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A pretty inequality (14/5/90)

This is a nice inequality which Art Pittenger showed me. If $X \geq 0$ is square integrable, $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies $\varphi(0) = 0$ (wlog) and

$$\psi(x) \equiv \varphi(x)/x \quad \text{is convex,}$$

then

$$E\,\varphi(X) \geq \mu\varphi(a)/a \qquad \text{where } a \equiv EX^2/\mu, \ \mu = EX.$$

The proof is v. simple. If Y has dist^n $\mu^{-1} x F(dx)$ (F is law of X) then

$$E\,\psi(Y) = \mu^{-1} E\varphi(X) \geq \psi(EY) = \psi(a) = \varphi(a)/a.$$

In particular, for $1 \leq p \leq 2$, we have for $X \geq 0$

$$(EX)^p \leq EX^p \leq (EX^2)^{p-1}(EX)^{2-p}$$

Infinite divisibility of squares of Gaussian fields (14/5/90).

Steve Evans asks this question; if $X \equiv (X_1, \dots, X_n)^T \sim N(0,V)$, when can one say that $(X_1^2, X_2^2, \dots, X_n^2)$ is infinitely divisible?

Easy calculus gives us that

$$E\exp\{-\tfrac{1}{2}\sum_1^n \lambda_i X_i^2\} = \det(I + \Lambda V)^{-\frac{1}{2}},$$

where $\lambda_i \geq 0$ for all i, $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. We have infinite divisibility iff

$$-\tfrac{1}{2}\log\det(I + \Lambda V) = -\int(1 - e^{-\lambda \cdot x})\,\mu(dx) \qquad (*)$$

for some measure μ on $(\mathbb{R}^+)^n$. Now since $\frac{\partial}{\partial a_{ij}}\log\det A = a^{ji}$, we have that

$$\frac{\partial}{\partial\lambda_k}\log\det(I + \Lambda V) = C_{kk} \equiv C_{kk}(\lambda),$$

where $C \equiv C(\lambda) = V(I + \Lambda V)^{-1}$. Using $\partial a^{ij}/\partial a_{rs} = -a^{ir}a^{sj}$, we have that.

$$\frac{\partial}{\partial\lambda_k} c_{pq} = -c_{pk}c_{kq}.$$

Now we can re express $(*)$ in terms of the totally monotone criterion; if $f(\lambda) = \log\det(I + \Lambda V)$ then we have infinite div. iff for all n, for all $i_1, \dots, i_n$,

$$(-1)^{n-1}\frac{\partial^n}{\partial\lambda_{i_1}\cdots\partial\lambda_{i_n}} f(\lambda) \geq 0.$$

Certainly we have $\frac{\partial}{\partial\lambda_k} f = c_{kk} \geq 0$, because C is non-negative-definite; $C = (V^{-1} + \Lambda)^{-1}$.

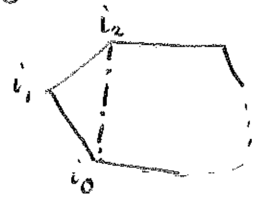
There is no trouble with the second-order derivatives:

$$\frac{\partial^2 f}{\partial \lambda_i \partial \lambda_j} = -c_{ij}c_{ji} = -c_{ij}^2 \le 0,$$

and higher-order derivatives will behave correctly if and only if the following criterion is satisfied. Consider any i, j, k; the condition is that for all $\lambda_1, \dots, \lambda_n$

$$\boxed{c_{ij}(\lambda)\, c_{jk}(\lambda)\, c_{ki}(\lambda) \ge 0 .}$$

Call this the "3-cycle condition". If this condition holds, then for any k-cycle $i_0, i_1, \dots, i_k = i_0$, then



$$c_{i_0 i_1} c_{i_1 i_2} \cdots c_{i_{k-1} i_k} \ge 0,$$

by induction on the length k of the cycle. Indeed, by replacing $c_{i_0 i_1} c_{i_1 i_2}$ by $c_{i_0 i_2} = c_{i_0 i_1} c_{i_1 i_2} c_{i_2 i_0} / c_{i_0 i_1} c_{i_1 i_2}$ we do not change the sign.

Hence when we differentiate c_{pq} k times, we get a sum of products round $(k+1)$ cycles, but multiplied by $(-1)^k$. Thus the 3-cycle condition implies that the law of $(X_1^2, \dots, X_n^2)$ is infinitely divisible, and conversely.

Notice that, since $C(\lambda) \to V$ as $\lambda \downarrow 0$, the 3-cycle condition for each $C(\lambda)$ implies the 3-cycle condition for V; thus a necessary condition for infinite divisibility is

$$V_{ij} V_{jk} V_{ki} \ge 0 \quad \forall i, j, k.$$

It's not hard to see that this is equivalent to the condition that there exists diagonal S, $S_{ii} = \pm 1$, such that, if $\tilde{V} \equiv S^T V S$, then $\tilde{v}_{ij} \ge 0 \ \forall i, j$.

<u>So let's assume infinite divisibility, and that $v_{ij} \ge 0$ for all</u> i, j. Now if we consider the (12) entry of $C(\lambda)$, we see a partitioning of $C(\lambda)$

$$C(\lambda) = \left(\begin{array}{c|c} A & B \\ \hline B^T & D \end{array}\right)^{-1}, \qquad A = \begin{pmatrix} \tilde{C}_{11}(\lambda) & \tilde{C}_{12}(\lambda) \\ \tilde{C}_{21}(\lambda) & \tilde{C}_{22}(\lambda) \end{pmatrix} \equiv \begin{pmatrix} v^{11} + \lambda_1 & v^{12} \\ v^{21} & v^{22} + \lambda_2 \end{pmatrix}$$

And so

$$\begin{pmatrix} c_{11}(\lambda) & c_{12}(\lambda) \\ c_{21}(\lambda) & c_{22}(\lambda) \end{pmatrix} = (A - BD^{-1}B^T)^{-1} \qquad BD^{-1}B^T = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$$

$$= \begin{pmatrix} v^{11} + \lambda_1 - \alpha & v^{12} - \beta \\ v^{21} - \beta & v^{22} + \lambda_2 - \gamma \end{pmatrix}^{-1}$$

$$= (\det)^{-1} \begin{pmatrix} v^{22} + \lambda_2 - \gamma & \beta - v^{12} \\ \beta - v^{21} & v^{11} + \lambda_1 - \alpha \end{pmatrix}$$

and letting $\lambda_3, \dots, \lambda_n \uparrow \infty$, we get $D^{-1} \to 0$ and $c_{12}(\lambda) \to -v^{12}$. But now if we let $\lambda_3 = \lambda_4 = \dots = \lambda_n = \lambda$, with λ_1, λ_2 fixed, we get

$$\frac{\partial}{\partial \lambda}\left(c_{12}(\lambda)^2\right) = \sum_{k=3}^{n} \frac{\partial}{\partial \lambda_k}\left(c_{12}(\lambda)^2\right) = 2 \sum_{k=3}^{n} -c_{12}(\lambda) c_{2k}(\lambda) c_{k1}(\lambda)$$
$$\leq 0$$

by the 3-cycle condition; hence, <u>once</u> $c_{12}(\lambda)$ <u>hits</u> 0, <u>it must remain there!</u> Since $c_{12}(\lambda_1, \lambda_2, 0, \dots, 0)$ is non-negative (because when we consider

$\dfrac{\partial c_{12}}{\partial \lambda_1} = -c_{12} c_{11}$, we can solve explicitly for c_{12} and see that $c_{12}(\lambda_1, \lambda_2, 0 \dots 0)$ is nonneg for all λ_1, for λ_2 fixed; and then repeat argument for λ_2, and use the fact that $c_{12}(0) = v_{12} \geq 0$ by hypothesis) we conclude that $c_{ij}(\lambda) \geq 0 \ \forall \lambda$, and

$$\boxed{-v^{ij} \geq 0 \quad \text{for all } i \neq j.}$$

It is tempting to guess that $-V^{-1}$ must be a Q-matrix – after all, the diagonal terms are nonpositive. Consider the example

$$V^{-1} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 3\varepsilon+1 & \varepsilon\sqrt{3} - \sqrt{3} \\ \varepsilon\sqrt{3} - \sqrt{3} & \varepsilon + 3 \end{pmatrix}.$$

This certainly <u>is</u> pds, the 3-cycle condition for $C(\lambda)$ is trivially satisfied, and so the infinite divisibility of (X_1^2, X_2^2) is assured, but for small $\varepsilon > 0$ the top row sum of V^{-1} is negative, the bottom row sum is positive, so $-V^{-1}$ is <u>not</u> a Q matrix

Nonetheless, $-V^{-1} - \theta I$ will be a Q-matrix for large enough θ.

The converse is also true; if $-V^{-1} = Q + \varepsilon I$ for some $\varepsilon > 0$, we can represent $C = (V^{-1} + \Lambda)^{-1} = (\Lambda - Q - \varepsilon I)^{-1}$ as

$$c_{ij}(\lambda) = c_{ij} = E^i \left[\int_0^\infty I_{\{j\}}(X_s) \exp\left(-\int_0^s f(X_u)\, du\right) ds \right],$$

where $f(j) = \lambda_j - \varepsilon$, and X is a Mkv chain with Q-matrix Q.

SDE for BM conditioned by value of $\int_0^1 B_s\,ds$ (28/5/90)

For BM B in $\mathbb{R}^d$, let $Y_t \equiv \int_0^t B_s\,ds$. We have $Y_1 = Y_t + (1-t)B_t + \int_t^1 (B_s - B_t)\,ds$, so $(Y_1 / \mathcal{F}_t) \overset{\mathcal{D}}{=} N(Y_t + (1-t)B_t, \frac{1}{3}(1-t)^3 I)$. Thus conditioning on $Y_1 = \eta$ gives us the harmonic function $h(t,x,y) = p(\frac{1}{3}(1-t)^3; \eta - y - (1-t)x)$, and if we h-transform using this we get drift $\nabla h / h$. Explicitly,

$$\begin{cases} dX_t = dW_t + \dfrac{3}{\cancel{2}(1-t)^2}\left(\eta - Y_t - (1-t)X_t\right)dt, & X_0 = 0 \\ dY_t = X_t\,dt. \end{cases}$$

If we now write $\tilde{X}_t \equiv X_t - \eta$ for BM centred about its centroid, $\tilde{Y}_t \equiv \int_0^t \tilde{X}_s\,ds$, then the SDE becomes

$$\boxed{\begin{aligned} d\tilde{X}_t &= dW_t - \frac{3}{(1-t)^2}\left\{\tilde{Y}_t + (1-t)\tilde{X}_t\right\}dt, \quad X_0 = -\eta \\ d\tilde{Y}_t &= \tilde{X}_t\,dt, \quad Y_0 = 0. \end{aligned}}$$

[Trying to do similar for exponentially-killed piece of BM doesn't work.]

