

$$p_2(0,0,u) = n^2(\beta-1)^2(1-\pi)\alpha\beta\left\{ -\beta + (\alpha+1)(\alpha+\beta)u \right\}$$

$$p_3(0,0,u) = n(\beta-1)^2(1-\pi)(1+\pi)\alpha\beta\left\{ \beta - (\alpha+1)(\alpha+\beta)u \right\}$$

$$p_4(0,0,u) = (\beta-1)^2\pi(1-\pi)\alpha\beta\left(-\beta + (\alpha+1)(\alpha+\beta)u \right)$$

Observe that everything in $Y \equiv Q(m,\theta)\frac{dm}{d\theta} - m\,P(m,\theta)$ has a factor of θ^{-3}, so if we consider instead $\tilde{Y} \equiv \theta^{-3}Y$, and let $\theta = s = 0$, writing $\tilde{m} \equiv \theta^{-1}m$ we find that we must have

$$0 = \left\{ u\tilde{q}_0(u) + u\tilde{q}_1(u)\tilde{m} + \tilde{q}_2(u)\tilde{m}^2 \right\}\frac{dm}{d\theta} - \tilde{m}\left(u\tilde{p}_0(u) + \tilde{p}_1(u)\tilde{m} + \tilde{p}_2(u)\tilde{m}^2 \right)$$

where we write $\tilde{q}_j(u)$ for $\tilde{q}_j(0,0,u)$ for brevity, and we understand $\tilde{m}$, $\frac{dm}{d\theta}$ to be evaluated at $\theta = s = 0$.

Thus we are understanding $\tilde{m} = \sum_{k\geq 0} a_{00k}u^k$, $\frac{dm}{d\theta} = \sum_{k\geq 0} a_{00k}(1+k\alpha)u^k$

It can be verified that using $\tilde{m} = a_{000} + a_{001}u$ actually solves the ODE, so this confirms the conjecture that $a_{00k} = 0 \quad \forall k \geq 2$.

The reason this is important is that if we use the approximation

$$m_1 = \theta\left(a_{000} + a_{100}\theta + a_{010}s + a_{001}u\right)$$

then $m - m_1$ is $O(\theta^3)$, since $s \equiv \theta^\beta = o(\theta)$. If we didn't know $a_{00k} = 0\ \forall k \geq 2$, then for small α, the contribution of $u^k = \theta^{k\alpha}$ could be of larger order.

__OR__ write $t = \dfrac{r\{(\beta-1)w^{-\alpha} + (\alpha+1)w^{\beta}\}}{\beta\rho' w^{-\alpha} + \alpha\rho' w^{\beta} - (\rho' - rK)(\alpha+\beta)} = \dfrac{r\{(\beta-1)w^{-\alpha} + (\alpha+1)w^{\beta}\}}{\rho'\psi_0(w) + rK(\alpha+\beta)}$

giving $\frac{\beta}{\alpha} w^{\alpha+\beta} = \dfrac{\rho' w^{\beta} + (\beta-1)(\rho' - rK)}{-\rho' w^{-\alpha} + (\alpha+1)(\rho' - rK)}$ ✓

and leading to

$$(\rho' - rK)\left[\beta(\alpha+1)w^{\alpha} - \alpha(\beta-1)w^{-\beta}\right] = \rho'(\alpha+\beta)$$

The unique solution of this is > 1 (as we require) iff $\rho' > rK$.

Callable convertible bonds again (5/9/03)

1) If now we allow the firm to call the bonds for price K each, we have that always

$$S \geq \underline{S}(m,V) \equiv \min\left\{ \frac{V}{n}, \frac{V-mK}{n-m} \right\} \vee 0$$

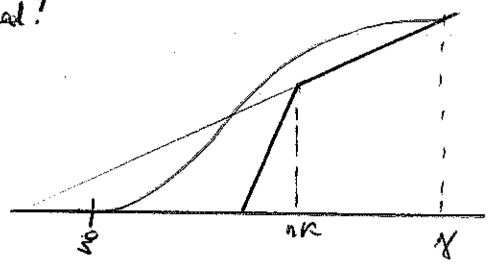
by the obvious reasoning. Let's suppose that

$$nK < \eta_0 = \frac{n\beta\rho}{r(\beta-1)}$$

otherwise the calling option will never get used.

