Some thoughts on policies with guaranteed annuity options (22/10/03)

1) There is a working paper 'Reserving, pricing and hedging for policies with guaranteed annuity options' by A.D. Wilkie, H.R. Waters + S. Yang presented to the IoA, which discusses a rather hybrid approach to the pricing + hedging of guaranteed annuity options. These effectively are contingent claims paid out at time T to surviving policy holders; their value is a function of interest rates and market index at that time.

2) In effect, the situation is that there is some Markov process X with generator $\mathcal{G}$, (space-time process $\tilde{X}$ with generator $\tilde{\mathcal{G}} = \frac{\partial}{\partial t} + \mathcal{G}$) and we have to pay out $\xi(X_T)$ at time T. We plan to make a hedging portfolio from some vector S of marketed assets, $S_t = S(\tilde{X}_t)$, with the value of the hedge given by

$$H_t = H_0 + \int_0^t \theta_u \cdot dS_u$$

for (previsible) portfolio process θ. What to do?

3) Let

$$V(\tilde{x}) \equiv V(t,x) = \inf E\left[\exp\{-\gamma (H_T - \xi(X_T))\} \mid X_t = x, H_t = 0 \right]$$

as usual, so that

$$\exp(-\gamma H_t)\, V(\tilde{X}_t) \text{ is a supermartingale, and a martingale under optimal control.}$$

How do we make this more explicit?

4) If we were requiring θ to be a simple process, or even initially <u>constant</u>, then we are looking at

$$e^{-\gamma \theta \cdot S(\tilde{X}_t)}\, V(\tilde{X}_t) = \varphi(\theta, \tilde{X}_t)$$

so

$$d\,\varphi(\theta, \tilde{X}_t) \doteq \Phi(\theta, \tilde{X}_t)\,dt \equiv \tilde{\mathcal{G}}\varphi(\theta, \tilde{X}_t)\,dt$$

so what we would find in general (first for simple θ, then general previsible θ) is

$$d\left(e^{-\gamma H_t} V(\tilde{X}_t)\right) \doteq e^{-\gamma H_t + \gamma \theta_t \cdot S_t}\, \tilde{\mathcal{G}}\varphi(\theta_t, \tilde{X}_t)\,dt$$

so our (HJB) equation for the value is

$$\boxed{\sup_\theta \tilde{\mathcal{G}}\varphi(\theta, \tilde{x}) = 0}$$

Solve this for a particular structure and get a reservation price!

A very very simple model for international trade (22/10/03)

1) Suppose that in country j $(j=1,\dots,J)$ there is a production process making 'stuff'. Output of a single unit of stuff requires input of α_j units of stuff and β_j units of labour. Suppose that x_{ij} units of stuff are exported from country i to country j ($x_{ij}<0$ corresponds to importation) with $x_{ij} = -x_{ji}$, of course. There are shipping costs $s_{ij} \geq 0$ for shipping from country i to country j, measured in world currency per unit of stuff. In country j, stuff trades freely at price p_j per unit.

Suppose that in country j there are L_j available units of labour. If production is set at ξ_j units of stuff, the utility to the people of country j is

$$U_j(c_j,\ L_j - \xi_j/\beta_j)$$

where c_j is the quantity of stuff consumed in country j.

What equilibria can there be for this system?

2) We have certain equations to be satisfied;

$$\begin{cases} \xi_j(1-\alpha_j) = c_j + \sum_k x_{jk} & \text{(conservation of stuff)} \\ p_j\,\xi_j(1-\alpha_j) = \sum_k x_{jk}^+ (p_k - s_{jk}) - \sum_k x_{jk}^- p_j + p_j c_j & \text{(balanced budget equation)} \end{cases}$$

where we assume for now that no budget deficit is allowed. Thus we get

$$c_j - \xi_j(1-\alpha_j) = -\sum_k x_{jk}$$

$$= -\sum_k x_{jk}^+ \left(\frac{p_k - s_{jk}}{p_j} - 1\right) - \sum_k x_{jk}$$

Let $\pi_j = \max_k \left(\frac{p_k - s_{jk}}{p_j} - 1\right)$, which is the best profitability of country j.

We shall want that each country sends all its exports to the most profitable country.

If we suppose that no country will export if it involves making a loss, then the conclusion would be that $x_{jk} > 0 \Rightarrow p_k = p_j + s_{jk} > p_j$. This seems a bit improbable...

3) Peppe observes that the above model assumes that all profits go to the exporting country, and this isn't realistic - better would be to split the profits in some ratio - but it's still impossible.

Divergence of opinions (31/10/03)

1) Suppose we have some underlying random vector $z \sim N(0, I)$, and agent j sees a private signal $a_j z$, together with a public signal $\sum_\ell c_\ell \hat{z}_\ell$, where $\hat{z}_\ell$ is agent ℓ's estimate of z. Each agent's best estimate $\hat{z}_\ell$ is of the form $\hat{z}_\ell = P_\ell z$, where P_ℓ is orthogonal projection onto the rows of

$$B_\ell = \begin{pmatrix} a_\ell \\ \Sigma c_i P_i \end{pmatrix} \quad , \text{ therefore } \quad P_\ell = B_\ell^T (B_\ell B_\ell^T)^{-1} B_\ell .$$

Given the a_j, c_ℓ, is there always a solution? Can there be more than one solution?

2) It seems we can have more than one solution. Take the case where z is 3-dimensional, $a_1 = (\cos\alpha, \sin\alpha, 0)$, $a_2 = (\cos\alpha, -\sin\alpha, 0)$ and c_1, c_2 are both 1×3. Suppose that we set

$$\xi = c_1 P_1 + c_2 P_2 , \quad e_i = a_i^T$$

and let y_i be unit vector orthogonal to e_i and ξ. The range of P_i is $\text{lin}(e_i, \xi)$, so we have to have

$$\xi = P_1 c_1^T + P_2 c_2^T \quad \in \mathcal{R}(P_1) \cap \mathcal{R}(P_2)$$

$$\therefore \quad P_2 c_2^T \in \mathcal{R}(P_1) , \quad P_1 c_1^T \in \mathcal{R}(P_2)$$

(i) If $P_1 = P_2$, then the projection P must be orthogonal projection onto $\text{lin}(e_1, e_2)$,

$$P = (e_1 \; e_2) \left(\begin{pmatrix} e_1^T \\ e_2^T \end{pmatrix} (e_1 \; e_2) \right)^{-1} \begin{pmatrix} e_1^T \\ e_2^T \end{pmatrix} = \frac{1}{\sin^2 2\alpha} (e_1 \; e_2) \begin{pmatrix} 1 & -\cos 2\alpha \\ -\cos 2\alpha & 1 \end{pmatrix} \begin{pmatrix} e_1^T \\ e_2^T \end{pmatrix}$$

and if $c \equiv (c_1 + c_2)^T$ we shall need that $\{Pc, e_1\}$ spans $\mathcal{R}(P)$, $\{Pc, e_2\}$ spans $\mathcal{R}(P)$, which means that neither of the components of

$$\begin{pmatrix} 1 & -\cos 2\alpha \\ -\cos 2\alpha & 1 \end{pmatrix} \begin{pmatrix} e_1^T c \\ e_2^T c \end{pmatrix}$$

may vanish - but that will be OK generically.

(ii) More interesting is the possibility that $P_1 \neq P_2$. Since we must then have $P_1 c_1^T \in \mathcal{R}(P_2)$, $P_2 c_2^T \in \mathcal{R}(P_1)$ it is clear that $P_1 c_1^T$ and $P_2 c_2^T$ must both be multiples of ξ. But

$$P_i = \text{const.} \, (e_i \; \xi) \begin{pmatrix} |\xi|^2 & -e_i^T \xi \\ -e_i^T \xi & 1 \end{pmatrix} \begin{pmatrix} e_i^T \\ \xi^T \end{pmatrix}$$

$$\therefore \quad P_i c_i^T \propto e_i \left\{ |\xi|^2 e_i^T c_i^T - e_i^T \xi \, \xi^T c_i^T \right\} + \text{const.} \, \xi .$$

Thus $\tilde{\xi} \equiv \xi / |\xi|$ must have the property $\boxed{(e_i^T \tilde{\xi})(\tilde{\xi}^T c_i^T) = e_i^T c_i^T}$ $\quad (i = 1, 2)$

Here's something which might be useful here. If P, Q are orthogonal projections, then the orthogonal projection onto the span of their ranges can be expressed in a formal sense as

$$P(I-QP)^{-1}(I-Q) + Q(I-PQ)^{-1}(I-P)$$

Though $I-QP$ isn't invertible, we have $(I-QP)x=0 \Rightarrow x = QPx = Q^2Px = Qx$, so x is in the range of Q. However, we do have it expressed as

$$\lim_{\lambda \uparrow 1} P(I-\lambda QP)^{-1}(I-Q) + Q(I-\lambda PQ)^{-1}(I-P)$$

Another observation. If we have $n \times r$ matrix B, $r<n$, full rank, then projection onto the subspace spanned by the columns of B is

$$\Pi = B(B^TB)^{-1}B^T$$

If we have <u>two</u> matrices B_1, B_2, proj^n onto the subspace spanned by $(B_1\ B_2)$ is expressible as

$$B_1(B_1^T(I-\Pi_2)B_1)^{-1}B_1^T(I-\Pi_2) + B_2(B_2^T(I-\Pi_1)B_2)^{-1}B_2^T(I-\Pi_1)$$

This the sum of two projections (not evidently orthogonal) and has to be symmetric, another property that's not obvious!

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We can wlog assume that $|c_i| = |e_i| = |\xi| = 1$. The equations to be satisfied by ξ can as well be expressed as

$$\xi^T (c_i^T e_i^T)\xi = c_i e_i \qquad (i=1,2).$$

Thus we are looking for the level sets of a quadratic form. If we consider the quadratic form given by the symmetric matrix

$$S = \tfrac{1}{2}(ab^T + ba^T)$$

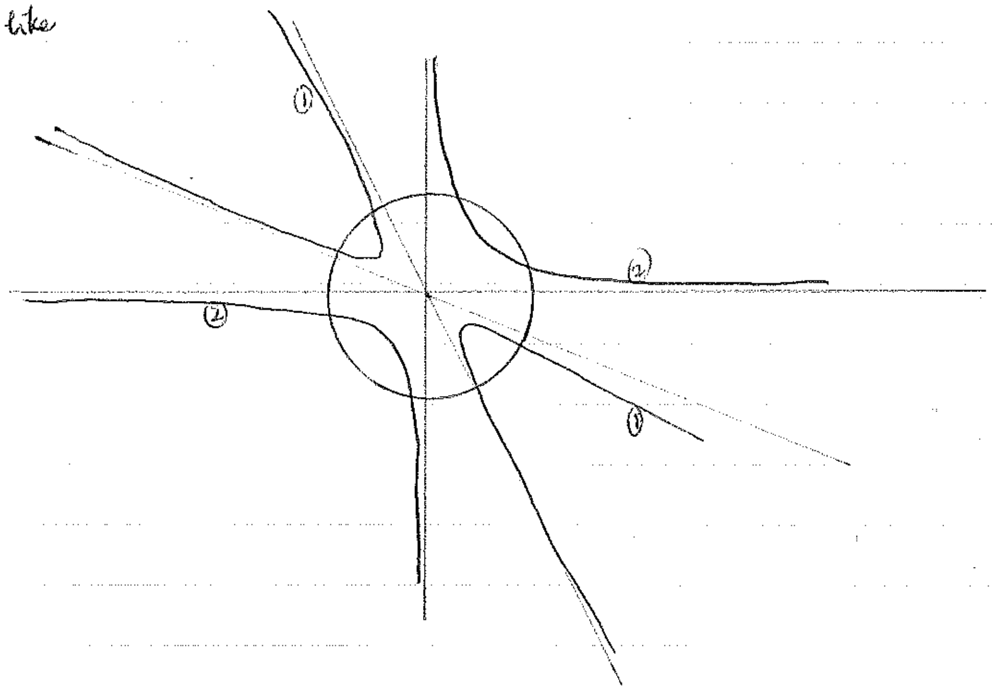
where a, b are unit vectors, then S has evectors (assuming a, b both unit vectors)

$a+b$, evalue $(1+a^Tb)$ $\in (0,2)$

$a-b$, evalue $(b^Ta - 1)$ $\in (-2,0)$

assuming a, b not collinear. Thus the surface where $x^TSx = 1$ is a hyperbola in the plane spanned by (a,b), and in $\mathbb{R}^3$ is the set of points whose orthogonal projection onto $\mathrm{lin}(a,b)$ is that hyperbola.

Various Maple plots show that there are often 4 points ξ for which $|\xi| = 1$ and $\xi^T(c_i^T e_i^T)\xi = c_i e_i$ $(i=1,2)$. However, it may also happen that there can be none; for consider the situation where c_1^T, c_2^T, e_1, e_2 all lie in the same plane; we could have a picture like



So we cannot say with certainty (of course, a tiny perturbation of one of those would break the condition that the 4 vectors are coplanar, **but** not the conclusion that there are no common points)

3) Suppose we get $P_1 \neq P_2$. If the agents pool their knowledge, they would know $a_1\xi$, $a_2\xi$, which is (in effect) two 'components' of ξ. Is it the case that the agents agree on these two components? This would mean that $a_1 P_2 = a_1$, $a_2 P_1 = a_2$. If this were the case, it would follow that $a_2^T \in \mathcal{R}(P_1)$, and thus $\mathcal{R}(P_2) \subseteq \mathcal{R}(P_1)$, a contradiction; so in this case the agents do not agree on what they would know by pooling their knowledge!!

A very simple model for international trade (6/11/03)

1) It seems that we need to be able to incorporate more than one commodity as well as multiple countries etc. So let's suppose we index countries by $j = 1, \dots, J$, and commodities by $m = 1, \dots, M$. In each country, there are R productive activities, and carrying on industry r at intensity θ produces (net) $a_{mr}\,\theta$ of commodity m, and requires $\beta_r\,\theta$ of labour. The constants a_{mr}, β_r may depend on the country j; we use superscript notation for the country when appropriate to make it explicit. In country j, there is a maximum available workforce $\bar{L}^j$, and the population has utility $U^j(c, L)$ for a bundle c of commodities consumed while exerting labour L.

2) Suppose shipping commodity m from i to j costs s_m^{ij} per unit shipped, where we'll assume $s_m^{ii} = 0$ $\forall i, \forall m$, $s_m^{ij} \geq 0$. Consider the optimisation problem

$$\max \sum_{j=1}^{J} \lambda^j\, U^j(c^j, L^j) - \sum\sum\sum s_m^{ij}\, x_m^{ij}$$

where x_m^{ij} denotes the quantity of asset m shipped from country i to country j. There are a number of constraints to be satisfied

$$\begin{cases} x_m^{ij} \geq 0, \quad \theta_r^j \geq 0 \\ A^j \theta^j - c^j = \sum_\ell \left(x^{j\ell} - x^{\ell j}\right) \\ L^j = \sum_r \beta_r^j\, \theta_r^j \end{cases}$$

(net output less consumption = net exports)

Thus the Lagrangian formulation is

$$\max \sum_j \lambda^j U^j(c^j, L^j) - \sum_{ij} s^{ij}\cdot x^{ij} + \sum_j \eta^j\cdot\Big(A^j\theta^j - c^j - \sum_\ell (x^{j\ell} - x^{\ell j})\Big) + \sum_j y^j \left(L^j - \beta^j\cdot\theta^j\right)$$

So we get conditions

$$\boxed{\begin{array}{l} \lambda^j \dfrac{\partial U^j}{\partial c} = \eta^j, \quad \lambda^j \dfrac{\partial U^j}{\partial L} = -y^j \\ \eta^j \leq \eta^\ell + s^{\ell j}, \quad \text{equal when } x^{\ell j} > 0 \\ \eta^j\cdot A^j \leq y^j \beta^j, \quad \text{equal where } \theta^j > 0 \end{array}}$$

– nothing shipped $\ell \to j$ if not profitable

– production processes not active if cost of labour exceeds profit...

3) It would not be difficult to include some modelling of the effect of capital, which could be explained as reducing the amount of labour needed to effect unit activity in the different productive activities:

$\beta^j_r(\kappa)$ units of labour required if κ is capital assigned to that process.

We'd suppose country j had total capital K^j, assigning κ^j_r to production process r. It seems natural to suppose each $\beta^j_r(\cdot)$ is convex and decreasing. We add

$$+\sum_j v_j\left(K^j - \sum_r \kappa^j_r\right)$$

to the Lagrangian, and get the condition

$$-y_j \; D\beta^j_r(\kappa^j_r)\,\theta^j_r = v_j \qquad \text{if } \theta^j_r > 0$$
$$\le v_j \qquad \text{if } \theta^j_r = 0$$

(19/12)

4) The original formulation has the silly consequence that solutions typically have inefficient countries exporting nothing and importing everything ... so we need to make it costly to do this. Without introducing notions of capital accounts, deficits etc, we could modify the specification of losses in trading by saying that if x^{ij}_m is amount sent out from country i to country j, then country j actually receives $\tau^{ij}_m x^{ij}_m$, where $0 < \tau^{ij}_m < 1$. Thus the equation for conservation of quantity is

$$\boxed{A^j\theta^j - c^j = \sum_\ell \left(x^{j\ell} - \tau^{\ell j} x^{\ell j}\right)}$$

and we shall demand that the consumers in each country can only consume what their wages entitle them to:

$$\boxed{p^j c^j = w^j L^j}$$

where we have

$$p^j = \frac{\partial U^j}{\partial c}(c^j, L^j), \quad w^j = -\frac{\partial U^j}{\partial L}(c^j, L^j)$$

Thus the constraint on consumption is actually a non-linear constraint; it also can be combined with the conservation of quantity constraint to give a zero-profit condition for the perfectly competitive industry in country j:

$$p^j A^j\theta^j - w^j L^j = p^j \sum_\ell \left(x^{j\ell} - \tau^{\ell j} x^{\ell j}\right).$$

__NB__ : must ensure that $H_T \geq 0$; this can be done by some BC on V,

such as

$$V(T, s, H) = \begin{cases} \exp\{\gamma \tilde{\beta}(T,s)\} & \text{if } H \geq 0 \\ M & \text{if } H < 0 \end{cases}$$

where M is v. big

Perturbation of the price of a put option in the liquidity model (17/11/03)

1) Recall the liquidity model of WN XXII, pp 25–28, in which the evolution of the wealth of the investor is given by

$$dW_t = rW_t\,dt + H_t(dS_t - rS_t\,dt) - h_t f(h_t) S_t\,dt$$

where $h_t = \dot H_t$, and H_t is the number of shares held at time t. If y_t denotes the value of the cash account at time t, then $y_t = W_t - H_t S_t$, so

$$dy_t = r y_t\,dt - S_t(h_t + h_t f(h_t))\,dt$$

So if $\tilde y_t \equiv e^{-rt} y_t$, $A_t = e^{-rt} S_t$, we shall have

$$\boxed{d\tilde y_t = -A_t\left(h_t + h_t f(h_t)\right)\,dt}$$

We have $dA = (\sigma\,dW_t + (\mu - r)\,dt)A_t$.

2) Suppose we consider a put option with strike K; let $p(t, S_t)$ be the time-t price of the option (Black-Scholes) if the current stock price is S_t (expiry $T > t$).