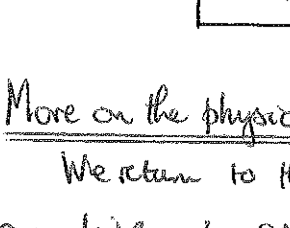
More on the physicists' self-repellent BM. (2/6/90)

We return to the situation where Wiener measure on $C([0,T])$ is weighted by a multiple of $\exp\{-\int_0^T ds \int_0^T du\, \psi(x_s - x_u)\}$. Here, we assume ψ is non-negative, even, decreasing in $\mathbb{R}^+$, together with any other properties which will make it easier.

Define

$$J(x) \equiv \int_0^T \dot{x}_s^2\,ds + \int_0^T ds \int_0^T du\, \psi(x_s - x_u).$$

It looks like we shall get a lot of insight from the form of the paths minimising J. Previously, I argued that an up-down-up segment of path could be straightened into an increasing piece of path without increasing J. Now notice that a path like:



can be replaced by the (shifted) path which starts with the dotted reflected initial piece and then continues as before — so the $\int \dot{x}^2$ is unchanged but $\iint \psi(x_s - x_u)$ is decreased.

Thus we may wlog restrict the search to increasing paths x.

Let $\ell_y \equiv 1/\dot{x}(x_y^{-1})$ be the local time at level y. We can express this otherwise by saying that ℓ is the derivative of the inverse function to x. In terms of this, if $x_T = \xi$, we can express J as

$$\boxed{J(x) = \int_0^{\xi} \ell_y^{-1}\,dy + \int_0^{\xi} dx \int_0^{\xi} dy\, \ell_x \ell_y\, \psi(x-y).}$$

Now we could consider minimising this subject to the constraint $\int_0^{\xi} l_y\,dy = T$, but it makes life easier to do an unconstrained minimum, and then work out what T was from the solution.

A simple variational argument shows that for the optimum,

$$\boxed{\frac{1}{l_a^2} = 2\int_0^{\xi} \psi(a-y)\, l_y\, dy}$$

which implies that $\int_0^{\xi} \frac{da}{l_a} = 2\int_0^{\xi} dx \int_0^{\xi} dy\ \psi(x-y)\, l_x\, l_y$ for optimum.

Trying to solve explicitly looks completely hopeless.

Beware: if one fixes ξ and heuristically optimises over $l_y \equiv \bar{l}$, then the best slope is <u>not</u> the same as if one fixes T in $J(\cdot)$ and optimises over $x_s = cs$!

Beware also: paper by Kusuoka in the volume "Infinite dimensional analysis and Stochastic processes" ed. S. Albeverio, Pitman Lecture Notes vol 124, Boston 85, gives an expression for the limiting drift which looks different.

<u>Integral of BM(S^2) as a model for a polymer?</u> (17/6/90)

Let Y_t be BM(S^{d-1}), and let $X_t \equiv \int_0^t Y_s\,ds$. Then X is a C^1 curve, and a C^1 curve in the plane has zero Lebesgue measure (Use the inverse function thm to make the curve locally a straight line). Let $v \in S^{d-1}$ be the starting point of Y, and consider the projection γ_t onto the plane corresponding to the first two coordinates. Take $d=3$ to avoid spurious generality. Then

$$\int dy\ I_{\{\gamma_t = y \text{ for some } t \le n\}} = 0 \quad P_v\text{-a.s.},$$

by the fact that γ has zero Lebesgue measure. Hence

$$P_v(\gamma_t = y \text{ for some } t \le n) = 0 \quad \text{for a.e. } y$$

But now the pair (Y_t, X_t) has a density wrt Lebesgue measure, so

$$P_v(\gamma_t = y \text{ for some } \underset{t\ge\varepsilon}{t \le n}) = \int P_v(Y_\varepsilon \in du, X_\varepsilon \in dx)\ P_u(\gamma_t + \pi x = y \text{ for some } t \le n-\varepsilon)$$

where πx is the 2d projection of x;

$$= 0 \quad \text{for } \underline{\text{all}}\ y$$

using the density. Hence for each v

$$P_v(\gamma_t = y \text{ for some } 0<t<\infty) = 0.$$

Joint exponent: $\varphi(\xi, \lambda) = \varphi(1,0) \exp \int_0^\infty dt \int_0^\infty \left(e^{-t} - e^{-\xi t - \lambda x}\right) P(X_t \in dx)$

So that's good – the path is non-self-intersecting. However, we have

$$dY_t = (I - Y_t Y_t^T)\, dW_t - \frac{n-1}{2} Y_t\, dt$$

so that $e^{(n-1)t/2} Y_t$ is a martingale, and from this can easily compute

$$E|X_t|^2 = \frac{8}{(n-1)^2}\left\{\frac{(n-1)t}{2} - 1 + e^{-(n-1)t/2}\right\}$$

– so the end-to-end distance grows like for Brownian motion, and the physicists won't buy it...

__Invariant σ-field of two independent processes__ (18/6/90)

Let $(\Omega_i, \mathcal{F}^i, P^i)$ be equipped with shifts $(\theta_t^i)_{t\geq 0}$, $i=1,2$; let $\mathcal{I}_j$ be inv. σ-field for the jth probability space. Let $\Omega = \Omega_1 \times \Omega_2$, $\mathcal{F} = \mathcal{F}^1 \otimes \mathcal{F}^2$, $P = P^1 \times P^2$ and let

$$\theta_t(\omega_1, \omega_2) \equiv (\theta_t^1 \omega_1, \theta_t^2 \omega_2)$$

This defines a semigp of shifts on $(\Omega, \mathcal{F}, P)$, and hence an invariant σ-field $\mathcal{I}$. Evidently $\mathcal{I} \supseteq \mathcal{I}_1 \vee \mathcal{I}_2$, but do we have equality, in view of independence?

The answer is "No". Take the case where each $(\Omega_i, \mathcal{F}^i, P^i)$ is an indpt copy of the canonical space of

$$dX_t = \sigma(X_t)\, dB_t + dt \qquad, X_0 = 0,$$

where $\sigma \in C_K^\infty$. Now the invariant σ-field is trivial (since this is a Mkv pr, the invariant events are characterised by bdd harmonic f^{ns}, and since $X_t \to \infty$ a.s. all such must be constant.) We have $X_t = t + Y$ for all large enough t. Thus we have $\mathcal{I}_1, \mathcal{I}_2$ trivial, yet $\mathcal{I}$ containing the non-trivial r.v. $\lim (X_t^1 - X_t^2)$.

__Skew-reflecting BM in $\mathbb{H}$__ (3/7/90)

DW suggests one tries the half-winding problem for BM in $\mathbb{H}$ with reflection constant on each of $\mathbb{R}^+$, $\mathbb{R}^-$. If one starts on $\mathbb{R}^-$, and looks at the process only when in $\mathbb{R}$, then we see a Cauchy process with constant drift c, at least until first entry into $\mathbb{R}^+$. Now if we time change this process by the inverse to local time at the max, we see a subordinator whose Laplace exponent is

$$\text{Const.} \exp \int_0^\infty dt \int_{[0,\infty)} P(X_t \in dx)\, \frac{e^{-t} - e^{-\lambda x}}{t}$$

(this is Fristedt's formula – see Greenwood+Pitman AAP 12), In this case, the integral is

$$\int_0^\infty dt \int_0^\infty \frac{t\,dx}{t^2+(x-ct)^2} \quad \frac{e^{-t}-e^{-\lambda x}}{\pi t}$$

$x = ty$

$$= \int_0^\infty dt \int_0^\infty \frac{dy}{1+(y-c)^2} \quad \frac{e^{-t}-e^{-\lambda y t}}{\pi t}$$

$$= \int_0^\infty \frac{dy}{1+(y-c)^2} \quad \frac{\log(\lambda y)}{\pi}$$

(Frullani!)

$$= \text{const} + \frac{\log \lambda}{\pi} \left\{ \frac{\pi}{2} + \tan^{-1}(c) \right\}.$$

Thus when we look at the ladder process, we see a stable subordinator index $\alpha = \left(\frac{\pi}{2} + \tan^{-1}(c)\right)/\pi$.

Now for a stable subordinator started at 0, the undershoot $\xi_- = \sup\{x \leq 1 : x \in R\}$ and overshoot $\xi_+ = \inf\{x \geq 1 : x \in R\}$ (where R is range of X) has joint density

$$P\left(\xi_- \in dt, \xi_+ \in du\right) = \frac{\alpha \sin \pi\alpha}{\pi} \quad t^{-(1-\alpha)} (u-t)^{-1-\alpha} \, du\,dt$$

(See Fristedt's article in Adv Prob 3, for example) Integrating out the t-variable gives

$$P\left(\xi_+ \in du\right)/du = \frac{\sin \pi\alpha}{\pi} \quad u^{-1}(u-1)^{-\alpha}$$

Thus if we start the process at -1, the first place it hits $\mathbb{R}^+$ has density

$$\frac{\sin \pi\alpha}{\pi} \quad (1+x)^{-1} x^{-\alpha}.$$

We could compute $E \log(\xi_+ - 1)$ by integration (if we could see how), but it seems much easier to transform by taking logs to $\{x+iy : x \in \mathbb{R}, 0<y<\pi\}$ and then work it out from there; the expected local time on one side of the strip before cross to the other is just $1/\pi$, so that's really all there is to it.

