Merton problem with liquidity effects again (7/9/04)

(i) Returning to the Merton problem with liquidity costs, wealth dynamics

$$dw = rw\,dt + H\,(\sigma dW + (\mu - r)dt)S - C\,dt - h\,f(h)\,S\,dt$$

with H denoting the # units stock held, we shall have in terms of the variable $(w - HS)/S \equiv Y$, the cash holdings in units of the stock the value function is

$$F(Y, H)\, S^{1-R}$$

where F solves the HJB equation

$$\boxed{\sup_{c,H}\Big[\,U(c) - \tilde{\rho}F + \tfrac{1}{2}\sigma^2 Y^2 F_{YY} - (h + hf(h) + c + \alpha Y)\,F_Y + hF_H\,\Big] = 0}$$

with

$$\tilde{\rho} \equiv \rho + (R-1)(\mu - \tfrac{1}{2}\sigma^2 R), \qquad \alpha \equiv \mu - r - \sigma^2 R.$$

(ii) If V_M is the Merton value function, $V_M(w) \equiv \gamma^{-R}\,U(w)$, we shall try to write

$$F(Y,H) = V_M\,(Y + H + \Delta(Y,H)).$$

If we do this, treat the perturbation Δ as small, and expand out to leading order in Δ, we shall obtain (with $f(x) = x/2$, though this doesn't appear critical)

$$\boxed{\begin{aligned} 0 = & -\frac{\sigma^2 R(\pi Y - (1-\pi)H)^2}{2\,(Y+H)} + \tfrac{1}{2}\sigma^2 Y^2 \Delta_{YY} \\ & - \frac{\{\sigma^2 RY\,(\pi Y - (1-\pi)H) + \gamma\,(Y+H)^2\}}{Y+H}\,\Delta_Y + \frac{\sigma^2 YR\{\pi Y - (1-\pi)H\}}{(Y+H)^2}\,\Delta \end{aligned}}$$

Notice that no derivatives wrt H appear, so we really only have an ODE in Y for each H. If we write

$$\Delta(Y,H) = H\,\varphi(Y/H),$$

then we find

$$\boxed{\tfrac{1}{2}\sigma^2 x^2 \varphi'' - \frac{\{\sigma^2 xR\,(\pi x - (1-\pi)\;) + \gamma(1+x)^2\}}{1+x}\,\varphi' + \frac{\sigma^2 xR\{\pi x - (1-\pi)\}}{(1+x)^2}\,\varphi - \frac{\sigma^2 R\,(\pi x - (1-\pi))^2}{2(1+x)} = 0}$$

SAME ODE for each H (though not necessarily same solution ...)

(iii) For small x, we are seeing $(\pi' \equiv 1-\pi)$

$$\frac{\sigma^2}{2}x^2\varphi'' - \gamma\varphi' - \tfrac{1}{2}\sigma^2 R\,\pi'^2 \doteq 0$$

which would have solution
$$\varphi'(x) = -\frac{R\sigma^2\pi'^2}{2\gamma} + A e^{-2\gamma/\sigma^2 x}.$$

For large x, we see approximately

$$\tfrac{1}{2}\sigma^2 x^2\varphi'' - (\gamma + \sigma^2 R\pi)x\varphi' + \sigma^2 R\pi\,\varphi - \tfrac{1}{2}\sigma^2\pi^2 R\,x \doteq 0$$

which is solved by

$$\varphi = -\frac{\sigma^2 R\pi^2}{2\gamma}x + Ax^\alpha + Bx^\beta$$

where α, β are roots of

$$\tfrac{1}{2}\sigma^2 t(t-1) - (\gamma + \sigma^2 R\pi)t + \sigma^2 R\pi = 0,$$

where $\alpha < 1 < \beta$.

The behaviour at small values of x suggest we may find it worth writing $\varphi(x) = \psi(1/x)$, which leads to the form $(s \equiv 1/x)$

$$\boxed{\tfrac{1}{2}\sigma^2 s^2\psi''(s) + \frac{\{\gamma(1+s)^2 + \sigma^2(1+s + R(\pi - \pi' s))\}}{1+s}\, s\psi'(s) + \frac{\sigma^2 R(\pi - s\pi')}{(1+s)^2}\psi(s) - \frac{\sigma^2 R(\pi - s\pi')^2}{2s(1+s)} = 0}$$

Maple can't find a closed form for this either....

Asymptotics for the HR stochastic vol model (8/9/04)

(i) In the HR model, the dynamics of the log discounted price are given by

$$dX_t = \sigma(Z_t)\,dW_t - \tfrac{1}{2}\sigma(Z_t)^2\,dt$$

while

$$dZ_t = dX_t - \lambda Z_t\,dt \equiv dX_t - \delta^{-2} Z_t\,dt$$

What is the effect of this stochastic volatility term as $\delta \downarrow 0$?

(ii) If we consider the price of a European option,

$$f(\tau, x, z) = E\left[\varphi(X_\tau) \mid X_0 = x,\ Z_0 = z\right]$$

then we shall have the pricing PDE

$$-rf - f_\tau + \tfrac{1}{2}\sigma(z)^2\left[f_{xx} + 2f_{xz} + f_{zz} - f_x - f_z\right] - \frac{z f_z}{\delta^2} = 0$$

and one form of the asymptotics we could seek is to find how

$$f(\tau/\delta^2, x, z/\delta\,;\delta)$$

behaves as $\delta \downarrow 0$, keeping τ, z both $O(1)$. Seeking an expansion

$$f(\tau/\delta^2, x, z/\delta\,;\delta) = \sum_{n\geq 0} \delta^n g_n(t, x, \zeta) \qquad \begin{pmatrix} t \equiv \tau/\delta^2 \\ \zeta \equiv z/\delta \end{pmatrix}$$

gives us some expansion (Maple!)

$$g_0 = \varphi(x)$$

$$g_1 = 0$$

$$g_2(t,x,\zeta) = t\left\{-r\varphi(x) + \tfrac{1}{2}\sigma(0)^2\left(\varphi''(x) - \varphi'(x)\right)\right\}$$

$$g_3(t,x,\zeta) = \zeta(1-e^{-t})\left\{\varphi''(x) - \sigma(0)\sigma'(0)\varphi'(x)\right\}$$

$$g_4(t,x,\zeta) = \tfrac{1}{2}\left\{\zeta^2(1-e^{-2t})/2 + \frac{\sigma(0)^2}{4}\left(e^{-2t}-1+2t\right)\right\}\left\{\varphi''(x) - \varphi'(x)\right\}\left(\sigma_0\sigma_0'' + \sigma_0'^2\right)$$

$$+ \tfrac{1}{2}t^2\left\{r^2\varphi(x) + r\sigma_0^2\varphi'(x) + \left(\tfrac{1}{4}\sigma_0^4 - r\sigma_0^2\right)\varphi''(x) - \tfrac{1}{2}\sigma(0)^4\varphi'''(x) + \frac{\sigma_0^4}{4}\varphi^{(4)}(x)\right\}$$

$$+ \left(1-e^{-t}-t\right)\left\{3\varphi''(x) - 2\varphi'''(x) - \varphi'(x)\right\}\sigma_0^3\sigma_0'/2$$

GRAD/ALESSANDRO/ HRasymp.mws

(iii) But there is perhaps a more interesting asymptotic here. Writing $v(z) = \frac{1}{2}\sigma(z)^2$, we have

$$-rf - f_\tau + v(z)\{f_{zz} + 2f_{zx} + f_{xx} - f_z - f_x\} - z f_z/\delta^2 = 0$$

and if we now write

$$f(\tau, x, z; \delta) = g(\tau, x, z/\delta; \delta)$$

we shall be finding out about the option price at times that are large relative to the mean mixing time of Z, at values of the offset that are typical. We thus have the PDE

$$-rg - g_\tau + v(\delta\zeta)\{g_{xx} + \frac{2}{\delta} g_{x\zeta} + \delta^{-2} g_{\zeta\zeta} - g_x - \delta^{-1} g_\zeta\} - \zeta g_\zeta/\delta^2 = 0.$$

Now the idea is to try to expand

$$g(\tau, x, \zeta; \delta) = \sum_{k \geq 0} \delta^k g_k(\tau, x, \zeta),$$

and pick off the terms one by one. We get

δ^{-2}]
$$v(0) D_{\zeta\zeta} g_0 - \zeta D_\zeta g_0 = 0$$

Assuming we work with a bounded expiry value φ which is also reasonably smooth, this means that for each (τ, x), $g(\tau, x, \cdot)$ is a bdd harmonic f^n for the OU operator, therefore constant so $g_0(\tau, x, \zeta) = g_0(\tau, x)$.

δ^{-1}]
$$v(0) D_{\zeta\zeta} g_1 - \zeta D_\zeta g_1 = 0$$

implies as before that $g_1(\tau, x, \zeta) = g_1(\tau, x)$ (assuming g_1 is suitably bounded...)

δ^0] We have

$$v(0) D_{\zeta\zeta} g_2 - \zeta D_\zeta g_2 = -\{v(0)(D_{xx} g_0 - D_x g_0) - D_1 g_0 - r g_0\}$$

and the RHS is independent of ζ. Fixing (τ, x), if we integrate both sides with respect to the invariant law of the OU, we get 0 on LHS, so RHS must be zero, that is,

$$\boxed{-r g_0 - D_\tau g_0 + v(0)(D_{xx} g_0 - D_x g_0) = 0}$$

that is, g_0 solves the BS PDE with constant vol $\sigma(0)$, and terminal condition $g_0 = \varphi$. Moreover, we get $g_2 = g_2(\tau, x)$ as before.

δ^1] This time we get

$$v(0) D_{\zeta\zeta} g_3 - \zeta D_\zeta g_3 + (-r g_1 - D_\tau g_1 + v(0)(D_{xx} g_1 - D_x g_1)) + \zeta v'(0)(D_{xx} g_0 - D_x g_0) = 0$$

Fixing (τ, x), we thus see $v(0) D_{\zeta\zeta} g_3 - \zeta D_\zeta g_3 + a + b\zeta = 0$, and for this to hold we must

$$b = \frac{1}{2} v''(0) \left\{ D_{\zeta\zeta}\, g_0 - D_{\zeta}\, g_0 \right\}$$

again have the mean wrt the invariant law of $a+b\zeta$ to be zero. Thus we deduce $a=0$, that is, g_1 also satisfies the BS PDE with volatility $\sigma(0)$. Assuming that the expansion is good down to and including $\tau=0$ would give us $g_1 \equiv 0$, and the equation

$$v(0)\, D_{\zeta\zeta}\, g_3 - \zeta D_\zeta g_3 + \zeta\, v'(0) \{ D_{xx} g_0 - D_x g_0 \} = 0$$

This gives us

$$g_3(\tau,x,\zeta) = \bar{g}_3(\tau,x) + v'(0)\Big(D_{xx} g_0 - D_x g_0 \Big)\zeta$$

for some function $\bar{g}_3$ to be determined.

δ^2 | This time we arrive at

$$v(0)\, D_{\zeta\zeta}\, g_4 - \zeta D_\zeta g_4$$

$$-r g_2 - D_\tau g_2 + v(0)\{ D_{xx} g_2 - D_x g_2 \}$$

$$+ \tfrac{1}{2} \zeta^2 v''(0) \{ D_{xx} g_0 - D_x g_0 \} - v_0 v_0' \{ 3 D_{xx} g_0 - D_x g_0 - 2 D_{xxx} g_0 \} = 0$$

and once again fixing (τ, x) we see an equation for g_4 of the form

$$v(0)\, D_{\zeta\zeta}\, g_4 - \zeta D_\zeta g_4 + a + b\zeta^2 = 0$$

which can only be solved if $a + \frac{\sigma(0)^2}{2} b \equiv a + v(0)\, b = 0$; this gives us a PDE for g_2

$$\boxed{\begin{aligned} &-r g_2 - D_\tau g_2 + v(0)\{ D_{xx} g_2 - D_x g_2 \} \\ &\quad + \tfrac{1}{2} v_0 v_0'' \{ D_{xx} g_0 - D_x g_0 \} - v_0 v_0' \{ 3 D_{xx} g_0 - D_x g_0 - 2 D_{xxx} g_0 \} = 0 \end{aligned}}$$

and the form

$$g_4(\tau,x,\zeta) = \bar{g}_4(\tau,x) + \tfrac{1}{2}\, b(\tau,x)\, \zeta^2$$

with

$$\boxed{\begin{aligned} & b(\tau,x) \\ \equiv \;& v_0'' \Big(D_{xx} g_0 - D_x g_0 \Big) \end{aligned}}$$

$\underline{\delta^3}$ Going down to the next term gives us

$$v_0 D_{\zeta\zeta} g_5 - \zeta D_\zeta g_5$$
$$+ \left\{ -r\bar{g}_3 - D_\tau \bar{g}_3 + v(0)\left(D_{xx}\bar{g}_3 - D_x \bar{g}_3\right)\right\}$$
$$+ \quad \alpha(\tau,x)\,\zeta^3 \quad + \quad \beta(\tau,x)\,\zeta$$

where

$$\alpha(\tau,x) = \frac{1}{6} v'''(0)\left\{ D_{xx} g_0 - D_x g_0 \right\}$$

$$(v_0')^{-1}\beta(\tau,x) = -D_{\tau xx} g_0 + D_{\tau x} g_0 - 2v_0 D_{xxx} g_0 + v_0 D_{xx} g_0 - r D_{xx} g_0 - 3v_0' D_{xx} g_0$$
$$+ v_0' D_x g_0 + r D_x g_0 + v_0 D_{xxxx} g_0 + 2 v_0' D_{xxx} g_0$$
$$- D_x g_2 + D_{xx} g_2$$

For there to be a solution, we must have that $\bar{g}_3$ solves the BS PDE, with BC zero, and so we get $\boxed{\bar{g}_3 = 0.}$

$\underline{\delta^4}$ A similar analysis gives (with $\langle \cdot \rangle$ denoting mean w.r.t. invariant law of the OU process)

$$\Big\langle -D_\tau \bar{g}_4 - r\bar{g}_4 + v(0)\left(D_{xx}\bar{g}_4 - D_x g_4\right)$$
$$+ a_4(\tau,x)\,\zeta^4 + b_4(\tau,x)\,\zeta^2 + c_4(\tau,x) \Big\rangle = 0$$

with

$$a_4(\tau,x) = v^{(4)}(0)\left(D_{xx} g_0 - D_x g_0\right)/4!$$

$$b_4(\tau,x) = v'''(0)v'(0)\left(D_{xx} g_0 - D_x g_0\right)/3 + v'''(0)v(0)\left(\tfrac{1}{3} D_{xxx} g_0 - \tfrac{1}{2} D_{xx} g_0 + \tfrac{1}{6} D_x g_0\right)$$
$$+ v''(0)v'(0)\left(D_{xxx} g_0 - \tfrac{3}{2} D_{xx} g_0 + \tfrac{1}{2} D_x g_0\right) + v(0)v''(0)\left(\tfrac{1}{4} D_{xxxx} g_0 - \tfrac{1}{2} D_{xxx} g_0 + \tfrac{1}{4} D_{xx} g_0\right)$$
$$+ \tfrac{1}{4} v''(0)\left(2(D_{xx} g_2 - D_x g_2) - D_{\tau xx} g_0 + D_{\tau x} g_0 + r(D_x g_0 - D_{xx} g_0)\right)$$

$$c_4(\tau,x) = \tfrac{1}{3} v'''(0)v(0)^2\left(2D_{xxx} g_0 - 3D_{xx} g_0 + D_x g_0\right) + v'(0)^2 v(0)\left\{4D_{xxxx} g_0 - 8D_{xxx} g_0\right.$$
$$\left. + 5D_{xx} g_0 - D_x g_0\right\} + v(0)^2 v'(0)\left\{2D_{xxxxx} g_0 - 5D_{xxxx} g_0 + 4D_{xxx} g_0 - D_{xx} g_0\right\}$$
$$+ v(0)v'(0)\left\{2D_{xxx} g_2 + D_x g_2 - 3D_{xx} g_2 - 2r D_{xxx} g_0 - 2D_{\tau xxx} g_0\right.$$
$$\left. + 3D_{\tau xx} g_0 + 3r D_{xx} g_0 - D_{\tau x} g_0 - r D_x g_0\right\}$$

$$\left[\langle \zeta^2 \rangle = \frac{\sigma(0)^2}{2}, \quad \langle \zeta^4 \rangle = 3\sigma(0)^4/4 .\right]$$

Money-in-the-bank example (16/9/04)

1) Here is an example discussed briefly with Phil, + taken up with Sevo. There are the usual two assets, riskless (rate r, const) and risky ($\sigma dW + \mu dt$), and the objective of the investor is to

$$\max E\left[\int_0^\infty e^{-\rho t}\, U(x_t, c_t)\, dt \,\Big|\, w_0 = w\right] \equiv V(w),$$

where x_t is amount in the bank account,

$$\begin{cases} dx_t = r x_t\, dt - c_t\, dt + dM_t - dL_t \\ dy_t = (\sigma dW_t + \mu dt) y_t - dM_t + dL_t. \end{cases}$$

The usual HJB analysis gets us to

$$\sup_{y,c}\left[U(w-y, c) - \rho V + \tfrac{1}{2}\sigma^2 y^2 V'' + \{(\mu-r)y + rw - c\} V' \right] = 0$$

with corresponding FOCs

$$\begin{cases} U_c = V' \\ U_x = \sigma^2 y V'' + (\mu - r) V' \end{cases}$$

2) One thing we could do would be to bound

$$-\tfrac{1}{2} y^2 \le -\tfrac{1}{2}\eta^2 - \eta(y-\eta)$$

to get

$$\sup_{y,c}\,[\text{—}] \le \sup\Big[U(w-y,c) - \rho V + \tfrac{1}{2}\sigma^2\eta^2 V'' + \sigma^2 V'' \eta(y-\eta) + ((\mu-r)y + rw - c) V' \Big]$$

$$= \sup\Big[U(w-y,c) - cV' - (w-y)\left(\sigma^2\eta V'' + (\mu-r)V'\right)\Big] + w\left(\sigma^2\eta V'' + (\mu-r)V'\right) - \tfrac{1}{2}\sigma^2\eta^2 V'' + rwV'$$

$$= \tilde{U}\left(\sigma^2\eta V'' + (\mu-r)V',\ V'\right) - \tfrac{1}{2}\sigma^2\eta^2 V'' + w\left(\sigma^2\eta V'' + (\mu-r)V'\right) + rwV'$$

which we may now proceed to minimise over η, giving FOCs

$$\sigma^2 V'' \,\tilde{U}_1'(\sigma^2\eta V''+(\mu-r)V', V') - \sigma^2\eta V'' + \sigma^2 w V'' = 0$$

or more simply

$$\boxed{\tilde{U}_1(\sigma^2\eta V''+(\mu-r)V', V') = \eta - w}$$

Solving this in closed form will rarely be possible, though numerical solution should be OK. This isn't really very different from directly optimising in HJB, and the drawback is that we get a non-linear ODE, and no idea what solution to choose..