Let
$$\tilde p(t, A_t) = e^{-rt} p(t, e^{rt} A_t)$$

and let

$$V(t, A, H) = \min E\left[\exp\left\{-\gamma\left(\tilde y_T - (Ke^{-rT} - A_T)^+\right)\right\} \,\Big|\, A_t = A,\ H_t = H,\ \tilde y_t = 0\right]$$

$$= \exp\left\{\gamma \tilde p(t, A_t) - \gamma A H\right\} \text{ in the Black-Scholes setting}$$

Standard BS theory says that

$$\tilde p_t + \tfrac12 \sigma^2 A^2 \tilde p_{AA} = 0$$

Thus we propose to present $V(t,A,H)$ as

$$V(t,A,H) = \exp\left\{\gamma \tilde p(t, A) + \gamma\varphi(t,A,H) - \gamma A H\right\}$$

where φ is going to be small, and develop the HJB equation;

$$0 = \min\left\{V_t + (\mu - r)A V_A + \tfrac12 \sigma^2 A^2 V_{AA} + h V_H + \gamma A h(1 + f(h)) V\right\}$$

This leads to

If we write

$$V(t,s,H) = \exp\left[\gamma \tilde{p}(t,s) - \gamma s H - \tfrac{1}{2}\theta^2 (T-t) \right] \left\{ 1 + \Phi(t,s,H) \right\} \qquad \theta \equiv (\mu - r)/\sigma$$

we find the HJB becomes

$$0 = \min_h \Big[\tfrac{1}{2}(1+\Phi)\left(\theta + \gamma\sigma s(\tilde{p}_s - H)\right)^2 + \Phi_t + \tfrac{1}{2}\sigma^2 s^2 \Phi_{ss} + \sigma s \left(\theta + \gamma\sigma s(\tilde{p}_s - H)\right)\Phi_s$$
$$+ h\Phi_H + \varepsilon\gamma s h^2 (1+\Phi) \Big]$$

$$0 = \min_{h} \Big[\tilde{p}_t + \varphi_t + (\mu - r)s(\tilde{p}_s + \varphi_s - H) + \tfrac{1}{2}\sigma^2 s^2 \Big\{ \tilde{p}_{ss} + \varphi_{ss} + \gamma(\tilde{p}_s + \varphi_s - H)^2 \Big\}$$
$$+ h(\varphi_H - s) + s\,h(1 + f(h)) \Big]$$

Assuming now $f(h) = \varepsilon h$, and using the PDE satisfied by $\tilde{p}$, we get more simply

$$\boxed{0 = \varphi_t + (\mu - r)s(\tilde{p}_s + \varphi_s - H) + \tfrac{1}{2}\sigma^2 s^2 \Big\{ \varphi_{ss} + \gamma(\tilde{p}_s + \varphi_s - H)^2 \Big\} - \frac{\varphi_H^2}{4\varepsilon s}}$$

Thus we expect that to leading order

$$\boxed{\varphi(t, s, H) = \sqrt{\varepsilon}\, g(t, s, H)}$$

where

$$0 = (\mu - r)s(\tilde{p}_s - H) + \tfrac{1}{2}\sigma^2 s^2 \gamma(\tilde{p}_s - H)^2 - g_H^2 / 4s \quad \ldots$$

but this cannot be solved if $\mu \neq r$! Two possible resolutions suggest themselves:

(a) Assume that we take expectations in the risk-neutral prob'y (though this seems a bit of a fake !) Then we get

$$g_H = \sigma(\tilde{p}_s - H)\sqrt{2s^3}$$

with the conclusion that $g(t, s, H) = \sigma(\tilde{p}_s - H)^2 \sqrt{\tfrac{1}{2}s^3} + g_0(t, s)$

where g_0 remains to be determined ...

(b) Seek a solution of the form

$$\varphi(t, s, H) = \psi(t, s) + \sqrt{\varepsilon}\, g(t, s, H)$$

Then to leading order we would see

$$0 = \psi_t + (\mu - r)s(\tilde{p}_s + \psi_s - H) + \tfrac{1}{2}\sigma^2 s^2 \big(\psi_{ss} + \gamma(\tilde{p}_s + \psi_s - H)^2 \big) - \frac{g_H^2}{4s}$$

Now we expect that for fixed t, s, $g(t, s, \cdot)$ will be decreasing then increasing; so there will be a unique point where $g_H = 0$; and this must be where the above quadratic in H is minimised. The quadratic is minimised where

$$\boxed{H = \tilde{p}_s + \psi_s + \frac{\mu - r}{s\gamma\sigma^2}}$$

We end up expanding φ as

$$\varphi(t,s,H) = -\frac{(\mu-r)^2}{2\gamma\sigma^2}(T-t) + \sum_{n\geq 1} \varepsilon^{n/2}\, g_n(t,s,H)$$

where $g_1 = g$ for brevity.

and the resulting PDE for ψ will be

$$\psi_t - \frac{(\mu-r)^2}{2\gamma\sigma^2} + \tfrac{1}{2}\sigma^2 s^2 \psi_{ss} = 0$$

solved by

$$\psi(t,s) = -\frac{(\mu-r)^2}{2\gamma\sigma^2}(T-t).$$

Does this make sense? Yes!! If the agent was trying to maximise the expected utility in the absence of hedging, and without the put option, then the optimal portfolio would be

$$H = \frac{\mu-r}{\gamma_s \sigma^2}$$

and the optimised value of the objective would be

$$\exp\left\{-\frac{\mu-r}{2\gamma\sigma^2}T\right\}.$$

Thus the leading-order effect, given by g, is found by solving the PDE for g; this leads to

$$\boxed{g(t,s,H) = \sqrt{\frac{\sigma^2 s^3 \gamma}{2}}\left(H - \tilde{p}_s - \frac{\mu-r}{\gamma_s\sigma^2}\right)^2 + F(t,s)}$$

What can we say of the function F?

Various Maple calculations (/WORK/SURB/LIQUID/liq-put1.mws) show that F solves

$$\boxed{\tfrac{1}{2}\sigma^2 s^2 F_{ss} + F_t + \left(\tilde{p}_{ss} - \frac{\mu-r}{\gamma\sigma^3 s^2}\right)^2 \sigma^3 s^{7/2}\sqrt{\frac{\gamma}{2}} = 0}$$

As boundary condition we expect $g(T,s,0) = 0$, so

$$\boxed{F(T,s) = -\sqrt{\frac{\sigma^2 s^3 \gamma}{2}}\left(\tilde{p}_s + \frac{\mu-r}{\gamma_s\sigma^2}\right)^2}$$

A remark on a seminar of Ben Hambly (26/11/03)

1) Ben Hambly worked with a student on a multiple-exercise American option problem, using the dual approach. So the problem is to calculate

$$Y_0^n \equiv \sup_{0 \le \tau_1 < \tau_2 < \dots < \tau_n \le T} E\left[\sum_1^n Z(\tau_j)\right]$$

(so really this makes more sense as a Bermudan problem). The case $n=1$ is the base case.

2) A natural conjecture here is that

$$Y_0^n = \inf_{M \in H_0^1} E\left[\sup_{0 \le t_1 < t_2 < \dots < t_n \le T} \sum_{j=1}^n \left(Z(t_j) - M(t_j)\right)\right]$$

which would be nicer than what Ben got – that expression involves taking an inf over policies also. It's clear that Y_0^n is bounded above by the RHS, but is there equality?

3) It looks like the answer is 'No'. If we follow through the kind of duality arguments in the appendix to 'Monte Carlo valuation of American options', taking

$$\mathcal{A} = \{\text{raw increasing processes } C,\ C_{0-}=0,\ C_T = n,\ \Delta C_j = 1\}$$

(so let's admit we're looking at discrete time now) then we get

$$\inf_{M \in H_0^1} \sup_{C \in \mathcal{A}} \left\{ \Phi(C) - (M,C) \right\} = \inf_{M \in H_0^1} E\left[\sup_{0 \le t_1 < \dots < t_n \le T} \sum_1^n \{Z(t_j) - M(t_j)\}\right]$$

$$\ge \sup_{C \in \mathcal{A}} \inf_{M \in H_0^1} E\, \Phi(C) - (M,C)$$

Now $\sup_M E \sum_1^T M_j \Delta C_j = \sup_M E \sum_1^T M_j \Delta \tilde{C}_j = \sup_M E(M_T \tilde{C}_T)$, where $\tilde{C}$ is the dual optional projection, $E(\Delta C_j | \mathcal{F}_j) = \Delta \tilde{C}_j$. We must therefore have $\tilde{C}_T = \text{const} = n$. If we take such an optional increasing process, it is clear that $\Delta \tilde{C}_j \le 1$, since $\Delta C_j \le 1$, however the idea of using the stopping times from the inverse of $\tilde{C}$ will fail because we might find two stopping times land at the same instant ...

False discovery rates: a Bayesian version (27/11/03)

1) There's a paper of Storey et al which considers the problem of false discovery; you perform N tests, one on each of N different objects, and try to determine whether an object is a 'positive'. Suppose you perform an observation X_j on object j, where X_j has density f_0 if it is a negative, f_1 if a positive. Suppose the prior prob'y of being a positive is p: then the posterior prob'y that X_j is a positive is

$$\pi_j = \frac{p f_1(X_j)}{p f_1(X_j) + q f_0(X_j)}$$

Let $(\theta_j)_{j=1}^N \equiv \theta$ be the vector of true values, $\theta_j = 0$ for negative, $=1$ for positive, and suppose we use a decision rule $d = d(X) = (d_j(X))_{j=1}^N$. If we define

$$R = \sum_1^N d_j \equiv \#\text{ of declared positives}$$

$$V = \sum_1^N d_j(1-\theta_j) = \#\text{ of false positives}$$

$$Y = \sum_1^N (1-d_j)\theta_j = \#\text{ of false negatives},$$

we propose a loss function

$$L(\theta, d) = \alpha V + \beta Y + \gamma V/(R \vee 1)$$

– the final term is the interesting one, being a penalty for the <u>false discovery rate</u>.

2) What is the Bayes rule here?

$$E\left[L(\theta,d)\right] = \alpha \sum_1^N d_j(1-\pi_j) + \beta \sum_1^N (1-d_j)\pi_j + \gamma\left(1 - \frac{\sum_1^N \pi_j d_j}{\sum_1^N d_j}\right) I_{\{\sum d_j > 0\}}$$

So if we differentiate wrto d_j we get

$$\alpha(1-\pi_j) - \beta\pi_j - \gamma\left\{\frac{\pi_j}{\sum d_i} - \frac{\sum \pi_i d_i}{(\sum d_i)^2}\right\}$$

$$= -\pi_j\left\{\alpha + \beta + \frac{\gamma}{\sum d_i}\right\} + \alpha + \frac{\gamma \sum \pi_i d_i}{\sum d_i^2}$$

Thus the Bayes rule is of a threshold form; we decide 1 if π_j is large enough.

3) How to determine the threshold? Suppose wmlog that we have ordered the observations so that $\pi_1 \geq \pi_2 \geq \dots \geq \pi_N$. Then we choose 1 for all k for which the derivative wrto d_k is negative; unpacking this gives the condition

Find largest k for which

$$\pi_k > \frac{\alpha k + \gamma\left(\sum_{1}^{k} \pi_j\right)/k}{k(\alpha+\beta) + \gamma}$$

A simple model of liquidity effects (1/12/03)

1) During Umut Cetin's visit, we considered the following (discrete-time) model. At each time $n = 1, \dots, N$, the price of a stock jumps, from $S_{n-} \equiv S_{n-1}$ to S_n. At time $n - \frac{1}{2}$, an agent adjusts his portfolio (holding) in the stock from H_{n-1} to H_n; the change in his bank account is given by

$$Y_n - Y_{n-1} = -\Delta H_n \, \tilde{\varphi}(\Delta H_n) \, S_{n-1} \equiv -\varphi(\Delta H_n) \, S_{n-1}$$

where we suppose φ is convex, with $0 < \tilde{\varphi}$, $\tilde{\varphi}(0) = 1$.

Suppose the objective is to

$$\max \; E \, U(Y_N)$$

where we insist $H_N = 0$, and Y_0, H_0 are given to us. What can we say about the optimal solution? (assumed for now to exist).

2) If H^* is optimal portfolio process, and we perturb to $H^* + \eta$, then we get the first-order condition

$$0 = E^*\left[\sum_{n=1}^N \varphi'(\Delta H_n) S_{n-1} \Delta\eta_n \right] \qquad \left[\frac{dP^*}{dP} \propto U'(Y_N^*) \right]$$

$$\equiv E^*\left[\sum_{n=1}^N \tilde{S}_{n-1} \Delta\eta_n \right] \qquad \left[\tilde{S}_n \equiv \varphi'(\Delta H_{n+1}) S_n \right]$$

$$= E^*\left[\sum_{n=1}^N \left(\tilde{S}_0 + \sum_{r=1}^{n-1} \Delta\tilde{S}_r \right) \Delta\eta_n \right]$$

$$= E^* \sum_{r=1}^N \Delta\tilde{S}_r \, (-\eta_r) \qquad \text{since } \sum \Delta\eta_n = 0$$

Since η is an arbitrary previsible process, we conclude that

$$\boxed{\text{Under } P^*, \text{ the marginal price } \tilde{S} \text{ is a martingale.}}$$

<u>Abandoning assets: how is it for general diffusions? (3/12/03)</u>

Suppose that the share delivers a dividend stream δ which is a one-dimensional diffusion with generator

$$\mathcal{G} = \frac{1}{2}\frac{d}{dm}\frac{d}{ds} = \frac{1}{2}\frac{1}{m'}D\left(\frac{1}{s'}D\right)$$

If we run with this until we reach level b, then dump the asset and revert to a subsistence consumption level ε, we get payoff

$$E^x\left[\int_0^{H_b} e^{-\lambda t}U(\delta_t)\,dt + \int_{H_b}^\infty e^{-\lambda t}U(\varepsilon)\,dt\right]$$

$$= \frac{1}{\lambda}U(\varepsilon) + E^x\int_0^{H_b}\{U(\delta_t)-U(\varepsilon)\}e^{-\lambda t}\,dt$$

$$= \frac{1}{\lambda}U(\varepsilon) + \int_b^\infty r_\lambda^b(x,y)\,\big(U(y)-U(\varepsilon)\big)\,m(dy)$$

where the resolvent density with killing at b is given by

$$r_\lambda^b(x,y) = c_\lambda\left[\psi_\lambda^+(x\wedge y)\,\psi_\lambda^-(x\vee y) - \psi_\lambda^-(x)\,\frac{\psi_\lambda^+(b)\,\psi_\lambda^-(y)}{\psi_\lambda^-(b)}\right]$$

What is the best choice of b? As before, we want the derivative wrto x to vanish at $x=b$, so we look at

$$\frac{1}{x-b}\left[\int_b^x r_\lambda^b(x,y)\big(U(y)-U(\varepsilon)\big)\,m(dy) + \int_x^\infty r_\lambda^b(x,y)\big(U(y)-U(\varepsilon)\big)\,m(dy)\right]$$

$$\to \frac{W(b)}{\psi_\lambda^-(b)}\int_b^\infty\big(U(y)-U(\varepsilon)\big)\,\psi_\lambda^-(y)\,m(dy)$$

So our criterion is

$$\boxed{\int_b^\infty\{U(y)-U(\varepsilon)\}\,\psi_\lambda^-(y)\,m(dy) = 0}$$

(This agrees with WN $\underline{\text{XXI}}$ p 24 for case of drifting BM where $\mathcal{G} = \frac{1}{2}\sigma^2D^2+\mu D$ and the test is

$$\int_b^\infty\{U(y)-U(\varepsilon)\}\,e^{-\beta y}\,dy = 0, \qquad \beta \equiv \left(\sqrt{\mu^2+2\lambda\sigma^2}-\mu\right)/\sigma^2$$
)

Care! If we are seeing prices from an equilibrium of CARA agents, the c's could themselves depend on the P_i

Some remarks on combining projections (5/12/03)

1) Suppose we have matrices B_1, B_2 which are $n \times r_1$, $n \times r_2$ respectively, with $r_1, r_2 < n$, both of B_1 and B_2 of full rank, and rank $(B_1, B_2) = r_1 + r_2$ (so the subspaces spanned by the columns of B_i intersect only at 0).

The orthogonal projection onto $\mathrm{lin}(B_i)$ is given by

$$\Pi_i \equiv B_i (B_i^T B_i)^{-1} B_i^T$$

The orthogonal proj^n onto the subspace spanned by the columns of both is expressible as

$$\Pi \equiv \beta_1 (I - \Pi_2) + \beta_2 (I - \Pi_1)$$

where $\beta_i \equiv B_i \left(B_i^T (I - \Pi_{3-i}) B_i \right)^{-1} B_i^T$, $i = 1, 2$, a symmetric matrix. Notice that β_i is not a projection, but

$$\beta_1 (I - \Pi_2) \quad , \quad \beta_2 (I - \Pi_1) \quad \text{are both proj}^{ns}.$$

Now observe that Π is symmetric, so we have to have

$$\beta_1 \Pi_2 + \beta_2 \Pi_1 = \Pi_1 \beta_2 + \Pi_2 \beta_1$$

and this can only happen if (recall that B_1, B_2 have trivial intersection)

$$\Pi_1 \beta_2 = \beta_1 \Pi_2 \quad , \quad \Pi_2 \beta_1 = \beta_2 \Pi_1 .$$

Thus

$$\Pi = \beta_1 + \beta_2 - \beta_1 \Pi_2 - \beta_2 \Pi_1 = \beta_1 + \beta_2 - \Pi_1 \beta_2 - \Pi_2 \beta_1 = (I - \Pi_1)\beta_2 + (I - \Pi_2)\beta_1$$
$$= (\beta_2 (I - \Pi_1))^T + (\beta_1 (I - \Pi_2))^T$$

2) Returning to the divergence of opinion example, with 3-dimensional noise $\xi \sim N(0, I)$ and with a pair of agents, each seeing a one-dimensional signal $a_j^T \xi$ ($|a_j| = 1$ wlog) as well as a 1-dimensional price of the form $c^T (P_1 + \lambda P_2) \equiv \zeta$, where λ is given, and c is given; the idea is to see what ζ can satisfy this, where P_i is proj^n onto the subspace of a_i, ζ. Maybe neatest to work with unit vector $u \propto \zeta$, and then look in the end for $u \propto c^T (P_1 + \lambda P_2)$. By the preceding, we find

$$P_1 = \frac{a_1 a_1^T (1 - u u^T)}{1 - (a_1^T u)^2} + \frac{u u^T (1 - a_1 a_1^T)}{1 - (a_1^T u)^2}$$

Maple?

<u>More on MC valuation of American options (8/12/03)</u>

1) We have been thinking of the 'good' martingales as ones which keep the mean of $\sup_t (Z_t - M_t)$ low, but we can instead think of the 'good' martingales as ones which keep the variance of $\sup_t (Z_t - M_t)$ small

<u>Proposition</u> Suppose that for some UIMG$_0$ $\bar{M}$ we have for some $x \in \mathbb{R}$

$$\sup_t (Z_t - \bar{M}_t) = x \quad \text{a.s.}$$

Then $x = Y_0^*$, the value of the American option.