For the record, if one has $BM(\mathbb{R})$, H the first time hit $\pm a$, we get

$$E^0 e^{-\frac{1}{2}\theta^2 H - \gamma L(H)} = \theta\{\theta \cosh \theta a + \gamma \sinh \theta a\}^{-1}.$$

RBM again (7/7/90)

In the usual notation, the Green f^n G for the RBM satisfies

$$G(z,z_0) - G^{\partial}(z,z_0) = \iint \tilde{\mu}(dv)\, \mathrm{Im}\left(\frac{1}{v-z_0}\right) \frac{\mu(dx)}{x-v}\left[\tan^{-1}\left(\frac{x-a}{b}\right) - \tan^{-1}\left(\frac{v-a}{b}\right)\right]$$

$$= \int \tilde{\mu}(dv)\, \mathrm{Im}\left(\frac{1}{v-z_0}\right) \pi(z,v),$$

say, where

$$\pi(z,v) \equiv \int \frac{\mu(dx)}{x-v} \int_v^x \frac{b\,dy}{b^2+(y-a)^2}.$$

To understand the limit behaviour of $G(\cdot,\cdot)$ as $z\to\infty$, have to understand limit behaviour of π.
Notice that

$$\liminf_{|z|\to\infty} \frac{|z|^2}{b}\pi(z,v) \equiv \liminf \int \frac{\mu(dx)}{x-v}\int_v^x \left(\frac{|z|}{|z-y|}\right)^2 dy$$

$$\geq \int \frac{\mu(dx)}{x-v}\int_v^x dy = +\infty \qquad \left(\underline{\text{only if}}\ \mu(\mathbb{R})=+\infty\right)$$

whereas

$$\lim_{|z|\to\infty} \frac{|z|^2}{b}\int_{-N}^{N}\frac{\mu(dx)}{x-v}\int_v^x \frac{b\,dy}{b^2+(y-a)^2} = \mu[-N,N],$$

so that we can chop out any interval, and the asymptotics of π won't be altered; only the tails of μ matter. So for $\underline{\text{fixed}}$ v, can choose N so large that $\left|\frac{x}{x-v}\right| \in [1-\varepsilon, 1+\varepsilon]$ for all $|x|>N$, and hence $\boxed{\pi(z,v)/\pi(z,0) \to 1}$ $(|z|\to\infty)$

It's worth noticing that

$$G^{\partial}(z,z_0) = \mathrm{Const}\cdot \log\frac{|z-\bar{z}_0|}{|z-z_0|} \propto \log\left(1+\frac{4by_0}{|z-z_0|^2}\right) \leq \frac{4by_0}{|z-z_0|^2}$$

and so $G^{\partial}(z,z_0)/\pi(z,0) \to 0$ $(|z|\to\infty)$ — no real surprise, this, but perhaps worth recording.

On the maximum of a Branching Brownian Motion (9/7/90)

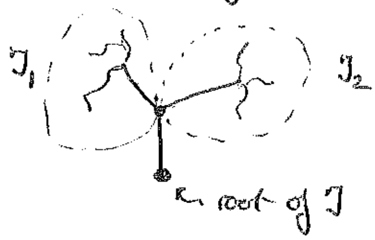
1) Consider a Brownian particle which branches at rate λ into two indept particles, and dies at rate $\mu \geq \lambda$. Until the first branching/death, we see an $\exp(\lambda+\mu)$ piece of Brownian path, whose max is $\exp(\theta)$, $\theta = \sqrt{2\lambda+2\mu}$. We can represent the history of the branching process as a ternary branching tree, and on each edge the Brownian increments are indept.

If there are k internal nodes (linked to 3 nbrs) and j external nodes, then the prob'y of this tree is $q^{j-1}p^k$ $\quad (p \equiv 1-q = \lambda/(\lambda+\mu))$. Notice also that if there are N vertices in the tree, then there are $N-1$ edges, and $2(\text{no of edges}) = 2N-2 = 3k+j$, $N = j+k \Rightarrow k = \frac{N}{2}-1$.

Where does the BBM take on its max value, + how big is it?

2) Take one of the above trees, or indeed any tree with valence ≤ 3 at each vertex, $\mathcal{T}$, say, and let $\varphi_{\mathcal{T}}(s) \equiv E\exp\{-sM(\mathcal{T})\}$, where $M(\mathcal{T})$ is the maximum value assumed on the tree. For this to make sense, must designate one of the external vertices of $\mathcal{T}$ as the root, set to be at level 0.

Now if we have the situation shown here, then



$$P[M(\mathcal{T})\geq x] = e^{-\theta x} + \int_0^x \theta e^{-\theta y}\,dy \int_0^\infty \theta e^{-\theta v}\,dv\; P[M(\mathcal{T}_1)\vee M(\mathcal{T}_2) \geq x-y+v]$$

so that

$$\int_0^\infty e^{-sx} P[M(\mathcal{T})\geq x]\,dx = \frac{1}{\theta+s} + \int_0^\infty \theta e^{-\theta y}\,dy \int_0^\infty \theta e^{-\theta v}\,dv \int_y^\infty e^{-sx}\,dx\; P[M(\mathcal{T}_1)\vee M(\mathcal{T}_2) \geq x-y+v]$$

$$\equiv \tilde{\varphi}_{\mathcal{T}}(s) = \frac{1}{\theta+s} + \frac{\theta}{\theta+s}\int_0^\infty \theta e^{-\theta v}\,dv \int_0^\infty e^{-ts}\,dt\; P[M(\mathcal{T}_1)\vee M(\mathcal{T}_2)\geq t+v]$$

$$= \frac{1}{\theta+s} + \frac{\theta^2}{\theta+s}\int_0^\infty \frac{e^{-\theta y}-e^{-sy}}{s-\theta}\, P[M(\mathcal{T}_1)\vee M(\mathcal{T}_2)\geq y]\,dy$$

$$= \frac{1}{\theta+s} + \frac{\theta^2}{s^2-\theta^2}\int_0^\infty (e^{-\theta y}-e^{-sy})\Big\{P(M(\mathcal{T}_1)\geq y) + P(M(\mathcal{T}_2)\geq y) - P(M(\mathcal{T}_1)\geq y)\,P(M(\mathcal{T}_2)\geq y)\Big\}\,dy$$

There are two special cases of interest, $\mathcal{T}_2 = \emptyset$ (when $P(M(\mathcal{T}_2)\geq y) = 0$), and $\mathcal{T}_2 =$ —•, when $P(M(\mathcal{T}_2)\geq y) = e^{-\theta y}$.

<u>Case 1</u> Here, the above identity simplifies to

$$\tilde{\varphi}_J(s) = (\theta+s)^{-1} + \frac{\theta^2}{s^2-\theta^2}\left\{ \tilde{\varphi}_{J_1}(\theta) - \tilde{\varphi}_{J_1}(s) \right\}.$$

<u>Case 2</u>. This time, the expression reduces to

$$\tilde{\varphi}_J(s) = \frac{1}{\theta+s} + \frac{\theta^2}{s^2-\theta^2}\left\{ \tilde{\varphi}_{J_1}(\theta) - \tilde{\varphi}_{J_1}(2\theta) - \tilde{\varphi}_{J_1}(s) + \tilde{\varphi}_{J_1}(s+\theta) + \frac{1}{2\theta} - \frac{1}{\theta+s} \right\}.$$

Now we have easily $\varphi_J(s) = 1 - s\tilde{\varphi}_J(s)$, so we can rephrase the identity for case 1 in terms of this.

$$\boxed{\theta\,\varphi_J(s) = \frac{\theta^2}{s^2-\theta^2}\left\{ s\,\varphi_{J_1}(\theta) - \theta\,\varphi_{J_1}(s) \right\}}$$

and the identity for case 2 becomes

$$\boxed{\frac{1}{s}\varphi_J(s) = \frac{\theta^2}{s^2-\theta^2}\left\{ \frac{\varphi_1(\theta)}{\theta} - \frac{\varphi_1(2\theta)}{2\theta} - \frac{\varphi_1(s)}{s} + \frac{\varphi_1(s+\theta)}{s+\theta} \right\}}.$$

3) If $f(x) = P^x$(no particle ever gets down to 0), $x \geq 0$, then we have that $0 \leq f \leq 1$, $f(0)=0$, f is increasing, and f solves

$$\frac{1}{2}f'' + (1-f)(\mu-\lambda f) = 0 \qquad \left(\frac{1}{2}f'' + cf' + (1-f)(\mu-\lambda f) = 0 \text{ in case of BM, drift } c \right)$$

If we set ψ to be inverse to f, then

$$\psi''(x) = -\psi'(x)^3 f''(\psi(x))$$
$$= 2\psi'(x)^3 (1-x)(\mu-\lambda x)$$

whence

$$D\left[-\frac{1}{2\psi'(x)^2} \right] = 2(1-x)(\mu-\lambda x)$$

implying

$$\frac{1}{2\psi'(x)^2} = 2\int_x^1 (1-t)(\mu-\lambda t)\,dt = 2\left\{ \mu(1-x) - \frac{\lambda+\mu}{2}(1-x^2) + \frac{\lambda}{3}(1-x^3) \right\}$$

so that we get the explicit solution

$$\boxed{\psi(x) = \frac{1}{2}\int_0^x \left[\mu(1-t) - \frac{\lambda+\mu}{2}(1-t^2) + \frac{\lambda}{3}(1-t^3) \right]^{-\frac{1}{2}} dt}.$$

Try to mix over $\gamma > 0$ on each side according to $\nu_\theta(d\gamma)$ so as to get $\int \mu(d\gamma)\, e^{-\theta\gamma}$ on LHS?

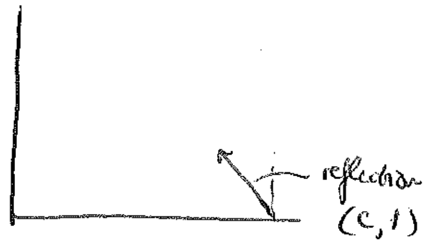
This would require $\forall \gamma > 0$

$$e^{-\theta\gamma} = c\gamma \operatorname{Re} \hat{\nu}_\theta(\gamma) - \sqrt{\gamma^2 + 2\lambda}\, \operatorname{Im} \hat{\nu}_\theta(\gamma)$$

and for this would like to build analytic f^n on $\mathbb{H}$ with argument $\tan^{-1}\left(\frac{\sqrt{\gamma^2+2\lambda}}{c\gamma} \right)$ at $\gamma > 0$. This seems hard to do explicitly.

Another route to RBM$((\mathbb{R}^+)^3)$? (9/7/90)

Let's consider RBM in $(\mathbb{R}^+)^2$ run until it hits the imaginary axis, at time τ, say. If we could get the joint law of (τ, Y_τ) then we could work out what had been happening to a third component meantime, and, in particular, we could work out by reflection principle the law of (Y_τ, Z_τ) on the event that the plane $z=0$ had not been hit by time τ.



We seek φ to solve

$$\tfrac{1}{2}\Delta\varphi - \lambda\varphi = 0, \qquad \varphi(iy) = e^{-\alpha y}$$

$$v\cdot\nabla\varphi = 0 \text{ on } \mathbb{R}, \quad v^T \equiv (c,1), \qquad 0 \le \varphi \le 1,$$

and then $\varphi(z) \equiv E^z\left[e^{-\alpha Y_\tau - \lambda\tau}\right]$.