2) What would the shareholder do if no conversion happened? Put another way, suppose we fix $m>0$, and ask where the firm would default/call on the assumption that the bondholders would let them do this. If the shareholders default at $\xi \in (0, mK)$ and call at $\gamma > mK$, how are ξ, γ determined?

(i) Could it be that $\gamma > nK$?



As we can add multiples of $(\frac{V}{\xi})^\beta - (\frac{V}{\xi})^{-\alpha}$ to any solution, we see that we would have to have S smooth-pasted to V/n at γ. This would mean that at at least three places in $(\xi, \gamma]$ the slope $\partial S/\partial V$ would be equal to $1/n$... but this can't happen... so we must have $\gamma \leq nK$ if there's no intervention by bond holders.

(ii) Suppose we sought a solution where $\xi \in (0, mK)$, $\gamma \in (mK, nK)$. Now if S smooth pastes to $(V-mK)/(n-m)$ at γ, we have

$$S(m,V) = \frac{m\kappa_0}{n-m}\psi_0(V/\gamma) + \frac{V-mK}{n-m} \qquad \left[\kappa_0 \equiv \frac{\rho' - rK}{r(\alpha+\beta)}\right]$$

To get smooth pasting to 0 at ξ we have to have

$$\begin{cases} m\kappa_0\, \psi_0(\xi/\gamma) + \xi - mK = 0 \\ \frac{m}{\xi}\kappa_0\, \alpha\beta\left[(\xi/\gamma)^\beta - (\xi/\gamma)^{-\alpha}\right] + 1 = 0 \end{cases}$$

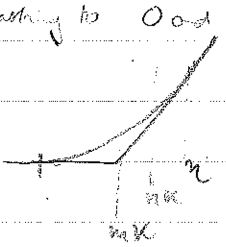
This suggests we look for a solution $\xi = mu$, $\gamma = w\xi$ for some $u>0$, $w>1$. The equations become

$$\begin{cases} \kappa_0\, \psi_0(1/w) + u - K = 0 \\ \kappa_0\, \alpha\beta(w^{-\beta} - w^{+\alpha}) + u = 0 \end{cases}$$

and eliminating u gives an equation for w:

$$\boxed{\kappa_0\left[\beta(\alpha+1)w^\alpha - \alpha(\beta-1)w^{-\beta} - \alpha - \beta\right] = K}$$

If we had $mt^* < nK$, then we have a sol^n smooth pasting to 0 and $(V-mK)/(n-m)$ at $\xi = mt^*$, $\eta = mw^*t^*$, but this η is too far right.
Pushing η down to make $S = K$ at nK, while keeping $S(\xi)=0$, we get $S'(\xi)<0$, so next holding $S(nK)=K$ fixed we raise S in (ξ, nK), shifting the point of bankruptcy right, but still in $(0, mK)$ — so it seems that we can actually go all the way up to $m = nK/t^*$



The bracket $[\cdot]$ increases monotonically from $-\infty$ to $+\infty$, so there is exactly one root w^*. If $\kappa_0 > 0$, the root is > 1, if $\kappa_0 < 0$ the root is < 1, so we deduce

$$\boxed{\exists \text{ solution of the desired form iff } \kappa_0 > 0 \text{ iff } \rho' > rK}$$

Then easily we get the solution u^*, which is clearly positive.

(iii) Let $m_1 = nK / u^* w^*$.