<u>Proof</u>. We have

$$Z_t \le x + \bar{M}_t \qquad \forall t$$

so it follows that $Y_t^* \le x + \bar{M}_t$, and since $Z_t \le Y_t^*$ we conclude that

$$\sup_t (Y_t^* - \bar{M}_t) = x \quad \text{a.s.}$$

Suppose that $Y_0^* < x$, and take $a \in (Y_0^*, x)$. Let $\tau = \inf\{t : Y_t^* - \bar{M}_t > a\}$, a time that is a.s. strictly less than T, and a time at which the value of the supermartingale $Y_t^* - \bar{M}_t$ is $\ge a$. Then we have

$$a \le E(Y_\tau^* - \bar{M}_\tau) = E(Y_\tau^*) \le Y_0^* \quad \text{※}. \qquad \square$$

<u>Remark</u>. We do not necessarily have that $\bar{M} = M^*$, because we could consider a situation where at time $T/2$ the process Z jumps to a large negative value and stays there. Now clearly M^* will be constant in $[T/2, T]$, but we could allow $\bar{M}$ to move around a bit during that interval. **<u>MORAL</u> : the dual prob is to minimise the variance of $\sup_t (Z_t - M_t)$.**

2) A good martingale makes a good stopping rule. Suppose we have found some UI martingale M such that

$$\xi \equiv \sup_t (Z_t - M_t)$$

is well bunched around its mean - small variance. If we also make the assumption

Z is bounded below (by zero, wlog),

we can use a stopping time of the form

$$\tau = \inf\{t : Z_t - M_t > a\}$$

and then $P(\tau = T) = F(a)$ (F is CDF of ξ) and

$$E Z_\tau = E(Z_\tau - M_\tau) \ge a\, P(\tau < T) + E[-M_T : \tau = T]$$

We will make the final term small by getting $P(\tau = T) = F(a)$ small - but because F is 'close to' the mean, we can actually make a just a little below the mean. Best choice is going to depend on the law of M_T, and of ξ. In practice, probably the best thing will be to simulate, and on each path carry out the stopping rule for a set of different a-values, then pick the best.

3) A good stopping rule gives a good martingale. This was something that Terry Lyons guessed would happen. However the original conjecture that we could use $M_t = E(Z_\tau | \mathcal{F}_t)$ seems not to be so good, because if we were to take $\tau = \tau^* = \inf\{t : A^*_t > 0\}$, we would find

$$E \sup_t (Z_t - M_t - Y^*_0) = E \sup_{t \geq \tau^*} (Z_t - M_{\tau^*} - Y^*_0)$$

because $Z_{\tau^*} = Y_{\tau^*} = Y^*_0 + M^*_{\tau^*}$ (assuming A^* is continuous, as will often be the case) so that $Y^*_0 + M_t = E_t Z_{\tau^*} = Y^*_0 + M^*_t = Y_t$ for $t \leq \tau^*$, so that $Z_t - M_t - Y^*_0$ is bounded above by 0 on $[0, \tau^*]$. The process $Z_t - M_{\tau^*} - Y^*_0$ takes value 0 at $t = \tau^*$, but there's no reason why it has to stay close to zero — the moral is, we can't give up hedging as soon as the holder should have stopped, we have to keep going, and this is why it's not enough just to do $M_t = E(Z_\tau | \mathcal{F}_t)$ for a good stopping rule.

Maybe all we can do is to look for some Y which is 'close' to Y^* in some sense, then if

$$Y_t = Y_0 + M_t + C_t$$

then

$$E \sup_t (Z_t - M_t) = E \sup_t (Z_t - Y_t + Y_0 + C_t)$$

$$\leq E \sup_t (Y^*_t - Y_t + Y_0 + C_t)$$

and so if $Y^* - Y$ is small, and C doesn't increase too much, we have a viable martingale. This is pretty much what Andreasen + Broadie do.

Dumping assets again (12/1/04)

(i) The story that John + I have been developing in cts time is not really giving us what we want, because when an asset gets dumped, its market price is zero - but the market prices of the remaining assets will typically not be zero, so they don't get dumped at the same time - they may be more likely to be dumped in the near future, but that isn't what we want - it's hard to quantify this effect.

(ii) Maybe we need to go back to discrete time, with (initially) M assets, and dividends $\delta_t \equiv (\delta_t^1, \dots, \delta_t^M)^T$ at time t. We could make δ_t some AR(1) process, but for now let's imagine they are IID $N(\mu, \Sigma)$, and that (μ, Σ) are known (could do a filtering story as well, but let's not just for the moment)

The representative agent with CARA utility sees the vector δ_t, and then decides which assets (if any) to dump, and proceeds. If $V(\delta; A)$ denotes the value to the representative agent if $A \subseteq \{1, \dots, M\}$ denotes the set of assets which are alive currently, then we shall have

$$V(\delta; A) = \sup_{a \subseteq A} \left[-\exp(-\gamma\, \delta \cdot 1_a) + \beta\, E\, V(\tilde{\delta}; a) \right]$$

as the equations to solve.

(iii) If $A = a = \{1\}$ is a singleton, we have with $v(a) \equiv E\, V(\tilde{\delta}; a)$ that $v(\emptyset) = -1/(1-\beta)$ and

$$V(\delta; a) = \max\left[-e^{-\gamma\delta} + \beta\, v(a) \;,\; -\frac{1}{1-\beta} \right]$$

Thus

$$V(\delta; a) = \begin{cases} -\dfrac{1}{1-\beta} & (\delta < \delta_*) \\[2ex] -e^{-\gamma\delta} + \beta\, v(a) & (\delta \geq \delta_*) \end{cases}$$

Now we see the form of $V(\cdot\,; a)$ (it's neither convex nor concave) and we can determine it exactly by computing δ_* from

$$v(a) = -\frac{1}{1-\beta}\, \Phi\!\left(\frac{\delta_* - \mu}{\sigma}\right) + \beta\, v(a)\, \bar{\Phi}\!\left(\frac{\delta_* - \mu}{\sigma}\right) - e^{-\gamma\mu + \frac{1}{2}\sigma^2\gamma^2}\, \bar{\Phi}\!\left(\frac{\delta_* - \mu + \sigma^2\gamma}{\sigma}\right)$$

It should be possible to solve the 2-asset and 3-asset problems numerically. Monte Carlo integration could be a tool for attacking the higher-dimensional problems.

Cash in the utility (13/1/04)

(i) Arguably we derive happiness / satisfaction from the security of ready cash. So if we had the conventional dynamics

$$dw_t = (rw_t - c_t)\,dt + \theta_t\,(\sigma dW_t + (\mu - r)\,dt)$$

with the objective

$$\max E\left[\int_0^\infty e^{-\rho t}\, U(c_t, x_t)\,dt \;\middle|\; w_0 = w\right] \equiv V(w)$$

where $x_t = w_t - \theta_t$ is value of cash holding, then we would have as HJB

$$\sup_{c,\theta}\left[U(c, w-\theta) - \rho V + \tfrac{1}{2}\sigma^2\theta^2 V'' + (rw - c + (\mu - r)\theta) V' \right] = 0$$

and the first-order conditions

$$\begin{cases} U_c = V' \\ -U_x + \sigma\theta V'' + (\mu - r) V' = 0 \end{cases}$$

Thus we have that the optimal c, θ are functions of V', V'', leading to a substantially non-linear equation for V.

(ii) Special case: no consumption

Maybe a bit extreme, but then we'd have

$$U'(w-\theta) = \sigma\theta V''(w) + (\mu - r) V'(w)$$

Not too easy in general, but if U were CRRA, $U'(x) = x^{-R}$, we'd try $V(w) = A\, w^{1-R}/1-R$ and try $\theta = \gamma w$, to end up trying to solve

$$\begin{cases} (1-\gamma)^{-R} = -\sigma\gamma R A + (\mu - r) A \\ \dfrac{(1-\gamma)^{1-R}}{1-R} - \rho \dfrac{A}{1-R} + \tfrac{1}{2}\sigma^2\gamma^2(-R)A + (r + (\mu - r)\gamma) A = 0 \end{cases}$$

for A, γ; this gives a quadratic for γ

Perpetual American put on Markov-modulated asset (4/2/04)

Suppose there's a finite-state Markov chain with Q-matrix Q, and an asset whose log-price X_t solves

$$dX_t = \sigma(\xi_t)\,dW_t + (r(\xi_t) - \tfrac{1}{2}\sigma(\xi_t)^2)\,dt$$

where ξ is the chain.

How do we price and exercise a perpetual American put? It's back to the good old noisy W-H factorisation; if we set

$$V(x,j) = \sup_\tau E\left[\exp\left(-\int_0^\tau r(\xi_u)\,du\right)(K - e^{X_\tau})^+ \;\middle|\; \xi_0 = j,\ X_0 = x\right]$$

then we have to have

$$\tfrac{1}{2}\sigma^2 V_{xx} + (r - \tfrac{1}{2}\sigma^2)V_x - rV + QV = 0$$

in the continue region. Looking for separable solutions $f(\xi_t)e^{\lambda X_t}$ which are martingales gives us

$$\tfrac{1}{2}\sigma^2\lambda^2 f + (r - \tfrac{1}{2}\sigma^2)\lambda f - rf + Qf = 0,$$

so with $g = \lambda f$ we get

$$\lambda\begin{pmatrix} f \\ g\end{pmatrix} = \begin{pmatrix} 0 & I \\ 2\sigma^{-2}(r - Q) & I - 2\sigma^{-2}r\end{pmatrix}\begin{pmatrix} f \\ g\end{pmatrix}$$

We get the usual e-vectors $f_1, \dots, f_n$ with $\lambda_1, \dots, \lambda_n$, $\mathrm{Re}(\lambda_i) < 0$, and now combine to express

$$V(x,\cdot) = \sum_{i=1}^n w_i\, e^{\lambda_i x} f_i$$

and the barrier levels $\{b_j,\ j \in I\}$ and weights $\{w_i\ ;\ i \in I\}$ must be chosen to guarantee

$$\left.\begin{aligned} \sum_{i=1}^n w_i\, e^{\lambda_i b_j} f_i(j) &= (K - e^{b_j})^+ \\ \sum_{i=1}^n \lambda_i w_i\, e^{\lambda_i b_j} f_i(j) &= -e^{b_j} \end{aligned}\right\} \quad \text{for } j = 1, \dots, n$$

However, not all is plain sailing; we can have these conditions satisfied and also there are values x s.t.

$$V(x,j) < K - e^x \qquad \text{for some } j \text{ !!}$$

So how <u>do</u> we find boundary values?

Not so easy!! Sometimes when a jump occurs, the form of V is <u>not</u> $\sum w_i e^{\lambda_i x} f_i(\xi)$ ... if we have already exercised at the new value of ξ at the present X level!!

(Simulation?)

<u>Towards quantiles of BM (5/2/04)</u>

1) Suppose we look to find (for $a < b$)

$$f(x) = E^x\left[\int_0^\infty \exp\left\{-\rho t - \alpha \int_0^t I_{\{X_u \le a\}}\,du - \beta \int_0^t I_{\{X_u \ge b\}}\,du\right\} dt\right]$$

We can stick this together as $\left(\tfrac{1}{2}u^2 = \rho,\ \tfrac{1}{2}\theta_1^2 = \rho+\alpha,\ \tfrac{1}{2}\theta_2^2 = \rho+\beta\right)$

$$f(x) = \begin{cases} \dfrac{2}{\theta_1^2} + A\, e^{\theta_1 (x-a)} & x \le a \\[2ex] \dfrac{2}{u^2} + B\, e^{ux} + C\, e^{-ux} & a \le x \le b \\[2ex] \dfrac{2}{\theta_2^2} + D\, e^{-\theta_2 (x-b)} & x \ge b \end{cases}$$

and use C^1 at $\{a, b\}$ to determine the coefficients A, B, C, D. Denoting $e^{u(b-a)} \equiv Z$, we get (assuming $a < 0 < b$) that

$$f(0) = \Big\{ 2\theta_1\theta_2(u+\theta_1)(u+\theta_2)Z^2 + \big[4(\theta_2 \cosh bu + \theta_1 \cosh au)u^3 + 4(\theta_2^2 \sinh bu - \theta_1^2 \sinh au)u^2 - 4\theta_1\theta_2(\theta_1 \cosh bu + \theta_2 \cosh au)u - 4\theta_1^2\theta_2^2(\sinh bu - \sinh au)\big] Z - 2\theta_1\theta_2(u-\theta_1)(u-\theta_2)\Big\} \Big/ \Big\{ \theta_1\theta_2 u^2 (u+\theta_1)(u+\theta_2)Z^2 - \theta_1\theta_2 u^2 (u-\theta_1)(u-\theta_2)\Big\}$$

(See WORK/JOINT_WORK/OOMEN/fun.mws)

With a bit of effort, we could extract the joint LT of $\left(\int_0^t I_{(X_u \le a)}\,du,\ \int_0^t I_{(X_u \ge b)}\,du\right)$ perhaps -- but this is not the quantile.

<u>Remark</u>: with $\theta_2 = u$ (i.e., $\beta = 0$), we get more simply

$$f(0) = 2\left(\theta_1 + (u-\theta_1)e^{ua}\right) / u^2\theta_1$$

Tree model with CARA agents (17/2/04)

(i) Agents $1, \dots, J$ have CARA utilities $U_j(x) = -e^{-\gamma_j x}$, and each agent tries to

$$\max \; E\Big[\sum_{n=0}^{\infty} \beta_j^n \, U_j(c_j(n)) \Big]$$

Suppose there's a single productive asset, whose dividends are IID r.v.s $(X_n)_{n\ge 0}$. We shall expect

$$\beta_j^n \, U_j'(c_j(n)) = \lambda_j \zeta_n$$

so that

$$\lambda_j^{-1} \beta_j^n \gamma_j / \zeta_n = e^{\gamma_j c_j(n)}$$

whence market clearing tells us

$$X_n = -\Gamma^{-1} \log \zeta_n + \Gamma^{-1} A + \Gamma^{-1} n \log \beta$$

$$\left(\begin{aligned} \Gamma^{-1} &= \sum_1^J \gamma_j^{-1} \\ \Gamma^{-1} \log \beta &= \sum \tfrac{1}{\gamma_j} \log \beta_j \\ \Gamma^{-1} A &= \sum \tfrac{1}{\gamma_j} \log(\gamma_j / \lambda_j) \end{aligned} \right)$$

so that

$$\zeta_n = \beta^n e^{-\Gamma X_n + A}$$

Hence the ex-dividend stock price S_n in period n is

$$S_n = \frac{1}{\zeta_n} E\Big[\sum_{j>n} \zeta_j X_j \Big] = e^{\Gamma X_n} \cdot \frac{\beta}{1-\beta} \cdot K \qquad \left(K \equiv E(X_1 e^{-\Gamma X_1}) \right)$$

in units of the consumption good.

What we find is that agent j consumes

$$\boxed{c_j(n) = \frac{1}{\gamma_j} \Gamma X_n + n \frac{\log \beta_j / \beta}{\gamma_j} + \frac{1}{\gamma_j} \left(\log \frac{\gamma_j}{\lambda_j} - A \right)}$$

NPV of agent j's future consumption is

$$E_n \Big\{ \sum_{m>n} \zeta_m c_j(m) \Big\} / \zeta_n = \frac{e^{\Gamma X_n} \beta}{1-\beta} \left\{ \frac{\Gamma E(X_1 e^{-\Gamma X_1})}{\gamma_j} + E(e^{-\Gamma X_1}) \left(n b_j + k_j + \frac{b_j}{1-\beta} \right) \right\}$$

where $b_j \equiv \gamma_j^{-1} \log \beta_j / \beta$, $k_j \equiv \gamma_j^{-1} \left(-A + \log(\gamma_j / \lambda_j) \right)$ (so $\sum_j b_j = 0 = \sum_j k_j$). From this, we see that at the end of period n, the number of shares held by agent j will be

$$\frac{\Gamma}{\gamma_j} + \frac{E(e^{-\Gamma X_1})}{E(X_1 e^{-\Gamma X_1})} \left\{ n b_j + k_j + \frac{b_j}{1-\beta} \right\}$$

In particular, if $\beta_j = \beta$ for all j, the no. of shares held by agents don't change, as you would expect.

(ii) How would this story change if there were limited liability? There comes a time τ when X_τ is so bad (very negative?) that we refuse to accept it, and discard the asset. Then exactly as before we have

$$\beta_j^n U_j'(q(n)) = \lambda_j \zeta_n$$

but now market clearing says

$$X_n I_{\{n<\tau\}} = \sum_j q(n) = \frac{1}{\gamma_j}\left[\log\left(\frac{\gamma_j}{\lambda_j}\right) + n\log\beta_j - \log\zeta_n\right]$$

so that

$$\zeta_n = \beta^n e^{A - \Gamma X_n I_{\{n<\tau\}}}$$

How do we determine what τ should be? The decision is to default at time n if current dividend X_n plus NPV of all future dividends is <0:

$$X_n + \frac{1}{\zeta_n} E_n\left[\sum_{m>n} \zeta_m X_m I_{\{m<\tau\}}\right] < 0$$

Thus the bankruptcy rule has to be something of the form: default at the first time that $X_n < \kappa$, for some κ to be determined. Suppose $p = p(\kappa) = P(X_n > \kappa)$, so that

$$\begin{aligned}
E_n\left[\sum_{m>n} \zeta_m X_m I_{\{m<\tau\}}\right] &= \sum_{m>n} p^{m-n-1} E_n\left(\zeta_m X_m I_{\{X_m>\kappa\}}\right) \\
&= \sum_{m>n} p^{m-n-1} \beta^m e^A E\left(X_m e^{-\Gamma X_m}; X_m > \kappa\right) \\
&= \beta^{n+1} e^A E\left(X_1 e^{-\Gamma X_1}; X_1 > \kappa\right) / (1-\beta p)
\end{aligned}$$

so our decision is based on

$$X_n + e^{\Gamma X_n} \frac{\beta}{1-\beta p} E\left[X_1 e^{-\Gamma X_1}; X_1 > \kappa\right]$$

Clearly this is <0 if X_n is negative enough. The critical condition for κ is

$$\frac{\beta p}{1-\beta p} E\left[X_1 e^{-\Gamma X_1} \,\middle|\, X_1 > \kappa\right] = -\kappa e^{-\Gamma\kappa}$$

There's always at least one root for this.

To match a given BS model, require

$$\left.\begin{aligned} \delta &= \rho - \mu(1-R) - \tfrac{1}{2}\sigma^2(1-R)^2 \\ r &= \rho + \mu R \quad - \tfrac{1}{2}\sigma^2 R^2 \end{aligned}\right\}$$

Maybe have $\sigma(\xi_t)$, ξ a MKV chain?

Equilibrium pricing of assets (18/2/04)

(i) Suppose we have single asset with dividend process $\Delta_t = \exp(\sigma W_t + \mu t)$ and a single (representative!) CRRA agent trying to max $E\int_0^\infty e^{-\rho t} U(c_t)\,dt$. We have

$$\zeta_t = e^{-\rho t} U'(\Delta_t) = \exp(-\rho t - (\sigma W_t + \mu t) R)$$

so that

$$S_t = \Delta_t E_t\left[\int_t^\infty (\zeta_s \Delta_s)/\zeta_t \Delta_t \quad ds\right]$$

$$= \Delta_t \; E\int_0^\infty \exp\{-\rho s + (1-R)(\sigma W_s + \mu s)\}\,ds$$

$$= \Delta_t / \{\rho - \mu(1-R) - \tfrac{1}{2}\sigma^2(1-R)^2\} \qquad [\therefore \text{ need } \rho > \mu(1-R) + \tfrac{1}{2}\sigma^2(1-R)^2]$$

So if we set

$$\delta = \rho - \mu(1-R) - \tfrac{1}{2}\sigma^2(1-R)^2 > 0$$

we get that the dividend stream is δS_t, and the stock price is log-Brownian. The riskless rate is $\rho + \mu R - \frac{1}{2}\sigma^2 R^2$, constant, and if $dW = d\tilde{W} - \sigma R\,dt$, where $\tilde{W}$ is risk-neutral BM, we shall have

$$dS = S(\sigma\, d\tilde{W} + (r-\delta)\,dt)$$

as it should be.