3) We could try Lagrangian methods, with Lagrangian process $Y_t = e^{-\rho t} z_t$, $dz_t = z_t(a_t dW_t + b_t dt)$. By the usual means, we get

$$\sup E\Big[\int_0^\infty e^{-\rho t}\{U(x_t,c_t) + z(rw + (\mu-r)y - c) + wz(b-\rho) + az\sigma y\}dt - [Yw]_0^\infty\Big]$$

$$= \sup E\int_0^\infty e^{-\rho t}\{U(x_t,c_t) - z_t c_t - x_t z_t(\mu-r+a\sigma) + w_t z_t(\mu+b-\rho+a\sigma)\}dt + Y_0 w_0$$

if transversality OK

$$= E\int_0^\infty e^{-\rho t}\tilde{U}(z_t(\mu-r+a\sigma), z_t)\,dt + Y_0 w_0$$

with the conditions

$$\begin{cases} \sigma a_t + \mu - \rho + b_t = 0 \\ \sigma a_t + \mu - r \geq 0 \end{cases} \qquad (\text{so } b_t \leq \rho - r)$$

We will have to have

$$\begin{cases} U_c(x_t,c_t) = z_t = V'(w_t) \\ U_x(x_t,c_t) = z_t(\mu-r+a_t\sigma) = \sigma y_t V''(w_t) + (\mu-r)V'(w_t) \end{cases}$$

So if we take the special form

$$U(x,c) = u(\alpha x + c)$$

we have $\alpha U_c = U_x$, and so a_t must be <u>constant</u>, which should make things simpler. (introducing $\tilde{c} \equiv c + \alpha x$ and rewriting dynamics of w in terms of $\tilde{c}$ gets us back to usual story).

4) Another special case that should be OK would be if $U(x,c)\lambda^{1-R} = U(\lambda x, \lambda c)$ . .

Heterogeneous officials?
Officials saving embezzled funds somewhere?
Govt borrowing from overseas?

Modelling corruption (17/9/04)

1) Studying a paper "Public expenditures, bureaucratic corruption and economic development" by Keith Blackburn, Niloy Bose + Emranul Haque, we find a simple model that exhibits some stylised behaviour that one would expect of corruption, but leaves open lots of questions, in particular, what can govt. do about it? Here is a variant of the (discrete-time) model they present which may permit us to go further in answering such questions.

Output in period t is

$$Y_t = A\,(l_t K_t)^{\alpha}\, k_t^{1-\alpha}\, G_t^{\beta}$$

where $\alpha, \beta \in (0,1)$, A are constants, l_t is labour supplied in period t, K_t is aggregate capital level, k_t is capital supplied to production, G_t is level of govt goods + services. In equilibrium, we would have $k_t = K_t$, and the wage rate w_t would be the marginal product of labour, $w_t = \alpha A\, l^{\alpha-1} G^{\beta} k_t$, with interest rate the marginal product of capital, $\tilde{r}_t = (1-\alpha) A\, l_t^{\alpha} G_t^{\beta}$. Let's assume that the wage rate, interest rate are evaluated at a fixed value of l (that is, population is constant). In the workforce, there are l workers and n officials, who are paid exactly the same as the workers. Output is split

$$(1) \qquad Y_t = l\,C_t + n\,c_t + i_t + j_t$$

where i_t is the amount invested in govt goods + services, j_t amount invested in capital growth, C_t is consumption per worker, c_t is consumption per official. The government's level evolves as

$$(2) \qquad G_{t+1} = \rho\, G_t + i_t$$

and capital evolves as

$$(3) \qquad K_{t+1} = \theta K_t + j_t .$$

A worker's budget constraint for his wealth x_t evolves as

$$(4) \qquad x_{t+1} = x_t(1+r_t) + w_t - \tau_t - C_t \qquad (r_t \equiv \tilde{r}_t + \theta - 1)$$

where τ_t is the taxes paid per worker in period t; if we insist government runs a balanced budget, then the amount govt spends on goods + services is

$$(5) \qquad I_t = (n+l)\,\tau_t - n w_t$$

but in fact $i_t = (1-\Lambda_t) I_t$, where Λ_t is average level of corruption in period t – the officials siphon off govt money. At the start of period t, an official has honest wealth h_t and dishonest wealth d_t. He is given I_t/n to administer for provision of govt goods + services, and he steals λ_t of it. He consumes c_t^h from his honest wealth, c_t^d from his dishonest wealth, so at time $t+1$ he starts with

$$\begin{cases} h_{t+1} = (1+r_t)h_t + w_t - \tau_t - c_t^h & (6) \\ d_{t+1} = d_t + \lambda_t I_t/n - c_t^d & (7) \end{cases}$$

If he steals proportion λ, his prob^{y} of being found out is λ (let's say), and if he is found out he gets amount u_0 of utility. Thus his expected gain is

$$\sum_{t\geq 0} \beta^t (1-\lambda)^t U(c_t^h + ac_t^d) + \sum_{t\geq 1} \beta^t (1-\lambda)^{t-1} \lambda u_0$$

if he is stealing constant fraction λ.

2) To make things easy, suppose dishonest gains have to be consumed immediately; and that we're after a stationary strategy. Then the official gets

$$\frac{U(c^h + ac^d) + \beta\lambda u_0}{1-\beta(1-\lambda)} = u_0 + \frac{U(c^h + ac^d) + (1+\beta)u_0}{(1-\beta+\beta\lambda)}$$

$$= u_0 + \frac{U(c^h + b\lambda) + (\beta-1)u_0}{1-\beta+\beta\lambda}$$

where $b = aI/n$. This he will maximise over λ, so we look at the slope

$$(1-\beta+\beta\lambda)^{-2}\Big[(1-\beta+\beta\lambda)\, b\, U'(c^h + b\lambda) - \beta\big(U(c^h + b\lambda) + (\beta-1)u_0\big)\Big]$$

So at an <u>interior</u> solution

(8) $$(1-\beta+\beta\lambda)\, b\, U'(s) = \beta\, U(s) - \beta(1-\beta)u_0$$

where $s \equiv c^h + b\lambda$. Thus for an interior solution, we may choose s, then deduce λ, and hence c_h

3) So how does this all fit together? The Cobb-Douglas form of production is irrelevant, we could use $Y = F(\ell, k; G)$ for quite general F. Assume workers and officials have same felicity U and impatience parameter β. In equilibrium, must have $\beta = 1/(1+r)$, so this determines what r must be. Let's look for a steady-state solution. The govt chooses a level for I, and then we go for a solution as follows:

Officials propose a value of λ, and use (8) to find s, hence c^h, $c^d = \lambda I/n$, c, $i = (1-\lambda)I$, $G = i/(1-\rho)$. Now that ℓ, G are known, the condition $F_k = \tilde{r} = r + (1-\theta)$ determines k, w, Y, whence (5) we deduce τ, $J = (1-\theta)k$, C from (1), x from (4), h from (6) and then we have to check $\ell x + nh = k$, adjusting λ until this happens. <u>BUT THIS STORY STINKS; for steady-state, r fixed by β, yet r depends on G ...</u>

Risk measures and pricing axiomatics (24/9/04)

1) Let's represent the stochastic structure by a finite tree $\mathcal{T}$ (non-recombining). We identify Ω with the terminal vertices of the tree, which is of depth T. For any $x \in \mathcal{T}$, we have a pricing operator $\pi_x : \mathcal{C}_x \to \mathbb{R}$, where $\mathcal{C}_x$ is the v. space of cash flows strictly descendant from x. Assume that (A1) π_x is monotone, concave, positive, $\pi_x(0) = 0$, and

(A2) if accumulated cash flow down every path from x to the end is $\leq K$ then $\pi_x(C) \leq K$

(A3) if $\bar{C}, C \in \mathcal{C}_x$ are two cashflows such that $\bar{C}_T - \bar{C}_x = a + (C_T - C_x)$, then $\pi_x(\bar{C}) = a + \pi_x(C)$*

(A4) for any stopping time τ, $\tau \geq x$, we shall have

$$\pi_x\left(C_{(t,T]}\right) \leq \pi_x\left(C_{(t,\tau]} + \pi_\tau\left(C_{(\tau,T]}\right)\right)$$

(So the cash-flow which delivers the same as C at each vertex of $\mathcal{T}$ which follows x and precedes τ, and then on $[\![\tau]\!]$ delivers the value at that vertex of the cashflows to come, will be worth at least as much as $C_{(t,T]}$ (t is the time-index of x)).

2) Is there some representation of such a family of pricing operators? Introduce the space Λ_x of (signed) measures on the strict descendants of $x \in \mathcal{T}$, and define for $\lambda \in \Lambda_x$

$$\tilde{\pi}_x(\lambda) = \sup_{C \in \mathcal{C}_x} \left\{ \pi_x(C) - \lambda \cdot C \right\}$$

Observe that if $\lambda_y < 0$ for some $y > x$, then by considering cashflow $C \in \mathcal{C}_x$ which pays ξ at vertex y ($\xi > 0$), zero else, we see (since π_x is positive) that $\tilde{\pi}_x(\lambda) = +\infty$. It is also easy to see (using (A3) and positivity) that $\tilde{\pi}_x(\lambda) = +\infty$ if $\lambda_y > 1$ for any y, and indeed this also holds under (A3)$'$.

Working for now with (A3), we see on reflection that $\tilde{\pi}_x(\lambda) = +\infty$ unless

$$\lambda_y = \sum_{z \in y+1} \lambda_z$$

where $y+1$ is the set of immediate successors of y, and $\lambda_x = 1$. Thus we see λ as a prob'y spreading down the tree. Now let's try to use (A4) to shed more light on $\tilde{\pi}_x$. We get from (A4) that for $\lambda \in \Lambda_x^* = \{\lambda \in \Lambda_x : \tilde{\pi}_x(\lambda) < \infty\}$

$$\tilde{\pi}_x(\lambda) \leq \sup_{C \in \mathcal{C}_x} \left\{ \pi_x\left(C_{(x,\tau]} + \pi_\tau(C_{(\tau,T]})\right) - \lambda_{(x,\tau]} \cdot C_{(x,\tau]} - \lambda_{(\tau,T]} \cdot C_{(\tau,T]} \right\}$$

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* This is actually quite restrictive; (A3)$'$ $\pi_x(\bar{C}) \leq a + \pi_x(C)$ is completely justifiable, but the equality is a big assumption!

$$= \sup_{C\in\mathcal{C}_x} \inf_{\alpha\in\Lambda_x^*} \Big[\tilde{\pi}_x(\alpha) + \alpha\cdot\big(C_{(x,\tau]} + \pi_\tau(C_{(\tau,T]})\big) - \lambda_{(x,\tau]}\cdot C_{(x,\tau]} - \lambda_{(\tau,T]}\cdot C_{(\tau,T]} \Big]$$

$$= \sup_{C\in\mathcal{C}_x} \inf_{\alpha\in\Lambda_x^*} \Big[\tilde{\pi}_x(\alpha) + (\alpha-\lambda)_{(x,\tau]}\cdot C_{(x,\tau]} + \alpha\cdot\pi_\tau(C_{(\tau,T]}) - \lambda\cdot C_{(\tau,T]} \Big]$$

$$= \inf_{\alpha\in\Lambda_x^*} \sup_{C\in\mathcal{C}_x} \Big[\tilde{\pi}_x(\alpha) + (\alpha-\lambda)\cdot C_{(x,\tau]} + \alpha\cdot\pi_\tau(C_{(\tau,T]}) - \lambda\cdot C_{(\tau,T]} \Big]$$

(minimax thm - Λ_x^* is compact + convex.)

Now the only way we can get a finite sup over C is if we have $\alpha=\lambda$ on $(x,\tau]$, written $\alpha \sim_\tau \lambda$, and we shall then have that this expression is

$$\boxed{= \inf_{\alpha\sim_\tau\lambda} \tilde{\pi}_x(\alpha) + \sum_{y\in[\![\tau]\!]} \lambda_y\, \tilde{\pi}_y(\lambda_y^{-1}\lambda_{(y,T]})}$$

The first term here is a bit mysterious; we have (with minimax)

$$\inf_{\alpha\sim_\tau\lambda} \tilde{\pi}_x(\alpha) = \sup_{C\in\mathcal{C}_x} \inf_{\alpha\sim_\tau\lambda} \Big\{ \pi_x(C_{(x,T]}) - \alpha\cdot C_{(x,T]} \Big\}$$

$$= \sup_{C\in\mathcal{C}_x} \inf_{\alpha\sim_\tau\lambda} \Big\{ \pi_x(C_{(x,T]}) - \lambda\cdot C_{(x,\tau]} - \alpha_{(\tau,T]}\cdot C_{(\tau,T]} \Big\}$$

Now if you think how to max $\alpha_{(\tau,T]}\cdot C_{(\tau,T]}$ over α which start like λ, what we are trying to do is choose a prob'y dist^n on the ends of the tree, and which respects the constraints imposed by $\lambda_{(x,\tau]}$; clearly what we do is to pick out from each $y\in[\![\tau]\!]$ the path which accumulates most. Having done that, when we take the sup over $C\in\mathcal{C}_x$, we may as well raise $C_{(\tau,T]}$ so that <u>all</u> paths descending from $y\in[\![\tau]\!]$ accumulate the same amount. By property (A3), there is no loss of generality in taking this amount to be 0, so in fact we get

$$\boxed{\inf_{\alpha\sim_\tau\lambda} \tilde{\pi}_x(\alpha) = \sup_{C\in\mathcal{C}_x} \Big\{ \pi_x(C_{(x,\tau]}) - \lambda\cdot C_{(x,\tau]} \Big\}}$$

Note that the RHS is clearly convex in λ, a fact which is not obvious for the LHS.

Optimal investment in an uncertain market (29/9/04)

1) Suppose we are able to invest in some vector of stocks,

$$dS = S(\sigma dW + \mu_t dt)$$

with constant riskless rate r, constant vol matrix σ, but rates of return being driven by some OU dynamics:

$$d\mu_t = \sigma_\mu dW' + b(m - \mu_t)dt.$$

We do not observe μ, but simply try to filter it from the log-prices X;

$$dX = \sigma dW + (\mu_t - \tfrac{1}{2}\mathrm{diag}(\sigma\sigma^T))dt$$

If we set $y_t = X_t + \{\frac{1}{2}\mathrm{diag}(\sigma\sigma^T) - m\}t$, then $dy = \sigma dW + (\mu_t - m)dt$ is a zero-mean Gaussian process, and we've set up for Kalman filtering.

2) If we have

$$\left.\begin{aligned} dX &= \sigma dW + BX dt \\ dY &= A\, dW + CX dt \end{aligned}\right\}$$

observe

then the innovations process is $d\nu \equiv dY - C\hat{X}dt$, and we have a representation

$$d\hat{X} = \Theta d\nu + B\hat{X}dt.$$

To find Θ,

$$d(\hat{X}Y^T) = \hat{X}(d\nu + C\hat{X}dt)^T + (\Theta d\nu + B\hat{X}dt)Y^T + \Theta d\nu d\nu^T$$

$$\doteq \{\hat{X}\hat{X}^T C^T + B\hat{X}Y^T + \Theta A A^T\}dt$$

and

$$d(XY^T) = X(AdW + CXdt)^T + (\sigma dW + BXdt)Y^T + \sigma A^T dt$$

Projecting and comparing the FV terms leaves us with

$$\Theta_t(AA^T) = V_t C^T + \sigma A^T$$

where $V_t = (X_t X_t^T)^\wedge - \hat{X}_t\hat{X}_t^T$ is the conditional variance of X_t given Y_t. Now from the SDE for X we have

$$d(XX^T) \doteq \{XX^T B^T + BXX^T + \sigma\sigma^T\}dt$$

and

$$d(\hat{X}\hat{X}^T) \doteq \{\hat{X}\hat{X}^T B^T + B\hat{X}\hat{X}^T + \Theta AA^T\Theta^T\}dt$$

so projecting the first onto Y_t and subtracting the second gives us

$$\dot{V}_t = V_t B^T + B V_t + \sigma\sigma^T - \theta A A^T \theta^T$$

$$= V_t B^T + B V_t + \sigma\sigma^T - \left(V_t C^T + \sigma A^T\right)(AA^T)^{-1}\left(C V_t + A\sigma^T\right)$$

3) Back to the investment example. Here we shall use $z = \mu - m$, $X = \begin{pmatrix} z \\ y \end{pmatrix}$, $B = \begin{pmatrix} -b & 0 \\ I & 0 \end{pmatrix}$, $A = (0\ I)\sigma$, $C = (0\ I)B$. Because the vector y is observed, its conditional covariance is zero, so we only want to get hold of v_t, which we use to denote the covariance mx of z. Thus

$$\dot{v} = -vb^T - bv + \left(\sigma_{zz}^2 + \sigma_{zy}\sigma_{yz}\right)$$
$$- \left(v + \sigma_{zz}\sigma_{zy} + \sigma_{zy}\sigma_{yy}\right)\left(\sigma_{yy}^2 + \sigma_{yz}\sigma_{zy}\right)^{-1}\left(v + \sigma_{zz}\sigma_{zy} + \sigma_{zy}\sigma_{yy}\right)^T$$

In principle, this determines a steady-state solution for v (though explicitly soluble only in 1 dimension.) The bottom line is that in the observation filtration

$$dy = \sigma_y \, dB + \hat{z}\, dt$$

where

$$d\hat{z} = \sigma_z \, dB - b\hat{z}\, dt$$

for some matrix σ_z that we determine from Kalman filter solution.

Can we do the optimisation for such dynamics? Must be possible!

4) In the Mehra & Prescott paper, the variability of 'riskless' rates is estimated from data, so we cannot get away with treating r as a constant. So let's suppose that r is in fact a Vasicek process, and for mathematical convenience, let's suppose that there is some variance ε on the riskless bond account, so that we can now describe the dynamics of all the assets (subsuming r into S) via

$$\begin{cases} dS = S\,(\sigma\, dW + \mu_t\, dt) \\ d\mu_t = \sigma_\mu\, dW + b\,(m - \mu_t)\,dt \end{cases}$$

This is notationally compact, and we can always let $\varepsilon \downarrow 0$ in the end. Abbreviating $m - \frac{1}{2}\mathrm{diag}(\sigma\sigma^T)$ to a, we have the wealth dynamics

$$dw = w\left\{\pi \cdot \left(\sigma_y\, dB + (\hat{z} + a)\,dt\right) - \gamma_t\, dt\right\} \qquad \gamma_t \equiv c_t / w_t .$$

For CRRA investor, the value is $V(w, \hat{z}, r) = h(\hat{z}, r)\,U(w)$ for some function h to be elucidated.... Looks quite complicated... Market incompleteness!!