What happens as m rises beyond m_1? We have to keep the point of contact with $\underline{S}$ equal to nK, but we lose the smooth contact; in fact, we shall have

$$K = \mathcal{S}(m, nK) = \frac{nK - m\rho'/r}{n-m} + \frac{\beta m\rho' - r(\beta-1)\xi}{r(\alpha+\beta)(n-m)}\left(\frac{nK}{\xi}\right)^{-\alpha} + \frac{\alpha m\rho' - r(\alpha+1)\xi}{r(\alpha+\beta)(n-m)}\left(\frac{nK}{\xi}\right)^{\beta}$$

This allows us to express m as a function of ξ

$$m\left\{\rho'\psi_0\left(\frac{nK}{\xi}\right) + r(\alpha+\beta)K\right\} = rnK\left\{\psi_1\left(\frac{nK}{\xi}\right) + \alpha+\beta\right\}$$

How high does m rise before this breaks down? Why does it break down? do we even get this far - might the bondholders not wish to be converting in this region? As for the last question, once we have $\xi(m)$ we can work out the value of $\eta(m)$ at which bondholders would be converting, and provided this is bigger than $\gamma(m) = m w^* \xi^*$ we know bondholders won't join in.

$$S = \frac{V - mp'/r}{n - m} + A\left(\frac{V}{S}\right)^{-\alpha} + B\left(\frac{V}{S}\right)^{\beta}$$

3) <u>And what might be happening above nK?</u>

(i) What would S look like if smooth pasted to V/n at the point ζ?

$$\left.\begin{aligned} S(m,\zeta) &= \zeta/n = \frac{\zeta - m\rho'/r}{n-m} + A + B \\ \frac{\partial S}{\partial V}(m,\zeta) &= \frac{1}{n} = \frac{1}{n-m} - \alpha A/\zeta + \beta B/\zeta \end{aligned}\right\}$$

Hence

$$\binom{A}{B} = \frac{m}{r\,n(n-m)(\alpha+\beta)} \begin{pmatrix} n\beta\rho' - (\beta-1)\zeta r \\ n\alpha\rho' - (\alpha+1)\zeta r \end{pmatrix}$$

or again

$$\boxed{S = \frac{V}{n} + \left\{n\rho'\psi_0(V/\zeta) - rV\psi_1(V/\zeta)\right\}\frac{m}{n(n-m)r(\alpha+\beta)}}$$

If the bondholders choose to convert at ω, we shall have

$$\boxed{B-S = \frac{rV\psi_1(V/\omega) - \rho(n-mc)\psi_0(V/\omega)}{r(\alpha+\beta)(n-m)}}$$

(Automatically, $\frac{\partial}{\partial V}(B-S)=0$ at $V=\omega$)

together with the condition $(B-S)(m,\zeta)=0$. The missing condition to find (ζ,ω) is the condition

$$\frac{\partial S}{\partial m} = \frac{\partial B}{\partial m} = 0 \qquad \text{at } V=\omega$$

However,

$$\begin{aligned} \frac{\partial S}{\partial m} = &\left\{n\rho'\psi_0(\omega/\zeta) - r\omega\psi_1(\omega/\zeta)\right\} / r(\alpha+\beta)(n-m)^2 \\ &- \frac{\omega}{\zeta}\cdot\frac{\zeta'}{\zeta}\cdot\alpha\beta(n\rho'-\delta\zeta)\left\{\left(\frac{\omega}{\zeta}\right)^{\beta-1} - \left(\frac{\omega}{\zeta}\right)^{-\alpha-1}\right\} m/n(n-m)r(\alpha+\beta) \end{aligned}$$

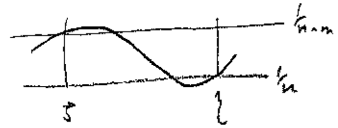
Simplifying this gives

$$\boxed{\frac{\zeta'}{\zeta} = \frac{n\{n\rho'\psi_0 - r\omega\psi_1\}\,\zeta/\omega}{m(n-m)(n\rho'-\delta\zeta)\psi_0'}}$$

Together with the condition $(B-S)(m,\zeta)=0$, this gives us a way to find ζ,ω rather like what we did before.