Fundamental sol^{ns} of the pde are given by $\sin\gamma x\,.\,e^{-\sqrt{\gamma^2+2\lambda}\,y}$, $\cos\gamma x\,.\,e^{-\sqrt{\gamma^2+2\lambda}\,y}$ so we try to form φ as a mixture of these. Bc on $i\mathbb{R}^+$ suggests we want

$$\varphi(x+iy) = \cancel{e^{-\alpha y}} + \int_0^\infty \mu(d\gamma)\,\sin\gamma x\,.\,e^{-\sqrt{\gamma^2+2\lambda}\,y} + e^{-\alpha y}\cos x\sqrt{\alpha^2-2\lambda}$$

assuming $\alpha^2 > 2\lambda$. The boundary condition on $\mathbb{R}^+$ would now say

$$0 = c\left(\int_0^\infty \mu(d\gamma)\,\gamma\cos\gamma x - \sqrt{\alpha^2-2\lambda}\,\sin x\sqrt{\alpha^2-2\lambda}\right) - \left(\alpha\cos x\sqrt{\alpha^2-2\lambda} + \int_0^\infty \mu(d\gamma)\sqrt{\gamma^2+2\lambda}\,\sin\gamma x\right)$$

or again

$$\boxed{\int \mu(d\gamma)\left\{c\gamma\cos\gamma x - \sqrt{\gamma^2+2\lambda}\,\sin\gamma x\right\} = c\sqrt{\alpha^2-2\lambda}\,\sin x\sqrt{\alpha^2-2\lambda} + \alpha\cos x\sqrt{\alpha^2-2\lambda}} \qquad (x>0).$$

Can one obtain μ from this?

More on Branching Brownian Motion (9/7/90)

The integral on p.10 can be evaluated more explicitly. We have

$$2\psi(x) = \int_0^x \frac{dt}{\sqrt{1-t}}\left\{\mu - \frac{\lambda+\mu}{2}(1+t) + \frac{\lambda}{3}(1+t+t^2)\right\}^{-\frac{1}{2}} \qquad 1-t = y^2$$

$$= \int_{\sqrt{1-x}}^1 2dy\left\{\mu - \frac{\lambda+\mu}{2}(2-y^2) + \frac{\lambda}{3}(2-y^2+1-2y^2+y^4)\right\}^{-\frac{1}{2}}$$

$$= \int_{\sqrt{1-x}}^1 2dy\left\{\frac{\lambda y^4}{3} + \frac{\mu-\lambda}{2}y^2\right\}^{-\frac{1}{2}}$$

$$= \int_{\sqrt{1-x}}^{1} \frac{2\,dy}{y} \left(\frac{\lambda y^2}{3} + \frac{\mu-\lambda}{2} \right)^{-\frac{1}{2}} \qquad a^2 \equiv 3(\mu-\lambda)/2\lambda$$

$$= \sqrt{\frac{3}{\lambda}} \cdot 2 \cdot \int_{\sqrt{1-x}}^{1} \frac{dy}{y\sqrt{y^2+a^2}}$$

$$= 2\sqrt{\frac{3}{\lambda}}\ \frac{1}{a} \left[\log y - \log\left(a + \sqrt{a^2+y^2}\right) \right]_{\sqrt{1-x}}^{1}$$

$$= \frac{2}{a}\sqrt{\frac{3}{\lambda}} \left\{ -\tfrac{1}{2}\log(1-x) - \log\left(a+\sqrt{1+a^2}\right) + \log\left(a + \sqrt{1-x+a^2}\right) \right\}.$$

Thus if we set

$$\eta \equiv \left(a+\sqrt{1+a^2}\right) \exp\left(a\psi\sqrt{\frac{\lambda}{3}}\right), \quad \text{we have}$$

$$\eta = \frac{a+\sqrt{1-x+a^2}}{\sqrt{1-x}} \qquad 1-x = t^2$$

$$\Rightarrow \quad \eta t - a = \sqrt{t^2+a^2}$$

$$\Rightarrow \eta^2 t^2 - 2a\eta t + a^2 = t^2 + a^2 \Rightarrow t = 2a\eta/(\eta^2-1) \Rightarrow \boxed{x = 1 - \frac{4a^2\eta^2}{(\eta^2-1)^2}}$$

A special case of interest is the critical branching case $\mu=\lambda$, $a=0$. This gives that the function f of p.10 is

$$\boxed{f(t) = 1 - \left(1 + \sqrt{\frac{\lambda}{3}}\, t\right)^{-2}} \qquad \text{(This must be well known, I think).}$$

Combining with the results on p.37 of previous book, this gives us that the particles diffuse like BM with drift $(c \equiv \sqrt{\lambda\beta})$

$$\frac{2}{x(1+cx)(1+2cx)}$$

killed at rate $\lambda/f(x)$, splitting at rate $\lambda f(x)$.

Generally, $f'(0) = 2\sqrt{\frac{\lambda}{3}} \cdot \left(\frac{3\mu-\lambda}{2\lambda}\right)^{\frac{1}{2}} = 2\left(\frac{3\mu-\lambda}{6}\right)^{\frac{1}{2}}$,

$$f(x) = 1 - \frac{4a^2\left(a+\sqrt{1+a^2}\right)^2 e^{2acx}}{\left(\left(a+\sqrt{1+a^2}\right)^2 e^{2acx} - 1\right)^2}$$

More on the 2-dimensional RBM (12/7/90).

1) We want to analyse the kernel $\tau(z,v)$ which appears in the integrals for the Green function, and so on. We know that $\tau(z,v)/\tau(z,0) \to 1$ as $z \to \infty$ for each v, but would like to know when this convergence is bounded, so that we can use dominated convergence.

We have

$$\tau(z,v) \equiv \int \frac{\mu(dx)}{x-v} \int_v^x \frac{b\,dy}{b^2+(y-a)^2} \leq \int \mu(dx) \frac{c}{b+|x-a|+|v-a|}$$

so we shall have the bounded convergence if we can control $\int \mu(dx) \{b+|x-a|\}^{-1}$.

[Indeed,

$$\tau(z,a) = \int \frac{\mu(dx)}{x-a} \int_0^{x-a} \frac{b\,dt}{b^2+t^2} = \int \frac{\mu(dx)}{b} \frac{b}{x-a} \int_0^{(x-a)/b} \frac{dt}{1+t^2},$$

and we can prove easily that $\forall\, x>0$

$$\frac{1}{1+x} \leq \frac{1}{x}\int_0^x \frac{dt}{1+t^2} \leq \frac{2}{1+x}$$

which shows that it's also necessary to control $\int \mu(dx) \{b+|x-a|\}^{-1}$]

Now if we allow tangential approach to ∞, can have $\tau(z,v)/\tau(z,0)$ unbounded, but if we restrict to nontangential, it's ok, because if $\lambda \equiv a/b$ remains bdd, we have

$$\tau(z,0) = \int \frac{\mu(dx)}{x} \int_0^{x/b} \frac{dy}{1+(y-\lambda)^2}$$

and since λ is bdd, $\exists\ \varepsilon>0$ s.t. $\frac{1}{x}\int_0^x \frac{dy}{1+(y-\lambda)^2} \geq \frac{\varepsilon}{1+x}$ $\forall\, x>0$, and then for x real. Hence

$$\tau(z,0) \geq \varepsilon \int \mu(dx) \frac{1}{b+|x|}$$

and $\frac{|b+|x-a||}{b+|x|}$ stays bounded below if λ is bdd, which gives

$$\tau(z,0) \geq \varepsilon\, \tau(z,v) \quad \forall v.$$

2) Now in the situation we've considering at the moment, we chose the b.v.'s so that $\theta(\tfrac{1}{x}) = -\theta(x)$, to enable us to use the reflection principle to calculate the Green f^n of the process in $\mathbb{H}$ killed when it hits unit circle [see p. 52 of previous notebook]. We also have that the exit kernel at a point z_0 of the body is the normal derivative

Note also that

$$\tau(\bar{z}^{-1}, v) = \int \frac{\mu(dy)}{yv(v^{-1}-y)} \int_{y}^{v^{-1}} \mathrm{Im}\left(\frac{1}{t-z}\right) dt$$

so that

$$\tau(\bar{z}^{-1}, v) - \tau(z, v^{-1}) = \int \frac{\mu(dy)}{y} \int_{y}^{v^{-1}} \mathrm{Im}\left(\frac{1}{t-z}\right) dt$$

$$= \int \frac{\mu(dy)}{y} \int_{y^{-1}}^{v^{-1}} \mathrm{Im}\left(\frac{1}{t-z}\right) dt$$

[Beware! This expression is no longer correct; it doesn't go to zero as $z \to 0$.]

(wrto the 2nd variable) of the Dirichlet Green's f^{n}.

As before, we have expression for G (without killing on unit circle):

$$\pi G(z,z_0) = \underbrace{\log\left|\frac{z-\bar{z}_0}{z-z_0}\right|}_{\equiv \pi G^{\partial}(z,z_0)} + \int \tilde{\mu}(dv) \int \mu(dx)\ \mathrm{Im}\left(\frac{1}{v-z_0}\right) \frac{1}{x-v} \int_v^x dy\ \mathrm{Im}\left(\frac{1}{y-z}\right).$$

From this, if $\tilde{G}$ is G.fn. for process killed on unit circle, we have

$$\pi\tilde{G}(z,z_0) = \pi(G(z,z_0) - G(z,\bar{z}_0^{-1})) = \pi(G(z,z_0) - G(\bar{z}^{-1},z_0)).$$

$[\mu(dx) \equiv \mathrm{Re}\,\psi(x)\ dx/\pi, \quad \tilde{\mu}(dv) = \mathrm{Re}(1/\psi(v))\ dv/\pi.]$

Now it's ~~not~~ hard to compute, using $\psi(1/x) = \psi(x)$, that

$$\boxed{\int_1^\infty \tilde{\mu}(dv)\ \mathrm{Im}\left(\frac{1}{v-z_0}\right) = \int_0^1 \tilde{\mu}(dv)\ \mathrm{Im}\left(\frac{1}{v-\bar{z}_0^{-1}}\right)}$$

from which

$$\pi(\tilde{G}(z,z_0) - \tilde{G}^{\partial}(z,z_0)) = \int \tilde{\mu}(dv)\ \mathrm{Im}\left(\frac{1}{v-z_0} - \frac{1}{v-\bar{z}_0^{-1}}\right) \tau(z,v)$$

$$= \int \tilde{\mu}(dv)\ \mathrm{Im}\left(\frac{1}{v-z_0}\right)\{\tau(z,v) - \tau(z,v^{-1})\}.$$

Similarly

$$\frac{\partial}{\partial r}\ \pi(G(z,re^{i\alpha}) - G^{\partial}(z,re^{i\alpha}))\Big|_{r=1} = \int \tilde{\mu}(dv)\ \frac{v^2-1}{|v-e^{i\alpha}|^4}\sin\alpha\ \ \tau(z,v)$$

and

$$\boxed{\int_1^\infty \tilde{\mu}(dv)\ \frac{v^2-1}{|v-e^{i\alpha}|^4} = \int_0^1 \tilde{\mu}(dv)\ \frac{1-v^2}{|v-e^{i\alpha}|^4}}$$

so once again the (signed) measure wrto which we integrate $\tau(z,v)$ has zero integral. Thus we need to know about the asymptotics as $z \to \infty$ of $\tau(z,v) - \tau(z,v^{-1})$.