(ii) What if we took a stochastic volatility model for Δ? More precisely, suppose that

$$\begin{cases} d\Delta = \Delta\{\sigma_t\, dW_t' + \alpha\, dt\} \\ dv_t = a\sqrt{v_t}\, dW_t + \beta(\kappa - v_t)\,dt \end{cases}$$

where $v_t \equiv \sigma_t^2$, $dW' = \rho\, dW + \sqrt{1-\rho^2}\, dZ$, W and Z independent BMs. Again with a CRRA representative investor, we would find

$$\zeta_t = e^{-\lambda t}\Delta_t^{-R}$$

and

$$S_t\zeta_t + \int_0^t \zeta_u \Delta_u\, du \qquad \text{is a martingale.}$$

Now it's not hard to guess that $S_t = \Delta_t\, h(v_t)$ and if we do this, we get a differential equation for h:

$$0 = \tfrac{1}{2}a^2 v h'' + \left(\beta(\kappa - v) + \rho a(1-R)\right)h' - \left(\tfrac{1}{2}\rho^2 R(1-R)v - \alpha(1-R) + \lambda\right)h + 1$$

Maple gives some solution in terms of Whittaker f^ns.

Some thoughts on particle filtering (24/2/04)

(i) Suppose that we have some Markov process (X_n) in discrete time, observed via Y_n, where the density of Y_n given X_n is $f(y|x)$. In a particle filtering story, we have at time n some particles $\xi_n^{(i)}$, $i=1,\dots,N_n$ and associated weights $w_n^{(i)}$, forming an atomic measure

$$\hat{\mu}_n(dx) = \sum_{i=1}^{N_n} w_n^{(i)} \, \delta_{\xi_n^{(i)}}(dx)$$

and this is usually taken to represent the posterior law of X_n given $\mathcal{Y}_n$. However, it seems we may have advantage in using a measure of the form

$$\tilde{\mu}_n(dz) = \int \hat{\mu}_n(dx)\, \kappa(x,dz)$$

(for some Markov kernel κ) as our estimate of the posterior of X_n given $\mathcal{Y}_n$ (this would mean that if some components of X represent parameters that don't change, we would allow a density for them, rather than insist on point masses)

If P is the transition kernel of the Markov process, then the prediction step would be to form

$$\tilde{\mu}_n P(dz) = \iint \hat{\mu}_n(dx)\, \kappa(x,dx')\, P(x',dz)$$

and then do the Bayesian step of rejection sampling; if $\bar{f}(y) = \max_x f(y|x)$ then we generate particles according to the law $\tilde{\mu}_n P(dz)$ and accept with prob'y $f(y|z)/\bar{f}(y)$ until we have enough particles. Weights attached to particles will be posterior likelihoods.

(ii) Might be worth doing the Markov kernel before the smoothing kernel κ? The point is that when computing the weights, if we have to do the integration

$$\iint \hat{\mu}_n(dx)\, \kappa(x,dx')\, P(x',dz)$$

this could be a lot harder in some applications than

$$\iint \hat{\mu}_n(dx)\, P(x,dx')\, \kappa(x',dz).$$

Some thoughts on Metastability (5/3/04)

Anton Bovier gave an interesting talk on metastability of Markov processes; the definition is the first issue, and the one being used depended on some threshold parameter; this seems to be a bit non-canonical. Could we have some characterisation for metastability in terms of something continuous?

(i) How about mean passage times?

(a) If we set $F_{ij} = E^i \tau_j \quad (i \neq j)$, $F_{ii} = 0 \ \forall i$, then we have

$$(QF + 11^T)_{ij} = 0 \quad \text{if } i \neq j$$

so in fact

$$QF + 11^T = \Delta, \quad \text{a diagonal matrix.}$$

If we left multiply by the invariant law π, we discover that in fact

$$\boxed{QF + 11^T = M^{-1} \equiv (\operatorname{diag} \pi_j)^{-1}}$$

This would allow us to recover Q from F;

$$Q = (M^{-1} - 11^T)F^{-1}$$

(the unknown M being recovered from the condition $Q1=0$) provided F is non singular. But if for some x we have $Fx=0$, then from the boxed equation above it has to be that $11^T x = M^{-1} x$, so $x \propto \pi^T$, and then since $x \geq 0$ we have $x=0$ (since $F \geq 0$). So F really is nonsingular, and we do have

$$\boxed{Q = (M^{-1} - 11^T)F^{-1}}$$

Right multiply by 1, we learn that $\boxed{F^{-1}1 \propto \pi^T}$ or equivalently $F\pi^T = \text{const}$, which is remarkable in its own right:

$$\boxed{\sum_{j \neq i} (E^i \tau_j)\pi_j = c, \quad \text{same for all } i}$$

(b) Resolvents? If $F_\lambda(i,j) = E^i e^{-\lambda \tau_j}$, $F_\lambda(i,i) = 1$, we shall have

$$(Q-\lambda)F_\lambda = -\Delta, \quad \text{a diagonal matrix,}$$

where $\Delta_{ii} = -\sum_{j \neq i} q_{ij} F_\lambda(j,i) + (\lambda + q_i) = (\lambda+q_i)\{-E^i e^{-\lambda \tilde{\tau}_i} + 1\} > 0$, where $\tilde{\tau}_i = \inf\{t > \sigma : X_t = i\}$, $\sigma = \inf\{s>0 : X_s \neq X_0\}$. So once again we can recover Q from F_λ, by the recipe

$$Q = \lambda I - \Delta F_\lambda^{-1}$$

(c) Symmetry? If Q is symmetrisable, $MQ = Q^T M$ and so

$$MQ = (I - \pi 1^T) F^{-1} = F^{-1} - \gamma \pi\pi^T \qquad (\text{for some } \gamma)$$

is symmetric, whence F is symmetric: $E^i \tau_j = E^j \tau_i \ \forall i \neq j$ if chain is symmetrisable. So in the symmetrisable case

$$\boxed{(i,j) \longmapsto E^i \tau_j \quad \text{is a metric!}}$$

(d) Mean hitting times to sets. If we partition the state space as $I = I_0 \cup I_1$, with Q partitioned as

$$Q = \begin{array}{c} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{array} \begin{array}{l} I_0 \\ I_1 \end{array} \qquad F = \begin{pmatrix} F_{00} & F_{01} \\ F_{10} & F_{11} \end{pmatrix}$$

then we have

$$Q = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = M^{-1} F^{-1} - \gamma 1 \pi^T$$

so that $D = M_1^{-1} (F_{11} - F_{10} F_{00}^{-1} F_{01})^{-1} - \gamma 1 \pi_1^T \equiv K - K z \pi_1^T$, say. Thus in terms of these things

$$D^{-1} = \left(I + \frac{z \pi_1^T}{1 - \pi_1 z} \right) K^{-1}$$

so that the mean first passage times from I_1 to I_0 are given by

$$-D^{-1} 1 = -\gamma^{-1} \left(I + \frac{z\pi_1^T}{1 - \pi_1 z} \right) z = \frac{-\gamma^{-1} z}{1 - \pi_1 z}$$

Now as $z = \gamma (F_{11} - F_{10} F_{00}^{-1} F_{01}) M_1 1 = \gamma (F_{11} - F_{10} F_{00}^{-1} F_{01}) \pi_1^T = 1 - F_{10} F_{00}^{-1} 1$ after simplifying.

Equilibria with agents who have random lifetimes (12/3/04)

Suppose there's a single productive asset with dividend stream $\Delta_t = \sigma W_t + \mu t$, and we consider an economy with CARA agents, but with the difference that each agent lives for an exponential length of time, then is replaced by a randomly-chosen agent who inherits his holding of the asset. Can we solve the equilibrium problem for this?

(i) Let's first consider a much simpler problem with just two agents, and one of these gets replaced at an $\exp(\lambda)$ time τ. To understand this, we first have to solve the post-τ behaviour, in terms of agents' holdings.

Standard story post-τ gives

$$\gamma_j e^{\rho_j t - \gamma_j g(t)} = \lambda_j \zeta_t$$

so

$$g(t) = -\tfrac{1}{\gamma_j} \log \frac{\lambda_j \zeta_t}{\gamma_j} - \frac{\rho_j}{\gamma_j} t$$

$$\begin{cases} \Gamma^{-1} = \sum_1^J \gamma_j^{-1} \\ \Gamma^{-1}\rho = \sum \gamma_j^{-1}\rho_j \\ \Gamma^{-1}\log A = \sum \gamma_j^{-1} \log \gamma_j/\lambda_j \end{cases}$$

Summing,

$$\Delta_t = -\Gamma^{-1}\log \zeta_t - \Gamma^{-1}\rho t + \Gamma^{-1}\log A$$

Market clearing gives us

$$\zeta_t = A\, e^{-\Gamma\Delta_t - \rho t}$$

To compute the NPV of agent j's future consumption, we compute

$$\frac{1}{\zeta_0} E \int_0^\infty \zeta_t\, g(t)\,dt = E\int_0^\infty e^{-\Gamma(\Delta_t - \Delta_0) - \rho t}\left\{\frac{\Gamma}{\gamma_j}\Delta_t + \frac{\rho - \rho_j}{\gamma_j} t - \frac{1}{\gamma_j}\left\{\log A - \log(\gamma_j/\lambda_j)\right\}\right\}dt$$

Now we have $E\int_0^\infty \exp(-s\Delta_t - vt)\,dt = \int_0^\infty \exp(-(v + s\mu - \frac12 s^2\sigma^2)t)\,dt = (v + s\mu - \frac12 s^2\sigma^2)^{-1}$

so that

$$E\int_0^\infty e^{-s\Delta_t - vt}\Delta_t\,dt = \frac{\mu - \sigma^2 s}{(v + s\mu - \frac12\sigma^2 s^2)^2}$$

$$E\int_0^\infty e^{-s\Delta_t - vt}\, t\,dt = \frac{1}{(v + s\mu - \frac12\sigma^2 s^2)^2}$$

and the NPV of g is

$$\frac{\Gamma}{\gamma_j}\frac{\mu - \sigma^2\Gamma}{(\rho + \Gamma\mu - \frac12\sigma^2\Gamma^2)^2} + \frac{\rho - \rho_j}{\gamma_j}\frac{1}{(\rho + \Gamma\mu - \frac12\sigma^2\Gamma^2)^2} - \frac{\log A - \log(\gamma_j/\lambda_j)}{\gamma_j(\rho + \Gamma\mu - \frac12\sigma^2\Gamma^2)}$$

$$= \frac{\theta_j(0)\;(\mu - \sigma^2\Gamma)}{(\rho + \Gamma\mu - \frac12\sigma^2\Gamma^2)^2}$$

where $\theta_j(0)$ is agent j's holding of the stock at time 0. This then gives

$$\lambda_j = \gamma_j A^{-1} \exp\left[\rho - \rho_j + \Gamma(\mu - \sigma^2\Gamma) - \gamma_j \theta_j(0)(\mu - \sigma^2\Gamma) \right]$$

So up to an irrelevant multiplicative constant we can take

$$\boxed{\lambda_j \propto \gamma_j \exp\{-\rho_j - \gamma_j \theta_j(0)(\mu - \sigma^2\Gamma)\}}$$

and the value to agent j of his future dividend stream is

$$E\int_0^\infty -e^{-\gamma_j c_j(t)}\,dt = -\frac{\lambda_j}{\gamma_j} E\int_0^\infty \zeta_t\,dt$$

$$= \frac{-\lambda_j A}{\gamma_j} \cdot \frac{1}{(\rho + \Gamma\mu - \frac{1}{2}\sigma^2\Gamma^2)}$$

If we use $\lambda_j = \gamma_j \exp\{-\rho_j - \gamma_j \theta_j(0)(\mu - \sigma^2\Gamma)\}$, the value to agent j of his consumption stream will be

$$\boxed{-e^{-\rho_j - \gamma_j \theta_j(0)(\mu - \sigma^2\Gamma)} \frac{\rho + \mu\Gamma - \sigma^2\Gamma^2}{\rho + \mu\Gamma - \frac{1}{2}\sigma^2\Gamma^2}}$$

Notice that this depends on agent j's holdings, but not on the holdings of the other agents. However, randomising this over (γ_1, ρ_1) say isn't going to go very nicely.

Equilibria for Markov-modulated dividend processes (23/3/04)

(i) Suppose there is an irreducible finite-state Markov chain ξ with transition rate matrix Q, and a productive asset generating dividend δ_t, where

$$d\delta_t = \sigma(\xi_t)\,dW_t + \mu(\xi_t)\,dt.$$

We shall consider a market with J CARA agents, each trying to maximise an objective of the form

$$E\left[\int_0^\infty -e^{-\beta_j t}\, e^{-\gamma_j c_j(t)}\,dt\right]$$

Can we compute equilibrium prices for this problem?

(ii) Suppose that we work just with one agent, and drop the j subscripts. Suppose that the price process is of the form

$$S_t = f(\xi_t, \delta_t)$$

for some function f (to be determined later by market clearing). The wealth equation is

$$dw_t = rw_t\,dt + \theta_t\left(dS_t - rS_t\,dt + \delta_t\,dt\right) - c_t\,dt$$

where the constant interest rate r is exogenously given. Let $e_t^k \equiv I_{\{\xi_t = k\}}$, and N_t^{ij} be no of jumps from i to j before time t. We have a value function $V(w, \xi, \delta)$ and need to do Itô on this. For this, we decompose S, w into their continuous and jumping parts:

$$\begin{aligned} dS &= dS^c + \Delta S \\ &= \left\{f_\delta\,d\delta + \tfrac{1}{2}\sigma^2 f_{\delta\delta}\,dt\right\} + \sum_{i\neq j}\left\{f(j,\delta_t) - f(i,\delta_t)\right\}dN_t^{ij}, \end{aligned}$$

$$\begin{aligned} dw &= \left\{rw\,dt + \theta\left(dS^c - rS\,dt + \delta\,dt\right) - c\,dt\right\} + \theta\sum_{i\neq j}\left\{f(j,\delta_t) - f(i,\delta_t)\right\}dN_t^{ij} \\ &= dw^c + \theta\,\Delta S, \end{aligned}$$

and consider

$$Y_t = e^{-\rho t}V(w_t, \xi_t, \delta_t) + \int_0^t e^{-\rho s}U(c_s)\,ds$$

which we want to be a supermartingale, and a martingale under optimal control. Then

$$\begin{aligned} e^{\rho t}dY_t &= \left(U(c_t) - \rho V\right)dt + V_w\,dw^c + \tfrac{1}{2}V_{ww}\,d\langle w^c\rangle + V_{w\delta}\,d\langle w^c, \delta\rangle + \tfrac{1}{2}V_{\delta\delta}\,d\langle\delta\rangle + V_\delta\,d\delta \\ &\quad + \sum_i\sum_{\neq j}\left\{V\left(w_{t-} + \theta_t\left(f(j,\delta_t) - f(i,\delta_t)\right), j, \delta_t\right) - V(w_{t-}, i, \delta_t)\right\}dN_t^{ij} \end{aligned}$$

$$\doteq \Big[\,U(c) - \rho V + V_w\left(rw + \theta\left(\tfrac{1}{2}\sigma^2 f_{\delta\delta} + \mu f_\delta - rf + \delta\right) - c\right) + \mu V_\delta$$

$$+\tfrac{1}{2}\sigma^2\left(\theta^2 f_\delta^2 V_{ww} + 2\theta f_\delta V_{w\delta} + V_{\delta\delta}\right)$$

$$+\sum_{i\neq j} I_{\{\xi_t=i\}}\Big(V(w_t+\theta_t(f(j,\delta_t)-f(i,\delta_t)),j,\delta_t)-V(w_t,i,\delta_t)\Big)q_{ij}\Big]dt$$

If we propose

$$V(w,\xi,\delta) = e^{-\lambda w}v(\xi,\delta) \qquad (\lambda=\gamma r)$$

as usual, we get

$$e^{\rho t}dY_t \doteq e^{-\lambda w}\Big[-e^{-\gamma c+\lambda w} - \rho v - \lambda v\{rw-c+\theta(\tfrac{1}{2}\sigma^2 f_{\delta\delta}+\mu f_\delta - rf+\delta)\} + \mu v_\delta$$

$$+\tfrac{1}{2}\sigma^2\left(\theta^2\lambda^2 f_\delta^2 v - 2\theta\lambda f_\delta v_\delta + v_{\delta\delta}\right)$$

$$+\sum_{i\neq j} e_t^i\left[\exp\{-\lambda\theta(f(j,\delta)-f(i,\delta))\}v(j,\delta)-v(i,\delta)\right]q_{ij}\Big]dt$$

Maxing over c, we get

$$\boxed{c-rw = -\tfrac{1}{\gamma}\log\left(-\lambda\gamma^{-1}v\right)}$$

The maximisation over θ is to max

$$\boxed{\begin{aligned}&rv - rv\log(-rv) - \rho v - \lambda v\,\theta\left(\tfrac{1}{2}\sigma^2 f_{\delta\delta}+\mu f_\delta - rf+\delta\right)+\mu v_\delta\\ &\quad+\tfrac{1}{2}\sigma^2\left(\theta^2\lambda^2 f_\delta^2 v - 2\theta\lambda f_\delta v_\delta + v_{\delta\delta}\right)\\ &\qquad+\sum\sum_{i\neq j} e^i\left[\exp\{-\lambda\theta(f(j,\delta)-f(i,\delta))\}v(j,\delta)-v(i,\delta)\right]q_{ij}\end{aligned}}$$

(iii) In view of what happens when we don't have the Markov-modulated dividend, we might

$$\boxed{\underline{\text{GUESS}}:\quad f(\xi,\delta) = a(\xi)\delta + b(\xi)}$$

and if you think how changing δ_0 would change the value of the share, it's clear that we must have $\boxed{a(\xi) = r^{-1} \text{ for all } \xi.}$

Illiquidity and the Merton problem: reworking of the expressions (27/3/04)

1) In the Merton problem with liquidity costs, we find that the value function as a function of current wealth w, portfolio H of stock and stock price can be scaled as

$$V(w,H,S) = S^{1-R}\, v(z,H) \qquad (z \equiv w/S)$$

and that in terms of v the HJB equation is

$$\sup_{c,h}\Big[U(c) - \tilde\rho v + \tfrac12 \sigma^2 (H-z)^2 v_{zz} - (h f(\varepsilon h) + c + \alpha(z-H))\, v_z + h v_H \Big] = 0.$$

Carrying out the optimisation gives

$$\tilde U(v_z) - \tilde\rho v + \tfrac12\sigma^2 (H-z)^2 v_{zz} - \alpha(z-H) v_z + \frac{v_z}{\varepsilon}\,\Phi\left(v_H/v_z\right) = 0$$

$$\left[\Phi(a) = \sup_t\,(at - t f(t)),\ \tilde\rho = \rho + (R-1)(\mu - \tfrac12\sigma^2 R) = \gamma R + \tfrac12\sigma^2 R(1-R)(1-\pi)^2,\quad \alpha = \sigma^2 R(\pi - 1)\right]$$

2) Now we propose the form

$$v(z,H) = V_M\left(z\, e^{-G(p,\varepsilon z)}\right) = \frac{\gamma^{-R}}{1-R}\, e^{\eta(z,H)}$$

where $p \equiv H/z$, $V_M(x) = \gamma^{-R} U(x)$ is the value function for the basic Merton problem.
The HJB in terms of η now reads (after dividing by v)

$$\boxed{v^{-1/R}\,\tilde U(\eta_z) - \tilde\rho + \tfrac12\sigma^2 (H-z)^2(\eta_{zz} + \eta_z^2) - \alpha(z-H)\eta_z + \frac{\eta_z}{\varepsilon}\,\Phi\left(\eta_H/\eta_z\right) = 0}$$

Notice that $\eta = (1-R)\left(\log z - G(p, \varepsilon z)\right)$, and hence

$$\begin{cases} \eta_z = \dfrac{(1-R)}{z}\left\{1 - \varepsilon z G_2 + p G_1\right\} \equiv \dfrac{1-R}{z}\, X, \text{ say} \\[2ex] \eta_H = -\dfrac{(1-R)}{z}\, G_1 \\[2ex] \eta_{zz} = \dfrac{(1-R)}{z^2}\left[z\dfrac{dX}{dz} - X\right] \end{cases}$$

whence the HJB expression is

$$\boxed{\begin{aligned} &\gamma(1-R)\, e^{(1-R)G/R}\,\tilde U(X) - \tilde\rho + \tfrac12\sigma^2(1-p)^2(1-R)\left\{ z\frac{dX}{dz} - X + (1-R)X^2\right\} \\ &\qquad - \alpha(1-p)(1-R)X + \frac{(1-R)X}{\varepsilon z}\,\Phi\left(\frac{-G_1}{X}\right) = 0 \end{aligned}}$$

$\alpha = \sigma^2 R(\pi - 1), \quad \tilde{\rho} = \gamma R + \frac{1}{2}\sigma^2 R(1-R)(1-\pi)^2$

Illiquidity and the Merton problem again (31/3/04)

i) We have the equation

$$U(c) - \tilde{\rho} v + \tfrac{1}{2}\sigma^2 (H-z)^2 v_{zz} - \{c + h f(h) + \alpha(z-H)\} v_z + h v_H \qquad (*)$$

to be maxed over c and h; this is just a reworking of HJB. It appears that this can't be solved, but if we can find an approximate solution, we can exhibit a near-optimal policy.