(ii) What would things look like if S were smooth-pasted to $(V-mK)/(n-m)$ at ξ, with smooth pasting of $B-S$ to zero at η? Running through the story as before, we find

$$\boxed{S(m,V) = \frac{m(\rho'/r - K)}{(n-m)(\alpha+\beta)}\psi_0(V/\xi) + \frac{V-mK}{n-m}}$$

$\xi \backslash \eta$	$(0, mK)$	$\{mK\}$	(mK, nK)	$\{nK\}$	(nK, ∞)
$(0, mK)$	**No**: $\frac{\partial S}{\partial V}$ cannot have 3 zeros	**No** would have to have $\frac{\partial S}{\partial V} < 0$ at $\eta = mK$, and then shifting η to the right would improve	**Possible** will have $\xi(m) = mt^*$, $\eta(m) = mt^*w^*$ for constants t^*, w^*. Needs $m < \frac{nK}{w^* t^*}$	**Possible** for each value of m in $\left(\frac{nK}{t^* w^*}, \frac{nK}{t^*}\right)$ we have a solution	**No**: otherwise there would be three roots of $\frac{\partial S}{\partial V} = \frac{1}{n}$ in $(0, \eta]$
$\{mK\}$	X	**No** live interval is $\emptyset$	**No**: shifting ξ left a bit would lead to improved S.	**No**: shifting ξ left would improve S.	**No**: shifting ξ left would improve S
(mK, nK)	X	X	**No** cannot have three roots for $\frac{dS}{dV} = \frac{1}{n-m}$	**No**: would need $\psi_0(nK/\xi) = 0$ so this can't happen.	**No**: would see graph of $\frac{\partial S}{\partial V}$:  and this is impossible, as $\frac{\partial S}{\partial V}$ is either monotone, or convex or concave.
$\{nK\}$	X	X	X	**No**: live interval is $\emptyset$	**No**: argument is a bit involved (p50)
(nK, ∞)	X	X	X	X	**No**: cannot have 3 roots to $\frac{dS}{dV} = \frac{1}{n}$

So (since S must be convex at ξ) we conclude that we can only have such a situation if

$$K \leq \rho'/r$$

The expression for B–S is the same as before, and the boundary condition at ξ is

$$(B-S)(m,\xi) = \frac{nK-\xi}{n-m}$$

Observe: If B–S smooth pastes at η to zero, we must have that B–S is convex at η; this gives the condition

$$\boxed{\delta\eta \geq \rho(n-m\tau)}$$

[Consistent with $\eta_0 = n\rho\beta/r(\beta-1)$ in the no-calling case? Yes – easy to check that $\eta_0 > n\rho/\delta$]

Similarly, if we have S smooth pasting to zero at ξ, we have to have

$$\boxed{\xi \leq \frac{m\rho'}{\delta}}$$

If S smooth pastes to V/n at some point ξ we learn that

$$\boxed{\xi \leq n\rho'/\delta}$$

and if there is smooth pasting at ξ to $(V-mK)/(n-m)$ we get also (as above)

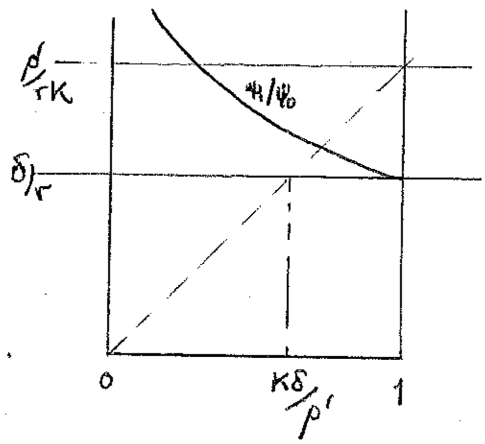
$$\boxed{\frac{\rho'}{r} \geq K}$$

<u>Cannot have a live interval</u> (ξ,η) <u>with</u> $\xi = nK < \eta$, <u>and shareholders control both ends</u>

Convexity of S at $\eta \Rightarrow \eta \leq n\rho'/\delta$.