5) So let's at a first pass suppose that the riskless rate is <u>constant</u>, and the rate of return on the single risky asset is μ_t, an OU process. In the observation filtration

$$\left.\begin{aligned} dS_t/S_t &= \sigma_s\, dW + \hat{\mu}_t dt \\ d\hat{\mu}_t &= \sigma_m\, dW + b(m-\hat{\mu}_t)\,dt \end{aligned}\right\}$$

The value function $V(w,\hat{\mu}) = h(\mu)\,U(w)$ solves the HJB equation; after some calculations, it can be reduced to

$$\tfrac{1}{2}\sigma_m^2\left(h'' + \frac{1-R}{R}\frac{h'^2}{h}\right) + \left(\frac{\sigma_m(1-R) - b\sigma_s R}{\sigma_s R}\mu + \frac{bm\sigma_s R - \sigma_m r(1-R)}{\sigma_s R}\right)h'$$

$$+ \left\{\frac{(1-R)\mu^2}{2\sigma_s^2 R} - \frac{\mu r(1-R)}{\sigma_s^2 R} + (1-R)r^2/2\sigma_s^2 R + r(1-R) - \rho\right\}h + R\,h^{1-1/R} = 0$$

Substituting $h(x) = g(x)^R$ <u>linearises</u> this to

$$\boxed{\begin{aligned} &\tfrac{1}{2}\sigma_m^2 R g'' + \left\{\left(\frac{\sigma_m(1-R)}{\sigma_s} - bR\right)\mu + bmR - \sigma_m r(1-R)/\sigma_s\right\}g' \\ &+ \left\{\frac{(1-R)\mu^2}{2\sigma_s^2 R} - \frac{\mu r(1-R)}{\sigma_s^2 R} + r(1-R) - \rho + \frac{(1-R)r^2}{2\sigma_s^2 R}\right\}g + R = 0 \end{aligned}}$$

Alternatively, we could observe that in this instance the SPD process will be

$$d\zeta_t = \zeta_t\left\{-r\,dt + \frac{r-\hat{\mu}_t}{\sigma_s}dW_t\right\}, \qquad \zeta_0 = 1,$$

multiplied by some positive constant. Thus we get

$$c_t = (\lambda e^{\rho t}\zeta_t)^{-1/R},$$

with

$$w_0 = E\left[\int_0^\infty \zeta_t c_t\, dt \,\Big|\, \mu_0\right] = \lambda^{-1/R} E\left[\int_0^\infty e^{-\rho t/R}\zeta_t^{1-1/R}dt \,\Big|\, \mu_0\right] \equiv \lambda^{-1/R} g(\mu_0)$$

$$V_0 = E\left[\int_0^\infty e^{-\rho t}U\big((\lambda e^{\rho t}\zeta_t)^{-1/R}\big)dt \,\Big|\, \mu_0\right] = \lambda^{(R-1)/R} E\left[\int_0^\infty e^{-\rho t/R}\zeta_t^{1-1/R}\frac{dt}{1-R} \,\Big|\, \mu_0\right]$$

$$= \lambda^{(R-1)/R} g(\mu_0)/(1-R).$$

Hence

$$\boxed{\frac{V_0}{U(w_0)} = g(\mu_0)^R.}$$

To take this further, notice that the equation for g can be re-expressed as

$$\tfrac{1}{2}\sigma_z^2\, g'' \;+\; \beta(a-\mu)\, g' \;+\;\left(-\tfrac{1}{2}\kappa(\mu-\alpha)^2 \;-\rho - r(R-1)\right) g \;+\; R = 0$$

where we now assume $\boxed{R>1}$ and set $\sigma_z^2 = R\sigma_M^2$, $\beta = bR + \sigma_M(R-1)/\sigma_S$,
$a = \left(b m R\sigma_S + r\sigma_M(R-1)\right)/\sigma_S\beta$, $\alpha = r$, $\kappa = (R-1)/R\sigma_S^2$. Thus if z is the OU process solving

$$dz = \sigma_z\, dW + \beta(a-z)\, dt$$

then we have the expression

$$\boxed{g(z) = E\left[\int_0^\infty R\exp\left\{-(\rho+r(R-1))t \;-\tfrac{1}{2}\kappa\int_0^t (z_s-\alpha)^2\,ds\right\}\,dt \;\Big|\; z_0 = z\right]}$$

for g. Now to do this we'll need to compute

$$E\left[\exp\left(-\tfrac{1}{2}\kappa\int_0^t (z_s-\alpha)^2\,ds\right)\;\Big|\; z_0 = z\right]$$

$$= \exp\left[-\tfrac{1}{2}Q(t)(z-B(t))^2 \;-\tfrac{1}{2}\theta(t)\right]$$

by the usual quadratic functionals methodology. Chucking this into Maple gives

$$\boxed{\begin{aligned} Q(t) &= \frac{\dot\psi(t)}{\sigma_z^2\,\psi(t)}, \qquad \psi(t) = e^{-\beta t}\left((\nu+\beta)e^{\nu t} + (\nu-\beta)e^{-\nu t}\right), \\ &\qquad\qquad\qquad\qquad \nu = \sqrt{\beta^2+\kappa\sigma_z^2}, \\ B(t) &= \frac{\beta(\alpha-a)\left(\cosh(\nu t)-1\right)}{\nu\sinh\nu t} + \alpha \end{aligned}}$$

with θ given by some great long expression that it's not worth writing down here.
(~/WORK/GRADS/SURB/EPP/epp3.mws)

6) <u>Doing the calculations more efficiently.</u> After doing the filtering, we have dynamics

$$\begin{cases} S_t^{-1} dS_t = \sigma_S \, dW + \hat{\mu} \, dt \\ d\hat{\mu} = \sigma_M \, dW + \beta_0 (m_0 - \hat{\mu}) \, dt \end{cases}$$

so we have state-price density process

$$\zeta_t^{-1} d\zeta_t = -r \, dt + \frac{r - \hat{\mu}}{\sigma_S} dW, \quad \zeta_0 = \lambda, \quad \zeta_t \equiv \lambda Z_t,$$

with budget constraint

$$w_0 = E\left(\int_0^\infty \zeta_t c_t \, dt\right) / \zeta_0 \qquad c_t = e^{-\rho t/R} \zeta_t^{-1/R}$$

$$= E \int_0^\infty e^{-\rho t/R} \zeta_t^{1-1/R} \, dt \, \lambda^{-1}$$

$$= \lambda^{-1/R} E\left[\int_0^\infty e^{-\rho t/R} Z_t^{1-1/R} dt \,\Big|\, \hat{\mu}_0 = x\right]$$

$$\equiv \lambda^{-1/R} g(x)$$

and value function

$$V(w_0, x) = E\left[\int_0^\infty e^{-\rho t} c_t^{1-R} \frac{dt}{1-R} \,\Big|\, \hat{\mu}_0 = x, \, w_0 = w\right]$$

$$= \lambda^{(R-1)/R} E\left[\int_0^\infty e^{-\rho t/R} Z_t^{1-1/R} dt \,\Big|\, \hat{\mu}_0 = x\right] / (1-R)$$

$$= \lambda^{(R-1)/R} g(x) / (1-R)$$

$$= U(w_0) \, g(x)^R.$$

Under optimal control, we shall have

$$\boxed{\begin{aligned} c_t^* &= w_t / g(\hat{\mu}_t) \\ \pi_t^* &= \frac{\hat{\mu}_t - r - \sigma_M \sigma_S R \, g'(\hat{\mu}_t)/g(\hat{\mu}_t)}{\sigma_S^2 R} \end{aligned}}$$

and so

$$dw_t = w_t \left[r \, dt + \pi_t^* \left(\sigma_S \, dW + (\hat{\mu}_t - r) \, dt \right) - \frac{dt}{g(\hat{\mu}_t)} \right].$$

There are now two problems; to evaluate g, and then to compute $\lim \frac{1}{t} \log E \, c_t^*$, as well as the SD of growth of consumption.

<u>Computing g</u> We have

$$g(x) = E\left[\int_0^\infty e^{-\rho t/R} Z_t^{(R-1)/R} dt \,\Big|\, \hat{\mu}_0 = x\right]$$

$$= E\int_0^\infty \exp\Big(-\rho t/_R - \frac{r(R-1)}{R}t + \frac{R-1}{R}\int_0^t \frac{r-\hat\mu}{\sigma_S}\,dW - \frac{1}{2}\Big(\frac{R-1}{R}\Big)^2\int_0^t\Big(\frac{r-\hat\mu}{\sigma_S}\Big)^2 ds$$
$$- \frac{1}{2}\,\frac{R-1}{R^2}\int_0^t\Big(\frac{r-\hat\mu}{\sigma_S}\Big)^2 ds\Big)\,dt$$

$$= E_Q\int_0^\infty \exp\Big\{-\frac{\rho + r(R-1)}{R}t - \frac{R-1}{2R^2}\int_0^t\Big(\frac{r-\hat\mu}{\sigma_S}\Big)^2 ds\Big\}\,dt$$

where $\Lambda_t \equiv \frac{dQ}{dP}\Big|_{\mathcal{G}_t}$ satisfies $d\Lambda_t = \Lambda_t\,\frac{R-1}{R}\,\frac{r-\hat\mu}{\sigma_S}\,dW$, and so under Q we have that $\hat\mu$ satisfies

$$\boxed{d\hat\mu = \sigma_M\,d\tilde W + \beta\,(m-\hat\mu)\,dt}$$

where $\boxed{\beta = \beta_0 + \frac{R-1}{R}\frac{\sigma_M}{\sigma_S}, \quad \beta m = \beta_0 m_0 + \frac{R-1}{R}\frac{\sigma_M}{\sigma_S}r.}$

We therefore need to compute

$$E\Big[\exp\Big\{-\tfrac{1}{2}\kappa\int_0^t(r-\hat\mu_s)^2 ds\Big\}\Big|\,\hat\mu_0 = x\Big] \equiv \varphi(t,x) \qquad \Big[\kappa \equiv \frac{R-1}{R^2\sigma_S^2}\Big]$$

$$= \exp\Big[-\tfrac{1}{2}Q(t)x^2 - B(t)x - \tfrac{1}{2}\theta(t)\Big]$$

we guess. It must be that φ solves

$$-\dot\varphi + \tfrac{1}{2}\sigma_M^2\varphi_{xx} + \beta(m-x)\varphi_x - \tfrac{1}{2}\kappa(r-x)^2\varphi = 0,$$

which is

$$-\tfrac{1}{2}\dot Q x^2 - \dot B x - \tfrac{1}{2}\dot\theta + \tfrac{1}{2}\sigma_M^2\Big((Qx+B)^2 - Q\Big) - \beta(m-x)(Qx+B) - \tfrac{1}{2}\kappa(r-x)^2 = 0$$

so we have

$$\boxed{\begin{aligned} &+\dot Q + \sigma_M^2 Q^2 + 2\beta Q - \kappa = 0 \\ &+\dot B + \sigma_M^2 QB - \beta m Q + \beta B + \kappa r = 0 \\ &+\dot\theta + \sigma_M^2(B^2 - Q) - 2\beta m B - \kappa r^2 = 0 \end{aligned}} \qquad Q(0) = B(0) = \theta(0) = 0$$

Solving the Ricatti equation for Q gives us

$$Q(\tau) = \frac{(\nu^2-\beta^2)\sinh\nu\tau}{\sigma_M^2(\nu\cosh\nu\tau + \beta\sinh\nu\tau)} \qquad [\nu^2 = \beta^2 + \sigma_M^2\kappa]$$

$$= \frac{\kappa\,\sinh\nu\tau}{\nu\cosh\nu\tau + \beta\sinh\nu\tau}$$

so in terms of $\psi(t) = \{(\nu+\beta)e^{\nu t} + (\nu-\beta)e^{-\nu t}\}e^{-\beta t}$ we find that

$$\boxed{Q(t) = \dot{\psi}(t)/\sigma_M^2\psi(t)}$$

$$\boxed{B(t) = e^{-\beta t}\{\beta(m-r)\kappa\,(e^{\nu t}-2+e^{-\nu t}) - \nu\kappa r\,(e^{\nu t}-e^{-\nu t})\}/\nu\psi(t)}$$

and writing $Y \equiv e^{\nu t}$ we have $[\delta \equiv (m-r)]$

$$\theta(t) = \log\frac{(\nu+\beta)Y^2+\nu-\beta}{2\nu} - \frac{(\nu+\beta)(\delta^2\beta^2(\beta-\nu)+\nu^2\sigma^2)}{\sigma_M^2\nu^2}\,t$$
$$+\frac{\kappa}{\nu^3}\,\frac{Y^2\{\nu^3r - 2\delta r\nu^2\beta - \beta^2\delta^2(2\beta+\nu)\} + 4\delta\beta(\delta\beta^2+r\nu^2)Y - \beta^2\delta^2(2\beta-\nu) - 2\delta\beta r\nu^2 - \nu^3r^2}{(\nu+\beta)Y^2+\nu-\beta}$$

$$= \log\frac{(\nu+\beta)Y^2+\nu-\beta}{2\nu} - \frac{(\nu+\beta)(\delta^2\beta^2(\beta-\nu)+\nu^2\sigma_M^2)}{\sigma_M^2\nu^2}\,t$$
$$+\frac{\kappa}{\nu^3}(Y-1)\frac{\{(\nu^3r^2 - 2\delta r\beta\nu^2 - \beta^2\delta^2(2\beta+\nu))Y + (\nu^3r^2 + 2\delta r\beta\nu^2 + (2\beta-\nu)\beta^2\delta^2)\}}{(\nu+\beta)Y^2+\nu-\beta}$$

This checks out with Maple (/WORK/GRADS/SURB/EPP/epp4.mws)

7) __The Kalman filter__ Before filtering, we have (W, W' indept BMs)

$$\begin{cases} S^{-1}dS = \sigma_S\,dW + \mu\,dt \\ d\mu = \sigma_{\mu S}\,dW + \sigma_{\mu\mu}\,dW' + \beta_0\,(m_0-\mu)\,dt \end{cases}$$

so if we set

$$\begin{cases} y_t = \log S_t + (\tfrac12\sigma_S^2 - m_0)\,t \\ z_t = \mu_t - m_0 \end{cases}$$

then

$$dX_t \equiv \begin{pmatrix} dz_t \\ dy_t \end{pmatrix} = \begin{pmatrix} \sigma_{\mu S} & \sigma_{\mu\mu} \\ \sigma_S & 0 \end{pmatrix}\begin{pmatrix} dW \\ dW' \end{pmatrix} + \begin{pmatrix} -\beta_0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} z_t \\ y_t \end{pmatrix}dt \equiv \Sigma\begin{pmatrix} dW \\ dW' \end{pmatrix} + B\begin{pmatrix} z \\ y \end{pmatrix}dt,$$

a zero-mean linear Gaussian process, where y is observed, z is not. If we set $M \equiv \Sigma\Sigma^T$, then if we go through the Kalman filtering, the variance of z given y is (asymptotically)

$$\boxed{V_{zz} = \{(\beta_0 M_{yy} + M_{zy})^2 + M_{yy}M_{zz} - M_{zy}^2\}^{\frac12} - \beta_0 M_{yy} - M_{zy}}$$

From the earlier KF stuff, $d\hat{X} = \theta\,d\nu + B\hat{X}\,dt$, where $d\nu = \sigma_S\,d\tilde{W}$ in this instance, and $\theta(AA^T) = VC^T + \Xi A^T$, with $A = P\Xi$, $C = PB$, $P \equiv (0,1)$.

Thus

$$\theta M_{yy} = (VB^T + M)P^T$$
$$= \left\{ M + \begin{pmatrix} V_{zz} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -\beta_0 & 1 \\ 0 & 0 \end{pmatrix} \right\} P^T$$
$$= \left\{ M + \begin{pmatrix} -\beta_0 V_{zz} & V_{zz} \\ 0 & 0 \end{pmatrix} \right\} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} M_{zy} \\ M_{yy} \end{pmatrix} + \begin{pmatrix} V_{zz} \\ 0 \end{pmatrix}$$

so $\theta = \begin{pmatrix} M_{zy}/M_{yy} \\ 1 \end{pmatrix} + \begin{pmatrix} V_{zz}/M_{yy} \\ 0 \end{pmatrix}$

and $M_{yy} = \sigma_S^2$. The first component of θ is therefore

$$M_{yy}^{-1}(V_{zz} + M_{zy}) = M_{yy}^{-1}\left(\left\{ (\beta_0 M_{yy} + M_{zy})^2 + M_{yy}M_{zz} - M_{zy}^2 \right\}^{\frac{1}{2}} - \beta_0 M_{yy} \right)$$

and so when we've done the filtering we find

$$\sigma_M = \frac{1}{\sigma_S}\left(\left\{ (\beta_0 M_{yy} + M_{zy})^2 + M_{yy}M_{zz} - M_{yz}^2 \right\}^{\frac{1}{2}} - \beta_0 M_{yy} \right)$$

8) There is also need to compute

$$E\left[Z_t^\alpha \mid \mu_0 = x \right]$$

for values of α different from $(1-R)/R$. The change of measure story goes through similarly to give

$$e^{-\alpha r t} E_Q \exp\left\{ -\tfrac{1}{2}\alpha(1-\alpha)\int_0^t \left(\frac{r - \hat{\mu}_s}{\sigma_S}\right)^2 ds \right\}$$

where under Q

$$d\hat{\mu}_t = \sigma_M d\tilde{W} + \beta(m - \hat{\mu}_t)dt \qquad \beta = \beta_0 + \alpha\sigma_M/\sigma_S$$
$$\beta m = \beta_0 m_0 + \alpha r \sigma_M/\sigma_S.$$

9) As an alternative, we could consider the situation in the relaxed investor paper, where $\lambda \equiv \mu/\sigma$ is given a Gaussian prior, $N(\lambda_0, v_0)$. Then we have

$$S_t^{-1} dS_t = \sigma\, dX_t \equiv \sigma(dW_t + \lambda dt) = \sigma\left(d\tilde{W}_t + \frac{v_0 X_t + \lambda_0}{1 + v_0 t} dt \right)$$

in the observation filtration. The change of measure martingale is now

$$Z_t = \sqrt{1+v_0 t}\, \exp\left[-\frac{1}{2}\frac{v_0}{1+v_0 t}X_t^2 + \left(\frac{r}{\sigma} - \frac{\lambda_0}{1+v_0 t}\right)X_t + \frac{t}{2}\left(\frac{\lambda_0^2}{1+v_0 t} - \frac{r^2}{\sigma^2}\right) \right]$$

in the observation filtration, so $\zeta_t = e^{-rt} Z_t$, and

$$c_t = e^{-\rho t/R} (\gamma \zeta_t)^{-1/R}, \qquad \text{for suitable } \gamma,$$

and therefore we have the budget-matching condition

$$w_0 = E\Big[\int_0^\infty \zeta_t c_t \, dt\Big] = \gamma^{-1/R} E\Big[\int_0^\infty e^{-\rho t/R} \zeta_t^{(R-1)/R} \, dt\Big]$$

$$\equiv \gamma^{-1/R} \varphi(\lambda_0, v_0)$$

where $\varphi(\lambda_0, v_0) = \int_0^\infty \frac{(1+v_0 t)^{(R-1)/2R}}{(1+\kappa v_0 t)^{1/2}} \exp\Big[-\frac{(\lambda_0 - r/\sigma)^2 \kappa t}{2R(1+\kappa v_0 t)} - \frac{\rho + r(R-1)}{R} t \Big] dt \qquad \Big(\kappa = \frac{R-1}{R}\Big)$

from the relaxed investor paper.