2) Suppose we try taking

$$\tilde{v}(z,H) = a z^{1-R} - b z^{\frac{3}{2}-R} \{ \kappa + \tfrac{1}{2}(p-\pi)^2 \} \qquad (p \equiv H/z)$$

for suitable ~~constants~~ $a, b, \kappa > 0$, and use the policy $c = (\gamma + \lambda\sqrt{z})z$. Then we shall have

$$\tilde{v}_z = z^{-R} \left\{ a(1-R) - b(\tfrac{3}{2}-R)\sqrt{z}\left(\kappa + \tfrac{1}{2}(p-\pi)^2\right) + b\sqrt{z}\, p(p-\pi) \right\}$$

$$\tilde{v}_{zz} = z^{-1-R} \left\{ -aR(1-R) - b\sqrt{z} \left\{ (\tfrac{3}{2}-R)(\tfrac{1}{2}-R)\left(\kappa + \tfrac{1}{2}(p-\pi)^2\right) - 2(1-R)p(p-\pi) + p(2p-\pi) \right\} \right\}$$

$$\tilde{v}_H = -b z^{\frac{1}{2}-R}(p-\pi)$$

and going back to substitute into $(*)$, we shall have z^{1-R} times

$$U(\gamma+\lambda\sqrt{z}) - \tilde{\rho}\left(a - b\sqrt{z}\left(\kappa + \tfrac{1}{2}(p-\pi)^2\right)\right) - (\gamma+\lambda\sqrt{z})\left(a(1-R) - b\{(\tfrac{3}{2}-R)(\kappa+\tfrac{1}{2}(p-\pi)^2) - p(p-\pi)\}\sqrt{z}\right)$$

$$- \tfrac{1}{2}\sigma^2(1-p)^2 \left(aR(1-R) + b\sqrt{z}\left\{ (\tfrac{3}{2}-R)(\tfrac{1}{2}-R)\left(\kappa + \tfrac{1}{2}(p-\pi)^2\right) - 2(1-R)p(p-\pi) + p(2p-\pi) \right\} \right)$$

$$- \alpha(1-p)\left(a(1-R) - b\sqrt{z}\left\{ (\tfrac{3}{2}-R)\left(\kappa + \tfrac{1}{2}(p-\pi)^2\right) - p(p-\pi) \right\}\right)$$

$$+ \frac{b^2 (p-\pi)^2}{2\left\{ a(1-R) - b\sqrt{z}\left((\tfrac{3}{2}-R)\left(\kappa+\tfrac{1}{2}(p-\pi)^2\right) - p(p-\pi)\right) \right\}}$$

3) We'd obviously like this all to vanish if we let $z=0$, when we find

$$U(\gamma) - \tilde{\rho} a - \tfrac{1}{2}\sigma^2(1-p)^2 aR(1-R) - \alpha a(1-R)(1-p) + \frac{b^2(p-\pi)^2}{2a(1-R)} - \gamma a(1-R)$$

Looking at the c/e of p^2, we find

$$\frac{b^2}{2a(1-R)} = \frac{\sigma^2 aR(1-R)}{2} \Rightarrow \boxed{b = a\sigma(1-R)\sqrt{R}}$$

From the c/e of p, we need

$$0 = \sigma^2 aR(1-R) + \alpha a(1-R) - \frac{b^2\pi}{a(1-R)} = \sigma^2 aR(1-R)\left[1 + \frac{\alpha}{\sigma^2 R} - \pi\right]$$

which is OK, by the special value of α!

For the constant term to vanish, we put $p = \pi$ and see that we must have

$$0 = U(\gamma) - \tilde{\rho} a - \tfrac{1}{2}\sigma^2(1-\pi)^2 a R(1-R) + \sigma^2 R a(1-R)(1-\pi)^2 - \gamma a(1-R)$$

$$= U(\gamma) - \tilde{\rho} a + \tfrac{1}{2}\sigma^2 a R(1-R)(1-\pi)^2 - \gamma a(1-R)$$

$$= U(\gamma) - a\gamma R - \gamma a(1-R) \quad = U(\gamma) - a\gamma$$

so $\boxed{a = \gamma^{-R}/(1-R)}$ is what this has to be, all in accordance with the original Merton problem.

4) What if we now look at the next term in the expansion, which is to $O(\sqrt{z})$ $(s \equiv \sqrt{z})$

$$U(\gamma)\frac{\lambda s}{\gamma}(1-R) - \lambda a s(1-R) + \gamma s b\left\{(\tfrac{3}{2}-R)(\kappa + \tfrac{1}{2}(p-\pi)^2) - p(p-\pi)\right\}$$

$$+ \tilde{\rho} b s\left(\kappa + \tfrac{1}{2}(p-\pi)^2\right) - \tfrac{1}{2}\sigma^2(1-p)^2 b s\left\{(\tfrac{3}{2}-R)(\tfrac{1}{2}-R)(\kappa+\tfrac{1}{2}(p-\pi)^2) - 2(1-R)p(p-\pi) + p(2p-\pi)\right\}$$

$$+ \alpha(1-p) b s\left\{(\tfrac{3}{2}-R)(\kappa + \tfrac{1}{2}(p-\pi)^2) - p(p-\pi)\right\}$$

$$+ \frac{b^2(p-\pi)^2}{2a(1-R)} \quad \frac{bs}{a(1-R)}\left\{(\tfrac{3}{2}-R)(\kappa + \tfrac{1}{2}(p-\pi)^2) - p(p-\pi)\right\}$$

We would certainly like this to vanish at $p = \pi$, which would give us

$$0 = \gamma(\tfrac{3}{2}-R)\kappa + \tilde{\rho}\kappa - \tfrac{1}{2}\sigma^2(1-\pi)^2\left[(\tfrac{3}{2}-R)(\tfrac{1}{2}-R)\kappa + \pi^2\right] + \alpha(1-\pi)(\tfrac{3}{2}-R)\kappa$$

which tells us that

$$\tfrac{1}{2}\sigma^2\pi^2(1-\pi)^2 = \kappa\left[\gamma(\tfrac{3}{2}-R) + \tilde{\rho} - \tfrac{1}{2}\sigma^2(1-\pi)^2(\tfrac{3}{2}-R)(\tfrac{1}{2}-R) + \alpha(1-\pi)(\tfrac{3}{2}-R)\right]$$

$$= \kappa\left[\frac{3\gamma}{2} - \frac{3}{8}\sigma^2(1-\pi)^2\right] = \frac{3\kappa}{8}\left(4\gamma - \sigma^2(1-\pi)^2\right)$$

This is the same as the Maple calculation gave us, and is problematic for the same reason – if $4\gamma - \sigma^2(1-\pi)^2 \le 0$, κ is negative (or infinite!)

5) OK, well maybe we should more generally take

$$\boxed{\tilde{v}(z, H) = a z^{1-R} - b z^{\frac{3}{2}-R}\varphi(p, z)}$$

and see whether we can get more useful asymptotics. We have

$$\tilde{v}_z = a(1-R)z^{-R} - b(\tfrac{3}{2}-R)z^{\frac{1}{2}-R}\varphi - b z^{\frac{1}{2}-R}\left\{z\varphi_z - p\varphi_p\right\}$$

$$\tilde{v}_{zz} = -R(1-R)a z^{-1-R} - b z^{-\frac{1}{2}-R}\left\{(\tfrac{1}{2}-R)(\varphi(\tfrac{3}{2}-R) + z\varphi_z - p\varphi_p) + (\tfrac{3}{2}-R)(z\varphi_z - p\varphi_p)\right.$$
$$\left. + z^2\varphi_{zz} + z\varphi_z - 2zp\varphi_{zp} + p^2\varphi_{pp} + p\varphi_p\right\}$$

and $\tilde{v}_H = -b z^{\frac{1}{2}-R} \varphi_p$

$$\tilde{v}_z = z^{-R}\left[a(1-R) - b\sqrt{z}\left(\left(\tfrac{3}{2}-R\right)\varphi + z\varphi_z - p\varphi_p\right)\right] \equiv z^{-R}\left[a(1-R) - b\sqrt{z}\, X\right], \text{ say}$$

$$\tilde{v}_{zz} = z^{-1-R}\left[-aR(1-R) - b\sqrt{z}\left(\left(\tfrac{3}{2}-R\right)\left(\tfrac{1}{2}-R\right)\varphi + (3-2R)(z\varphi_z - p\varphi_p) + 2p\varphi_p + z^2\varphi_{zz} - 2pz\varphi_{pz} + p^2\varphi_{pp}\right)\right]$$

Putting it all together, assuming that we have $c = \gamma + \nu\sqrt{z}$, we get z^{1-R} times

$$U(\gamma+\nu\sqrt{z}) - (\gamma+\nu\sqrt{z})\left[a(1-R) - b\sqrt{z}\left(\left(\tfrac{3}{2}-R\right)\varphi + z\varphi_z - p\varphi_p\right)\right] - \tilde{\rho}\,(a - b\sqrt{z}\,\varphi)$$

$$-\tfrac{1}{2}\sigma^2(1-p)^2\left[aR(1-R) + b\sqrt{z}\,Y\right] + \sigma^2 R(1-\pi)(1-p)\left[a(1-R) - b\sqrt{z}\,X\right]$$

$$+ \frac{b^2\varphi_p^2}{2\left[a(1-R) - b\sqrt{z}\,X\right]}$$

As before, to deal with the leading order terms we require $a = \gamma^{-R}/(1-R)$ and

$$b^2\varphi_p^2 = a^2\sigma^2(1-R)^2 R\,(p-\pi)^2 + \varepsilon$$

and then what remains (the terms of order $\sqrt{z}$) will be $b\sqrt{z}$ times

$$X\left(\gamma - \sigma^2 R(1-\pi)(1-p) + \tfrac{1}{2}\sigma^2 R\varphi_p^2\right) + \tilde{\rho}\varphi$$

$$-\tfrac{1}{2}\sigma^2(1-p)^2\, Y + \frac{\varepsilon}{2a(1-R)\, b\sqrt{z}}$$

where of course

$$Y = \left(\tfrac{3}{2}-R\right)\left(\tfrac{1}{2}-R\right)\varphi + (3-2R)(z\varphi_z - p\varphi_p) + 2p\varphi_p + z^2\varphi_{zz} - 2zp\varphi_{zp} + p^2\varphi_{pp},$$

and we think ε must be $O(\sqrt{z})$.

Can we propose good forms for φ, which don't give the same problems as $K + \frac{1}{2}(p-\pi)^2$??

Some thoughts on a talk by Thorsten Oest (3/4/04)

(i) Let $\mathcal{X}$ be the collection of contingent claims attainable at time T from 0 initial wealth, a vector space. For any $\mathcal{F}_T$-measurable V_T, define

$$\varphi(V_T, \mu) \equiv \inf\{ E(V_T - X)^2 ; \; X \in \mathcal{X}, \; EX = \mu \}$$

Clearly φ is convex.

(ii) There is a notion of 'hedging' developed by Bouchaud + Sornette where one approximates V_T by the $X \in \mathcal{X}$ for which $E(V_T - X)^2$ is minimised (assuming inf is attained). Oest proposes instead that we should try to minimise

$$F\left(E(V_T - X)^2 \right) - \alpha EX$$

for some suitably chosen increasing F (in the application of interest, $F(x) = \sqrt{x}$) and some $\alpha > 0$ to be chosen. Numerical results suggested that this was the same thing as B+S where the contingent claim was changed to $V_T + \lambda S_T$ for some λ. Assuming a little differentiability ($\mu \to \varphi(V_T, \mu)$ is C^1 $\forall V_T$) then this is correct.

(iii) If Π is some $\mathcal{F}_T$-meas. contingent claim attainable from initial wealth p, then

$$\begin{aligned}\varphi(V_T + \Pi, \mu) &= \inf\{ E(V_T + \Pi - X)^2 ; \; X \in \mathcal{X}, \; EX = \mu \} \\ &= \inf\{ E(V_T + p - \tilde{X})^2 ; \; \tilde{X} \equiv X - \Pi + p \in \mathcal{X}, \; E\tilde{X} = \mu - E\Pi + p \} \\ &= \inf\{ E(V_T - \tilde{X})^2 + 2p\, EV_T + p^2 - 2p(\mu - E\Pi + p) ; \; \tilde{X} \in \mathcal{X}, \; E\tilde{X} = \mu - E\Pi + p \} \\ &= \varphi(V_T, \mu + p - E\Pi) + p^2 + 2p\, EV_T - 2p(\mu - E\Pi + p).\end{aligned}$$

The problem proposed by Oest is to

$$\min_\mu F(\varphi(V_T, \mu)) - \alpha\mu$$

so this gets solved when

$$(*) \qquad F'(\varphi(V_T, \mu))\, \varphi'(V_T, \mu) = \alpha$$

If we do B+S 'hedging' on the claim $V_T + \lambda\Pi$, the optimal μ gives $\varphi'(V_T, \tilde{\mu}) = 2\lambda p \tilde{\mu}$, where $\tilde{\mu} \equiv \mu + \lambda p - \lambda E\Pi$, and by adjusting λ we can get $(*)$ satisfied. The optimal X for the problem of Oest differs from the optimal X for $V_T + \lambda\Pi$ (in the B+S sense) by $\lambda(\Pi - p)$.

Leading-order small-m behaviour of convertible bonds (23/4/04)

If m is extremely small, and $V \in (0, \eta_0)$ is held fixed, then

$$E^V\left[e^{-rH_\xi};\ H_\xi < H_\eta\right] \le E^V\left[e^{-rH_\xi}\right] = (V/\xi)^{-\alpha} = \xi^\alpha V^{-\alpha}$$

As the losses due to default are $O(m^{1+\alpha})$ since $\xi(m) \sim \text{const.}\, m$, thus the losses due to default can be ignored, and we get the budget equation

$$(n-m)\, S(m,V) + m\, B(m,V) = V + \text{NPV of tax rebates} + o(m)$$

Now NPV of tax rebates is to leading order

$$E^V\left[\int_0^{H_\eta} m\rho\tau\, e^{-rs}\, ds\right] = \frac{m\rho\tau}{r}\left\{1 - \left(\frac{V}{\eta}\right)^\beta\right\}$$

and each bond is worth (to leading order)

$$E^V\left[\int_0^{H_\eta} \rho e^{-rs}\, ds + e^{-rH_\eta}\frac{\eta}{n}\right] = \frac{\rho}{r}\left(1 - \left(\frac{V}{\eta}\right)^\beta\right) + \frac{\eta}{n}\left(\frac{V}{\eta}\right)^\beta,$$

(the recovery on default gives an amount which is $O(1)$, but with prob$^{\text{ty}}$ $o(m^\alpha)$). Thus

$$\begin{aligned}(n-m)\, S &= V + m\left[\frac{\rho\tau}{r}\left(1-\left(\frac{V}{\eta}\right)^\beta\right) - \frac{\rho}{r}\left(1-\left(\frac{V}{\eta}\right)^\beta\right) - \frac{\eta}{n}\left(\frac{V}{\eta}\right)^\beta\right]\\ &= V - m\left[\frac{\rho'}{r}\left(1-\left(\frac{V}{\eta}\right)^\beta\right) + \frac{\eta}{n}\left(\frac{V}{\eta}\right)^\beta\right]\end{aligned}$$

and hence

$$\begin{aligned}\frac{1}{m}\left\{S - \frac{V}{n}\right\} &= \frac{V}{n(n-m)} - \frac{1}{n-m}\left[\frac{\rho'}{r}\left(1-\left(\frac{V}{\eta}\right)^\beta\right) + \frac{\eta}{n}\left(\frac{V}{\eta}\right)^\beta\right]\\ &\to \frac{V}{n^2} - \frac{1}{n}\left[\frac{\rho'}{r}\left(1-\left(\frac{V}{\eta_0}\right)^\beta\right) + \frac{\eta_0}{n}\left(\frac{V}{\eta_0}\right)^\beta\right] \qquad (m \downarrow 0)\end{aligned}$$

This is in agreement with what we had earlier, but is a lot simpler!