$S(m,nK)=K \Rightarrow \rho'\psi_0(nK/\eta) = rK\,\psi_1(nK/\eta)$. Now for $x \in (0,1)$, we find

$$\frac{\psi_1(x)}{\psi_0(x)} = \frac{\int_x^1 (t^{-\alpha-1}-t^{\beta-1})\frac{dt}{t}}{\int_x^1 (t^{-\alpha-1}-t^{\beta-1})\,dt}\;\frac{(\alpha+1)(\beta-1)}{\alpha\beta} \downarrow \frac{(\alpha+1)(\beta-1)}{\alpha\beta} = \frac{\delta}{r} \quad \text{as } x \uparrow 1$$



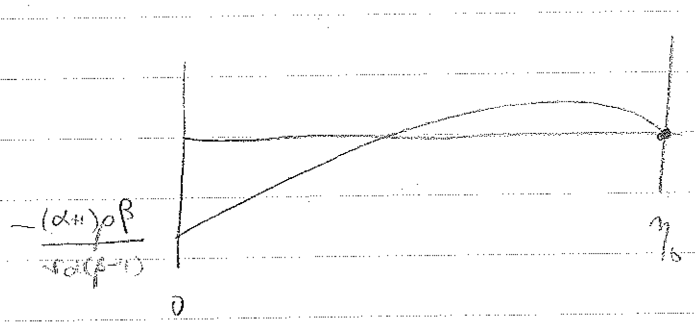
We see (rather suspiciously!) that η doesn't depend on m; and that the critical value $x = nK/\eta \in (0,1)$ must satisfy

$$\frac{\psi_1(x)}{\psi_0(x)} = \frac{\rho'}{rK}$$

which makes $\rho' > \delta K$ necessary – not inconsistent with $\eta \leq n\rho'/\delta$.

But if $\eta \leq n\rho'/\delta$, we have that $nK/\eta \geq K\delta/\rho'$, and it is clear that

$$\frac{\psi_1(y)}{\psi_0(y)} < \frac{1}{y}\cdot\frac{\delta}{r} \text{ for } y \in (0,1), \text{ so } \frac{\psi_1(K\delta/\rho')}{\psi_0(K\delta/\rho')} < \frac{\rho'}{rK}, \text{ so } \underline{\text{no}} \text{ solution.}$$



$-(\alpha+1)\rho\beta$
0
η_b

In the no-calling case, we can exploit our asymptotic expansion of $m(\theta)$ near zero to get the asymptotics of $S(m,V)$ for small m; we have to first order

$$S(m,V) = \frac{V}{n} + \theta\left\{\left(\frac{\beta(\alpha+1)}{\alpha(\beta-1)(1-rc)n}\right)V - \frac{(\alpha+1)\rho\beta}{\alpha(\beta-1)r} - \frac{(\alpha+1)\rho\beta(\beta c+1-c)}{\alpha(\beta-1)^2(1-c)r}\left(\frac{V}{\eta_0}\right)^\beta\right\} + \text{—}$$

The term in θ vanishes when $V=\eta_0$, and the coefficient of θ, viewed as a function of V, starts at $-(\alpha+1)\rho\beta/r\alpha(\beta-1)<0$, is concave, so has one root in $(0,\eta_0)$. Since we have to leading order that

$$m \approx \frac{(\alpha+1)\beta n}{\alpha(1-c)(\beta-1)}\theta,$$

our asymptotic for S in terms of m is

$$S(m,V) = \frac{V}{n} + m\left\{\frac{V}{n^2} - \frac{\rho(\beta c+1-c)}{nr(\beta-1)}\left(\frac{V}{\eta_0}\right)^\beta - \frac{\rho(1-c)}{nr}\right\} + \text{—} \equiv \frac{V}{n} + m\,h(V) + \sim$$