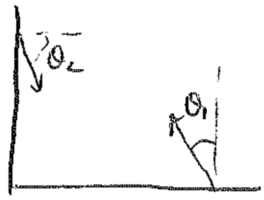
<u>The physicists' model</u> (16/7/90). If we have some nice ψ and try to minimise

$$\int_0^T \dot{\varphi}_s^2\, ds + \int_0^T ds \int_0^T du\ \psi(\varphi_s - \varphi_u)$$

then if we take $\dot{\varphi} \in L^2([0,T])$ as the primitive, it's clear that the second term is a continuous functional of $\dot\varphi$, and the first is also. However, though the second is weakly cts, the first is not. Nonetheless, the first is lsc, so this gives the existence of a minimising $\dot\varphi$. Uniqueness cannot be expected; if ψ is even, and $\dot\varphi$ minimises, so does $-\dot\varphi$. This argument was supplied by L. Gross.

<u>Reflecting BM in $(\mathbb{R}^+)^3$</u> (17/7/90)

(i) Let's look at a special case where reflection on $z=0$ is normal, and reflection on each of the other two faces has no z-component, so it's really just a 2d problem.



Now if we look at local time L on the axis $x=y=0$, its inverse τ is a stable subordinator of index $\frac{1}{2}(\theta_1+\theta_2)/\xi = (\theta_1+\theta_2)/\pi$ in this case.

Then the process $Z\circ\tau$ is again a Lévy process, nice + symmetric, and also stable;

$$E\, e^{i\theta Z(\tau_t)} = e^{-c\,\theta^{2\gamma}} \qquad \gamma \equiv (\theta_1+\theta_2)/\pi$$

The Kesten-Bretagnolle criterion for hitting points, $\int Re\left(\frac{1}{1+\psi(\theta)}\right)d\theta < \infty$, is satisfied iff $2\gamma > 1$

<u>— so the RBM in $(\mathbb{R}^+)^3$ will hit zero in this example iff</u> $\frac{\theta_1+\theta_2}{\xi} > 1$

[Recall result that RBM is a semimartingale iff $(\theta_1+\theta_2)/\xi < 1$].

(ii) Could it be that SRBMs cannot hit O? Let us show that this is not true by taking

$$R = \begin{pmatrix} 1 & -a & -a \\ -a & 1 & -a \\ -a & -a & 1 \end{pmatrix}$$

where $0<a<\frac{1}{2}$. In the case $a=\frac{1}{2}$, it's evident that the process reaches O, because $1\cdot X_t$ is a BM; but the case $a=1/2$ is not a SRBM, by results of Reiman, Williams, + L. Taylor.

We shall prove that <u>for</u> $a>\frac{1}{3}$, <u>it is certain that the corner can be reached</u>, by constructing a semimartingale $f(X_t)$, where for suitable $\alpha,\beta>0$

$$f(x) = (1\cdot x)^\alpha \quad |x|^\beta,$$

such that $f(X_t)$ is a supermartingale ≥ 0; convergence $\Rightarrow X$ reaches O.

Straightforward Itô formula gives (with $V \equiv 1.X$)

$$df(X) = dmg + |X|^\beta V^{\alpha-1}\left[\alpha 1 + \beta \frac{VX}{|X|^2}\right] R\, dL$$

$$+ \tfrac{1}{2} |X|^{\beta-2} V^{\alpha-2} dt \left[3\alpha(\alpha-1)|X|^2 + (\beta(\beta+1) + 2\alpha\beta)V^2\right] \circ$$

Taking the dL^3 term in the singular drift, X must be of form $r(\cos\theta, \sin\theta, 0)$, so

$$\left(\alpha 1 + \beta \frac{VX}{|X|^2}\right) R^{(3)} = \alpha(1-2a) + \beta(\cos\theta + \sin\theta)^2(-a)$$

$$\leq 0 \qquad \text{for all } \theta \in [0, \pi/2]$$

iff $\boxed{\alpha(1-2a) \leq a\beta}$

Looking at the dt piece, we shall have this ≤ 0 if always:

$$3\alpha(\alpha-1) + (2\alpha+\beta+1)\beta \cdot \left(\frac{X.1}{|X|}\right)^2 \leq 0.$$

But $(x.1)/|x| \leq \sqrt{3}$, so this inequality is

$$\alpha(\alpha-1) + (2\alpha+\beta+1)\beta \leq 0$$

i.e. $\boxed{(\alpha-\beta) \geq (\alpha+\beta)^2}$

If we choose $\beta = (1-2a)\alpha/a$ to make the first inequality just OK, we get

$$\boxed{\alpha \leq \frac{(3a-1)a}{(1-a)^2}}$$

So we can choose a positive α provided $a > 1/3$.
Can similarly analyse $(1.X)^{-\gamma}|x|^{-\delta}$, and find that the inequalities are

$$\gamma(1-2a) \geq 2a\delta, \qquad 3\gamma(\gamma+1) \leq \delta(1-\delta-2\gamma).$$

Taking equality in the first, the second is reducible to $\gamma \leq 2a(1-8a)/(1+8a^2)$, so if $a < 1/8$, we can make a superharmonic with pole at 0

— so for $a < 1/8$, the corner is not reached.

Some cross thoughts on economics (17/7/90)

One problem about equilibrium theory in economics which I find very unsatisfying is that if one takes 2 agents trading 2 commodities, with given utilities, and initial endowments, then all the theory tells you is the collection of points where they might end up, but not where they will actually end up. The mechanism of trading is not considered.

I thought one way to model this would be to consider a cts trajectory along which the agents move. Let agent j have allocation $x_j(t)$ at time t, where $\sum_j x_j(t) = \sum_j x_j(0)$ = initial endowments, a vector quantity. So why not let $(x_1(t), \ldots, x_N(t))$ evolve in such a way as to continue to satisfy this constraint, yet increase everyone's utility at the same rate?

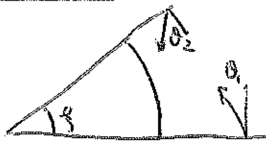
Sounds nice, but look at the trivial example of 2 agents, one commodity, $u_1(x) = x$, $u_2(x) = -2x$. If $x_1(0) = 0$, $x_2(0) = 1$, then the above rule would prevent any trading, though commonsense $\Rightarrow$ 2 gives it all to 1! The business of looking at trading infinitesimally is essentially linearising utility; but one still doesn't know how to handle things for this very simple situation!

This can still happen in the more interesting higher-dimensional setting, for example if choice space is $\mathbb{R}^2$ (so could have an Edgeworth box, say), with

$$u_1(x,y) = \alpha(2x+y), \quad u_2(x,y) = -x-2y.$$

Then any movement in direction $(b,-1)$, with $\frac{1}{2} \le b \le 2$ will improve the utility of both ($\alpha > 0$ is assumed) but there is in general no way to do this at equal rates.

Joint law of duration of exc^n + loc times on the two sides for RBM (19/7/90)



The map which takes the wedge as pictured intersected with unit disc into $\mathbb{H}$, taking the arc to $[-1,1]$, is simply

$$z \longmapsto -\tfrac{1}{2}\left(z^{-\pi/\xi} + z^{\pi/\xi}\right),$$

and the asymptotics of the exit density are not to hard to get; one has that the exit density (renormalised) $\longrightarrow$ $(1-t)^{\theta_2/\pi}(1+t)^{\theta_1/\pi}(1-t^2)^{-\frac{1}{2}}\,dt \qquad -1 < t < 1,$

which in terms of the pie-shaped piece we came from gives exit density for an excursion as

$$\left(1+\cos\gamma\theta\right)^{\theta_2/\pi}\left(1-\cos\gamma\theta\right)^{\theta_1/\pi} \qquad (\gamma\equiv\pi/\xi)$$

for $0<\theta<\xi$. Now I wondered whether one could take

$$\varphi(z)\equiv E^z\exp\{-\lambda H_0-\alpha_1 L^1(H_0)-\alpha_2 L^2(H_0)\}$$

and then rescale in some way (eg $(1-\varphi(\varepsilon z))/\varepsilon^\beta$) and get a limit which one could compute. The excursion interpretation says that the rescaling must be ε^α, where $\alpha=(\theta_1+\theta_2)/\xi$. The function φ satisfies

$$\tfrac{1}{2}\Delta\varphi=\lambda\varphi \text{ inside}, \quad v_j\cdot\nabla\varphi=\alpha_j\varphi \text{ on side } j.$$

However, rescaling shows that

$$1-\varphi(\varepsilon z)=1-E^z\exp(-\varepsilon^2\lambda H_0-\varepsilon\alpha_i L^i(H_0))$$

so if $\alpha>1$ (which ensures finiteness of the means) when we take

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}(1-\varphi(\varepsilon z))=E\,\alpha_i L^i(H_0)$$

which is completely the wrong scaling. This is happening because $L^1_{\tau_t}$ would have to be a stable (α) subordinator $\therefore$ explode; we are taking a bad functional of the excursion path to evaluate. What would be better would be to take

$$\varphi(z)=E^x\exp\left[-\lambda H_0-\alpha_1 L^1(H_0)^2-\alpha_2 L^2(H_0)^2\right]$$

which always scales correctly, but characterising this φ is a big mess. In fact we only need to characterise the integral of $\varphi(e^{i\theta})$ against the exit density, but is this any easier?

Limit laws of one-dimensional diffusions again. (11/8/90).

With the usual setting, transient diff^n in $(0,1]$ in natural scale, then I claim that (i) $\frac{c(X_t)}{t} \xrightarrow{a.s.} 1$ (ii) $\frac{H_x}{c(x)} \xrightarrow{a.s.} 1$ (iii) $\frac{\sigma_x}{c(x)} \xrightarrow{a.s.} 1$ are equivalent. Already know that (i) $\Rightarrow$ (ii), (iii).