The value of the objective is

$$E \int_0^\infty e^{-\rho t} U(c_t)\, dt = \frac{\gamma^{(R-1)/R}}{1-R} E\Big[\int_0^\infty e^{-\rho t/R} \zeta_t^{(R-1)/R} \, dt\Big]$$

$$\equiv \frac{\gamma^{(R-1)/R}}{1-R} \varphi(\lambda_0, v_0)$$

$$= \varphi(\lambda_0, v_0)^R \, U(w_0).$$

Comparing with the standard Merton problem, where the value is

$$\gamma_M^{-R} \, U(w_0)$$

$\Big[\gamma_M \equiv \big(\rho + (R-1)(r + \tfrac{1}{2}\sigma^2 \pi_M^2 R)\big) / R \Big]$, we see that the efficiency here is

$$\big(\varphi(\lambda_0, v_0)\gamma_M\big)^{R/(1-R)}.$$

Risk measures + pricing axiomatics again (10/11/04)

1) If we don't assume $\pi_x(C_{(x,T]}) = \pi_x(C_{(x,\tau]} + \pi_{[\tau]}(C_{(\tau,T]}))$, but only $\leq$, we have seen that

$$\tilde{\pi}_x(\lambda) \leq \sup_C \{ \pi_x(C_{(x,\tau]}) - \lambda \cdot C_{(x,\tau]} \} + \sum_{z \in [\tau]} \lambda_z \tilde{\pi}_z(\lambda_{z+}/\lambda_z) \qquad (1)$$

for any stopping time, and so we have $\tilde{\pi}_x(\lambda) \leq \inf_\tau \{ \cdots \}$. However, this inf is attained by $\tau = T$ and possibly other times as well, and for any time for which this holds with equality, the only inequality step in proving (1) must have been equality; that is,

$$\pi_x(C^*_{(x,T]}) = \pi_x(C^*_{(x,\tau]} + \pi_{[\tau]}(C^*_{(\tau,T]}))$$

where C^* is the extremal consumption process for dual variable λ.

2) Using the dual relationship of λ, C^* together with (1), we have for a stopping time τ which attains equality that

$$\begin{aligned}
\pi_x(C^*) &= \tilde{\pi}_x(\lambda) + \lambda \cdot C^* \\
&= \sup_C \{ \pi_x(C_{(x,\tau]}) - \lambda \cdot C_{(x,\tau]} \} + \sum_{z \in [\tau]} \lambda_z \tilde{\pi}_z(\lambda_{z+}/\lambda_z) + \lambda \cdot C^* \\
&\geq \pi_x(C^*_{(x,\tau]}) + \sum_{z \in [\tau]} \lambda_z \{ \tilde{\pi}_z(\lambda_{z+}/\lambda_z) + \frac{\lambda_{z+}}{\lambda_z} \cdot C^*_{(z,T]} \} \\
&\geq \pi_x(C^*_{(x,\tau]}) + \sum_{z \in [\tau]} \lambda_z \pi_z(C^*_{(z,T]}) = \pi_x(C^*_{(x,\tau]}) + \lambda \cdot \pi_{[\tau]}(C^*_{(\tau,T]}).
\end{aligned}$$

Could this actually be an <u>equality</u>? We have

$$\begin{aligned}
\pi_x(C^*) &= \pi_x(C^*_{(x,\tau]} + \pi_{[\tau]}(C^*_{(\tau,T]})) \\
&= \inf_\alpha \{ \tilde{\pi}_x(\alpha) + \alpha \cdot (C^*_{(x,\tau]} + \pi_{[\tau]}(C^*_{(\tau,T]})) \} \\
&\leq \inf_{\alpha \sim_\tau \lambda} \{ \tilde{\pi}_x(\alpha) + \lambda \cdot (C^*_{(x,\tau]} + \pi_{[\tau]}(C^*_{(\tau,T]})) \} \\
&= \sup_C \{ \pi_x(C_{(x,\tau]}) - \lambda \cdot C_{(x,\tau]} \} + \lambda \cdot \{ C^*_{(x,\tau]} + \pi_{[\tau]}(C^*_{(\tau,T]}) \}
\end{aligned}$$

so it would be <u>sufficient</u> to prove that this last sup is attained at $C = C^*$.

3) Using the concavity of π_x and the fact that λ is the gradient of π_x at C^*, we can deduce various inequalities, such as

$$\pi_x(C^*) = \pi_x(C^*_{(x,\tau]} + \pi_{[\tau]}(C^*_{(\tau,T]})) \leq \pi_x(C^*) + \lambda \cdot \{ \pi_{[\tau]}(C^*_{(\tau,T]}) - C^*_{(\tau,T]} \}$$

whence

$$\lambda \cdot \pi_{[\tau]} (C^*_{(\tau,T]}) \equiv \sum_{z \in [\tau]} \lambda_z \, \pi_z (C^*_{(\tau,T]})$$

$$\geq \lambda \cdot C^*_{(\tau,T]}$$

In a different direction, we have for any C

$$\pi_x (C_{(x,\tau]}) \leq \pi_x (C^*) + \lambda \cdot C_{(x,\tau]} - \lambda \cdot C^*,$$

but this is no better use.

4) Another observation; we have for λ, x fixed, optimal C^*, and τ which have the property

$$\tilde{\pi}_x(\lambda) = \inf_{\alpha \sim_\tau \lambda} \tilde{\pi}_x(\alpha) + \sum_{z \in [\![\tau]\!]} \lambda_z \, \tilde{\pi}_z (\lambda_{z+} / \lambda_z)$$

that

$$\tilde{\pi}_x(\lambda) = \pi_x (C^*) - \lambda \cdot C^* = \inf_{\alpha \sim_\tau \lambda} \tilde{\pi}_x(\alpha) + \sum_{z \in [\![\tau]\!]} \lambda_z \, \tilde{\pi}_z (\lambda_{z+} / \lambda_z)$$

$$\leq \pi_x (C^*_{(x,\tau]} + \pi_{[\![\tau]\!]} (C^*_{(\tau,T]})) - \lambda \cdot C^*$$

$$= \inf_{\alpha} \left\{ \tilde{\pi}_x(\alpha) + \alpha \cdot \left(C^*_{(x,\tau]} + \pi_{[\![\tau]\!]} (C^*_{(\tau,T]}) \right) - \lambda \cdot C^* \right\}$$

$$\leq \inf_{\alpha \sim_\tau \lambda} \tilde{\pi}_x(\alpha) + \sum_{z \in [\![\tau]\!]} \lambda_z \left\{ \pi_z (C^*_{(\tau,T]}) - \frac{\lambda_{z+}}{\lambda_z} C^*_{(z,T]} \right\}$$

$$\leq \inf_{\alpha \sim_\tau \lambda} \tilde{\pi}_x(\alpha) + \sum_{z \in [\![\tau]\!]} \lambda_z \, \tilde{\pi}_z (\lambda_{z+} / \lambda_z)$$

So in fact all of these things are <u>equal</u>, so in particular

$$\tilde{\pi}_z (\lambda_{z+} / \lambda_z) = \pi_z (C^*_{(\tau,T]}) - \frac{\lambda_{z+}}{\lambda_z} \cdot C^*_{(z,T]} \qquad \forall \; z \in [\![\tau]\!] .$$

Moreover, we saw earlier that

$$\inf_{\alpha \sim_\tau \lambda} \tilde{\pi}_x(\alpha) = \sup_C \left\{ \pi_x (C_{(x,\tau]}) - \lambda \cdot C_{(x,\tau]} \right\}$$

$$= \pi_x (\bar{C}_{(x,\tau]}) - \lambda \cdot \bar{C}_{(x,\tau]}$$

we learn, where

$$\bar{C}_{(x,\tau]} = C^*_{(x,\tau]} + \pi_{[\tau]} (C^*_{(\tau,T]}) .$$

One deduction we can make here is that the earlier conjecture that

$$\sup_{C} \left\{ \pi_x\left(C_{(x,\tau]}\right) - \lambda \cdot C_{(x,\tau]} \right\} = \pi_x\left(C^*_{(x,\tau]}\right) - \lambda \cdot C^*_{(x,\tau]}$$

will probably not be true; indeed, the sup is attained at $\bar{C}$, not C^*, and if π_x were strictly concave, then the conjecture would be false.

5) A further observation that may help; translation invariance implies that we would be indifferent between a cash flow C, and C' obtained from C by deferring a payment at vertex y to the terminal nodes descended from y. Thus we can consider the evaluation of π_x only on cash-flows which pay at time T.

6) Say that τ splits λ if

$$\tilde{\pi}_x(\lambda) = \inf_{\alpha \sim_\tau \lambda} \tilde{\pi}_x(\alpha) + \sum_{z \in [\tau]} \lambda_z \tilde{\pi}_z\left(\lambda_{z+}/\lambda_z\right)$$

Suppose that (λ^*, C^*) is a minimax pair, and τ splits λ^*. Then we have by convexity of $\tilde{\pi}_x$, and the fact that the gradient of $\tilde{\pi}_x$ at λ^* is $-C^*$

$$\begin{aligned}
\tilde{\pi}_x(\lambda^*) - (\lambda - \lambda^*) \cdot C^* &= \inf_{\alpha \sim_\tau \lambda^*} \tilde{\pi}_x(\alpha) + \lambda^* \cdot \pi_{[\tau]}\left(C^*_{(\tau,T]}\right) - (\lambda - \lambda^*) \cdot C^* \\
&\leq \tilde{\pi}_x(\lambda) \\
&\leq \inf_{\alpha \sim_\tau \lambda} \tilde{\pi}_x(\alpha) + \sum_{z \in [\tau]} \lambda_z \tilde{\pi}_z\left(\lambda_{z+}/\lambda_z\right) \\
&\leq \inf_{\alpha \sim_\tau \lambda} \tilde{\pi}_x(\alpha) + \sum_{z \in [\tau]} \lambda_z \left\{ \pi_z\left(C'_{(\tau,T]}\right) - \frac{\lambda_{z+}}{\lambda_z} \cdot C'_{(z,T]} \right\}
\end{aligned}$$

for the 'good' C', but it doesn't seem to help; if $\lambda \sim_\tau \lambda^*$, we might guess that τ will also split λ, but it's not clear from this

Discretizing the convertible bond problem? (25/11/04)

(i) Let's take the good old perpetual defaultable callable convertible bond question, and do a discretisation in $X \equiv \log V$, and m. We have

$$X_t = \sigma W_t + \mu t + X_0 \quad , \qquad \mu = r - \delta - \tfrac{1}{2}\sigma^2,$$

and we will propose putting X onto a grid with spacing Δx, and m onto a grid with spacing Δm. We'll approximate the diffusion by a continuous-time chain, with up/down intensities $\lambda_\pm$, where we want

$$\lambda_+ \left(f(x+\Delta x) - f(x) \right) + \lambda_- \left(f(x-\Delta x) - f(x) \right) \doteq \tfrac{1}{2}\sigma^2 f''(x) + \mu f'(x)$$

so we shall have

$$\left.\begin{aligned} \lambda_+ + \lambda_- &= \sigma^2/\Delta x^2 \\ \lambda_+ - \lambda_- &= \mu/\Delta x \end{aligned}\right\} \quad \Rightarrow \quad \begin{aligned} \lambda_+ &= \left(\frac{\sigma^2}{\Delta x^2} + \frac{\mu}{\Delta x} \right) \Big/ 2 \\ \lambda_- &= \left(\frac{\sigma^2}{\Delta x^2} - \frac{\mu}{\Delta x} \right) \Big/ 2 \end{aligned}$$

so we shall require $\sigma^2 > |\mu| \Delta x$ for these to be non-negative. We'll also be killing at rate r, so that the operator

$$\mathcal{L} \equiv \tfrac{1}{2}\sigma^2 D^2 + \mu D - r$$

is approximated by $L = Q - r$ where Q is the Q-matrix of the RW, and we have the analogous equations for S, B while there is no action.

(ii) The solution methodology is to build solutions for $m = \Delta m, 2\Delta m, \ldots$ recursively. We'll suppose that there's always default at the lowest x-value, always conversion at the highest x-value. Suppose we've computed the solutions $S(j\Delta m, \cdot)$, $B(j\Delta m, \cdot)$ for $j < k$, and set $s(x) \equiv S((k-1)\Delta m, x)$, $b(x) = B((k-1)\Delta m, x)$.

The bond holders will propose some vector $\gamma_x \geq 0$, representing the intensity with which they convert when in state x (in practical terms, γ_x will be bounded above by some large value.) The shareholders will then solve their optimal stopping problem; there is some lower-bound vector $\underline{s}(x)$ that they get if they stop, else

$$Qf(x) - rf(x) + \gamma_x \left(s(x) - f(x) \right) + \frac{\delta e^x - mp'}{n-m} = 0.$$

This is solved numerically; on the stopping set we have that $B = \underline{b}$, some specified function, so we now compute the value B, and optimise over γ.

When the bond-holders do jump across to the previous m-slice, one of them gets the stock, the rest get the bond.

Good-deal bounds (27/1/05)

(i) Tomas Björk has been doing some interesting work on good-deal bounds. He provides the following example to motivate it. If you were offered a gamble where you get £100 with prob $\frac{1}{2}$, £0 with prob $\frac{1}{2}$, would you be willing to pay 1p for it? It seems that anyone* would!

Such pricing would be possible in an arbitrage-free market, but would be unrealistic. It seems to me that the unacceptable feature is that the price and the expected value are so massively different. Of course scaling the gamble up makes it less obviously desirable, but this is to do with risk-aversion; if we consider infinitesimal gambles (and therefore marginal prices) then we see the unacceptable feature most clearly. Let's say our model satisfies a good-deal bound if for any Y

$$Y \geq 0 \quad \Rightarrow \quad EY \leq \frac{1}{a}\,\pi(Y) = \frac{1}{a}\,E[\zeta Y]$$

(where ζ is the state-price density) and

$$Y \geq 0 \quad \Rightarrow \quad EY \geq \frac{1}{b}\,\pi(Y)$$

for some $a \in (0,1)$, and some $b > 1$ (the first says you can't buy for very little some nonnegative claim with large mean, the second that you can't sell for a lot some non-negative claim with large mean)

What this says then is that ζ must satisfy

$$a \leq \zeta \leq b.$$

(ii) If the model satisfies a good-deal bound, and $Y \geq 0$ is some contingent claim, what's the most Y could cost? Our problem is

$$\max \; E[\zeta Y] \qquad \text{s.t.} \quad E\,\zeta = 1, \qquad a \leq \zeta \leq b$$

Lagrangian:

$$\max \; E[(Y-\lambda)\,\zeta] \qquad \text{s.t.} \quad a \leq \zeta \leq b$$

so we find that we take $\zeta = b$ if Y is big, $\zeta = a$ if Y is small, where we pick λ so that

$$1 = E\,\zeta = a + (b-a)\,P(Y > \lambda)$$

---

* But if you scale it up, and win £10^{10} with prob $\frac{1}{2}$, £0 with prob $\frac{1}{2}$, would you be prepared to pay £10^{6} for it? Probably not...

So $$\boxed{P(Y>\lambda) = \frac{1-a}{b-a}.}$$

Of course, this assumes that the distⁿ fⁿ of Y doesn't jump in the wrong place; if F jumps at $\lambda = \bar{F}^{-1}((1-a)/(b-a))$ then we take $\zeta = b$ if $Y>\lambda$, $\zeta = a$ if $Y<\lambda$ and ζ = whatever is needed to satisfy constraint if $Y=\lambda$.

When we've done this, we find the maximised price to be

$$\begin{aligned} E[\zeta Y] &= \lambda + E[\zeta(Y-\lambda)] \\ &= \lambda + b\,E(Y-\lambda)^+ - a\,E(Y-\lambda)^- \\ &= \lambda + b\,E(Y-\lambda)^+ - a\{E(Y-\lambda)^+ - E(Y-\lambda)\} \\ &= \lambda + a\,E(Y-\lambda) + (b-a)\,E(Y-\lambda)^+ \end{aligned}$$

For the minimal price, the argument is similar, and we get

$$E[\zeta Y] = \tilde{\lambda} + b\,E(Y-\tilde{\lambda}) - (b-a)\,E(Y-\tilde{\lambda})^+$$

where $\tilde{\lambda} = F^{-1}((1-a)/(b-a))$.

(iii) Here's a little example to illustrate, with a Black-Scholes flavour.

$$Y = \left(S_0 \exp\{\sigma W_T + (\mu - \tfrac{1}{2}\sigma^2)T\} - K\right)^+$$

Now if we select $\xi \equiv (\bar{\Phi})^{-1}((1-a)/(b-a))$ and then set

$$\lambda = \left(S_0 \exp\{\sigma\sqrt{T}\,\xi + (\mu - \tfrac{1}{2}\sigma^2)T\} - K\right)^+,$$

we have the situation above for the maximal price. Thus if we set

$C(S_0, K, \sigma, \mu, T) = E\left(S_0 e^{\sigma W_T + (\mu - \frac{1}{2}\sigma^2)T} - K\right)^+$, we get an upper bound of

$$(1-a)\lambda + C(S_0, K, \sigma, \mu, T) + (b-a)\,C(S_0, K+\lambda, \sigma, \mu, T)$$

and a lower bound of

$$(1-b)\tilde{\lambda} + C(S_0, K, \sigma, \mu, T) - (b-a)\,C(S_0, K+\tilde{\lambda}, \sigma, \mu, T).$$

Example?
Dynamic version?