Merton problem liquidity effects; some heuristics (29/4/04)

1) Suppose we look at the dynamics of the Merton problem

$$dw_t = (rw_t - c_t - \eta_t)\,dt + \theta_t(\sigma dW_t + (\mu - r)dt)$$

and think how much having θ away from πw, and having $\eta > 0$ will cost us. If we take

$$Y_t \equiv e^{-\rho t} V_M(w_t) + \int_0^t e^{-\rho s} U(C_s)\,ds$$

where $V_M(w) = \gamma^{-R} U(w)$ is the usual Merton value, we find that

$$e^{\rho t} dY_t \doteq \{U(c) - \rho V_M + \tfrac{1}{2}\theta^2\sigma^2 V_M'' + (rw - c - \eta + \theta(\mu - r)) V_M' \} dt$$

So if we use $C = \gamma w$ as Merton prescribes, we get

$$\tilde{U}(\gamma^{-R} w^{-R}) - \rho V_M - \eta V_M' + \tfrac{1}{2}\theta^2\sigma^2 V_M'' + (rw + \theta(\mu - r)) V_M'$$

$$= \frac{R}{1-R}(\gamma w)^{1-R} - \rho\gamma^{-R}\frac{w^{1-R}}{1-R} - \frac{\eta}{w}\gamma^{-R} w^{1-R} - \frac{R}{2}\theta^2\sigma^2\gamma^{-R} w^{-1-R} + \left(r + \frac{\theta}{w}(\mu - r)\right)\gamma^{-R} w^{1-R}$$

$$= \gamma^{-R} w^{1-R}\left[\frac{\gamma R - \rho}{1-R} - \frac{\eta}{w} - \frac{1}{2}\sigma^2 R\left(\frac{\theta}{w}\right)^2 + \frac{\theta}{w}(\mu - r) + r\right]$$

$$= -\gamma^{-R} w^{1-R}\left[\frac{\eta}{w} + \frac{1}{2}\sigma^2 R\left(\pi - \frac{\theta}{w}\right)^2\right]$$

So the loss of objective we suffer from using a general θ and paying out η will be

$$E\int_0^\infty e^{-\rho t}\gamma^{-R} w_t^{1-R}\left[\frac{\eta_t}{w_t} + \frac{1}{2}\sigma^2 R\left(\pi - \frac{\theta_t}{w_t}\right)^2\right] dt$$

2) In the liquidity problem, $\theta_t = H_t S_t$, $\eta_t = \frac{1}{2}\varepsilon \dot{h}_t^2 S_t$, so the loss we suffer is proportional to

$$E\int_0^\infty e^{-\rho t} w_t^{1-R}\left[\frac{\varepsilon \dot{h}_t^2}{2 z_t} + \frac{1}{2}\sigma^2 R\left(\pi - \frac{H_t}{z_t}\right)^2\right] dt \qquad (z \equiv w/S)$$

<u>ASSUME</u>: w, z are what they would be if the <u>exact</u> Merton problem was being solved.

$$\begin{cases} w_t = w_0 \exp(\sigma\pi W_t + at) \\ z_t = z_0 \exp\left(-\sigma(1-\pi)W_t + (a - (\mu - r) + \tfrac{1}{2}\sigma^2)t\right) \\ \quad = z_0 \exp(-\sigma(1-\pi)W_t + bt) \end{cases}$$

$a = r + \pi(\mu - r) - \gamma - \frac{1}{2}\sigma^2\pi^2$
$= (r - \rho + \sigma^2\pi^2 R^2/2)/R$

say

The objective fn becomes

$$E \int_0^\infty e^{-\rho t} \, w_t^{1-R} \, z_t^{-2} \left[\frac{\epsilon}{2} h_t^2 z_t + \frac{1}{2}\sigma^2 R \left(H_t - \pi z_t\right)^2 \right] dt$$

$$= \tilde{E} \int_0^\infty e^{-\tilde{\rho} t} \left[\frac{\epsilon}{2} h_t^2 z_t + \frac{1}{2}\sigma^2 R \left(H_t - \pi z_t\right)^2 \right] dt$$

where $\begin{cases} \left.\dfrac{d\tilde{P}}{dP}\right|_{\mathcal{F}_t} = \exp\left\{ \tilde{\sigma} W_t - \frac{1}{2}\tilde{\sigma}^2 t \right\} \\ \tilde{\rho} = \rho - \frac{1}{2}\tilde{\sigma}^2 + \sigma^2 + a(1+R) - 2(\mu - r) \end{cases}$ $\quad \left(\tilde{\sigma} = \sigma(2 - \pi - R\pi)\right)$

Diversity or Leverage? (7/5/04)

1) Yannis was speaking about models with 'diverse' markets; the definition leads to arbitrage opportunities, the impact of which is limited by the requirement that agents can only hold non-negative positions. The motivation is the empirical observation that rates of return on small stocks are generally larger than on the big stocks, so by imposing 'diversity' this in effect pushes the big stocks down somewhat. But could something similar happen without arbitrage if we think of the leverage of firms?

2) Suppose a firm has an output process δ_t, where

$$d\delta_t = \delta_t \{ \sigma \, dW_t + \mu \, dt \}$$

in risk-neutral prob'y, with spot rate r constant, tax rate $\tau \in [0,1]$, and the firm is funded by equity + bonds. If coupon payments to bonds are at rate $\rho \, dt$, the dividend process for the stock is $(\delta_t - \rho')\,dt$, $\rho' = \rho(1-\tau)$. The firm defaults when δ falls to some level b (to be chosen by firm). The valuation of the stock therefore is

$$S(x) = E^x \left[\int_0^{H_b} e^{-rt} (\delta_t - \rho')\,dt \right] \qquad \left[\underline{\text{Assume}} \quad r > \mu \right]$$

so that S will solve

$$\tfrac{1}{2}\sigma^2 x^2 S'' + \mu x S' - rS + (x - \rho') = 0, \qquad S(b) = 0$$

which turns out to be

$$S(x) = \frac{x}{r-\mu} - \frac{\rho'}{r} + \left(\frac{x}{b}\right)^{-\alpha} \left(\frac{\rho'}{r} - \frac{b}{r-\mu} \right)$$

where $-\alpha < 0$ solves the quadratic

$$\tfrac{1}{2}\sigma^2 t(t-1) + \mu t - r = 0.$$

The optimal choice of b is

$$b^* = \frac{\alpha \rho'(r-\mu)}{r(\alpha+1)} \qquad (< \rho' \text{ if } \mu > 0)$$

from which

$$\boxed{S(x) = \frac{x}{r-\mu} - \frac{\rho'}{r} + \left(\frac{x}{b^*}\right)^{-\alpha} \cdot \frac{\rho'}{r(\alpha+1)}}$$

We have $S(b^*) = S'(b^*) = 0$, $S''(b^*) = \alpha\rho' / r(\alpha+1)\, b^{*2} = 1/b^*(r-\mu)$. If we now look at the SDE for the stock, we get

$$\frac{dS(\delta_t)}{S(\delta_t)} = \sigma \frac{\delta_t S'(\delta_t)}{S(\delta_t)} \, dW_t + \left(r + \frac{\rho' - \delta}{S(\delta)} \right) dt$$

The debt can similarly be valued:

$$D(x) = E^x\left[\frac{\rho}{r}(1-e^{-rH_b}) + p\,D_0\, e^{-rH_b}\right] = \rho/r + (p D_0 - \rho/r)(x/b^*)^{-\alpha}$$

We would need to choose p so that at time 0 the debt was worth its face value:

$$D_0 = D(x_0) = \frac{\rho}{r} + (p D_0 - \rho/r)(x/b^*)^{-\alpha}.$$

(3) Another even simpler idea proposed by John is to try the Geske model for equity, where we model firm value as

$$dV_t = V_t(\sigma\, dW_t + r\,dt) - \delta V_t\, dt$$

and then treat equity as an option on V.

4) Another possible story: suppose a CRRA agent invests in stocks which satisfy

$$dS^i = S^i\left(\sigma_{ij}(S)\,dW^j + \mu_i(S)\,dt\right)$$

to maximise

$$E\left[\int_0^\infty e^{-\rho t}\, U(c_t)\,dt\right]. \qquad (U'(x) = x^{-R})$$

If we look at the value function $V(w,S) = U(w)\,v(S)$ by usual scaling arguments, we get the HJB story that the optimisation over θ ($\equiv$ cash value of holdings of stocks) leads to

$$\boxed{\theta^* = \frac{1}{R}(\sigma\sigma^T)^{-1}(\mu - r)\,w}$$

Now if we supposed that the risk premium was common over all the BMs, equal to λ, say; then

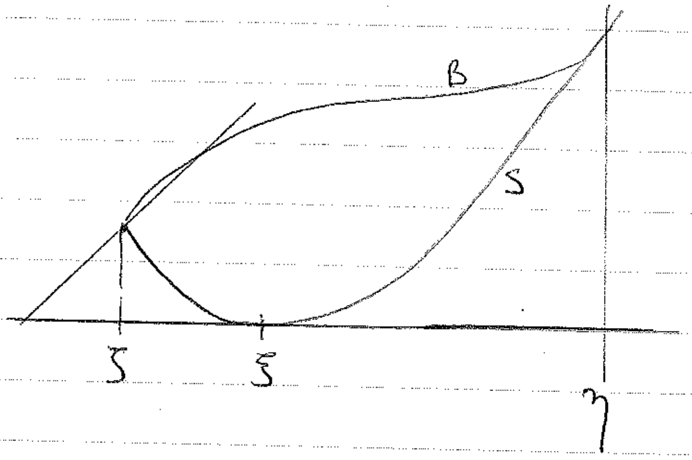
$$\theta^* = \frac{\lambda}{R}(\sigma\sigma^T)^{-1}\sigma 1 \;.\; w$$

While λ, R may be hard to estimate, we do get a mutual-fund theorem, that under this model the holdings are proportional to

$$(\sigma\sigma^T)^{-1}\sigma 1$$

For a diagonal σ, we have

$$\theta^*_i \propto 1/\sigma_{ii}(S)$$


B
S
ζ
ξ
η

Convertible bonds after $m = pn$ (2/6/04)

In the no-calling problem, we find that bankruptcy is impossible to the right of $m = m_1 \equiv pn$, and that the solution is characterised by $0 < \zeta(m) < \xi(m) < \eta(m)$, where we have S smooth pasting to 0 at ξ, $S(m, \zeta) = \zeta/n$ and B smooth pasting to S at η. We therefore have the equations

$$0 = m\rho' \psi_0(\zeta/\xi) - r\zeta\psi_1(\zeta/\xi) - r(\alpha+\beta)(n-m)\zeta/n$$

$$0 = r\zeta\, \psi_1(\zeta/\eta) - \rho(n-mrc)\psi_0(\zeta/\eta)$$

$$0 = n\rho' \psi_0(\eta/\xi) - r\eta\psi_1(\eta/\xi) - (n-m)\frac{\eta}{\xi^2}\frac{d\xi}{dm}\left\{m\rho'\psi_0'(\eta/\xi) - r\eta\psi_1'(\eta/\xi)\right\}$$

In preparation for using the Scilab routine dassl, we can think of this as $g(m, y, yd)$ where $y = [\zeta; \xi; \eta]$, and yd is the derivative of y. We shall also need the Jacobian:

$$\frac{\partial g_1}{\partial y_1} = \frac{1}{\xi}\left\{m\rho'\psi_0'(\zeta/\xi) - r\zeta\psi_1'(\zeta/\xi)\right\} - r\psi_1(\zeta/\xi) - r(\alpha+\beta)(n-m)/n$$

$$\frac{\partial g_1}{\partial y_2} = -\frac{\zeta}{\xi^2}\left\{m\rho'\psi_0'(\zeta/\xi) - r\zeta\psi_1'(\zeta/\xi)\right\}, \qquad \frac{\partial g_1}{\partial y_3} = 0,$$

$$\frac{\partial g_2}{\partial y_2} = 0, \quad \frac{\partial g_2}{\partial y_1} = r\psi_1(\zeta/\eta) + \frac{1}{\eta}\left\{r\zeta\psi_1'(\zeta/\eta) - \rho(n-mrc)\psi_0'(\zeta/\eta)\right\},$$

$$\frac{\partial g_2}{\partial y_3} = -\frac{\zeta}{\eta^2}\left\{r\zeta\psi_1'(\zeta/\eta) - \rho(n-mrc)\psi_0'(\zeta/\eta)\right\}, \qquad \frac{\partial g_3}{\partial y_1} = 0,$$

$$\frac{\partial g_3}{\partial y_2} = -\frac{\eta}{\xi^2}\left\{n\rho'\psi_0'(\eta/\xi) - r\eta\psi_1'(\eta/\xi) - (n-m)\frac{\eta}{\xi^2}\frac{d\xi}{dm}\left\{m\rho'\psi_0''(\eta/\xi) - r\eta\psi_1''(\eta/\xi)\right\}\right\}$$
$$+ \frac{2(n-m)\eta}{\xi^3}\frac{d\xi}{dm}\left\{m\rho'\psi_0'(\eta/\xi) - r\eta\psi_1'(\eta/\xi)\right\}$$

$$\frac{\partial g_3}{\partial y_3} = \frac{1}{\xi}\left\{n\rho'\psi_0'(\eta/\xi) - r\eta\psi_1'(\eta/\xi) - (n-m)\frac{\eta}{\xi^2}\frac{d\xi}{dm}\left(m\rho'\psi_0''(\eta/\xi) - r\eta\psi_1''(\eta/\xi)\right)\right\}$$
$$- r\psi_1(\eta/\xi) - \frac{n-m}{\xi^2}\frac{d\xi}{dm}\left\{m\rho'\psi_0'(\eta/\xi) - r\eta\psi_1'(\eta/\xi)\right\} + \frac{r(n-m)\eta}{\xi^2}\frac{d\xi}{dm}\psi_1'(\eta/\xi)$$

But dassl doesn't appear to like this ...

Convertible bonds: differentiating s (14/6/04)

(i) Gunther considers $s(\eta) \equiv S(m(\eta), \eta)$ and obtains a first-order ODE, but there may also be useful information to be gained from differentiating again. We have

$$s'(\eta) = \frac{\partial S}{\partial V}(m,\eta) + m' \frac{\partial S}{\partial m} = \frac{\partial S}{\partial V}(m,\eta)$$

$$s''(\eta) = \frac{\partial^2 S}{\partial V^2} + 2m' \frac{\partial^2 S}{\partial m \partial V} + m'' \frac{\partial S}{\partial m} + (m')^2 \frac{\partial^2 S}{\partial m^2}$$

$$= \frac{\partial^2 S}{\partial V^2} + 2m' \frac{\partial^2 S}{\partial m \partial V} + (m')^2 \frac{\partial^2 S}{\partial m^2}$$

Moreover, we have

$$0 = \frac{d}{d\eta}\left(\frac{\partial S}{\partial m}(m,\eta) \right) = m' \frac{\partial^2 S}{\partial m^2} + \frac{\partial^2 S}{\partial m \partial V}$$

so we have

$$\boxed{s''(\eta) = \frac{\partial^2 S}{\partial V^2}(m,\eta) + m' \frac{\partial^2 S}{\partial V \partial m}(m,\eta)}$$

Now we also have

$$\mathcal{L}\left(\frac{\partial S}{\partial m}\right) + \frac{\delta V - np'}{(n-m)^2} = 0$$

and we know $\quad \frac{\partial S}{\partial m}(m,\eta) = 0, \quad \frac{\partial S}{\partial m}(m,\xi) \leq 0.$

In fact, differentiating $0 = S(m, \xi(m))$ with respect to m gives us $\frac{\partial S}{\partial m}(m, \xi(m)) = 0.$

(ii) Similarly, if we consider $b(\eta) \equiv B(m(\eta), \eta) = s(\eta)$, we see that

$$\begin{cases} b'(\eta) = \dfrac{\partial B}{\partial V}(m,\eta) \\[2ex] b''(\eta) = \dfrac{\partial^2 B}{\partial V^2}(m,\eta) + m' \dfrac{\partial^2 B}{\partial m \partial V}(m,\eta) \end{cases}$$

From this we learn that

$$0 < \frac{\partial^2 Y}{\partial V^2}(m,\eta) = -m' \frac{\partial^2 Y}{\partial m \partial V} = (-m')\left(\frac{\partial^2 B}{\partial m \partial V} - \frac{\partial^2 S}{\partial m \partial V} \right)$$

so in particular

$$\boxed{\frac{\partial^2 B}{\partial m \partial V} > \frac{\partial^2 S}{\partial m \partial V}} \quad \text{at } (m,\eta)$$

(iii) Since we have

$$\mathcal{L}B + \rho = 0$$

we also have $(B_m \equiv \partial B/\partial m)$

$$\mathcal{L}B_m = 0$$

and since $B_m(m, \eta) = 0$, we must have

$$B_m(m, V) = \kappa(m)\left\{\left(\frac{V}{\eta(m)}\right)^{-\alpha} - \left(\frac{V}{\eta(m)}\right)^{\beta}\right\}$$

where $\kappa(m) < 0$, since $B_m(m, \xi(m)) < 0$. Thus $B_m(m, V) < 0$ for $\xi(m) \leq V < \eta(m)$, and we also see that

$$\frac{\partial B_m}{\partial V} \equiv \frac{\partial^2 B}{\partial m \partial V} > 0 \quad \ldots\ldots$$

but how can we <u>prove</u> $B_m(m, \xi(m)) < 0$?? (Equivalently, $Y_m(m, \xi(m)) < 0$?)

Some asymptotics for the Hobson-Rogers ~~stoch~~ vol model (21/7/04)

With the notation $Z_t = \log(e^{-rt} P_t)$ where P_t is price of underlying asset and $S_t = \int_0^\infty \lambda e^{-\lambda u}(Z_t - Z_{t-u})\,du$, we propose

$$dZ_t = \sigma(S_t)\,dW_t - \tfrac{1}{2}\sigma(S_t)^2\,dt$$

and have

$$dS_t = dZ_t - \lambda S_t\,dt$$

as in the original work. Now let's suppose $\lambda \equiv 1/\varepsilon$ is v. big; prices should be perturbations of BS prices ... can we do some asymptotics?

Suppose $C(t, Z, S)$ is the price at time t of a European-style derivative expiring at T. Then we have

$$-rC + C_t + \tfrac{1}{2}\sigma(S)^2\{C_{ZZ} + 2C_{ZS} + C_{SS}\} - \tfrac{1}{2}\sigma(S)^2\{C_Z + C_S\} - \tfrac{1}{\varepsilon} S C_S = 0$$

$$\equiv \left(\mathcal{L}_0 + \tfrac{1}{\varepsilon}\mathcal{L}_1\right) C$$

Say. Now we do some formal expansion: $C(t,Z,S) = \sum_{n\geq 0} \varepsilon^n C^{(n)}(t,Z,S)$ and deduce

$$0 = \tfrac{1}{\varepsilon}\mathcal{L}_1 C^{(0)} + \{\mathcal{L}_1 C^{(1)} + \mathcal{L}_0 C^{(0)}\} + \varepsilon\{\mathcal{L}_0 C^{(1)} + \mathcal{L}_1 C^{(2)}\} + \ldots$$

Each term has to be identically zero, so if we look at the first term, we learn that $C^{(0)}$ does not depend on S. The second term gives us

$$\mathcal{L}_0 C^{(0)} = S C^{(1)}_S$$

$$= \left\{-rC^{(0)} + C^{(0)}_t + \tfrac{1}{2}\sigma(S)^2 C^{(0)}_{ZZ} - \tfrac{1}{2}\sigma(S)^2 C^{(0)}_Z\right\}$$

so if we set $S=0$, we learn that $C^{(0)}$ is just the usual BS solution with $\sigma = \sigma(0)$.

Hence

$$S C^{(1)}_S = S C^{(1)}_S - 0.C^{(1)}_S = \tfrac{1}{2}\{\sigma(S)^2 - \sigma(0)^2\}\{C^{(0)}_{ZZ} - C^{(0)}_Z\}$$

so if we set $F(S) = \int_0^S \tfrac{1}{2}\{\sigma(x)^2 - \sigma(0)^2\}\frac{dx}{x}$, we have the expression

$$C^{(1)} = F(S)\{C^{(0)}_{ZZ}(t,Z) - C^{(0)}_Z(t,Z)\} + \gamma(t,Z)$$

for some function γ to be found. But this we get from the third term;

$$\mathcal{L}_0 C^{(1)} = \mathcal{L}_0\left(F(S)(C^{(0)}_{ZZ} - C^{(0)}_Z)\right) + \mathcal{L}_0\gamma = -S C^{(2)}_S$$

so setting $S=0$ we get a simple PDE for γ! Perhaps not surprisingly, the asymptotics depend only on the derivatives of σ at zero.