How does this compare with $\left(\frac{V-mK}{n-m}\right)^+ \wedge \frac{V}{n} = \underline{S}(m,V)$? If we consider $V\in(0,nK)$ we get $\underline{S}(m,V) = \frac{V}{n} + \frac{V-nK}{n^2}m + O(m^2)$

so we see that $\underline{S}(m,V) < S(m,V)$ for small m iff $K > \rho(1-c)/r + \frac{\rho(\beta c+1-c)}{r(\beta-1)}\left(\frac{V}{\eta_0}\right)^\beta$ ($V<nK$, of course). So the condition that there is an interval of m-values where the no-calling solution holds is

$$\boxed{h(nK) > 0}$$

This implies (but is strictly stronger than) $K > \rho'/r$.

As m rises, do we eventually reach some m where $\inf_{V\leq\eta(m)} S(m,V) - \underline{S}(m,V) = 0$?

If we do, there cannot be smooth pasting between (mK, nK) (else $\rho'/r \geq K$ – see previous page) and if there were smooth pasting to V/n at ζ, we'd have to have $\zeta \leq n\rho/\delta < \eta(m)$ – but this cannot happen for the same reason that S cannot control both ends of (ξ,ζ) with $\zeta<mK$, $\zeta>nK$ ... so the only possibility is that $\boxed{S=\underline{S} \text{ at } V=nK}$

- Because of Lemma 1, when we have a live interval (ξ,η) where S controls one end, B the other, it has to be that B controls the upper end η (since $B-S$ cannot have a min then a max)
- If $nK \geq \eta_0$, can we be sure that the no-calling solution is still correct? Assuming $\eta(\cdot)$ decreases, we could only get probs if at some m, S smooth pastes to $\underline{S}$ (impossible, as then $K\leq\rho'/r$), or if we got at some point $S(m,\eta(m)) = \underline{S}(m,\eta(m)) = (\eta(m)-mK)/(n-m) = B(m,\eta(m))$... but this couldn't happen, because when calling happens, bonds + shares $= V$.

S smooth pastes to V/n at $\xi > nK$, B−S smooth pastes to 0 at $\eta > \xi$?

As we've seen, the convexity of S at ξ and B−S at η forces

$$nK < \xi \leq \frac{n\rho'}{\delta} < \frac{\rho(n-m\tau)}{\delta} \leq \eta$$

So we shall need K to be fairly small. On p 49 we derived a differential equation for $\zeta \equiv \xi$, but what are the initial conditions? To understand this, notice that if firm calls at ξ, there is no bankruptcy, so for m small we can approximate

$$mB + (n-m)S = V + E^V(\text{tax repayments})$$

$$\approx V + \frac{m\rho\tau}{r} E^V(1-e^{-rH}) \equiv V + \frac{m\rho\tau}{r} g(V),$$

where H is the time we exit (ξ, η). We have $g(V) = 1 + A V^{-\alpha} + B V^{\beta}$, where we find A, B from

$$\begin{pmatrix} \xi^{-\alpha} & \xi^{\beta} \\ \eta^{-\alpha} & \eta^{\beta} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = -\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

In the limit as we let $m \downarrow 0$, we find the following equations for ξ, η:

$$\boxed{\begin{aligned} n\rho\, \psi_0(\xi/\eta) &= r\xi\, \psi_1(\xi/\eta) \\ n\rho'\, \psi_0(\eta/\xi) &= r\eta\, \psi_1(\eta/\xi) \end{aligned}}$$

To solve, write $t \equiv \xi/\eta$, take the ratio of the two equations and find

$$\frac{1}{1-\tau} = \frac{t\, \psi_1(t)\, \psi_0(1/t)}{\psi_0(t)\, \psi_1(1/t)}$$

needs checking!

Maple plots show that the RHS appears to be a decreasing function of $t \in (0,1)$, decreasing from ∞ to 1. Thus there would be just one solution for ξ/η, and the values of ξ, η can easily be recovered from that. The value of η should satisfy $\eta \geq n\rho/\delta$, and $\xi \leq n\rho'/\delta$; appears correct numerically.

What happens for very small m?