(iv) But if you have a good deal, then any multiple of it is a good deal, and if you add a constant, it will still be a good deal... and if you try to express this in terms of EY and $\pi(Y)$, you can't do it in a non-trivial way. Perhaps better is to require the definition of a good deal Y to be expressed in terms of the Sharpe ratio: Y is a good deal iff

$$\{EY - \pi(Y)\}^2 \geq a^2 \operatorname{var}(Y).$$

So to say that there is no good deal means that for all Y

$$\left(E\, Y(\zeta-1)\right)^2 \leq a^2 \operatorname{var}(Y).$$

It's enough to check this for zero-mean Y; and then by Cauchy Schwarz

$$\left(E\, Y(\zeta-1)\right)^2 \leq \operatorname{var}(Y)\; E((\zeta-1)^2) = \operatorname{var}(Y).\operatorname{var}(\zeta)$$

so the condition for no good deal is

$$\boxed{\operatorname{var}(\zeta) \leq a^2.}$$

(v) Now suppose that we are given some non-negative claim Y, and wish to find bounds for the price of Y given that the no-good-deal condition $\operatorname{var}(\zeta) \leq a^2$ is satisfied.

We have

$$\left(EY - \pi(Y)\right)^2 = E\left(Y(1-\zeta)\right)^2 \leq \operatorname{var}(Y)\operatorname{var}(\zeta)$$

with equality when $1-\zeta = \lambda Y + \alpha$ for some scalars α, λ. We shall then have

$$\zeta = 1 + \beta(Y - EY)$$

if equality is to hold. There are problems with $\zeta \geq 0$ however. If we want to find for $Y \geq 0$ [the] upper

$$\max \; E[\zeta Y] \qquad \text{s.t. } E\zeta = 1,\; \operatorname{var}(\zeta) = a^2$$

the Lagrangian form is

$$\max \; E\left[\zeta Y + \lambda\zeta - \tfrac{1}{2}\beta(\zeta-1)^2\right]$$

so doing a pointwise max over ζ we get $\boxed{\zeta = \left(1 + \beta^{-1}(Y+\lambda)\right)^+}$

for constants β, λ chosen to match mean and variance of ζ; if $\kappa = \beta + \lambda$, get

$$\boxed{\begin{aligned} \beta &= \int_{-\kappa}^{\infty} (y+\kappa)\, F(dy) \\ \beta^2(1+a^2) &= \int_{-\kappa}^{\infty} (y+\kappa)^2\, F(dy) \end{aligned}}$$

Bound is obtained with $\zeta = A\,(Y-b)^+$, $E[\zeta Y] = A\,E[(Y-b)^2 ; Y>b] + Ab\,E[Y-b : Y>b]$.

for the conditions. This means that we have to find K so as to make

$$\frac{\int_{-K}^{\infty} (y+K)^2 F(dy)}{\left\{ \int_{-K}^{\infty} (y+K) F(dy) \right\}^2} = 1+a^2$$

if possible. As $K \to \infty$, the LHS $\to 1$. Assuming F has a density f, we can differentiate w.r.t. K to get derivative

$$\frac{\int_{-K}^{\infty} (y+K) F(dy) \cdot 2\int_{-K}^{\infty}(y+K) F(dy) - 2\int_{-K}^{\infty}(y+K)^2 F(dy)\ \bar{F}(-K)}{\left\{ \int_{-K}^{\infty} (y+K) F(dy) \right\}^3} < 0$$

so if there is a root K, then it's unique.

On the other hand, if we consider for large $b>0$

$$\frac{E\left[(Y-b)^{+\,2}\right]}{\left[E(Y-b)^+\right]^2} = \frac{E\left[(Y-b)^2 \mid Y>b\right]}{E\left[Y-b \mid Y>b\right]^2}\ \frac{1}{\bar{F}(b)} \geq \frac{1}{\bar{F}(b)} \to \infty \text{ as } b \uparrow \infty,$$

so there certainly will be a root (unless for some $b_0 < \infty$ $F(b_0) = 1 > F(b_0-)$).

(vi) <u>An example</u>. Let's take once again the situation

$$Y = \left(S_0\ e^{\sigma W_T + (\mu - \frac{1}{2}\sigma^2)T} - K\right)^+,$$

where we shall need to compute $E\left[(Y-b)^n ;\ Y>b\right]$ for $n = 1, 2$. If $b \leq 0$ then it's immediate: $E[Y-b] = EY - b = C(S_0, K, \sigma, \mu, T) - b$, and $E(Y-b)^2 = EY^2 - 2bEY + b^2$, so this is easy enough, but if $b>0$ we have to calculate

$$E\left[(Y-b) : Y>b\right] = C(S_0, K+b, \sigma, \mu, T)$$

and

$$E\left[(Y-b)^2 : Y>b\right] = \int_{\alpha}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \left(S_0\, e^{\sigma\sqrt{T}x + (\mu-\frac{1}{2}\sigma^2)T} - K - b\right)^2 dx$$

$$\left[\alpha \equiv \frac{1}{\sigma\sqrt{T}}\left\{\log\frac{K+b}{S_0} - (\mu - \tfrac{1}{2}\sigma^2)T\right\}\right]$$

$$= S_0^2\, e^{(2\mu-\sigma^2)T} \int_{\alpha}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \left(e^{\sigma\sqrt{T}x} - e^{\sigma\sqrt{T}\alpha}\right)^2 dx$$

$$= S_0^2\, e^{(2\mu-\sigma^2 T)}\left\{ e^{2\sigma^2 T}\,\bar{\Phi}(\alpha - 2\sigma\sqrt{T}) - 2e^{\frac{1}{2}\sigma^2 T + \sigma\sqrt{T}\alpha}\,\bar{\Phi}(\alpha - \sigma\sqrt{T}) + e^{2\sigma\sqrt{T}\alpha}\,\bar{\Phi}(\alpha)\right\}.$$

The form of ζ which we need to minimize $E\,\zeta Y$ will be

$$\zeta = (\lambda - Y)^+ \beta$$

so we shall require

$$\left.\begin{aligned} \beta\, E[\lambda - Y \;;\; Y<\lambda] &= 1 \\ \beta^2 E[(\lambda-Y)^2 ; Y<\lambda] &= 1+a^2 \end{aligned}\right\}$$

so as before

$$\frac{E[(\lambda-Y)^2 ;\; Y<\lambda]}{E[\lambda - Y ;\; Y<\lambda]^2} = 1+a^2$$

One new feature should be noted; this expression decreases with λ, as before, with limit 1 as $\lambda \uparrow \infty$. However, as $\lambda \downarrow 0$ we get limit $1/P(Y=0)$; what happens if this is $< 1+a^2$? In that case, we use

$$\zeta = \frac{1}{P(Y=0)} I_{\{Y=0\}}$$

and then the variance of ζ is $\leqslant 1+a^2$, while the lower bound for the price $E[Y\zeta]$ is zero.

[The original Cochrane–Saa-Requejo paper does this last, but somewhat better, in that they ensure that the underlyings are also correctly priced by the proposed SPD ... looks like these guys did it earlier + better!]

More on pricing operators (9/2/05)

1) We return to the setting considered earlier of a finite Ω which is represented by a tree, and a family $\{\pi_x : x \in \mathcal{T}\}$ of pricing operators. Now we shall consider $\mathcal{C}_x$ to be the set of cashflows ~~at~~ or after x, with $\pi_x : \mathcal{C}_x \to \mathbb{R}$ satisfying certain axioms:

(C) π_x is strictly concave, and C^2 in the interior of its domain of finiteness

(M) π_x is monotone

(NA) $\pi_x(0) = 0$

(TI) If $K, K' \in \mathcal{C}_x$ with the property that along every path from x to T the cashflow K accumulates exactly a more than the cashflow K', then

$$\pi_x(K) = \pi_x(K') + a$$

(DC) If $K \in \mathcal{C}_x$, and τ is a stopping time, $\tau \geq x$, and if $\bar{K}_{[x,\tau]}$ denotes the cashflow which at $z \in [\![\tau]\!]$ is worth $\sum_{x \leq y \leq z} K_y$, and zero for $z \notin [\![\tau]\!]$, then

$$\pi_x(K) = \pi_x\left(\pi_{[\![\tau]\!]}\left(K_{(\tau,T]} + \bar{K}_{[x,\tau]}\right)\right)$$

Various properties follow from these assumptions. First, if $K_y = K'_y \;\forall y > x$, then by (TI)

$$\pi_x(K) - \pi_x(K') = K_x - K'_x .$$

Now define the convex duals

$$\tilde{\pi}_x(\lambda) \equiv \sup_K \left\{ \pi_x(K) - \lambda \cdot K \right\}$$

By (M) and (TI), $\tilde{\pi}_x(\lambda)$ is only finite if $\lambda_x = 1$, $\lambda_y \geq 0 \;\forall y$, $\sum_{z \in y+} \lambda_z = \lambda_y \;\forall y \geq x$.

Notice that $\{\pi_x : x \in \mathcal{T}\}$ induces a family of one-step pricing operators $\pi_{x,x+1}$ defined on cashflows K which pay only at vertices $z \in x+1$. These operators are again concave and monotone, and from (DC) we obtain

$$\pi_x(K) = \pi_{x,x+1}\left(\pi_{[\![x+1]\!]}\left(K_{(x+1,T]} + \bar{K}_{[x,x+1]}\right)\right)$$

allowing us to compute π_x recursively from all the one-step pricing operators $\pi_{y,y+1}$.

How does this look for the duals?

$$\begin{aligned}
\tilde{\pi}_x(\lambda) &= \sup_K \left[\pi_x(K) - \lambda \cdot K \right] \\
&= \sup_K \left[\pi_x\left(\pi_{[\![x+1]\!]}\left(K_{(x+1,T]} + \bar{K}_{[x,x+1]}\right)\right) - \lambda \cdot K \right] \\
&= \sup_K \left[\pi_{x,x+1}\left(\pi_{[\![x+1]\!]}\left(K_{[x+1,T]} + \bar{K}_{[x,x+1)}\right)\right) - \lambda \cdot K \right]
\end{aligned}$$

$$= \sup_{K} \inf_{\alpha} \left[\tilde{\pi}_{x,x+1}(\alpha) + \alpha \cdot \pi_{[x+1]}(K_{[x+1,T]} + \bar{K}_{[x,x+1)}) - \lambda \cdot K \right]$$

$$= \inf_{\alpha} \sup_{K} \left[\tilde{\pi}_{x,x+1}(\alpha) + \sum_{z \in x+1} \alpha_z \left(K_x + \pi_z(K_{[z,T]}) \right) - \lambda_x K_x - \sum_{z \in x+1} \lambda_z \frac{\lambda_{z+}}{\lambda_z} \cdot K_{[z,T]} \right]$$

so for the inner sup to be finite, we have to have $\sum_{z \in x+1} \alpha_z = \lambda_x$, and also $\alpha_z = \lambda_z$ for all $z \in x+1$; so altogether we have

$$= \tilde{\pi}_{x,x+1}(\lambda_{x,x+1}) + \sum_{z \in x+1} \lambda_z \tilde{\pi}_z\left(\lambda_{z+}/\lambda_z \right).$$

2) Nevertheless, it appears that the condition (TI) is too restrictive, and we require instead the weaker assumptions

(TI)′ If $K, K' \in \mathcal{C}_x$ are such that down every path from x to T the accumulated values of K and K' agree, then $\pi_x(K) = \pi_x(K')$

(CE) If K', K agree at all $y > x$, then $\pi_x(K) - K_x = \pi_x(K') - K'_x$.

Under these assumptions, the above recursive forms still hold good. [CHECK!]

3) Let's now move on to a situation with consumption and cashflows. What we've been looking at so far is quite simple, in effect, value cashflows only by their terminal values. Suppose now that we define π_x on $\mathcal{C}_x \times \mathcal{C}_x$, where we interpret $\pi_x(C, K)$ to be the greatest price we'd pay at x in order to receive cashflow K, and consumption stream C. The axioms we take now are

(C) $(C,K) \mapsto \pi_x(C,K)$ is strictly concave, C^2

(M) $(C,K) \mapsto \pi_x(C,K)$ is monotone in both arguments

(NA) $\pi_x(0,0) = 0$ (? necessary?)

(TI) If $K, K' \in \mathcal{C}_x$ are such that down each path from x to T the cumulative values of K, K' agree, then $\forall C$

$$\pi_x(C,K) = \pi_x(C,K')$$

(CE) If K, K' agree at all $y > x$, then for any

$$\pi_x(C,K) - K_x = \pi_x(C,K') - K'_x.$$

(DC) $\forall C, K$, for all stopping times τ,

$$\pi_x(C,K) = \pi_x\left(C_{[x,\tau)}, \pi_{[\tau]}(C_{[\tau,T]}; K_{[\tau,T]} + \bar{K}_{[x,\tau)}) \right).$$

As before, we can define one-step pricing operators

$$\pi_{x,x+1}(C, K) = \pi_x(C, K)$$

where C is a consumption stream paying just at x, and K is a cashflow paying at x and $x+1$. Then using (DC) we get

$$\pi_x(C,K) = \pi_{x,x+1}\left(C_{[x]}\ ,\ \pi_{[[x+1]]}\left(C_{[x+1,T]}\ ,\ K_{[x+1,T]} + \bar{K}_{[x,x+1]}\right)\right).$$

How about the convex dual functions? Define

$$\tilde{\pi}_x(\theta,\lambda) \equiv \sup_{C,K}\left\{\pi_x(C,K) - \theta\cdot C - \lambda\cdot K\right\}.$$

For finiteness, we need $\theta \geq 0$, $\lambda \geq 0$ (using (M)), $\lambda_x = 1$ (using (CE)), and for all $y \geq x$, $\sum_{z \in y+1} \lambda_z = \lambda_y$ (TI). To derive the recursive structure, we have

$$\begin{aligned}
\tilde{\pi}_x(\theta,\lambda) &\equiv \sup_{C,K}\left\{\pi_x(C,K) - \theta\cdot C - \lambda\cdot K\right\} \\
&= \sup_{C,K}\left\{\pi_{x,x+1}\left(C_{[x]}\ ,\ \pi_{[x+1]}\left(C_{[x+1,T]}\ ,\ K_{[x+1,T]} + \bar{K}_{[x,x+1]}\right)\right) - \theta\cdot C - \lambda\cdot K\right\} \\
&= \sup_{C,K}\ \inf_{\psi,\alpha}\left\{\tilde{\pi}_{x,x+1}(\psi,\alpha) + \psi\, C_x + \sum_{z\in x+1}\alpha_z\, \pi_z\left(C_{[z,T]}, K_{[z,T]} + K_x\right)\right. \\
&\qquad\qquad \left. - \theta\cdot C - \lambda\cdot K\right\}
\end{aligned}$$

$$\begin{aligned}
&= \sup_{C,K}\ \inf_{\psi,\alpha}\left[\tilde{\pi}_{x,x+1}(\psi,\alpha) + (\psi-\theta_x)C_x + \sum_{z\in x+1}\alpha_z\ \pi_z\left(C_{[z,T]}, K_{[z,T]} + K_x\right)\right. \\
&\qquad\qquad \left. - \theta\cdot C_{(x,T]} - \lambda\cdot K_{(x,T]} - K_x\right]
\end{aligned}$$

$$\begin{aligned}
\underset{(?!\ \text{Certainly} \leq)}{=}\ & \inf_{\psi,\alpha}\ \sup_{C,K}\left[\tilde{\pi}_{x,x+1}(\psi,\alpha) + (\psi-\theta_x)C_x + \sum_{z\in x+1}\alpha_z\,\pi_z\left(C_{[z,T]}, K_{[z,T]}\right)\right. \\
&\qquad\qquad \left. - \theta\cdot C_{(x,T]} + K_x\left(-1+\sum\alpha_z\right) - \lambda\cdot K_{(x,T]}\right]
\end{aligned}$$

As we only get finite value if $\psi = \theta_x$, $\sum\alpha_z = 1$, giving us

$$\begin{aligned}
= \inf_{\alpha}\ \sup_{C,K}\left[\tilde{\pi}_{x,x+1}(\psi,\alpha) + \sum_{z\in x+1}\alpha_z\left\{\pi_z\left(C_{[z,T]}, K_{[z,T]}\right) - \frac{\lambda_{z+}}{\alpha_z}K_{[z,T]}\right.\right. \\
\left.\left. - \frac{1}{\alpha_z}\theta_{[z,T]}\cdot C_{[z,T]}\right\}\right]
\end{aligned}$$

Once again, property (CE) gives us $\alpha_z = \lambda_z$ for all z, and we derive

$$\boxed{\tilde{\pi}_x(\theta,\lambda) = \tilde{\pi}_{x,x+1}(\theta_x, \lambda_{x,x+1}) + \sum_{z\in x+1} \lambda_z \, \tilde{\pi}_z\left(\frac{\theta_{[z,T]}}{\lambda_z}, \frac{\lambda_{[z,T]}}{\lambda_z}\right)}$$

assuming the minimax result is OK.

4) How does it look if we try utility-indifference pricing? Ignoring consumption, if $\mathcal{X}_x$ is the set of gains-from-trade achievable from vertex x, we would have $\pi_x(K)$ defined via

$$\sup_{X\in\mathcal{X}_x} E\, U(X_T) = \sup_{X\in\mathcal{X}_x} E\, U(X_T + \bar{K}_{[x,T]} - \pi_x(K)).$$

Now I believe that the (DC) property doesn't work for this, because if the first move is from x to $z \in x+1$, we find ourselves looking at a problem like

$$\sup_{X\in\mathcal{X}_z} E\, U(X_T + \bar{K}_{[z,T]} - b + X^*_z)$$

and we've no information on what X^*_z may be (and this matters, as the prices you'll get will depend on this)

Looks like in some cases we could cast the Epstein-Zin recursive utility into such a form...

Black-Scholes with jumps: a question. (9/2/05)

(i) Let's consider a world where $r=0$, and we have a single stock evolving as

$$dS_t = S_{t-}\{\sigma dW_t + dJ_t\}, \qquad S_0 = 1,$$

where σ is constant, and J is some zero-mean compound Poisson process, with $\Delta J_t > -1$ for all t. Then $\log S_t \equiv X_t$ is a Lévy process, with exponent $\psi(\cdot)$, say. Suppose we tried to hedge some European option using Black-Scholes technology, assuming a constant vol equal to σ_0 (not necessarily equal to σ); how bad would the hedge be? What choice of σ_0 is best?