Proof of (ii) $\Rightarrow$ (i). Fix $p \in (0,1)$, and let $y \equiv y(x) \equiv c^{-1}(p\, c(x)) > x$. Easy to see (since c must be slowly varying) that $x/y(x) \to 0$, and, in particular, $x < \frac{1}{2} y(x)$ eventually. Since $\frac{H_x}{c(x)} \xrightarrow{a.s.} 1$, must have $\frac{H_{x-} - H_x}{c(x)} \xrightarrow{a.s.} 0$. We also have $\frac{H_x - H_y}{c(x)} \xrightarrow{P} (1-p)$, so given $\varepsilon > 0$, pick x_0 so small that $\forall x \le x_0$, $x \le \frac{1}{2} y(x)$ and

$$P\left[\frac{H_x - H_{y(x)}}{c(x)} \ge (1-\varepsilon)(1-p)\right] \ge 1-\varepsilon .$$

Let $\underline{X}_t \equiv \inf\{X_s : s \le t\}$, $Y_t \equiv X_t - \underline{X}_t$. Then there is a sequence of excursions of Y from 0 where the excursion gets up to $\underline{X}_t$, and we need only consider these excursions, after X first reaches x_0. If there were infinitely many such excursions during which X got up to $y(\underline{X})$ then on infinitely many excursions $H_{x-} - H_x \ge (1-\varepsilon)(1-p)\, c(x)$ which contradicts $(H_{x-} - H_x)/c(x) \xrightarrow{a.s.} 0$. Thus ultimately $X_t \le y(\underline{X}_t)$, and we have

$$p\,\frac{c(\underline{X}_t)}{H(\underline{X}_t-)} \le \frac{c(y(\underline{X}_t))}{t} \le \frac{c(X_t)}{t} \le \frac{c(\underline{X}_t)}{t} \le \frac{c(\underline{X}_t)}{H(\underline{X}_t)} \xrightarrow{a.s.} 1 .$$

Proof of (iii) $\Rightarrow$ (i).

This is somewhat similar. Fix $\lambda > 1$, let $z \equiv z(x) \equiv c^{-1}(\lambda c(x)) < x$. Let $\sigma_x^\lambda \equiv \sup\{t < H_{z(x)} : X_t = x\}$, so that

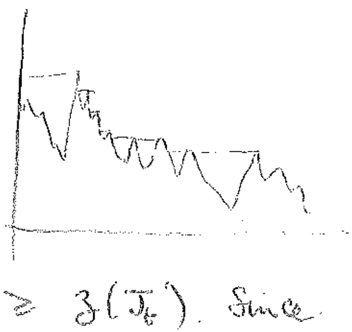
$$\frac{H_x}{c(x)} \le \frac{\sigma_x^\lambda}{c(x)} \le \frac{\sigma_x}{c(x)} \xrightarrow{a.s.} 1$$

and $H_x/c(x) \xrightarrow{P} 1$, so we conclude that $\sigma_x^\lambda / c(x) \xrightarrow{P} 1$, whence $\frac{H_z - \sigma_x^\lambda}{c(x)} \xrightarrow{P} (\lambda - 1)$, and for all small enough x,

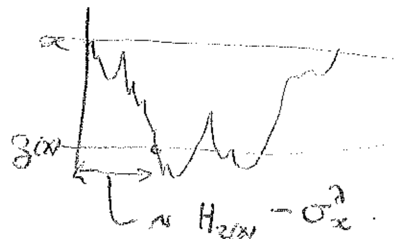
$$P\left[(H_z - \sigma_x^\lambda)/c(x) \ge (1-\varepsilon)(\lambda-1)\right] \ge 1-\varepsilon .$$

Also, as before,

$$\frac{\sigma_{x-} - \sigma_{x+}}{c(x)} \xrightarrow{a.s.} 0 ,$$



and we consider now excursions down from max of the reversed process (which is a time-changed BES(3)). The aim is to prove that if $J_t \equiv \sup\{X_s : s \ge t\}$, then ultimately $X_t \ge z(J_t)$. Since $z(x)/x \to 0$, we may as well restrict attention to those excursions of BES(3) down from level x which drop below $x/2$. Now on any excursion down from x which drops below $z(x)$, there's prob at least $1-\varepsilon$ that the excursion lasts at least $(1-\varepsilon)(\lambda-1)\,c(x)$ (coz the first bit has the same law as $H_{z(x)} - \sigma_x^\lambda$)



Hence if we see i.o. an excn dropping down from x to below $z(x)$ then i.o. we shall have $(\sigma_x - \sigma_{x+})/c(x) \ge (1-\varepsilon)(\lambda-1)$ ✻

Thus ultimately $X_t \ge z(J_t)$, and we have

$$1 \xleftarrow{a.s.} \frac{c(J_t)}{\sigma(J_t)} \le \frac{c(X_t)}{t} \le \frac{c(z(J_t))}{t} \le \frac{\lambda\, c(J_t)}{\sigma(J_t+)} \xrightarrow{a.s.} \lambda .$$

Since λ is arbitrary, we've got it.

Asymptotics of linear polymer model (11/8/90).

Looking at $dX_t = dB_t + c\left(\int_0^t (X_t - X_u)\,du\right)dt$, Pat Fitzsimmons offers a formula for X_t in the form

(*) $$X_t = \int_0^t g(t,s)\,dB_s .$$

The form he gives is not correct, but something like it is:

$$g(t,s) = 1 + cse^{-cs^2/2}\int_s^t e^{cu^2/2}\,du .$$

Proof runs as follows. From (*)

$$dX_t = g(t,t)\,dB_t + \left(\int_0^t g_1'(t,s)\,dB_s\right)dt = dB_t + \left(c\int_0^t (X_t - X_u)\,du\right)dt$$

$\therefore$ $g(t,t) = 1$, and

$$\int_0^t g_1'(t,s)\,dB_s = ctX_t - c\int_0^t X_u\,du$$

$$= ct\int_0^t g(t,s)\,dB_s - c\int_0^t du\int_0^u g(u,s)\,dB_s$$

$$\Rightarrow \quad g_1'(t,s) = ct\,g(t,s) - c\int_s^t g(u,s)\,du, \quad \text{and} \quad g_1'(s,s) = cs\,g(s,s) = cs.$$

Now differentiate wrt t and keep going!

Some thoughts on no-arbitrage and equivalent martingale measures (14/8/90).

Whatever definitions one may choose of "no-arbitrage" and "EMM", the definition should not be affected by changing to an equivalent measure, and there should be a theorem no-arbitrage $\Leftrightarrow \exists$ EMM. One possible pair of candidates is:

(i) No-arbitrage holds if whenever H is bdd prev, $\int_0^1 H_s dX_s \geq 0$, then $\int_0^1 H_s dX_s = 0$.

[We'll assume for now for simplicity that X is a cts semimg; obviously the integrand H has to satisfy some boundedness hypothesis, and this one seems to be the only one not changed when change to an equivalent meas.]

(ii) An equivalent mg measure is a law $\tilde{P} \sim P$ s.t. under $\tilde{P}$ X is a martingale.

I conjectured that if under $\tilde{P}$ X is a martingale, then for any bdd prev. H, $H\cdot X$ will be a $\tilde{P}$-martingale. This is false, as Marc Yor shows me, using results in a paper by ~~Dellacherie~~-Meyer-Yor in Sem Prob XII. If T_n reduce X strongly to X^n, and if the conjecture were true, then

$$(H\cdot X)_{T_n \wedge 1} \xrightarrow{L^1} (H\cdot X)_1$$

for each bdd prev H. Then Thm 9 of the above paper proves that (X^n) are bdd in H^1, at least if we stop at 1, and so $X \in H^1$. But there exist cts UI mg which are not in H^1.

Nonetheless, the paper of Azéma, Gundy + Yor in SP XIV can be used to prove

$$\exists \text{ EMM} \Rightarrow \text{no-arbitrage}$$

(with the above definitions.) How? Well, suppose that X is a cts mg, and $\int_0^1 H_s dX_s \leq 0$ (I prefer to work with this form of it). There is an $\varepsilon > 0$ such that

$$P\left[\int_0^1 H_s dX_s \leq -\varepsilon\right] > 0,$$

so by stopping when $\int_0^t H_s dX_s$ gets down to $-\varepsilon$, we may wlog assume that $\int_0^t H_s dX_s \geq -\varepsilon$ for all t. Thus $H\cdot X$ is a cts local mg, and a supermg, so is bdd in L^1. We can then apply AGY which says that if X is a cts loc mg bdd in L^1 then

$$X \text{ is UI} \Leftrightarrow \lambda P(X^* > \lambda) \to 0 \Leftrightarrow \lambda P([X]^{\frac{1}{2}} > \lambda) \to 0.$$

But X is a UI mg (we stop always at 1), so $\lambda P([X]^{\frac{1}{2}} > \lambda) \to 0$, and

$$P([H\cdot X]^{\frac{1}{2}} > \lambda) \leq P(c[X]^{\frac{1}{2}} > \lambda) = o(1/\lambda)$$

and hence $H\cdot X$ is a UI martingale, and the only way we can get $(H\cdot X)_1 \leq 0$ is if $(H\cdot X)_1 \equiv 0$.

An example. Let's take $X_t = B_t + \int_0^t (1-s)^{-\alpha} ds$, where $\alpha \in [\frac{1}{2}, 1)$. This is an example where there is no EMM, because the deterministic drift is not in L^2. Can there be an arbitrage opp.?

This is complete rubbish; if we solve

$$dX_t = (1+X_t)\, f_t\, (dB_t + f_t\, dt) \qquad f_t = (1-t)^{-\alpha} \text{ here}$$

we get

$$(1+X_t) = \exp\left(\int_0^t f_s\, dB_s + \tfrac{1}{2} \int_0^t f_s^2\, ds \right) \to \infty \text{ as } t \uparrow 1$$

so the wealth process X remains bdd below by -1, but explodes as $t \uparrow 1$; so we certainly have arbitrage here.

If there's an arbitrage opportunity (bdd prev H s.t. $\int_0^1 H\,dX \ge 0, \ne 0$), then for each $t<1$ we certainly have $P\left(\int_0^t H_s\,dX_s < 0\right) > 0$ (because there's no arb. on $[0,t]$, since there's an EMM up to t). So $\exists\, \varepsilon>0$ s.t. $P\left(\int_0^t H_s\,dX_s < -\varepsilon\right) > 0$, and so $\exists$ some bdd prev process θ (in fact, $\theta_s \equiv H_s I_{(s\ge t)} I_{(\int_0^t H_u dX_u < -\varepsilon)}$) such that $\int_0^1 \theta_u\,dX_u \ge \varepsilon$ a.s. on the event $\int_0^t H_s\,dX_s < -\varepsilon$.