Assuming that the no-calling solution doesn't work, we have to explore solutions with live intervals (ξ, η), bottom controlled by S, with $\xi > mK$, top controlled by B. Set

$$\tilde{S} = \lim_{m \to 0} \frac{1}{m}\left(S(m,V) - V/n\right)$$

which satisfies $\mathcal{L}\tilde{S} = \frac{\rho'}{n} - \frac{\delta V}{n^2}$, and set

$$\bar{B} \equiv B - V/n$$

which satisfies $\mathcal{L}\bar{B} = -\rho + \delta V/n$. We have also that

$$\tilde{S} \geq \frac{V - nK}{n^2} \wedge 0 \qquad \text{with equality at } \xi, \eta$$

The boundary conditions for $\bar{B}$ are that $\bar{B}(\eta) = 0$, $\bar{B}(\xi) = (K - V/n)^+$. We also require that $\bar{B}$ smooth-pastes to zero at η, and $\tilde{S}$ smooth-pastes to $((V-nK)\wedge 0)/n^2$ at ξ if $\xi \neq nK$. We find

$$\bar{B} = \frac{rV\psi_1(V/\eta) - n\rho\,\psi_0(V/\eta)}{r\,n(\alpha+\beta)}$$

If $\tilde{S}$ smooth pastes at $\xi > nK$, then we get

$$\tilde{S} = \frac{n\rho'\psi_0(V/\xi) - rV\psi_1(V/\xi)}{n^2 r\,(\alpha+\beta)}$$

and if it smooth pastes at $\xi < nK$ we get

$$\tilde{S} = \frac{V - nK}{n^2} + \frac{\rho' - rK}{r(\alpha+\beta)n}\psi_0\!\left(\frac{V}{\xi}\right)$$

conditions for (ξ, η) would then be $(t = \xi/\eta)$

$$\eta = nK - \frac{n(\rho' - rK)}{r(\alpha+\beta)}\psi_0(1/t)$$

$$(K - \tfrac{\xi}{n})\,n\,r(\alpha+\beta) = r\xi\psi_1(t) - n\rho\psi_0(t)$$

Thus

$$(nK - \eta t)\,r(\alpha+\beta) = rt\eta\,\psi_1(t) - n\rho\,\psi_0(t)$$

gives an equation for t.

Nailing $\tilde{S}$ to zero at $\xi = nK$, $\eta > nK$ gives us

$$\tilde{S} = \frac{rV - n\rho'}{r n^2} + aV^{-\alpha} + bV^{\beta}$$

where

$$\begin{pmatrix} \xi^{-\alpha} & \xi^{\beta} \\ \eta^{-\alpha} & \eta^{\beta} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} n\rho' - r\xi \\ n\rho' - r\eta \end{pmatrix} \cdot \frac{1}{n^2 r}$$

What's critical is the <u>slope</u> of $\tilde{S}$ at $\xi = nK$ in this situation; going through it all, we get

$$\frac{\partial \tilde{S}}{\partial V} = \frac{\left\{r\psi_1(1/\theta) - n\rho'\psi_0(1/\theta)/\eta\right\}}{n^2 r\,\theta\,(\theta^{-\beta} - \theta^{\alpha})} \qquad [\theta = nK/\eta]$$

If this is greater than $\frac{1}{n^2}$, we smooth paste to the left of nK, if it's < 0, we smooth paste to the right of

nK, and otherwise it's contact at nK, no smooth pasting. The critical value η^* of η is decided by solving $\bar{B}(nK)=0$;

$$r\,\psi_1(\theta)\,K = \rho\,\psi_0(\theta) \qquad\qquad (\theta = nK/\eta)$$

which only has a solution $\theta\in(0,1)$ if $\boxed{K<\rho/\delta}$

How do we find the critical values of ξ, η at $m=0$? When $\xi = nK$, we just did it. If $\xi > nK$, we have the conditions $\bar{B}(\xi)=0$, $\bar{S}(\eta)=0$. We saw on p 53 that this leads us to solve

$$(1-rc) = \frac{\psi_0(t)\,\psi_1(1/t)}{t\,\psi_1(t)\,\psi_0(1/t)} \qquad \left[t \equiv \xi/\eta \in (0,1)\right]$$