(ii) Of course, this depends on the option, and the measure of 'bad'. It would be nice to do a put option, say, but maybe easier to start with would be to take the claim to be hedged to be

$$\exp(-\alpha X_T) = S_T^{-\alpha}, \qquad (\alpha > 0)$$

which is in any case a mixture of puts of different strikes. The price of this option at time t when $X_t = x$ is just

$$F(t,x) \equiv \exp\{(T-t)\psi(-\alpha) - \alpha x\}$$

and so the claim can be represented as

$$F(T,X_T) = F(0,X_0) + \int_0^T F_x(t,X_t)\,\sigma dW_t + \int_0^T\int \{F(t,X_t+y) - F(t,X_t)\}\big(\nu(dt,dy) - \mu(dy)dt\big)$$

where ν is the Poisson random measure of jumps of X, μ the Lévy measure. If instead we chose to try hedging by using the portfolio process $H(t,X_t)$, then we would generate gains from trade equal to

$$G_T \equiv \int_0^T H(t,X_t)\sigma S_t dW_t + \int_0^T H(t,X_t) S_t (e^y - 1)\,\nu(dt,dy)$$

$$= \int_0^T H(t,X_t)\sigma S_t\, dW_t + \int_0^T H(t,X_t)\, S_t (e^y-1)\big(\nu(dt,dy) - \mu(dy)dt\big),$$

since S is a martingale. An easy thing to do now is to consider how close these are in L^2; by standard results, we get

$$\Delta = E\left|G_T + F(0,X_0) - F(T,X_T)\right|^2$$

$$= E\int_0^T \left\{H(t,X_t) - \frac{F_x(t,X_t)}{S_t}\right\}^2 \sigma^2 S_t^2\, dt$$

$$+ \int \mu(dy)\, E\int_0^T \left\{H(t,X_t) S_t (e^y - 1) - F(t,X_t+y) + F(t,X_t)\right\}^2 dt.$$

Now the Black-Scholes model is structurally very similar to the actual model; we shall have a price

$$\tilde{F}(t,x) = \exp\left\{-\alpha x + (T-t)\psi_0(-\alpha)\right\}$$

where $\psi_0(z) = \frac{1}{2}\sigma_0^2 z(z-1)$. This gives $H(t,X_t) = -\alpha \tilde{F}(t,X_t)/S_t = -\alpha S_t^{-\alpha-1} e^{(T-t)\psi_0(-\alpha)}$

Thus

$$\Delta = E\int_0^T \alpha^2 \left\{e^{(T-t)\psi_0(-\alpha)} - e^{(T-t)\psi(-\alpha)}\right\}^2 S_t^{-2\alpha}\, \sigma^2 dt$$

$$+ \int \mu(dy)\, E\int_0^T S_t^{-2\alpha}\left\{-\alpha(e^y-1)e^{(T-t)\psi_0(-\alpha)} - e^{(T-t)\psi(-\alpha)}(e^{-\alpha y}-1)\right\}^2 dt$$

$$= \int_0^T \alpha^2 \left\{e^{(T-t)\psi_0(-\alpha)} - e^{(T-t)\psi(-\alpha)}\right\}^2 \sigma^2\, e^{t\psi(-2\alpha)}\, dt$$

$$+ \int \mu(dy) \int_0^T e^{t\psi(-2\alpha)}\left\{\left(1-e^{-\alpha y}\right)e^{(T-t)\psi(-\alpha)} - \alpha(e^y-1)e^{(T-t)\psi_0(-\alpha)}\right\}^2 dt.$$

For any particular example, this would not be too hard to evaluate numerically!

A problem on optimal capital structure (14/2/05)

1) Let's return to the problem John + I studied a while back, where you have a production process which delivers a cashflow process

$$\delta_t \equiv \sigma W_t + \mu t$$

and an agent desires to choose default barrier b such that if $\tau = \inf\{t : \delta_t < b\}$ then

$$E\left[\int_0^\tau e^{-\rho t} U(\delta_t)\,dt + \frac{1}{\rho} e^{-\rho\tau} U(0)\right]$$

is maximised. Here, $U : \mathbb{R} \to \mathbb{R}$ is a C^2 utility. The resolvent density of the diffusion killed at b (density wrt speed measure dx/σ^2) will be

$$r_\lambda^b(x,y) \equiv c_\lambda \left[\psi_\lambda^+(x\wedge y)\psi_\lambda^-(x\vee y) - \psi_\lambda^-(x)\frac{\psi_\lambda^+(b)\psi_\lambda^-(y)}{\psi_\lambda^-(b)}\right] \qquad c_\lambda^{-1} \equiv \tfrac{1}{2}\left(\psi_\lambda^- D\psi_\lambda^+ - \psi_\lambda^+ D\psi_\lambda^-\right)$$

2) Suppose we put ourselves in the pricing measure; the stock is worth

$$E^x\left[\int_0^\tau e^{-rt} K\delta_t\,dt\right] \equiv K g(x)$$

where $\tau \equiv \tau_b$, and $\delta_t = \sigma W_t + \mu t$, and K is the number of units of the productive asset that we're dealing with. Solving for g, we get

$$\boxed{g(x) = \left\{\frac{x-b}{r} + \frac{rb+\mu}{r^2}\left(1 - e^{-\beta(x-b)}\right)\right\}}$$

where roots of $\frac{1}{2}\sigma^2\theta^2 + \mu\theta - r = 0$ are $-\beta < 0 < \alpha$. Optimising over b leads to

$$\boxed{b^* = -\frac{r + \mu\beta}{r\beta}}$$

So this is the value of the stock if there is no debt to repay.

3) If the firm is having to pay back a constant coupon $c\,dt$, then the value of the stock is

$$E^x\left[\int_0^\tau e^{-rt}(K\delta_t - c)\,dt\right] \equiv h(x)$$

solved by

$$\boxed{h(x) = \frac{K(\mu + rb) - rc}{r^2}\left(1 - e^{-\beta(x-b)}\right) + \frac{K}{r}(x-b)}$$

and in that case the best choice will be

$$\boxed{b = \frac{r\beta c - \kappa(r+\mu\beta)}{r\beta\kappa}} \qquad (1)$$

Notice that this is equal to $b^* + c/\kappa$.

Suppose that the management of the firm has capital K_0, and they propose to issue bonds paying coupon c, for an initial sum of D. The number of units of the productive asset they can buy is $\kappa = (K_0 + D)/g(x_0)$, where x_0 is the initial value of X.

The value of the bonds at time 0 will be

$$D = E^{x_0}\left[\frac{c}{r}(1-e^{-r\tau}) + \theta e^{-r\tau} \kappa g(b) \right]$$

$$= \frac{c}{r}\left(1-e^{-\beta(x_0-b)}\right) + \theta\kappa g(b) e^{-\beta(x_0-b)}$$

and since $D = \kappa g(x_0) - K_0$, we learn that

$$\boxed{\kappa\left[g(x_0) - \theta g(b) e^{-\beta(x_0-b)}\right] = K_0 + \frac{c}{r}\left\{1-e^{-\beta(x_0-b)}\right\},} \qquad (2)$$

where θ is the proportion of firm value that survives default. Thus the approach is to choose firstly b, then solve (1), (2) for κ and c, and then maximise the equity value $h(x_0)$ over choice of b.

Notice that we will probably need to allow tax rebates on the coupon payments, so that we replace c in (1) and the definition of h by ρc, where $(1-\rho)$ is the tax rate.

From (1) and (2), we get

$$\begin{cases} \kappa\left[g(x_0) - \theta g(b) e^{-\beta(x_0-b)} - \frac{b-b^*}{\rho r}\left(1-e^{-\beta(x_0-b)}\right)\right] = K_0, \\ \rho c = \kappa(b-b^*) \end{cases}$$

Maybe interesting... can we find maximising b analytically.

Some thoughts on the equity premium puzzle (17/2/05)

(i) The original analysis of Mehra + Prescott takes a discrete-time model with a CRRA representative agent, where the outputs y_t in period t are assumed to equal the consumption in period t (markets clear) and the ratio y_{t+1}/y_t is supposed to be controlled by a Markov chain (2-state in their analysis).

It appears that the agent is assumed to know the distributional properties of $\xi_t \equiv \log(y_t/y_{t-1})$ with certainty, but this assumption would be hard to defend in the face of the known imprecision in estimating rates of return. So let's instead suppose that the agent's model for ξ is

$$\xi_t - \mu = \theta(\xi_{t-1} - \mu) + \eta_t$$

where η_t are IID $N(0, \sigma_\eta^2)$, with θ, σ_η^2 both <u>known</u>, but μ in contrast being unknown, having prior $N(\hat{\mu}_0, v_0)$ distribution. The agent is (Kalman) filtering μ from the observed values of ξ, and this will affect the prices of equity and bonds.

(ii) Firstly the Kalman filter: calculations of a familiar nature lead to the recursions

$$\boxed{\begin{aligned} \hat{\mu}_{t+1} &= \frac{\sigma_\eta^2 \hat{\mu}_t + (1-\theta) v_t (\xi_{t+1} - \theta \xi_t)}{(1-\theta)^2 v_t + \sigma_\eta^2}, \\ \frac{1}{v_{t+1}} &= \frac{1}{v_t} + \frac{(1-\theta)^2}{\sigma_\eta^2}. \end{aligned}}$$

(iii) To value equity at time t, we need to compute $(\zeta_t \equiv \beta^t y_t^{-R})$ for cum-dividend price:

$$E_t\left[\sum_{s\geq t} \zeta_s y_s / \zeta_t\right] = y_t E_t\left[\sum_{s\geq t} \beta^{s-t} \left(\frac{y_s}{y_t}\right)^{1-R}\right]$$

$$= y_t \sum_{s\geq t} \beta^{s-t} e^{(1-R) m_{ts} + \frac{1}{2}(1-R)^2 v_{ts}}$$

where m_{ts} is the conditional mean of $\sum_{j=t+1}^{s} \xi_j$, v_{ts} its conditional variance. It's not hard to do the calculations:

$$\begin{cases} m_{ts} = (s-t)\hat{\mu}_t + \dfrac{\theta(1-\theta^{s-t})}{1-\theta}(\xi_t - \hat{\mu}_t) \\ v_{ts} = \displaystyle\sum_{\ell=t+1}^{s} \sigma_\eta^2 \left(\frac{1-\theta^{s-\ell+1}}{1-\theta}\right)^2 + \left\{s-t + \frac{\theta(1-\theta^{s-t})}{1-\theta}\right\}^2 v_t \end{cases}$$

For a single-period bond, we need to calculate

$$E_t\left[\zeta_{t+1}/\zeta_t\right] = E_t\left[\beta\left(\frac{y_{t+1}}{y_t}\right)^{-R}\right] = \beta\, E_t\left[e^{-\xi_{t+1} R}\right]$$

$$= \beta \exp\{-R m_{t,t+1} + \tfrac{1}{2}R^2 v_{t,t+1}\}$$

$$= \beta \exp\left[-R\left(\theta\xi_t + (1-\theta)\hat{\mu}_t\right) + \tfrac{1}{2}R^2(1-\theta)^2 v_t + \tfrac{1}{2}R^2\sigma_\eta^2\right].$$

(iv) Arnaud has dug out a paper by M. L. Weitzman "A Unified Bayesian theory of equity 'puzzles'" which appears very much on the same theme. He is taking $\theta = 0$ (so the ξ_t are IID) with (μ, σ_η^2) unknown, but having a normal-gamma prior, which has to be truncated at low precision in order for the expectations defining the equity to be finite. He then argues that some very similar values of δ can be used to match various equity premium puzzles, at least in mean.

Let's try to do this in a more wholeheartedly Bayesian way. The emphasis Weitzman attaches to the truncation level δ is rather unnatural to me — we should be finding that the action focuses on μ, not on σ_η^2. So suppose that the representative agent has a prior for (μ, τ) with density

$$\propto \exp\left[-\tfrac{1}{2}\kappa\tau\mu^2 - b\tau - c/\tau\right]\tau^{\alpha-1}$$

where κ, b, c, α are positive parameters. After observing $\xi_1, \dots, \xi_m$ (assumed IID for now, so that we are for now restricting to $\theta = 0$), the posterior density will be

$$\propto \exp\left[-\tfrac{1}{2}(m+\kappa)\tau\left(\mu - \frac{m}{m+\kappa}\bar{\xi}\right)^2 - \left(b + \tfrac{1}{2}S_{\xi\xi} + \tfrac{1}{2}\frac{m\kappa}{m+\kappa}\bar{\xi}^2\right)\tau - c/\tau\right]\tau^{\alpha+m/2-1}$$

Therefore conditional on $\mathcal{Y}_t$, we think $(\xi_j)_{j>t}$ are IID Gaussians with mean $t\bar{\xi}/(t+\kappa)$ and precision $(t+\kappa)\tau$, where τ has some inverse Gaussian distribution. To price the bond therefore, we must compute

$$E_t\left[\sum_{s\geq t}\beta^{s-t}\exp\left(\sum_{j=t+1}^{s}\xi_j\,(1-R)\right)\right]$$

$$= E_t\sum_{s\geq t}\beta^{s-t}\exp\left\{(1-R)(s-t)\frac{t}{t+\kappa}\bar{\xi}_t + \tfrac{(s-t)}{2}(1-R)^2/\tau(t+\kappa)\right\}$$

$$= E_t\left[1\Big/\left(1-\beta\exp\left\{(1-R)t\bar{\xi}_t/(t+\kappa) + \frac{(1-R)^2}{2\tau(t+\kappa)}\right\}\right)\right]$$

and if τ can get arbitrarily close to 0 (as it can in this model) then this expectation is <u>infinite</u>.... and, indeed, the convergence of this sum can be problematic for other reasons.

Note: $\int_0^\infty \exp(-bt - c/t)\, t^{\alpha-1}\,dt = 2(c/b)^{\alpha/2} K_\alpha(2\sqrt{bc})$

components of the observations. The year-t stock price is

$$S_t = y_t \sum_{j=0}^{N} \beta^j e^{(1-R)j m_t} \frac{K_{d'}(2\sqrt{b'g})}{K_{d'}(2\sqrt{b'c})} \equiv y_t \, F(t, \bar{\xi}_t, S_{\xi\xi}(t); \theta)$$

The observation $Y_t \equiv (Y_t^1, Y_t^2, Y_t^3)^T$ is just $\left(\xi_t, G(t, \bar{\xi}_t, S_{\xi\xi}(t); \theta), \frac{F(t, \bar{\xi}_t, S_{\xi\xi}(t); \theta)}{F(t-1, \bar{\xi}_{t-1}, S_{\xi\xi}(t-1); \theta)} e^{\xi_t}\right)^T$
plus some noise.

The 'hidden state' to be filtered in this example is just the parameter θ, which we shall allow to evolve (a little) according to transition density φ. If our approximation at time n to the posterior is

$$\sum_{i=1}^{N} w_n^i \, \delta_{z_n^i}$$

then the approximation to the posterior at time $n+1$ will be proportional to

$$x \longmapsto \sum_{i=1}^{N} w_n^i \, \varphi(z_n^i, x) \, f(Y_{n+1} | x)$$

where we suppose that $f(Y_{n+1}|x) = \exp\left[-\frac{1}{2} \sum_{j=1}^{3} (Y_{n+1}^j - \eta^j)^2 / v_j \right] (2\pi)^{-3/2} (v_1 v_2 v_3)^{-1/2}$
for concreteness (the noises don't have to be Gaussian, of course).

Here,

$$\eta = \begin{pmatrix} \xi_{n+1} \\ G(n+1, \bar{\xi}_{n+1}, S_{\xi\xi}(n+1); x) \\ e^{\xi_{n+1}} \dfrac{F(n+1, \bar{\xi}_{n+1}, S_{\xi\xi}(n+1); x)}{F(n, \bar{\xi}_n, S_{\xi\xi}(n); x)} \end{pmatrix}$$

is the mean of the observations given the parameters

(V) To get round the exploding infinities, let's value the equity by computing the sum to $t+N$

$$y_t\, E_t\left[\sum_{s=t}^{t+N} \beta^{s-t}\left(\frac{y_s}{y_t}\right)^{1-R}\right] = y_t \sum_{s=t}^{t+N} \beta^{s-t}\, E_t\left[\exp\left\{\sum_{k=t+1}^{s}(1-R)\xi_k\right\}\right].$$

Now $\xi_k \sim N(\mu, 1/\tau)$ and conditional on $\mathcal{F}_t$ the pair (μ,τ) has density

$$\propto \sqrt{\tau}\, \exp\left[-\tfrac{1}{2}\kappa'\tau(\mu-m)^2\right].\ \tau^{\alpha'-1}\exp\left[-b'\tau - c/\tau\right]$$

where $m = t\,\bar{\xi}_t/(t+\kappa)$, $\kappa' = \kappa+t$, $\alpha' = \alpha + t/2 - \frac{1}{2}$, $b' = b + \frac{1}{2}S_{\xi\xi} + \frac{1}{2}\frac{t\kappa}{t+\kappa}\bar{\xi}_t^2$. Therefore

$$E_t\, \exp\left(\sum_{k=t+1}^{t+j}(1-R)\xi_k\right) = E_t\, \exp\left\{(1-R)j\mu + \tfrac{1}{2}j(1-R)^2/\tau\right\}$$

$$= E_t\, \exp\left\{(1-R)jm + \tfrac{1}{2}j^2(1-R)^2/\kappa'\tau + \tfrac{1}{2}j(1-R)^2/\tau\right\}$$

$$= \text{const.}\, e^{(1-R)jm}\int_0^\infty \tau^{\alpha'-1}\exp\{-b'\tau - g/\tau\}\, d\tau$$

$$\left[g \equiv c - \frac{(1-R)^2}{2}\, j\left(1+ j/\kappa'\right)\right]$$

$$= \text{const}\, e^{(1-R)jm}\left(\frac{g}{b'}\right)^{\alpha'/2} K_{\alpha'}\left(2\sqrt{b'g}\right).$$

To determine the constant, note that if $R=1$ this expression should be 1, so we get

$$= e^{(1-R)jm}\left(\frac{g}{c}\right)^{\alpha'/2}\frac{K_{\alpha'}\left(2\sqrt{b'g}\right)}{K_{\alpha'}\left(2\sqrt{b'c}\right)}.$$

Summing gets us to the price of the stock.