But we could just as well use $\theta_s = H_s I_{(s\ge t)}$ and then get $\int_t^1 \theta_u\,dX_u \ge \varepsilon$ a.s. If we consider

$$\psi(t) = \sup\{\varepsilon>0 : \exists \text{ bdd prev. } \theta \text{ s.t. } \int_t^1 \theta_u\,dX_u \ge \varepsilon\}$$

then evidently $\psi\downarrow$, but if we take $0<s<t<1$, it's also clear that for any θ bdd

$$P\left[\int_s^t \theta_u\,dX_u < 0\right] > 0 \qquad \text{unless } \int_s^t \theta_u\,dX_u \equiv 0$$

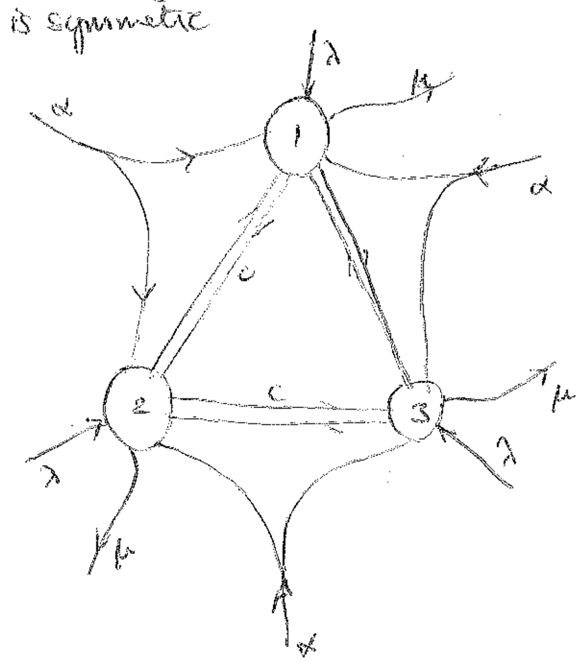
or else there's an arbitrage opportunity by time t. Hence $\psi(t) \ge \psi(s)$, and ψ is constant! But if θ is such that $\int_t^1 \theta_u\,dX_u \ge \varepsilon > 0$, by taking expectations, we get that

$$\varepsilon \le E\int_t^1 \frac{\theta_u\,du}{(1-u)^\alpha} \to 0 \text{ as } t\uparrow 1, \text{ so } \psi \equiv 0.$$

Careful! the θ chosen here will depend on t, so can't conclude this ....

So here we have a case where (ii) fails, yet (i) holds.

RBM in $(\mathbb{R}^+)^3$ – an example (6/9/90). Many of the RBMs can be realised as diffusion limits of queueing systems. Here is a realisation of example (ii) on p.15 in terms of a queueing system, at least for some values of a. There are 3 servers, + the thing is symmetric



There are arrival streams of rate λ for each of the 3 queues, then there are 3 indept arrival streams of pairs of customers, rate α; servers serve out of system rate μ, pass on to other servers rate c.

For zero-mean, we need

$$\boxed{\lambda + 2\alpha = \mu}$$

And for isotropic covariance, $\boxed{\alpha = 2c}$.

A.s. limit theorems for 1d diffusions (7/9/90)

If $v(x) = \operatorname{var} H_x$ ($X_0 = 1$ is assumed) in the Hobson-Rogers work, then we could let $\tau_t = v^{-1}(t)$, and then the process

$$M_t \equiv H_{\tau_t} - c(\tau_t)$$

is an L^2 martingale, $EM_t^2 = t$, with independent increments. The limit theorem we want $(H_x/c(x) \xrightarrow{a.s.} 1)$ amounts to proving $M_t/c(\tau_t) \xrightarrow{a.s.} 0$. This says that for each $\varepsilon > 0$, $\varepsilon c(\tau_t)$ is an upper function for M. Assuming we could replace M by B, then we could hope to use Kolmogorov's test to check whether or not $g(t) \equiv \varepsilon c(\tau_t)$ is an upper function. But to use K's test, need to know

$$\frac{g(t)}{\sqrt{t}} \uparrow \text{ ultimately}, \qquad \frac{g(t)}{t} \downarrow \text{ ultimately}$$

which works out as saying

$$\frac{c(x)}{v(x)} \downarrow \text{ as } x \downarrow 0, \qquad \frac{c(x)}{\sqrt{v(x)}} \uparrow \text{ as } x \downarrow 0$$

But notice that when J is non-trivial (ie. v bold), the first of these is impossible. Also, if $m(v) = \frac{1}{v}(\log \frac{1}{v})^{\alpha}$, where $2\alpha \leq 1$, we get unbold variance, but $\frac{c(x)}{v(x)} \downarrow$ still fails. So this whole thing looks quite futile.

Mean time to hit corner is infinite $\Rightarrow$ time to hit corner is a.s. infinite (9/9/90)

Referring to the formulae in the paper II, we have mean time to hit corner is infinite iff

$$\iint \frac{\tilde{\mu}(dv)}{1+|v|^{1+\eta}} \; \mu(dx) \; \frac{1}{x-v} \int_v^x \frac{1}{1+t^2}\, dt = +\infty$$

where $\eta = 2\xi/\pi < 1$. If now we let $h_n(\tilde{z}) \equiv E^{\tilde{z}}$ (time in G_n before hit corner) where G_n increase to whole statespace, but each G_n excludes a nbd of 0, then we have (in the upper half-plane formulation)

$$h_N(z) = \int \tilde{\mu}(dv) \; |v|^{-1-\eta} \Big(\int_{\frac{1}{|v|} \leq y \leq \frac{N}{|v|}} \big(\log|1+y| - \log|1-y| \big) \frac{dy}{y^{1+\eta}} \Big) \int \frac{\mu(dx)}{x-v} \int_v^x \operatorname{Im}\Big(\frac{1}{t-z}\Big)\, dt$$

$\rightarrow \infty$ as $N \rightarrow \infty$ by above assumption.

(i) Write $\tau_n(v) \equiv \int_{1\le|z|\le n} |z|^{-2-\eta}\, \mathrm{Im}\left(\frac{1}{v-z}\right) dx\,dy$. The asymptotics of this we know, and

$$h_n(z) = \iint \tilde{\mu}(dv)\,\tau_n(v)\;\mu(dx)\;\frac{1}{x-v}\int_v^x \mathrm{Im}\left(\frac{1}{t-z}\right) dt.$$

Assuming mean time to reach 0 is infinite, $h_n(z) \to \infty$ as $n\to\infty$ for each z. However, we shall prove firstly that

$$\lim_{n\to\infty}\{h_n(z) - h_n(i)\} \equiv g(z)$$

exists and is finite. Indeed $h_n(z) - h_n(i)$ is expressed as an integral, which we split into $x > v > 0$, $x > 0 > v$ (and the other two regions analogously), and for $x>v>0$ we have that

$$\frac{1}{x-v}\int_v^x \mathrm{Im}\left(\frac{1}{t-z}\right) dt \asymp \frac{1}{x-v}\int_v^x \frac{dt}{1+t^2}$$

[where $f \asymp g$ means f/g is bdd, bdd from 0]

$$\asymp \frac{1}{x-v}\int_v^x \frac{dt}{(1+t)^2} = \frac{1}{(1+x)(1+v)}$$

Now since $\tau(v) \equiv \uparrow\lim \tau_n(v) \sim (1+|v|^{(1+\eta)})^{-1}$ $\quad(\eta<1)$

$$\sim (\log v)/(1+v^2) \qquad (\eta = 1)$$

we certainly have

$$\int_0^\infty \tilde{\mu}(dv)\,\tau(v)\int_v^\infty \mu(dx)\;(1+x)^{-1}(1+v)^{-1} < \infty.$$

[using $\int \frac{\tilde{\mu}(dv)}{1+|v|}\,\tau(v) < \infty$]

The integral over $x > 0 > v = -y$ is treated differently because

$$\frac{1}{x-v}\int_v^x \mathrm{Im}\left(\frac{1}{t-z} - \frac{1}{t-i}\right) dt = \frac{1}{x-v}\left[\int_x^\infty \mathrm{Im}\left(\frac{1}{t-i} - \frac{1}{t-z}\right)dt + \int_{-\infty}^v \mathrm{Im}\left(\frac{1}{t-i} - \frac{1}{t-z}\right)dt\right]$$

and $\left|\int_x^\infty \mathrm{Im}\left(\frac{1}{t-i} - \frac{1}{t-z}\right)dt\right| \asymp \frac{1}{1+x}$, $\frac{1}{x-v} \le \frac{1}{|v|}$, so everything here is also integrable

(ii) The next step is to prove that $g(ib) \to -\infty$ as $b\to\infty$. Once again we split the integral up, and for $x>v>0$,

$$\frac{1}{x-v}\int_v^x dt\left(\frac{b}{b^2+t^2} - \frac{1}{1+t^2}\right) = O\left(\frac{1}{(1+x)(1+v)}\right) + \frac{1}{x-v}\int_v^x \frac{b\,dt}{b^2+t^2}$$

and the first piece is integrable, the second piece is

$$\asymp \frac{1}{(b+x)(b+v)}$$

, so also integrable, and asymptotically negligible.

The other piece is

$$\frac{1}{x-v}\int_v^x \left(\frac{b}{b^2+t^2} - \frac{1}{1+t^2}\right)dt = \frac{1}{x-v}\left\{\int_x^\infty \left(\frac{1}{1+t^2}-\frac{b}{b^2+t^2}\right)dt + \int_\infty^v \left(\frac{1}{1+t^2} - \frac{b}{b^2+t^2}\right)dt\right\}$$

$$= \frac{1}{x-v}\left\{-\frac{\pi}{2}+\tan^{-1}(x/b) - \frac{\pi}{2} + \tan^{-1}(|v|/b)\right\} + \frac{1}{x-v}\left(O\left(\frac{1}{1+x}\right) + O\left(\frac{1}{1+|v|}\right)\right)$$

The final terms contribute something integrable, the first give

$$-\frac{1}{x+|v|}\left[\tan^{-1}(b/x)+\tan^{-1}(b/|v|)\right] \downarrow -\frac{\pi}{x+|v|} \qquad (b\uparrow\infty)$$

So that

$$\iint \tilde\mu(dv)\,\pi(v)\, I_{(v<0)}\,\mu(dx)\ \frac{1}{x-v}\int_v^x\left(\frac{b}{b^2+t^2}-\frac{1}{1+t^2}\right)dt = -\infty$$

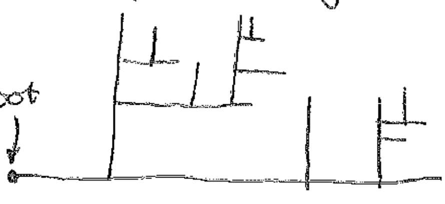
as required.

(iii) Now it's easy. We have $\frac{1}{2}\Delta g+1 = 0$, so that $g(Z_t)+t$ is a cts local mg. As Z approaches the corner, we know that Z keeps crossing the imaginary axis (one-sided escape is precluded, since you get in a.s. in finite time), so that $\liminf_{t\to H_0} g(Z_t) = -\infty$ by (ii) above. This means in particular that $H_0<\infty$ is impossible.