If we had $\xi < nK$, the conditions $\bar{B}(\xi) = (K-\xi/n)$, $\tilde{S}(\eta)=0$ apply, so we must have

$$\begin{cases} r\,\xi\,\psi_1(\xi/\eta) - n\rho\,\psi_0(\xi/\eta) = r(\alpha+\beta)(nK-\xi) \\ n\rho'\,\psi_0(\eta/\xi) = r\,\eta\,\psi_1(\eta/\xi) \end{cases}$$

Write $t\equiv \xi/\eta \in (0,1)$, and use the second to express $\xi = n\rho'\,t\,\psi_0(1/t)/r\,\psi_1(1/t)$, then reduce the first equation to something entirely in t.

How do the solutions continue out from $m=0$?

We find we need the boundary condition $\partial S/\partial m = 0$ at η. Three cases:

(a) Contact below nK

We have
$$S(m,V) = \frac{m(\rho'-rK)}{r(\alpha+\beta)(n-m)}\,\psi_0(V/\xi) + \frac{V-mK}{n-m}$$

so
$$0 = \frac{\partial S}{\partial m} = \frac{n(\rho'-rK)}{r(\alpha+\beta)(n-m)^2}\,\psi_0\!\left(\frac{\eta}{\xi}\right) + \frac{\eta-nK}{(n-m)^2} - \frac{\eta}{\xi^2}\,\frac{d\xi}{dm}\,\psi_0'\!\left(\frac{\eta}{\xi}\right)\frac{m(\rho'-rK)}{r(\alpha+\beta)(n-m)}$$

as well as

$$r(\alpha+\beta)(nK-\xi) = r\,\xi\,\psi_1(\xi/\eta) - \rho(n-mc)\,\psi_0(\xi/\eta)$$

from the condition $(B-S)(m,\xi) = (nK-\xi)/(n-m)$. This last gives

$$\xi = \frac{r(\alpha+\beta)nK + \rho(n-mc)\,\psi_0(\theta)}{r\,(\psi_1(\theta)+\alpha+\beta)}$$

from which we can express $\frac{d\xi}{dm}$ in terms of $\frac{d\theta}{dm}$

Nice questions.

1) Kalvis asks about the following: consider $X_{n+1} = f(X_n + Z_n)$, where Z_n are IID zero mean, and we might have (for example) $f(x) = ax(1-x)$. What can we say about the behaviour of this Markov pr? (6/5/03)

2) Bill Janeway makes the comment on asymmetric information studies that everyone gets the same data, what differs is the model they're using to analyse it.

3) Cadenillas, Cvitanić + Zapatero use a nice model of rewarding an executive, who applies effort u, and chooses volatility v to drive the stock price

$$dS = \left[\mu S + \delta u + \alpha S v\right]dt + S v \, dW$$

where μ, δ, α are constants. He is rewarded at time T with some function $f(S_T)$, and he wants to max $E\left[U(f(S_T)) - \int_0^T \tfrac{1}{2} u_t^2 \, dt\right]$. The firm wants to max $E\left[\lambda S_T - f(S_T)\right]$. For log utility, and f a call option payoff, they get some results, but it seems to me that you must replace u by uS in the dynamics (the whole problem should scale in S) ... but then if you don't change the penalty on effort, the problem is ill posed if we had (say) $f(S) = (S-K)^+$, U is CRRA...

better appears the penalty $u^2/(1-u)$ ... this can actually go quite a long way

4) Josef Teichmann reports an interesting conjecture; if A, B are $d \times d$ symmetric (real) matrices, $B \geq 0$, then $\exists$ nonnegative measure μ s.t.

$$\operatorname{tr}(\exp(A - zB)) = \int_0^\infty e^{-zx} \mu(dx).$$