(VI) For the bond, we need to compute

$$E_t\left(y_{t+1}/y_t\right)^{-R} = E_t\, e^{-R\xi_{t+1}}$$

which is like the preceding but with $j=1$, and $1-R$ replaced by $-R$. Thus if $\tilde{c} \equiv c - \frac{R^2}{2}(1+\frac{1}{\kappa'})$ we get

$$E_t\left(\frac{y_{t+1}}{y_t}\right)^{-R} = e^{-Rm}\left(\frac{\tilde{c}}{c}\right)^{\alpha'/2}\frac{K_{\alpha'}\left(2\sqrt{b'\tilde{c}}\right)}{K_{\alpha'}\left(2\sqrt{b'c}\right)}.$$

So the risk-free rate prevailing from time t to time $t+1$ is

$$\boxed{\rho_t = m_t R - \log\beta + \frac{\alpha'}{2}\log\left(c/\tilde{c}\right) + \log\frac{K_{\alpha'}\left(2\sqrt{b'c}\right)}{K_{\alpha'}\left(2\sqrt{b'\tilde{c}}\right)} \equiv G(t, \bar{\xi}_t, S_{\xi\xi}(t);\ \theta)}$$

where $\theta^T \equiv (R, \alpha, \kappa, b, c, \beta, v_1, v_2, v_3)$ is the parameter vector, where v_i are the variances in the

<u>A simple model for trading and information (23/2/05)</u>

(i) Let's suppose that the price p_t at time t of some asset can be represented as

$$p_t = \sigma W_t + \sum_{i=1}^{N} \theta_t^i$$

where θ_t^i is the demand of the ith agent. Each agent is a profit maximiser, so aims to

$$\max \quad E\left[\int_0^\infty e^{-rt}\,\theta_t\,dp_t - \tfrac{1}{2}\varepsilon_i \int_0^\infty e^{-rt}(\theta_t - \xi_t^i)^2\,dt\right]$$

where the second term represents inventory costs, and ξ_t^i is the desired target inventory level. Agent i's objective is therefore to maximise

$$E\left[\int_0^\infty e^{-rt}\,\theta_t\,\mu_t\,dt - \tfrac{1}{2}\varepsilon_i \int_0^\infty e^{-rt}(\theta_t - \xi_t^i)^2\,dt\right]$$

$$= E\left[\int_0^\infty e^{-rt}\,\theta_t\,\hat{\mu}_t^i\,dt - \tfrac{1}{2}\varepsilon_i \int_0^\infty e^{-rt}(\theta_t - \xi_t^i)^2\,dt\right],$$

where $dp_t \doteq \mu_t\,dt$, and $\hat{\mu}_t^i$ is agent i's estimate of μ. Agent i sees p_t, and ξ_t^i, but doesn't know the desired inventories of other agents. Agent i will therefore take

$$\boxed{\theta_t^i = \xi_t^i + \varepsilon_i^{-1}\,\hat{\mu}_t^i}$$

(ii) Let's now suppose that the ξ^i solve some linear Gaussian system

$$d\xi = \Sigma\,dZ + B\xi\,dt$$

where Z is a BM in $\mathbb{R}^N$, independent of W, and B is some known matrix. Now we adjoin p, μ to ξ to form state vector $X = (p, \mu, \xi^T)^T$, satisfying an equation of the form

$$d\begin{pmatrix} p \\ \mu \\ \xi \end{pmatrix} = \left(\begin{array}{c|c} a & A \\ \hline 0 & \Sigma \end{array}\right)\begin{pmatrix} dW \\ dZ \end{pmatrix} + \begin{pmatrix} b & \beta \\ 0 & B \end{pmatrix}\begin{pmatrix} p \\ \mu \\ \xi \end{pmatrix} dt$$

where a is 2×1, A is $2\times N$, b is 2×1, β is $2\times N$, to be determined. <u>Given</u> values of these unknowns, it is not hard to find the SDE satisfied by $\hat{\mu}^i$ (using the Kalman filter), hence we obtain the SDE for θ^i, and adding up we arrive at the SDE for p, which has to agree with the assumed form. So we are down to a numerical search for a set of values (perhaps more than one!?) that is consistent in this sense.

Holger Kraft "Optimal portfolios with stochastic interest rates and defaultable assets" has some results on this sort of problem (and some fairly explicit expressions for allied material) ISBN 3-540-21230-2

Investment/consumption: some illustrative examples (25/2/05)

1) The standard dynamics for the wealth equation

$$dw = rw\,dt + \theta\left(\sigma dW + (\mu - r)dt\right) - c\,dt$$

for agent with objective $E\int_0^\infty e^{-\rho t} U(c_t)\,dt$, with CRRA U, leads to the solution $\theta^* = \pi_M w$, value

$$V_M(w) = \gamma_*^{-R}\, U(w) \quad , \quad \gamma_* = \left(\rho + (R-1)\left(r + \tfrac{1}{2}\sigma^2 \pi_M^2 R\right)\right)/R .$$

2) Suppose you make the spot-rate follow an OU process $dr = \sigma_r dZ + \beta(\bar{r} - r)dt$; then the value function is of the form $V(r,w) = v(r)\, U(w)$, where v will solve the HJB equation:

$$R\, v^{1-1/R} - \rho v + \beta(\bar{r} - r)v' + r(1-R)v + \tfrac{1}{2}\sigma_r^2 v'' + \tfrac{1}{2}\sigma^2 R(1-R)\left(\frac{(\mu - r)v + \rho_0 \rho_r \sigma v'}{\sigma^2 R v}\right)^2 v = 0$$

where ρ_0 is correlation between W and Z, and

$$\theta^* = w\, \frac{(\mu - r)v + \rho_0 \rho_r \sigma\, v'}{\sigma^2 R v} .$$

This will be impossible to solve (Maple agrees.)

3) However, we could replace r by $r(\xi_t)$, where ξ is some Markov chain with jump rates Q. Now the value function is $v(\xi)\, U(w)$ (the vol + drift may depend on ξ), and we find

$$0 = \left[R v^{1-1/R} - \rho v + \frac{(1-R)}{R}\frac{(\mu - r)^2}{2\sigma^2} v + Qv + r(1-R)v\right] \equiv R v^{1-1/R} - Mv$$

with

$$\theta = (\mu - r)/\sigma^2 R ,$$

This nonlinear equation for v can it seems be solved recursively via $v_0 = 1$,

$$v_{n+1} = M^{-1} R v_n^{1-1/R} .$$

So this gets us a long way toward solving the problem for the Vasicek riskless rate, numerically.

4) Looking back at the relaxed investor story, I proposed there the filtering problem where you start with a prior $N(\lambda_0, v_0)$ for the parameter $\lambda \equiv \mu/\sigma$ in the standard model, and then worked out expressions for the value and efficiency, but these quantities were all computed under the assumption that the true value of λ was in fact λ_0.

How do things turn out if the true value of λ is actually λ^*? Wrong question!

5) How do the investment and consumption decisions of the uncertain investor differ from those of the Merton investor?

Concerning the consumption level, we have

$$e^{-\rho t}\, U'(c^*_t) = \gamma \zeta_t$$

where γ is related to the wealth via

$$w_0 = \gamma^{-1/R}\, \varphi(\lambda_0, v_0),$$

where φ is as given at section 7 of the relaxed investor paper:

$$\varphi(\lambda_0, v_0) = \int_0^\infty \frac{(1+v_0 t)^{(R-1)/2R}}{(1+k v_0 t)^{1/2}} \exp\left[-\frac{(\lambda_0 - r/\sigma)^2 k t}{2R(1+k v_0 t)} - \frac{\rho + r(R-1)}{R} t \right] dt \qquad \left(k = \frac{R-1}{R}\right)$$

Thus we shall find that

$$\boxed{c^*_t = w_t / \varphi(\hat\lambda_t, v_t).}$$

Then the value is

$$E\int_0^\infty e^{-\rho t}\, U(c^*_t)\, dt = E\int_0^\infty \frac{1}{1-R}\, \gamma^{(R-1)/R}\, e^{-\rho t/R}\, \zeta_t^{(R-1)/R}\, dt$$

$$= \frac{1}{1-R}\, \gamma^{(R-1)/R} \cdot \varphi(\lambda_0, v_0)$$

So we have that the value is

$$U(w_0)\, \varphi(\lambda_0, v_0)^R\;;$$

from here, the value at time t will be

$$U(w_t)\, \varphi(\hat\lambda_t, v_t)^R \equiv V(t, \hat\lambda_t, w_t)$$

where $d\hat\lambda_t = d\hat W_t / (\tau_0 + t)$, where $\tau_0 = 1/v_0$ is prior precision, and w_t solves the usual wealth equation

$$dw_t = r w_t\, dt + \theta\sigma\,(dX_t - (r/\sigma)\, dt) - c_t\, dt$$

$$= r w_t\, dt + \theta\sigma\,(d\hat W_t + (\hat\lambda_t - r/\sigma)\, dt) - c_t\, dt.$$

If we use the fact that V must solve the HJB equations, we find that the optimal θ is

$$\boxed{\frac{\theta^*}{w} = \frac{\sigma\hat\lambda - r}{\sigma^2 R} + \frac{\varphi_\lambda}{\sigma \cdot (\tau_0 + t)\,\varphi}}.$$

What would happen if you followed the Merton rule of investing proportion $\hat{\pi}_t \equiv \dfrac{\sigma\hat{\lambda}_t - r}{\sigma^2 R}$ of wealth in the risky asset, while consuming at rate $\hat{\gamma} \equiv \left(\rho + (R-1)(r + \frac{1}{2}\sigma^2 R\hat{\pi}^2)\right)/R$ times wealth? The wealth dynamics become

$$dw = rw\,dt + \hat{\pi} w\left(\sigma\,d\hat{W} + (\sigma\hat{\lambda} - r)\,dt\right) - \hat{\gamma}\, w\,dt$$

so that

$$w^{-1}dw = \sigma\hat{\pi}\,d\hat{W} + \left\{ r + \sigma^2 R\hat{\pi}^2 - \frac{\rho + (R-1)(r + \frac{1}{2}\sigma^2 R\hat{\pi}^2)}{R} \right\} dt$$

$$= \sigma\hat{\pi}\,d\hat{W} + \left\{ \frac{r}{R} - \frac{\rho}{R} + \frac{1}{2}\sigma^2\hat{\pi}^2(R+1) \right\} dt$$

Now observe that

$$d\left(\tfrac{1}{2}\sigma^2 R\hat{\pi}^2(\tau_0+t)\right) = \frac{\sigma^2 R}{2}(\tau_0+t)\left[2\hat{\pi}\frac{d\hat{\lambda}}{\sigma R} + \frac{1}{\sigma^2 R^2}\frac{dt}{(\tau_0+t)^2}\right] + \frac{\sigma^2 R}{2}\hat{\pi}^2\,dt$$

$$= \sigma\hat{\pi}\,dW + \frac{dt}{2R(\tau_0+t)} + \frac{\sigma^2 R}{2}\hat{\pi}^2\,dt$$

so that
$$w^{-1}dw = d\left(\tfrac{1}{2}\sigma^2 R\hat{\pi}^2(\tau_0+t)\right) - \frac{dt}{2R(\tau_0+t)} + \left(\frac{r}{R} - \frac{\rho}{R}\right)dt + \tfrac{1}{2}\sigma^2\hat{\pi}^2\,dt$$

giving us an expression

$$\boxed{w_t = w_0 \exp\left[\tfrac{1}{2}\sigma^2 R\,\hat{\pi}_t^2(\tau_0+t) - \tfrac{1}{2}\sigma^2 R\,\hat{\pi}_0^2\,\tau_0\right] e^{-(\rho-r)t/R}\cdot\left(\frac{\tau_0}{\tau_0+t}\right)^{1/2R}}$$

(Checked by Maple)

for the wealth at time t. The value of the objective is therefore

$$E\int_0^\infty \frac{e^{-\rho t}}{1-R}\,\hat{\gamma}_t^{1-R}\,w_t^{1-R}\,dt$$

which we can make some progress with, since $\hat{\pi}_t \sim N\left(\dfrac{\sigma\lambda_0 - r}{\sigma^2 R}, \dfrac{1}{\sigma^2 R^2}\cdot\dfrac{t}{\tau_0(\tau_0+t)}\right)$.

However, we do still have a double integration, because we have something of the form $(A + B\hat{\pi}^2)^{1-R}$ and there's nothing closed-form for this.

We have $\hat{\pi}_t \sim N(a_1, a_2)$, $\hat{\gamma}_t = (a_3 + a_4\hat{\pi}_t^2)$, $w_t = w_0\exp(a_5\hat{\pi}^2)\,a_6$, where

$$a_1 = \hat{\pi}_0 = \frac{\sigma\lambda_0 - r}{\sigma^2 R}, \quad a_2 = \frac{t}{\sigma^2 R^2\tau_0(\tau_0+t)}, \quad a_3 = \frac{\rho + (R-1)r}{R}, \quad a_4 = \frac{(R-1)\sigma^2}{2},$$

$$a_5 = \frac{\sigma^2 R}{2}(\tau_0+t), \quad a_6 = \exp\left[-(\rho-r)t/R - \tfrac{1}{2}\sigma^2 R\,\tau_0\,\hat{\pi}_0^2\right]\left(\tau_0/(t+\tau_0)\right)^{1/2R}$$

Phil makes the point that $\Lambda(K_{\tau}, \Delta_{\tau})$ would be more interesting ... $\longrightarrow$

Modelling of default: some remarks (14/3/05)

1) When a firm defaults, it may still have value, it just doesn't have (or doesn't want to hand over) cash. Suppose that x_t is the cash held by the firm at time t, K_t the total capitalisation of the firm at time t; then

$$dx_t = r_t x_t\,dt + \theta_t\,dy_t - b_0 K_t\,dt - p_t\,dK_t + d\Delta_t - (r_t + s_t)\Delta_t\,dt - \eta_t\,dt$$

where $\theta_t \in [0, K_t]$ is intensity of operation of the firm's capacity, b_0 denotes rate of fixed costs, p_t is the current purchase price of a unit of the productive asset, y_t is process of operating revenue, Δ_t is level of debt, s_t is the spread on borrowing. Let's suppose that

default happens when x_t hits zero.

(Even in the case of voluntary default, the management would make sure that there was no cash lying around for creditors to seize! The illiquid assets of the firm would get liquidated, so the shares might still have value then).

2) First special case Keep K, Δ, r, s constant, $\theta = K$, so we get dynamics (with changed consts)

$$dx = rx\,dt + (\sigma\,dW + \mu\,dt) - b\,dt - \eta\,dt,$$

and the objective of the management is to choose η so as to achieve

$$V(x) \equiv \sup E^x\left[\int_0^\tau e^{-\lambda t}\eta_t\,dt + e^{-\lambda\tau}A\right], \qquad [\lambda = r?]$$

where A is the liquidation value of the firm. Usual HJB story gives

$$\sup_\eta \left[\eta - \lambda V + \tfrac{1}{2}\sigma^2 V'' + (\mu - b + rx)V' - \eta V'\right] = 0$$

so we will need $V' \geq 1$, and to solve the ODE

$$\tfrac{1}{2}\sigma^2 V'' + (rx + \mu - b)V' - \lambda V = 0$$

which is the resolvent of an OU process, essentially undoable.

3) Could we assume $r=0$? This may not be as faked as it appears; revenues and costs would get scaled by inflation in any case. Assuming this, we get the problem

$$\tfrac{1}{2}\sigma^2 V'' + (\mu - b)V' - \lambda V = 0$$

solved by $e^{\alpha x}$, $e^{-\beta x}$, where α, β are roots of $\tfrac{1}{2}\sigma^2 t^2 + (\mu - b)t - \lambda = 0$. The dividend policy of the firm now will be to reflect the diffusion at some level k^*. Thus if

$$X_t = x_0 + \sigma W_t + (\mu - b)t$$

and $\bar{X}_t = \left(\sup_{u\le t} X_u - k\right)^+$, then $x_t = X_t - \bar{X}_t$, the drifting BM reflected down from k.

If we want

$$f(x) = E^x\left[\int_0^\tau e^{-\lambda s}\, d\bar{X}_s\right]$$

then we have to solve $\frac{1}{2}\sigma^2 f'' + (\mu - b) f' - \lambda f = 0$ with $f'(k) = 1$, $f(0) = 0$, which gives

$$f(x) = \frac{e^{\alpha x} - e^{-\beta x}}{\alpha e^{\alpha k} + \beta e^{-\beta k}},$$

and to find

$$E^x\left[e^{-\lambda\tau}\right] = g(x)$$

we just get $\frac{1}{2}\sigma^2 g'' + (\mu - b) g' - \lambda g = 0$ with $g(0) = 1$, $g'(k) = 0$, giving

$$g(x) = \frac{\beta e^{\alpha(x-k)} + \alpha e^{-\beta(x-k)}}{\beta e^{-\alpha k} + \alpha e^{\beta k}}$$

Combining,

$$E^x\left[\int_0^\tau e^{-\lambda t}\, \eta_t\, dt + A e^{-\lambda\tau}\right] = f(x) + A g(x) = \varphi(x)$$

and now we have to choose k optimally to ensure $\varphi'(x) \ge 1$ for all $x \in [0, k]$. This will happen when $\varphi''(k) = 0$, which is the condition

$$\boxed{\alpha^2 e^{\alpha k} - \beta^2 e^{-\beta k} + A\alpha\beta(\alpha+\beta)\, e^{(\alpha-\beta)k} = 0.}$$

Assuming $A \ge 0$, this can only be solved if $\beta^2 \ge \alpha^2$, that is $b \le \mu$. This has a natural interpretation; if $b < \mu$ then the productive activity is sufficiently profitable that you want to keep going and postpone dividends!

4) However, if we suppose $\lambda = r$ (so firm is maximising share value) then we can actually get something a bit better. The ODE for V,

$$\tfrac{1}{2}\sigma^2 V''(x) + (rx + \mu - b) V'(x) - r V(x) = 0$$

has one solution

$$V_0(x) = (rx + \mu - b)$$

so the other solution is expressible as $V_0(x)\, g(x)$, where

$$\tfrac{1}{2}\sigma^2 (g'' V_0 + 2 g' V_0') + (rx + \mu - b) V_0 g' = 0$$

so we shall have

$$g''(x) + \left\{\frac{2V_0'}{V_0} + 2(rx + \mu - b)/\sigma^2\right\} g'(x) = 0$$

which is solved by

$$\int^{x} e^{-y^2/2} \frac{dy}{y^2} = \int^{x} e^{-y^2/2} \, d\left(-\frac{1}{y}\right) = -\frac{e^{-x^2/2}}{x} - \Phi(x)\sqrt{2\pi}$$

$$g'(x) = K_1 \, (rx+\mu-b)^{-2} \exp\left[-(rx+\mu-b)^2/\sigma^2 r \right]$$

Therefore

$$g(x) = K_0 - K_1 \left\{ \frac{e^{-(rx+\mu-b)^2/\sigma^2 r}}{(rx+\mu-b)\sqrt{2/\sigma^2 r}} + \sqrt{2\pi}\, \Phi\left((rx+\mu-b)\sqrt{2/\sigma^2 r} \right) \right\},$$

and so the other solution to the ODE is

$$V_1(x) = e^{-(rx+\mu-b)^2/\sigma^2 r} \sqrt{\frac{\sigma^2 r}{2}} + \sqrt{2\pi}\, \Phi\left((rx+\mu-b)\sqrt{\frac{2}{\sigma^2 r}} \right) (rx+\mu-b)$$

Thus the solutions which are equal to A at zero are of the form

$$V(x) = A \, \frac{rx+\mu-b}{\mu-b} - K \left\{ V_1(x) - \frac{rx+\mu-b}{\mu-b} V_1(0) \right\}$$

for some K. Notice that $V_1'(x) = r\sqrt{2\pi}\, \Phi\left((rx+\mu-b)\sqrt{2/\sigma^2 r} \right)$, increasing, so for $V' \geq 1$ we shall have to have $K \geq 0$.