ACTUALLY, ALL OF THIS IS COMPLETE RUBBISH — NO SUCH EXPANSION CAN HOLD UNIFORMLY ..

Some portfolio things (24/7/04)

(i) Suppose we have an N-vector of assets, prices $X = (X^1, \dots, X^N)$, with market value process $\bar{X} \equiv 1 \cdot X \equiv \sum X^j$. Now suppose that

$$dX^i = X^i \left(dM^i + dA^i\right)$$

where M is cts martingale, A is cts FV process. If we use a portfolio π, then the wealth process Z^π generated satisfies

$$dZ^\pi / Z^\pi = \pi \cdot \frac{dX}{X} = \pi \cdot (dM + dA)$$

In the special case where

$$\pi = \theta \equiv X/\bar{X}$$

then we just have $Z^\pi = \bar{X}$.

(ii) Now if we are interested in the log of portfolio wealth relative to the market we have

$$\boxed{d \log(Z^\pi/\bar{X}) = (\pi - \theta) \cdot (dM + dA) - \pi \cdot d[M,M]\, \pi + \theta \cdot d[M,M]\, \theta}$$

Viewing M, A as the fundamentals, the market weights θ are derived quantities; we have

$$d\left(\frac{1}{\bar{X}}\right) = \frac{1}{\bar{X}} \left\{ -\theta \cdot (dM + dA) + \theta \cdot d[M,M]\, \theta \right\}$$

so that

$$d\theta = \frac{1}{\bar{X}}\, dX + X\, d\left(\tfrac{1}{\bar{X}}\right) + dX\, d\left(\tfrac{1}{\bar{X}}\right)$$

$$= \left(\mathrm{diag}(\theta) - \theta\theta^T\right)(dM + dA) - \left(\mathrm{diag}(\theta) - \theta\theta^T\right) d[M,M]\, \theta$$

(iii) The objective of the investor is to outperform the market. Fernholz + Karatzas study this problem, and one of the things they do is to propose some specific portfolio $\pi = F(\theta)$, in such a way that we can write

$$\log(Z^\pi/\bar{X}) = \varphi(\theta) + \text{FV process}$$

for some function φ. The attractions of this are (i) the portfolio chosen is based on the easily observable θ, not on poor estimates of drift (ii) the fluctuations of the martingale part are strongly controlled by φ. Matching the martingale parts of the two expressions for $\log(Z^\pi/\bar{X})$,

$$D\varphi \left(\mathrm{diag}\,\theta - \theta\theta^T\right) = \pi - \theta$$

which defines a π for which $\sum \pi_i = 1$, though some components of π could go negative in general.

(iv) We introduce the useful notation

$$\boxed{Q \equiv \mathrm{diag}\,\theta - \theta\,\theta^T}$$

So that we can more elegantly express

$$d\theta = Q\{dM + dA - d[M,M]\,\theta\} \qquad , \; d[\theta,\theta] = Q\,d[M,M]\,Q$$

$$D\varphi \,.\, Q = \pi - \theta.$$

After some calculations, we shall find

$$\boxed{d\log(Z^{\pi}/\bar{X}) = d\varphi - \tfrac{1}{2}\,\mathrm{tr}\left(D^2\varphi\, d[\theta,\theta]\right) - D\varphi\, d[\theta,\theta]\, D\varphi \; - D\varphi\, Q\, d[M,M]\,\theta}$$

Interestingly, only the qv of M (and <u>not</u> dA) appears in the terms after $d\varphi$.

Good choices of φ?

Phil remarks that the situation studied in the Fernholz-Karatzas paper cannot reasonably be expected to hold, as the relative arbitrage notion defined there is in fact an arbitrage.

Some simple models with consumption guarantees (25/7/04)

(i) As a first approximation to explaining the preference people show for unit trusts with guarantees on the final payout amount, we could suppose that people like to have money in the bank; this is more secure than the risky asset. This gives us the usual dynamics

$$dw_t = r w_t \, dt + \theta_t \{ \sigma \, dW_t + (\mu - r) \, dt \} - c_t \, dt$$

but now the objective

$$\max E \int_0^\infty e^{-\rho t} \{ u(w_t - \theta_t) + U(c_t) \} \, dt$$

The value function is a function only of w, and satisfies

$$\sup_{\theta, c} \; u(w-\theta) + U(c) - \rho V + \tfrac{1}{2}\sigma^2\theta^2 V'' + (rw + \theta(\mu - r) - c) V' = 0$$

If we want to do the CRRA[*] problem, with $U(x) = x^{1-R}/(1-R)$, $u(x) = \alpha U(x)$, then it's clear that $V(x) = A\,U(x)$ for some constant to be found, and $\theta = \pi w$, $c = \gamma w$. Hence we get the equations

$$\sup_{\gamma, \pi} \; \alpha U(1-\pi) + U(\gamma) - \rho A U(1) - A\tfrac{1}{2}\sigma^2\pi^2 R(1-R) U(1) + A(1-R)(r + \pi(\mu - r) - \gamma) U(1) = 0$$

whence

$$\boxed{U'(\gamma) = A, \qquad -\alpha U'(1-\pi) - A\sigma^2\pi R(1-R) U(1) + A(\mu - r)(1-R) U(1) = 0}$$

An interesting twist on this might be to see what u would be needed to cause the proportion of wealth in the bank to fall as w rises; this is easy enough to do, but just needs some appropriate specification of how the cash buffer should vary with w.

(ii) Suppose now we take a different problem, where the agent wants to provide himself with a lower bound L_t for consumption, and that L_t should be increasing. We shall clearly want $w_t \geq L_t / r$ always. Value is now a function of (w, L), and the aim is to

$$\max E \int_0^\infty e^{-\rho t} \{ u(L_t) + U(c_t) \} \, dt$$

giving HJB

$$\sup_{\theta, c \geq L} \{ u(L) + U(c) - \rho V + \tfrac{1}{2}\sigma^2\theta^2 V_{ww} + (rw + (\mu - r)\theta - c) V_w \} = 0, \qquad V_L \leq 0$$

Taking $U(x) = x^{1-R}/(1-R)$, $u(x) = \alpha U(x)$ gives $V(w, L) = U(w)\,\varphi(L/w)$ by usual scaling. Writing $x = L/w$ we get

$$\begin{cases} V_w = U'(w)\varphi(x) - \dfrac{U(w)}{w}\varphi'(x) \\[2ex] V_{ww} = U''(w)\varphi(x) - 2\dfrac{U'(w)}{w} x\varphi'(x) + \dfrac{U(w)}{w^2}\{x^2\varphi''(x) + 2x\varphi'(x)\} \end{cases}$$

---

[*] If $u(x) = \alpha \log x$, $U(x) = \log x$, we get $V(x) = a + b\log x$, $b = (1+\alpha)/\rho$, $\gamma = 1/b$, π maximises $\alpha\log(1-\pi) - \tfrac{1}{2}\sigma^2\pi^2 b + b(\mu - r)\pi$. Get $\pi < 1$ iff $(\mu - r)(1+\alpha) > \alpha\rho$. Finally,

$$\rho a = \alpha\log(1-\pi) + \log\gamma - \tfrac{1}{2}\sigma^2\pi^2 b + (r + \pi(\mu - r) - \gamma) b.$$

This takes us to the new form of HJB, viz

$$\sup_{\substack{\gamma,\pi \\ \gamma \geq x}} \Big[\alpha(1-R)U(x) + (1-R)U(\gamma) - \rho\varphi(x) + \tfrac{1}{2}\sigma^2\pi^2\{x^2\varphi''(x) + 2Rx\varphi'(x) - R(1-R)\varphi(x)\}$$
$$+ (r + (\mu-r)\pi - \gamma)\{(1-R)\varphi(x) - x\varphi'(x)\}\Big] = 0$$

and $\varphi'(x) \leq 0$, equal when guarantee is being raised. We have a boundary condition at $x = r$; $\varphi(r) = (1+\alpha)\, r^{(1-R)}/\rho$. Notice that γ, π will be dependent on x.

However, it seems a lot easier to try $V(w, L) = L^{1-R}\, h(w/L)$, whence we obtain $(y \equiv w/L)$

$$\sup_{\substack{c,\theta \\ c \geq L}} \Big[\alpha U(1) + U(\tfrac{c}{L}) - \rho h(y) + \tfrac{1}{2}\sigma^2 (\tfrac{\theta}{L})^2 h''(y) + (ry + (\mu-r)\tfrac{\theta}{L} - \tfrac{c}{L})h'(y) \Big] = 0$$

giving

$$\boxed{\frac{\alpha}{1-R} + \tilde{U}(h') - \rho h + ry h' - \frac{(\mu-r)^2 h'^2}{2\sigma^2 h''} = 0, \quad U'(\tfrac{c}{L}) = h'(y), \quad \frac{\theta}{L} = -(\mu-r)\, h'/\sigma^2 h''}$$

while $C > L$ ..... and

$$\boxed{\frac{(\alpha+1)}{1-R} - \rho h + (ry-1)h' - \frac{(\mu-r)^2 h'^2}{2\sigma^2 h''} = 0} \qquad \text{when } C = L$$

At the bottom end, where $y = \frac{1}{r}$, we get that the value is $(1+\alpha)\rho^{-1}U(1) \cdot L^{1-R} \Rightarrow h(\frac{1}{r}) = \frac{\alpha+1}{\rho(1-R)}$

Looking at our equation above, therefore, we learn the boundary conditions

$$h(\tfrac{1}{r}) = h'(\tfrac{1}{r}) = 0$$

(unless $h''(\frac{1}{r}) = +\infty$?) The boundary condition for increasing L, $V_L \leq 0$, becomes easily

$$y h'(y) \geq (1-R)\, h(y)$$

For $C = L$ we write $h(y) = \frac{\alpha+1}{\rho(1-R)} + g(t)$ $\quad (t \equiv ry - 1)$ and discover

$$-\rho g + r t g' - \tfrac{1}{2}\kappa^2 g'^2/g'' = 0$$

as the equation for g. We can look for a solution that is a power; $g(t) = a t^\lambda$, where we find

$$-\rho + r\lambda - \tfrac{1}{2}\kappa^2 \frac{\lambda}{\lambda-1} = 0 \qquad \text{i.e.} \qquad r\lambda^2 - \lambda(\rho + \tfrac{1}{2}\kappa^2 + r) + \rho = 0.$$

This has one root in $(0,1)$ and one in $(1,\infty)$. __If__ this is the correct form of the solution, we must select the root $\beta \in (0,1)$ for a concave function, and we conjecture the form

$$\boxed{h(y) = \frac{\alpha+1}{\rho(1-R)} + a(ry-1)^\beta}$$

for the solution for low values of y. The changeover point comes where $h'(y) = U'(1)$,

or more explicitly
$$y = y^* \equiv \left(1 + (a r \beta)^{1/1-\beta}\right)/r.$$

(iii) Notational reset: make the objective
$$E \int_0^\infty e^{-\rho t}\{(1-p)\,u(L_t) + p\,u(C_t)\}\,dt$$
so that HJB is
$$0 = \sup_{C \geq L,\,\theta}\left[(1-p)\,u(L) + p\,u(c) - \rho V + \tfrac{1}{2}\sigma^2\theta^2 V_{ww} + (rw + (\mu-r)\theta - c)V_w\right], \quad V_L \leq 0$$
$$= \sup_{C \geq L}\left[(1-p)\,u(L) + p\,u(c) - cV_w + rwV_w - \tfrac{1}{2}\kappa^2\frac{V_w^2}{V_{ww}} - \rho V\right] \qquad \kappa = \frac{\mu - r}{\sigma}$$

Case 1: $C = L$ Here we have
$$0 = u(L) - LV_w + rwV_w - \tfrac{1}{2}\kappa^2 V_w^2/V_{ww}$$
and now we do a change of variables to (z, L), where $z = V_w(w, L)$, and make a new function $J(z,L) = V(w,L) - zw$. (This is a trick I got from Hong Liu via Phil). We have various relations; $J_z = -w$, $J_{zz} = -1/V_{ww}$, so the HJB becomes
$$0 = u(L) - zL - \rho J + (\rho - r)\,zJ_z + \tfrac{1}{2}\kappa^2 z^2 J_{zz}$$

A particular solution is $J = \dfrac{u(L)}{\rho} - \dfrac{Lz}{r}$, and the solution of the homogeneous equation comes as linear combinations of two powers of z, where the powers are roots of the quadratic
$$-\rho + t(\rho - r - \tfrac{1}{2}\kappa^2) + \tfrac{1}{2}\kappa^2 t^2 = 0 \qquad \left[r = \tfrac{1}{2}\kappa^2(\alpha+1)(\beta-1),\ \rho = \tfrac{1}{2}\kappa^2\alpha\beta\right]$$
so there are two roots, $-\alpha < 0$ and $\beta > 1$. Thus the solution J has the form
$$J = \frac{u(L)}{\rho} - \frac{Lz}{r} + A(L)\,z^{-\alpha} + B(L)\,z^{\beta}.$$
Now J must be convex decreasing, so this forces $B(L) = 0$, and for large enough z we have
$$J(z,L) = \frac{u(L)}{\rho} - \frac{Lz}{r} + A(L)\,z^{-\alpha} \qquad (A \geq 0 \text{ necessary})$$
How large is large enough? For us, $V(w,L) = L^{1-R}h(y)$, $y \equiv w/L$, so $V_w = L^{-R}h'(y)$, and we get changeover to $C > L$ when $V_w = p\,u'(L) = pL^{-R}$. Thus the critical z is $z = pL^{-R}$, and
$$\boxed{J(z,L) = \frac{u(L)}{\rho} - \frac{Lz}{r} + A(L)\,z^{-\alpha} \qquad (z \geq pL^{-R})}$$

Case 2: $C > L$ This time
$$0 = (1-p)\,u(L) + p\,\tilde{u}(V_w/p) + rwV_w - \tfrac{1}{2}\kappa^2 V_w^2/V_{ww} - \rho V$$

$B_0 < 0$, $B_2 < 0$

where $\tilde{u}(x) = -x^{1-1/R}/(1-1/R)$ is the dual function of u. With the same change of variables, we arrive at

$$0 = (1-p)u(L) + p\tilde{u}(z/p) - \rho J + (\rho - r) z J_z + \tfrac{1}{2}\kappa^2 z^2 J_{zz}$$

which has particular solution

$$J = \frac{(1-p)u(L)}{\rho} + b\,\tilde{u}(z), \qquad b = \frac{R^2 p^{1/R}}{\rho R + \frac{1}{2}\kappa^2(R-1) + r R(R-1)}$$

Thus for $z \leq pL^{-R}$ we get

$$J(z,L) = \frac{(1-p)u(L)}{\rho} + b\tilde{u}(z) + B_1 z^{-\alpha} + B_2 z^{\beta}$$

By scaling, we can prove $J(z,L) = L^{1-R} J(zL^R, 1)$, so we can reduce to the case $L=1$, and find the solution (with slight abuse of notation) as

$$J(z) \equiv J(z,1) = \frac{u(1)}{\rho} - \frac{z}{r} + A\left(\frac{z}{p}\right)^{-\alpha} \qquad (z \geq p)$$

$$= \frac{(1-p)u(1)}{\rho} + b\tilde{u}(z) + B_1\left(\frac{z}{p}\right)^{-\alpha} + B_2\left(\frac{z}{p}\right)^{\beta} \qquad (z \leq p)$$

Matching values + first derivatives at p gives $B_1 = B_0 + A$, where

$$\boxed{B_0 = \frac{2p}{\kappa^2(\alpha+\beta)(1-R-\alpha R)\alpha(\alpha+1)} \qquad B_2 = \frac{-2p}{\kappa^2(\alpha+\beta)(1-R+\beta R)\beta(\beta-1)}}$$

[Maple].

The boundary condition $V_L = 0$ at the upper boundary of W/L can be reworked to the condition $(1-R)J + RzJ_z = 0$. If this holds at $z = px \in (0,p)$, then we deduce the form of A (which is the only part of the solution J which we don't yet have):

$$\boxed{A = \frac{-2\alpha p(\alpha+1)x^{\alpha+\beta} + 2(\alpha+\beta)(\beta-1)(\alpha+1)(1-p)x^{\alpha} + 2\beta p(\beta-1)}{\alpha\beta\kappa^2(\alpha+\beta)(\alpha+1)(\beta-1)(\alpha R + R - 1)}}$$

[Maple]

The derivative of this expression vanishes at the point

$$\left(\frac{(\beta-1)(1-p)}{p}\right)^{1/\beta},$$

[Maple].

which is in $(0,1)$ iff $\beta(1-p) < 1$. Notice that in order that J should be convex we will have to have $\boxed{A \geq 0.}$ When $x=0$, we have $A > 0$ (under non-degeneracy assumption)

We seek the critical value y^* of $y = w/L$. If $\frac{w}{L} > y^*$, then $V(w, L) = \left(\frac{w}{y^*}\right)^{1-R} v(y^*)$, and we shall have to have that it would be suboptimal to continue while in this region:

$$0 \geq \sup_c \left[(1-p)\, u(L) + p\, u(c) - \rho V - c V_w + r w V_w - \tfrac{1}{2}\kappa^2 V_w^2 / V_{ww} \right]$$

$$= (1-p)\, u(L) + p\, \tilde{u}(V_w/p) - \rho V + r w V_w - \tfrac{1}{2}\kappa^2 V_w^2 / V_{ww}$$

$$= (1-p)\, u(L) + p\, \tilde{u}(z/p) - \rho J + (\rho - r) z J_z + \tfrac{1}{2}\kappa^2 z^2 J_{zz}$$

Now we set $L = 1$ and see what happens if we smooth paste V to $\left(\frac{w}{y^*}\right)^{1-R} v(y^*)$ at $w = y^*$, equivalently, smooth paste J to $-a\, z^{1-1/R}/(1-1/R)$ at z^* for some $a > 0$. We shall need

$$0 \geq (1-p)\, u(1) + p^{1/R}\, \tilde{u}(z) - \rho a\, \tilde{u}(z) + (\rho - r)(1 - \tfrac{1}{R}) a\, \tilde{u}(z) - \tfrac{1}{2}\kappa^2 (1 - \tfrac{1}{R}) \tfrac{1}{R}\, a\, \tilde{u}(z)$$

with equality at $z = z_0$. This becomes

$$0 \geq (1-p)\, u(1) + \tilde{u}(z) \left[p^{1/R} - \rho a + (\rho - r)(1 - \tfrac{1}{R}) a - \tfrac{1}{2}\kappa^2 (1 - \tfrac{1}{R}) \tfrac{1}{R}\, a \right]$$

with equality at $z = z_0$, and the coefficient of $\tilde{u}(z)$ being < 0. Choosing our $z_0 = p x_0$ gives an expression for a:

$$a = \frac{2 R\, p^{1/R} \left\{ R + (p^{-1} - 1)\, x_0^{\,1/R - 1} \right\}}{\kappa^2 (\alpha R + R - 1)(\beta R - R + 1)} \qquad \text{[Maple]}$$

Thus if we have picked x_0, we can find the a which just matches the HJB thing out to the right of the critical value of y; we can also find the value of A at which the smooth pasting BC is satisfied. Thus we can compute the value of x_0 which joins the solution to the right of x_0 smoothly to the solution to the left of x_0; it is (of course!)

$$\left\{ \frac{(\beta - 1)(1 - p)}{p} \right\}^{1/\beta} \qquad \text{[Maple]}$$

(iv) <u>What happens if this value exceeds 1?</u> The presumption is that the interval in which $C > L$ shrinks away to the empty set, ...