Numerical study suggests you would keep on postponing dividends, at least in some circumstances. However, we can quickly get to this point, because if $\tilde{x}_t = e^{-rt} x_t$ we have

$$d\tilde{x}_t = -e^{-rt}\eta_t \, dt + e^{-rt}\left(\sigma \, dW + (\mu-b)\, dt \right)$$

$$\therefore \quad \tilde{x}_t - \tilde{x}_0 = -\int_0^t e^{-rs}\eta_s \, ds + \frac{\mu-b}{r}\left(1-e^{-rt}\right) + M_t$$

and so

$$E^x\left[\int_0^\tau e^{-rs}\eta_s \, ds + e^{-r\tau} A \right] = E^x\left[\frac{\mu-b}{r}\left(1-e^{-r\tau}\right) + e^{-r\tau} A + x \right]$$

since $\tilde{x}_\tau = 0$; thus if $(\mu-b)/r + A > 0$ we would want to keep going as long as we could, else liquidate immediately.

- Should we be looking at alternative revenue processes?
- When x is very small, bankruptcy is nearly certain, so the firm's management would actually default a bit before x gets down to 0 ....? No, the remote chance of dividends can't be ignored.

5) <u>Second special case</u>: K, Δ, r, ρ constant, but θ allowed to be chosen from $[0, K]$. I have a guess that this could be ill posed, but let's leave this for now

6) <u>Third special case</u>: K can be increased at price p, $0 \leq \theta_t \leq K_t$. What we expect here is that whenever θ_t bumps up to K_t we would want to raise K_t, except if we were at the level where dividends are to be paid out. The value function is now $V(x, K)$, and should satisfy

$$\sup_{0 \leq \theta \leq K} \left[\tfrac{1}{2}\sigma^2\theta^2 V_{xx} + (rx+\theta\mu-b)V_x - rV \right] = 0$$

together with $V_x \geq 1$

$$V_K - p V_x \leq 0.$$

Work now with dual variables $z = V_x(x, K)$, $J(z, K) = V(x,K) - xz$, which turns HJB into

$$\sup_{0 \leq \theta \leq K} \left[-\frac{1}{2} \frac{\sigma^2 \theta^2}{J''} + (\theta\mu - b) z - rJ \right] = 0 \quad , \quad z \geq 1, \quad J_K - pz \leq 0$$

to be optimised where

$$\theta = \frac{\mu z J''}{\sigma^2}$$

to give the ODE for J:

$$\frac{\mu^2 z^2}{2\sigma^2} J''(z) - rJ(z) - bz = 0$$

solved by $J(z, K) = -\frac{bz}{r} + a_1(K)\left(z/z_1(K)\right)^{-\alpha} + a_2(K)\left(z/z_1(K)\right)^{\beta}$

where $-\alpha < 0 < \beta$ are roots of $\frac{\mu^2}{2\sigma^2} t(t-1) - r = 0$, and $z_1(K)$ is the place where $J_z = 0$, which corresponds to $x = 0$. Using the conditions at $z_1(K)$, we have

$$\left.\begin{aligned} J_z(z_1(K), K) &= 0 \\ J(z_1(K), K) &= A \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} a_1(K) &= (\beta A + (\beta - 1)\, b z_1(K)/r) / (\alpha + \beta) \\ a_2(K) &= (\alpha A + (\alpha + 1)\, b z_1(K)/r) / (\alpha + \beta) \end{aligned}\right.$$

which expresses $a_1(K)$, $a_2(K)$ explicitly in terms of $z_1(K)$. The solution is good in some interval $z \in [z_0(K), z_1(K)]$ (or is it $[z_1(K), z_0(K)]$?!)

To find $z_0(K)$, we have to look for the value z where

$$\theta = \mu z J''(z, K)/\sigma^2 = K$$

(assuming it's unique), and then we have

$$J_K(z_0(K), K) = p z_0(K)$$

which gives us a nonlinear implicit first-order ODE for $z_1(K)$. We want to solve until $z_0(K) = 1$... but how do we find an initial condition? Is the problem well posed?

More thoughts on pricing operators (25/4/05)

(i) In the earlier story, the fact that $\pi_x(K)$ does not depend on current wealth level is a bit restrictive. Suppose instead we consider operators which depend on current cash balance a;

$\pi_x(K;a) = p$ means that if the current cash balance is a, then we would just be willing to pay p in order to receive K.

A related concept would be

$\bar\pi_x(K;a) = q$ meaning that if our current cash balance were augmented by q, then we would just be prepared to pay q for K; in other words,

$$\bar\pi_x(K;a) = q \iff \pi_x(K;a+q) = q.$$

(ii) What properties may/should we seek?

(C) $\forall a$, $\pi_x(\cdot\,;a)$ is concave (always, for a reservation bid price)

(M) $K' \geq K \Rightarrow \pi_x(K';a) \geq \pi_x(K;a)$ (not clear that this should be true for $\bar\pi$)

(L) $I_A\,\pi_\tau(K;a) = \pi_\tau(I_A I_{[\tau,\infty)}K\,;a)$

(TI) $\pi_x(K+b\,;a) = b + \pi_x(K;a)$ for constants b

(not clear this holds for $\bar\pi$ again)

(Z) $\pi_x(0\,;a) = 0$ (and this time also $\bar\pi_x(0\,;a) = 0$)

Notice from (TI) that $\pi_x(b\,;a) = b$ (using (Z)), and so $\bar\pi_x(b;a) = b$ for constants b.

Now what about (DC)? Various natural ideas suggest themselves:

(DC1) $p = \pi_x(K;a) = \pi_x\big(K I_{[x,\sigma)} + \bar\pi_\sigma(K;a-p) I_{[\sigma,\infty)}\,;a\big)$

(DC2) $p = \pi_x(K;a) = \pi_x\big(K^\sigma + \bar\pi_\sigma(K-K_\sigma\,;a+K_\sigma-p) I_{[\sigma,\infty)}\,;a\big)$

(DC3) $\bar\pi_x(K;a) = \bar\pi_x\big(K I_{[x,\sigma)} + \bar\pi_\sigma(K;a) I_{[\sigma,\infty)}\,;a\big)$

(DC4) $\bar\pi_x(K;a) = \bar\pi_x\big(K^\sigma + \bar\pi_\sigma(K-K_\sigma\,;a+K_\sigma) I_{[\sigma,\infty)}\,;a\big)$

It is an easy exercise to prove that (DC1) $\iff$ (DC2), and that (DC3) $\iff$ (DC4).

However, we do even have that (DC3) $\iff$ (DC1) .... how?

What (DC3) says is that for each K, x, a we have

$$p = \bar{\pi}_x(K;a) = \bar{\pi}_x(K';a), \quad \text{where } K' = KI_{[x,\sigma)} + \bar{\pi}_\sigma(K;a)I_{[\sigma,\infty)}$$

so

$$\pi_x(K;a+p) = p = \pi_x(K';a+p)$$

$$= \pi_x\left(KI_{[x,\sigma)} + \bar{\pi}_\sigma(K;a)I_{[\sigma,\infty)}\,;\,a+p\right)$$

so if we write $a' = a+p$ we get

$$p = \pi_x(K;a') = \pi_x\left(KI_{[x,\sigma)} + \bar{\pi}_\sigma(K;a'-p)I_{[\sigma,\infty)}\,;\,a'\right)$$

which is (DC1). Similarly, (DC3) $\Rightarrow$ (DC1), so all of these statements are equivalent.

Notice that if we use (DC3) we should be able to construct back the operators $\bar{\pi}$ in the same way as before ... the issue will be whether we can stick together one-step operators $\bar{\pi}_{x,x+1}$ in such a way as to give concavity of the resulting π_x ...

iii) <u>Proposition</u> Suppose that

(a) $\bar{\pi}_x(\cdot;\cdot)$ is concave in the two arguments

(b) $t \mapsto \bar{\pi}_x(K;a-t) - t$ is strictly decreasing for each K, a.

Then $\pi_x(\cdot;\cdot)$ is jointly concave in both arguments.

<u>Proof</u> For $i=1,2$, set $p_i = \pi_x(K_i;a_i)$, $\bar{p} \equiv \alpha_1 p_1 + \alpha_2 p_2$, $\alpha_i \geq 0$, $\alpha_1+\alpha_2=1$. Now $\pi_x(K_i;a_i) = \bar{\pi}_x(K_i;a_i-p_i)$, so we have

$$\bar{p} = \alpha_1 p_1 + \alpha_2 p_2 = \alpha_1\bar{\pi}_x(K_1;a_1-p_1) + \alpha_2\bar{\pi}_x(K_2;a_2-p_2)$$

$$\leq \bar{\pi}_x(\bar{K};\bar{a}-\bar{p}) \qquad \text{(by (a))}$$

so we have $\bar{\pi}_x(\bar{K};\bar{a}-\bar{p}) - \bar{p} \geq 0$, so if we seek the q for which $\bar{\pi}_x(\bar{K};\bar{a}-q) = q$ then by (b) we shall have $q \geq \bar{p}$. But this q is $\pi_x(\bar{K};\bar{a})$. □

What we now need is to find conditions on the one-step $\bar{\pi}_{x,x+1}$ which allow the properties (a) and (b) to pass back down the tree by the Lego principle.

If the $\bar{\pi}_{x,x+1}$ are concave and monotone, then property (a) passes back through the tree.

If we have the property

$$\boxed{\forall h>0, \quad \bar{\pi}_{x,x+1}(k_x, k_{x+1}-h\,;\,a+h) > \bar{\pi}_{x,x+1}(k_x, k_{x+1}\,;\,a) - h}$$

then the property (b) also passes back through the tree.

However, concavity would imply that $a \mapsto \bar{\pi}_x(K;a)$ must be <u>increasing</u> (otherwise as we raise a, the price ultimately goes to $-\infty$) — could be !! This would automatically give property (b).

Monte Carlo approach to utility-maximization etc. (8/5/05)

1) Let's consider some discrete-time controlled Markov decision process, with finite horizon and objective

$$E\left[\sum_{j=0}^{T-1} f_j(X_j, a_j) + F(X_T)\right]$$

where the actions (a_j) are chosen, and the transitions are given by some density $\varphi(X_{j+1}, X_j, a_j)$ with respect to some reference Markov transition structure. Thus the objective can be expressed as

$$E^*\left[\sum_{j=0}^{T-1} \Lambda_j(a) f_j(X_j, a_j) + \Lambda_T(a) F(X_T)\right]$$

where $\Lambda_j(a) \equiv \prod_{r=0}^{j-1} \varphi(X_{r+1}, X_r, a_r)$. The problem is to obtain

$$V_0(x_0) = \sup_{a\in\mathcal{A}} E^*\left[\sum_{j=0}^{T-1} \Lambda_j(a) f_j(X_j, a_j) + \Lambda_T(a) F(X_T)\right]$$

where $\mathcal{A}$ is the class of adapted controls, and x_0 is some given start value. Now for any adapted process a, it is clear that

$$E^*\left[\sum_{j=0}^{T-1} \Lambda_j(a) f_j(X_j, a_j) + \Lambda_T(a) F(X_T)\right]$$

$$= E^*\left[\sum_{j=0}^{T-1} \Lambda_j(a)\left(f_j(X_j, a_j) + \Delta M_{j+1}\right) + \Lambda_T(a) F(X_T)\right] \quad \text{for any martingale } M$$

$$= E^*\left[\sum_{j=0}^{T-1} \Lambda_j(a)\left\{f_j(X_j, a_j) - h_{j+1}(X_{j+1})\varphi(X_{j+1}, X_j, a_j) + E_j^*\big(h_{j+1}(X_{j+1})\varphi(X_{j+1}, X_j, a_j)\big)\right\} + \Lambda_T(a) F(X_T)\right]$$

for any functions (h_j). Thus

$$V_0(X_0) = \sup_{a\in\mathcal{A}} E^*\left[\sum_{j=0}^{T-1} \Lambda_j(a)\left\{f_j(X_j, a_j) - h_{j+1}(X_{j+1})\varphi(X_{j+1}, X_j, a_j) + E_j^*\big(h_{j+1}(X_{j+1})\varphi(X_{j+1}, X_j, a_j)\big)\right\} + \Lambda_T(a) F(X_T)\right]$$

$$\leq E^*\left[\sup_{(a_j)}\left\{\sum_{j=0}^{T-1} \Lambda_j(a)\Big(f_j(X_j, a_j) - h_{j+1}(X_{j+1})\varphi(X_{j+1}, X_j, a_j) + E_j^*\big[h_{j+1}(X_{j+1})\varphi(X_{j+1}, X_j, a_j)\big]\Big) + \Lambda_T(a) F(X_T)\right\}\right]$$

So if we take the infimum over possible sequences (h_j), we achieve an upper bound for the value. However, let's see what the Bellman equation tells us;

we have

$$V_j(X_j) \geq f_j(X_j, a_j) + E_j^*\Big[V_{j+1}(X_{j+1})\,\varphi(X_{j+1}, X_j, a_j)\Big]$$

for all a_j. So what we obviously should try is to use $h_j = V_j$, and we then get

$$\sum_{j=0}^{T-1} \Lambda_j(a)\Big\{f_j(X_j, a_j) - V_{j+1}(X_{j+1})\,\varphi(X_{j+1}, X_j, a_j) + E_j^*\Big[V_{j+1}(X_{j+1})\,\varphi(X_{j+1}, X_j, a_j)\Big]\Big\}$$
$$+ \Lambda_T(a)\,F(X_T)$$

$$\leq \sum_{j=0}^{T-1} \Lambda_j(a)\Big\{V_j(X_j) - V_{j+1}(X_{j+1})\,\varphi(X_{j+1}, X_j, a_j)\Big\} + \Lambda_T(a)\,F(X_T)$$

$$= V_0(X_0)$$

using the fact that $V_T = F$. So in fact the infimum is attained, and

$$\boxed{V(X_0) = \min_{(h_j)} E^*\Big[\sup_{(a_j)}\Big\{\sum_{j=0}^{T-1}\Lambda_j(a)\Big\{f_j(X_j, a_j) - h_{j+1}(X_{j+1})\,\varphi(X_{j+1}, X_j, a_j) + E_j^*\big(h_{j+1}(X_{j+1})\,\varphi(X_{j+1}, X_j, a_j)\big)\Big\} + \Lambda_T(a)\,F(X_T)\Big\}\Big]}$$

[Of course, we can make the minimum over arbitrary martingale-difference sequences just as well, but it's important to allow the martingale differences chosen depend on (a_j).]

As with the MC valuation of American options, this allows us to do an optimisation pathwise.

2) <u>How should this be handled numerically?</u> Once we've decided what sequence (h_j) we plan to use, the recipe is quite easy to describe; we generate repeatedly sample paths of X (using the law P^*), and then compute the sequence (a_j^*) which is optimal for that path just by working back through the tree. In more detail, we find a_{T-1}^* by maximising

$$\Big\{F(X_T) - h_T(X_T)\Big\}\,\varphi(X_T, X_{T-1}, a_{T-1}) + E_{T-1}^*\Big[h_T(X_T)\,\varphi(X_T, X_{T-1}, a_{T-1})\Big] + f(X_{T-1}, a_{T-1})$$

— so if we make the natural assumption that $\boxed{h_T = F}$ then we choose a_{T-1} to maximize this.

Working back, if we've got optimal $a_{m+1}^*, a_{m+2}^*, \dots, a_{T-1}^*$, the terms involving a_m are (after cancelling out $\Lambda_m(a)$)

$$f(X_m, a_m) - h_{m+1}(X_{m+1})\,\varphi(X_{m+1}, X_m, a_m) + E_m^*\Big[h_{m+1}(X_{m+1})\,\varphi(X_{m+1}, X_m, a_m)\Big]$$

$$+\,\varphi(X_{m+1}, X_m, a_m)\Big[\sum_{j=m+1}^{T-1}\prod_{r=m+1}^{j-1}\varphi(X_{r+1}, X_r, a_r^*)\Big\{f_j(X_j, a_j^*) - h_{j+1}(X_{j+1})\,\varphi(X_{j+1}, X_j, a_j^*)$$

$$+ E_j^*\Big(h_{j+1}(X_{j+1})\, \varphi(X_{j+1}, X_j, a_j^*)\Big) \Big\} \quad + \quad \prod_{r=m+1}^{T-1} \varphi(X_{r+1}, X_r, a_r^*)\; F(X_T) \Big].$$

The point is the everything in the square bracket is already known, so the optimisation over a_m is <u>comparatively</u> straightforward... but the questions which remain are

(i) How do we choose the (h_j)?

(ii) How do we compute the conditional expectations $E_j^*\big[h_{j+1}(X_{j+1})\,\varphi(X_{j+1}, X_j, a_j)\big]$?

(iii) How do we do the individual pathwise optimisations?

Interesting questions

1) Can we do credit modelling <u>without</u> discontinuous processes? Make the events happen because of different information effects? Or like William Perraudin do some randomized Nash eq^{m}? Modelling fundamentals gives handle on possible 'correlation' of defaults. If we take the view that default holds when the cash held by the firm hits zero, then you can have default at a time when the firm is still worth quite a lot. Could try to model cash x_t at time t by

$$dx_t = r x_t\,dt + \theta_t\,dy_t - bK_t\,dt - p_t\,dK_t + d\Delta_t - (r+s)\Delta_t\,dt \qquad 0 \le \theta_t \le K_t$$

where K_t is the size of the enterprise at time t, θ_t the intensity of production, y_t the revenue process from production, p_t the cost of adding a new unit of production, Δ_t level of debt, s the spread.

2) Alexander Schied looks at the problem

$$\sup_{H} \inf_{Q \in \mathcal{Q}} E_Q\, U\left(x_0 + \int_0^T H\,dS\right), \qquad (*)$$

where $\mathcal{Q}$ is some suitable family of probabilities. By Minimax, this is $\inf_{\mathcal{Q}} \sup_{H} E_Q\, U(x_0 + \int_0^T H\,dS)$, so by duality this becomes $\inf_{\mathcal{Q}} \inf_{y} \inf_{Y \in \mathcal{Y}(y)} E\left[V(Y) + x_0 y\right]$, and the inf can be taken in any order. Can we see $(*)$ as a sup of ______ the objective over some restricted class of H?

3) Talking to Aytac Ilhan about equilibrium pricing in the incomplete setting, we were exploring the question whether introducing new assets affects the equilibrium prices already present ... something I've wondered about in the past but never completely satisfied myself over.

4) Ales Cerny asks how you would choose investment strategy H so as to (for example)

$$\max E\, U(Y_T), \quad \text{where } Y_T = \int_0^T H_u\,dS_u \Big/ \left\{\int_0^T H_u^2\, d\langle S\rangle_u\right\}^{1/2} \quad \text{– realised Sharpe Ratio.}$$