One possibility would be that J smooth pastes to $a\, \tilde{u}(z)$ at some point $z_0 > p$; the point z_0 solves

$$g(z_0) \equiv -(1-R)(\beta R - R + 1)\, \beta\, z_0 - (\beta - 1)\left\{ z_0^{\,1 - 1/R}\, R^2 \beta\, p^{1/R} - (1-R) - \beta p R \right\} = 0, \qquad \text{[Maple]}$$

which cannot be solved in closed form. When $z_0 = p$, get $g(p) = (\beta - 1 - \beta p)(1 - R)$. Now we consider the two cases:

$\underline{0<R<1}$: Here g is strictly concave, positive at p, tending to $-\infty$ at ∞, so $\exists!$ root $\xi_0 > p$

$\underline{R>1}$: This time, g is strictly convex, negative at p, tending to ∞ at ∞, so $\exists!$ root $\xi_0 > p$.

Thus we do indeed get things smoothly joining up; what we have here is a solution to the familiar ratcheting problem!

(v) Observe that the original Merton problem can be ill posed when $0<R<1$; it is only well posed if

$$\rho > (1-R)\{r + \tfrac{1}{2}\kappa^2/R\}$$

which is the same as

$$\alpha R + R - 1 > 0$$

so we shall assume this, otherwise the problem is ill posed.

Minimum guarantee problems, general U (8/8/04)

(i) In general, the primal problem is

$$\sup_{C \geq L}\left[(1-p)u(L) + pU(C) - \rho V + (rw - C)V_w - \tfrac{1}{2}\kappa^2 V_w^2/V_{ww} \right] \leq 0$$

with equality in the continuation region, which we assume is of the form $\{(w,L) ; w \leq g(L)\}$ for some increasing function g. Maximising over C gives

$$c^* = I\left(\tfrac{V_w}{p} \wedge U'(L) \right)$$

and the equations $V_L \leq 0$,

$$(1-p)u(L) + p\, U(I(\tfrac{V_w}{p} \wedge U'(L))) - V_w\, I\left(\tfrac{V_w}{p} \wedge U'(L)\right) - \rho V + rwV_w - \tfrac{1}{2}\kappa^2 V_w^2/V_{ww} \leq 0.$$

In terms of dual variables $z \equiv V_w$, $J(z,L) \equiv V(w,L) - zw$, we have as before

$$0 = (1-p)u(L) + p\,U(I(\tfrac{z}{p} \wedge U'(L))) - z\,I(\tfrac{z}{p} \wedge U'(L)) - \rho J + (\rho - r) z J_z + \tfrac{1}{2}\kappa^2 z^2 J_{zz}$$

in the continuation region $z \leq h(L)$ for some function h. $J(\cdot, L)$ is convex decreasing for each value of L. We can reexpress the equation for J as

$$\left(\rho - (\rho - r)zD - \tfrac{1}{2}\kappa^2 z^2 D^2\right)J = f(z,L)$$

$$\equiv (1-p)u(L) + p\,U(I(\tfrac{z}{p} \wedge U'(L))) - z\,I(\tfrac{z}{p} \wedge U'(L)).$$

Now there's a probabilistic solution to this (not the only one, but if we require J decreasing convex I think it is unique - this needs checking);

$$\boxed{J(z,L) = E\left[\int_0^\infty e^{-\rho t} f(Z_t, L_t)\, dt \,\middle|\, Z_0 = z,\ L_0 = L\right]}$$

where Z is the geometric BM

$$dZ = (\kappa dW + (\rho - r)dt)Z$$

and $L_t = L_0 \wedge \varphi(\underline{Z}_t)$, $\underline{Z}_t \equiv \inf_{s \leq t} Z_s$, and φ is to be chosen.

(ii) It's perhaps more convenient to be working in terms of $X_t \equiv \log Z_t$,

$$dX_t = \kappa\, dW_t + m\, dt$$

for then various expressions for the resolvent and excursion measure can be written simply. The ρ-resolvent density is

$$r_\rho(0,x) = \frac{2e^{-bx^+ - ax^-}}{\kappa^2(a+b)}$$

where $-a < 0 < b$ are roots of $\tfrac{1}{2}\kappa^2 t^2 + mt - \rho = 0$. From this we can get the mean occupation measure of excursions up from 0

$$n\left(\int_0^\zeta e^{-\rho t}\, I_{\{x_t\in dy\}}\, dt\right)$$

$$= \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\left\{ r_\rho(\varepsilon,y) - e^{-a\varepsilon} r_\rho(0,y)\right\} = 2e^{-by}/\kappa^2,$$

and hence by decomposing in terms of excursions up from the minimum of X we get the expression

$$E\left[\int_0^\infty e^{-\rho t} F(X_t, \varphi(\underline{X}_t))\, dt \;\middle|\; X_0 = \underline{X}_0 = x\right]$$

$$= \int_{-\infty}^{x} e^{-a(x-y)} \left\{\int_0^\infty F(y+v, \varphi(y))\, \frac{2e^{-bv}}{\kappa^2}\, dv\right\} dy$$

(iii) How can we optimise this over φ? In general, we would need to do some variational argument, but an obvious approach would just to try a y-by-y optimisation;

$$\max_\eta \int_0^\infty F(y+v, \eta)\; 2e^{-bv}\, dv$$

If the optimising η turn out to be monotone in y, then we are done. Now we have

$$f(z,L) = (1-p)\,u(L) + p\,\tilde{U}\left(\tfrac{z}{p}\wedge U'(L)\right) - \left(z - pU'(L)\right)^+ L$$

$$= \begin{cases} (1-p)\,u(L) + p\,U(L) - zL & \text{if } z \geq pU'(L) \\ (1-p)\,u(L) + p\,\tilde{U}(z/p) & \text{if } z \leq pU'(L) \end{cases}$$

So $F(x,L) = f(e^x, L)$. Do we get required monotonicity here? Taking $\gamma \equiv e^y$,

$$\Psi(y,L) = \int_0^\infty b e^{-bv} f(e^{y+v}, L)\, dv$$

$$= (1-p)\,u(L) + \int_0^\infty b e^{-bv} \min\left\{p\tilde{u}(\gamma e^v/p),\ pU(L) - L\gamma e^v\right\} dv$$

$$= (1-p)\,u(L) + \int_1^\infty b x^{-b-1} \min\left\{p\tilde{U}(\gamma x/p),\ pU(L) - \gamma L x\right\} dx$$

$$= (1-p)\,u(L) + \int_1^{pU'(L)/\gamma} b x^{-b-1}\, p\,\tilde{U}(\gamma x/p)\, dx + \int_{pU'(L)/\gamma}^\infty b x^{-b-1}\left\{pU(L) - \gamma L x\right\} dx$$

$$\text{if } pU'(L)/\gamma > 1$$

$$= (1-p)\,u(L) + pU(L) - \gamma L\, b/(b-1) \quad \text{if } pU'(L)/\gamma \leq 1.$$

$$\left[\sup_{c \geq L} \left\{ p\,U(c) - \lambda c \right\} = p\,\tilde{U}(\lambda/p) - \int_0^L (\lambda - p\,U'(x))^+ \, dx . \right]$$

Differentiating wrt L gives

$$\Psi_L(y,L) = \begin{cases} (1-p)u'(L) + p\,U'(L) - \gamma b/(b-1) & \text{if } p\,U'(L) \le \gamma \\ (1-p)\,u'(L) - \dfrac{\gamma^b}{b-1}\left(p\,U'(L)\right)^{-b+1} & \text{if } p\,U'(L) > \gamma \end{cases}$$

Assuming Inada conditions for u, U, we get a unique L for which this vanishes, where

$$(1-p)\,u'(L) = \left\{\frac{\gamma\, b}{b-1} - p\,U'(L)\right\} I_{\{p U'(L) \le \gamma\}}$$
$$+ \frac{\gamma^b}{b-1}\left(p\,U'(L)\right)^{-b+1} I_{\{p\,U'(L) > \gamma\}}$$

(LHS is decreasing, RHS is increasing. Raising γ raises RHS, and so lowers the solution L).

(iv) Alternatively, as Phil proposes, we can view the optimisation as an optimisation over arbitrary $C_t \ge L_t$, subject to a budget constraint. His notes on this contain a few errors, and are for the special case $u = U$, so let's develop this approach here for $u \ne U$.

$$\max\ E\int_0^\infty e^{-\rho t}\{(1-p)\,u(L_t) + p\,U(C_t)\}\,dt$$

subject to

$$E\int_0^\infty \xi_t C_t\,dt = w_0, \qquad C_t \ge L_t, \quad L_0 \text{ given}, \ L \text{ increasing}.$$

The Lagrangian form is

$$\max\ E\int_0^\infty e^{-\rho t}\left[(1-p)\,u(L_t) + p\,U(C_t) - \lambda e^{\rho t}\xi_t C_t\right] dt$$

where $d\xi_t = \xi_t(-r\,dt - \kappa\,dW_t)$ is the state-price density, $\kappa = (\mu-r)/\sigma$. The objective can be maximised t by t to give

$$E\int_0^\infty e^{-\rho t}\Big[(1-p)\,u(L_t) + p\,\tilde U(\lambda e^{\rho t}\xi_t/p)$$
$$- \int_0^{L_t}\left(\lambda e^{\rho t}\xi_t - p\,U'(x)\right)^+ dx\Big]\,dt$$

$$= H(L_0) + E\int_0^\infty e^{-\rho t}(1-p)\left(u(L_t) - u(L_0)\right)dt - E\int_0^\infty e^{-\rho t}\int_{L_0}^{L_t}\left(\lambda e^{\rho t}\xi_t - p\,U'(x)\right)^+ dx\,dt$$

where

$$H(L_0) \equiv E\int_0^\infty e^{-\rho t}\left[(1-p)\,u(L_0) + p\,\tilde U(\lambda e^{\rho t}\xi_t/p) - \int_0^{L_0}\left(\lambda e^{\rho t}\xi_t - p\,U'(x)\right)^+ dx\right]$$

The terms to be optimised can be re-expressed conveniently as

$$E\int_{L_0}^{\infty}(1-p)u'(\ell)\, e^{-\rho\tau_\ell}/\rho\, d\ell \;-\; E\int_{L_0}^{\infty}dx \int_{\tau_x}^{\infty}\left(\lambda\xi_t - e^{-\rho t}p\,U'(x)\right)^+ dt$$

<u>Proposition</u> Defining

$$h(\xi_0) = E\int_0^\infty \left(\xi_t - e^{-\rho t}\right)^+ dt$$

we have

$$h(x) = \begin{cases} Bx^b & (x\le 1)\\ \frac{x}{r} - \frac{1}{\rho} + Ax^{-a} & (x\ge 1)\end{cases}$$

where $-a<0$, $b>1$ are roots of $\frac{1}{2}\kappa^2 y(y-1) + (\rho-r)y - \rho$, and A, B come from C^1 join at $x=1$;

$$\begin{pmatrix}A\\B\end{pmatrix} = \frac{1}{(a+b)\rho r}\begin{pmatrix}\rho - b(\rho-r)\\ \rho + a(\rho-r)\end{pmatrix}$$

<u>Proof</u>. Straightforward. $\square$

Using this, our goal is to maximise

$$E\int_{L_0}^{\infty}(1-p)u'(\ell)\,\frac{e^{-\rho\tau_\ell}}{\rho}\,d\ell \;-\; E\int_{L_0}^{\infty}d\ell\;\, pU'(\ell)e^{-\rho\tau_\ell}\,h\!\left(\lambda e^{\rho\tau_\ell}\xi(\tau_\ell)/pU'(\ell)\right)$$

$$= \int_{L_0}^{\infty} E\; e^{-\rho\tau_\ell}\left\{\frac{1-p}{\rho}u'(\ell) - pU'(\ell)\,h\!\left(\frac{\lambda e^{\rho\tau_\ell}\xi(\tau_\ell)}{pU'(\ell)}\right)\right\} d\ell$$

which it seems reasonable to attack as a collection of optimal stopping problems, one for each ℓ.

Fixing ℓ, we aim to choose stopping time τ so as to

$$\max\; E\; e^{-\rho\tau}\left(k - f(\tilde{\xi}_\tau)\right)$$

where $k = (1-p)u'(\ell)/\rho$, $f(x) = pU'(\ell)\,h(\lambda x/pU'(\ell))$ and $\tilde{\xi}_t = e^{\rho t}\xi_t$.

The form of the solution is

$$V(x) = \begin{cases} a_\ell\, x^{-a} & (x\ge x^*)\\ k - f(x) & (x\le x^*)\end{cases}$$

where x^*, a_ℓ are selected to make V be C^1 at x^*:

$$\left.\begin{aligned} a_\ell x^{-a} &= k - f(x)\\ a\,a_\ell x^{-a-1} &= f'(x)\end{aligned}\right\} \quad\Rightarrow\quad \boxed{k = f(x) + \frac{xf'(x)}{a}}$$

To find $x^* = x^*(\ell)$, we have to consider two regimes:

Case 1: $x > pU'(\ell)/\lambda$. Here the equation for x takes the form

$$k \equiv (1-p)\,u'(\ell)/\rho \;=\; pU'(\ell)\left\{\frac{\tilde{x}\,(1+a)}{ar} - \frac{1}{\rho}\right\} \qquad \tilde{x} \equiv \lambda x / pU'(\ell)$$

Case 2: $x \leq pU'(\ell)/\lambda$. This time the equation is

$$k \equiv (1-p)\,u'(\ell)/\rho \;=\; pU'(\ell)\,\frac{\rho + a(\rho - r)}{a\rho r}\,\tilde{x}^{\,b}$$

Comparing with the form of the solution from p56, we see that they are exactly the same, taking $\gamma\,[p56] \equiv \lambda x\,[\text{here}]$, using the identities $ab = 2\rho/\kappa^2$, $(a+1)(b-1) = 2r/\kappa^2$

(v) When is the problem well posed? The issue is whether the sup is unbounded (it may be alternatively that we are unable to satisfy the constraints, but that is easily fixed by increasing initial wealth). There are two things to deal with, $E\int_0^\infty e^{-\rho t} u(L_t)\,dt < \infty$ and $E\int_0^\infty e^{-\rho t} U(C_t)\,dt < \infty$. For the first, we need to decide 'Is the ratcheting problem well posed?' and for the second we have to decide 'Is the Merton problem well posed?' [If both are well posed, then neither component of our objective can become infinite; if the ratcheting problem is ill-posed, then we can make the payoff of our problem infinite, and if the Merton problem is ill posed the wealth grows exponentially so the consumption is eventually greater than any lower bound].

(a) Merton problem: We have $e^{-\rho t}U'(C_t) = \lambda \xi_t$, so we need

$$E\int_0^\infty e^{-\rho t}\, U(\lambda e^{\rho t}\xi_t)\,dt < \infty$$

We have $e^{\rho t}\xi_t = e^{x_t}$ where $dx_t = \kappa\,dW_t + m\,dt$ $(m \equiv \rho - r)$, so if we write $F(x) \equiv U(\lambda e^{x})$ then we want

$$E\int_0^\infty e^{-\rho t} F(x_t)\,dt < \infty$$

$$= R_\rho F(x_0)$$

where R_ρ is the ρ-resolvent operator, given explicitly on p 54; we shall need the condition

$$\boxed{\infty > \int^{\infty} e^{-bx} F(x)\,dx = \int^{\infty} U(y)\, y^{-1-b}\,dy}$$

where b is the positive root of $\frac{1}{2}\kappa^2 s^2 + ms - \rho = 0$.

(b) Ratcheting problem. In this case, from the Lagrangian story we'll get

$$e^{-\rho t} U'(c_t) - \lambda \xi_t \leq 0$$

so that $c_t \geq I(\lambda e^{\rho t} \xi_t)$. Thus if $\underline{x}_t = \inf_{u \leq t} x_u$ $\quad (x_u = \kappa W_u + mu)$

the optimal consumption for the ratcheting problem takes the form

$$I(\lambda e^{-\underline{x}_t})$$

and the payoff is

$$E \int_0^\infty e^{-\rho t} U(I(\lambda e^{\underline{x}_t}))\, dt$$

$$= E \int_0^\infty e^{-\rho t} G(\underline{x}_t)\, dt \qquad \text{say, } \boxed{G(x) = U(I(\lambda e^x))}$$

$$= E \int_0^\infty e^{-\rho t} \left\{ G(0) + \int_{-\infty}^0 I_{\{\underline{x}_t \leq z\}} (-G')(z)\, dz \right\} dt$$

so we need

$$E \int_{-\infty}^0 (-G')(z) \frac{1}{\rho} e^{-\rho H_z}\, dz < \infty$$

where $H_z \equiv \inf\{u : x_u \leq z\}$. However, the Laplace transform of the the time to hit low levels is known; we obtain the condition

$$\boxed{\int_{-\infty} (-G')(z)\, e^{az}\, dz < \infty}$$

where $-a < 0$ is the root of $\frac{1}{2}\kappa^2 s^2 + ms - \rho$ below 0.

Interesting questions

1) Terry says the following:

(i) If you take some Padé approximation to $\exp(x)$, this is actually a continued fraction expansion. When you use this to approximate $\exp(tG)$ for some Markovian generator, you get a Markov process generator again... (no known counterexample)

(ii) To simulate a Gaussian variable, take ± 1 step, then ± 2 step, ..., then $\pm n$ step; by renormalising to unit variance you get a much better approx to Gaussian than just the rw.

(iii) Work with Dan Crisan on particle filters - seems that this can be a v. good way to estimate a model

(iv) For the MC American, how about using the martingale $M_t = E_t Z_\tau$ where τ is some good stopping time?

2) How would we numerically delta hedge? If we have some tree method, then rather than start again from a nearby point, just reweight the probs on the next time-slice of the tree!

3) Dual methods for optimal control?

4) David + Vicky's study of q-optimal things gives us a BSDE...! Is it possible to characterise the solution to an expected-utility maximisation problem via a BSDE?

[Further to 1(i), if we do a Padé approximation to $\exp(z) \approx P(z)/P(-z)$ of degree L, then

$$P(z) = \sum_{j=0}^{L} \frac{(2L-j)!}{(2L)!} \frac{L!}{(L-j)!} \frac{z^j}{j!}$$

]

5) Conventionally we define the objective of an optimal investment/consumption problem as $E\int_0^\infty U(t, c_t)\,dt$, but maybe we should try to use $E\int_0^\infty e^{-\rho t}\, U(c_t, x_t)\,dt$, where x_t is cash in hand at time t??

6) If we have some utility $U: \mathbb{R} \to \mathbb{R}$, and we are concerned to ensure that $\tilde{E}\, U(\theta \cdot Z) < \infty$ for some $\tilde{P}$ equivalent to P, $\forall \theta$, we're effectively looking for some weight W such that for all $t \in \mathbb{R}$ $\sup_z |U(tx)| / W(x) < \infty$. We can take $W(x) = \max\{1, U(x^2), |U(x^2 \operatorname{sgn}(x))|\}$.

7) Yannis + Bob Fernholz look at markets where diversity is imposed in some exogenous way, + find arbitrage. The empirical observation is that small-cap stocks typically have higher rates of appreciation than large-cap stocks. It seems to me that it might be possible to model this by the higher leverage of small firms...

8) Peppe has done an equilibrium analysis of a Markov-modulated stock process. How does the equilibrium look if there's a <u>non-zero</u> supply of bonds??