Path decomposition of branching BM? (11/9/90)

(i) Let's consider a real-valued BBM as an p9, which is killed at rate μ, but gives birth to new particles at rate $\lambda \le \mu$. Let $\theta \equiv \sqrt{2\lambda+2\mu}$. Represent such a tree diagrammatically as:

root



Here, the line that keeps going is the old particle, the line at right angles is the new one. The lengths of branches have no significance.

Each edge e of this tree represents a piece of Brownian path killed at an independent $\exp(\lambda+\mu)$ time. Let $p \equiv \lambda/(\lambda+\mu)$, $q \equiv 1-p$ be the probabilities of birth, death, resp at the end of an edge.

Given the value of the process $\xi_{i(e)}$ at the initial end of the side, the value $\xi_{j(e)}$ at the final end $j(e)$ of the side e, and the minimum value η_e attained by the path on e, we can specify a r.c.d. for the path in the following way: run a BM with drift θ down from $\xi_{i(e)}$ til it hits η_e, run a BM with drift $-\theta$ down from $\xi_{j(e)}$ til it hits η_e, then take the first, followed by the reversal of the second.

The point is: we only need to consider the heights of the ends of the edges, and the minimal values on the edges.

Let's define

$$\rho(x) = \theta e^{-\theta x} I_{(x>0)} .$$

Let P^x denote the law of BBM started at x. Let's fix some binary tree τ with edge set E, vertex set V, m internal nodes (valence 3), m' external nodes (valence 1). Let $\mathcal{T}$ be the random tree-shape adopted by BBM. Then fixing $e^* \in E$

$$E^0\Big[\exp\Big(i \sum_{j\in V} \alpha_j (\xi_j - \eta_{e^*}) + i \sum_{e\in E} \beta_e (\eta_e - \eta_{e^*})\Big) : \mathcal{T} = \tau, \text{ min occurs on edge } e^* \Big]$$

$$= \int \exp\Big(i \sum_{j\in V} \alpha_j (\xi_j - \eta_{e^*}) + i \sum_e \beta_e (\eta_e - \eta_{e^*})\Big) \prod_{e\in E} \rho(\xi_{i(e)} - \eta_e)\rho(\xi_{j(e)} - \eta_e) I_{(\eta_e \geq \eta_{e^*})} \, d\underline{\xi}\, d\underline{\eta} .$$

$$p^m q^{m'-1},$$

where the root node 0 obviously satisfies $\xi_0 = 0$; the $\underline{\xi}$ integral is over ξ_j, $j \neq 0$. Now if we change variables to $x_j \equiv \xi_j - \eta_{e^*}$, $y_e \equiv \eta_e - \eta_{e^*}$, we can now re-express the integral as

$$\int \exp\Big(i \sum_j \alpha_j x_j + i \sum_{e\neq e^*} \beta_e y_e\Big) \Big[\prod_{e\in E} \rho(x_{i(e)} - y_e)\rho(x_{j(e)} - y_e) I_{(y_e \geq 0)}\Big] d\underline{x}\, d\underline{y}\; p^m q^{m'-1},$$

where now $y_{e^*} = 0$, and the integral excludes dy_{e^*}, of course.

(ii) Let F be the event {no particle ever reaches 0}, and let

$$\psi(x) \equiv P^x(F).$$

We computed on p 11–12 that

$$\psi^{-1}(x) = a^{-1}\sqrt{\frac{3}{\lambda}} \Big[\log\big(a + \sqrt{1-x+a^2}\big) - \tfrac{1}{2}\log(1-x) - \log\big(a + \sqrt{1+a^2}\big)\Big] \qquad \Big(a^2 \equiv \frac{3(\mu-\lambda)}{2\lambda}\Big)$$

when $\mu > \lambda$, and

$$\psi(x) = 1 - (1+cx)^{-2}, \qquad c \equiv \sqrt{\lambda/3} \quad \text{if } \lambda = \mu .$$

Hence $\boxed{\psi'(0) = 2\big((3\mu-\lambda)/6\big)^{\frac{1}{2}} .}$

Now define for $x>0$, $\tilde{P}^x(\cdot) \equiv P^x(\cdot \cap F)/\psi(x)$, the law of the BBM conditioned never to put a particle down to 0. In the limit as $x \downarrow 0$ we get

$$\lim_{\varepsilon\downarrow 0} \tilde{P}^\varepsilon\left[J=\tau,\ x_j,\ y_e\right] = \frac{\rho(0)}{\psi'(0)}\left[\prod_{e\neq e_0} \rho(x_{i(e)}-y_e)\,\rho(x_{j(e)}-y_e)\, I_{(y_e\geq 0)}\right] \cdot p^m q^{m'-1} \cdot \rho(x_1)$$

where e_0 is the edge emanating from the root, x_1 is height at first vertex after 0. We have

$$\frac{\rho(0)}{\psi'(0)} = \left[3(\mu+\lambda)/\beta(\mu-\lambda)\right]^{1/2}.$$

<u>Self-consistency for polymer carpet via Green f^n</u> (15/10/90)

Look again at p. 23 of previous book (2/2/90). We took a BM coming in from ∞ but conditioned not to be V-killed. This gives a diffusion with generator

$$\mathcal{G}f = \frac{1}{2\psi^2}\, D(\psi^2 Df),$$

where $\psi(x) \equiv E_x \exp\left(-\int_0^{H_0} V(B_s)\,ds\right)$. We can compute the expected local time at level $x_0>0$ without speaking of local time by working out the Green f^n, $G(\cdot, x_0)$. The harmonic f^{ns} for $\mathcal{G}$ are const and

$$h(x) = \int_0^x \psi(y)^{-2}\,dy$$

so

$$G(x,x_0) = \begin{cases} G(x_0,x_0), & x\geq x_0 \\ h(x)\,G(x_0,x_0)/h(x_0), & x\leq x_0 \end{cases}$$

and

$$-\mathcal{G}G(\cdot,x_0) = \delta_{x_0},$$

which fixes the value of $G(x_0,x_0)$, because the value of the jump of the derivative at x_0 is simply

$$-G(x_0,x_0)\,\frac{h'(x_0)}{h(x_0)} = -2$$

$$\Rightarrow \boxed{G(x_0,x_0) = \psi(x_0)^2 \int_0^{x_0} \psi(y)^{-2}\,dy}$$

as previously.

Mean occupation times for BBMs etc. (15/10/90)

Let's take a BM which is killed at rate μ, and at rate λ gives birth to a random number of offspring, distribution $(p_n)_{n\geq 1}$, each of which behaves like some other (possibly branching) Markov process. For example, we could have each descendant another BBM, or else a rod of fixed length, or a Brownian whisker of fixed length.

Let's define for some fixed $V \geq 0$, and $A \subseteq \mathbb{R}^d$,

$$\psi(x) = P_x(\text{no } V\text{-killing on the polymer})$$

$$h(x) = E_x[\text{total time spent in } A;\ \text{no } V\text{-killing on the polymer}]$$

$$\psi_0(x) = P_x(\text{no } V\text{-killing on the individual descendant polymer})$$

$$h_0(x) = E_x(\text{total time in } A;\ \text{no } V\text{-killing on individual descendant}).$$

Hence we have

$$h = R^V_{\lambda+\mu}(I_A\psi) + \lambda R^V_{\lambda+\mu}\Big(\sum_{n\geq 1} p_n\big(h\,\psi_0^n + n\,h_0\,\psi\,\psi_0^{n-1}\big)\Big)$$

$$= R^V_{\lambda+\mu}(I_A\psi) + \lambda R^V_{\lambda+\mu}\big(h\,P(\psi_0) + h_0\psi\,P'(\psi_0)\big)$$

where $P(s) = \sum s^n p_n$. Thus we have the equation

$$\boxed{(\lambda+\mu+V-\tfrac{1}{2}\Delta)\,h = I_A\psi + \lambda h\,P(\psi_0) + \lambda h_0\,\psi\,P'(\psi_0)}$$

We are interested in the situation where we take a Poisson process rate m in $\mathbb{R}^d$ for the starting points of independent polymers, and wish to calculate the total time in A for polymers which have no V-killing; it's just

$$m\int_{\mathbb{R}^d} dx\ h(x) = m\int_{\mathbb{R}^d} dx\ R_\mu^{V+\lambda(1-P(\psi_0))}\big(I_A\psi + \lambda h_0\,\psi\,P'(\psi_0)\big)(x)$$

The special case of BBM (so $P(s) = s$, $h_0 = h$, $\psi_0 = \psi$) yields

$$(\mu-\lambda+V+2\lambda(1-\psi)-\tfrac{1}{2}\Delta)\,h = I_A\psi,$$

so that expected total time in A is given by

$$m \int_{\mathbb{R}^d} dx \; R_{\mu-\lambda}^{V+2\lambda(1-\psi)} (I_A \psi)(x)$$

$$= m \int_A dy \, \frac{\psi(y)}{(\mu-\lambda)} \, P_y \left(\text{no } \{V + 2\lambda(1-\psi)\}\text{-killing of BM before } T_{\mu-\lambda} \right)$$

indept $\exp(\mu-\lambda)$.

The case $V \equiv 0$ gives $\psi \equiv 1$, and expected total time in A is $m\,|A|/(\mu-\lambda)$.

If we take

$$\tilde{\psi}(y) = P_y \left[\text{no } V + 2\lambda(1-\psi) \text{ -killing of BM before } T_{\mu-\lambda} \right]$$

then $\tilde{\psi} = (\mu-\lambda) R^{U}_{\mu-\lambda} 1$, $U \equiv V + 2\lambda(1-\psi)$

so that $\frac{1}{2}\Delta\tilde{\psi} - (\mu-\lambda+U)\tilde{\psi} + \mu-\lambda = 0$.

The monomer density at x is now $m\,\psi(x)\,\tilde{\psi}(x)/(\mu-\lambda) \equiv \rho$, so that if we had

$$V = V_0 + f(\rho),$$

we would have the relations

$$\boxed{\begin{aligned} &\tfrac{1}{2}\Delta\psi - \left(V_0 + f\left(\frac{\psi\tilde{\psi}\,m}{\mu-\lambda}\right)\right)\psi + (\lambda\psi-\mu)(\psi-1) = 0 \\ &\tfrac{1}{2}\Delta\tilde{\psi} - \left(V_0 + f\left(\frac{m\psi\tilde{\psi}}{\mu-\lambda}\right) + \mu+\lambda - 2\lambda\psi\right)\tilde{\psi} + \mu-\lambda = 0 \end{aligned}}$$

<u>Diffusion in a medium with low density of impurities</u> (24/10/90)

1) Let's suppose we have BM in $\mathbb{R}^3$, throughout which are distributed in some way balls of radius 1 (say) where the diffusion is at some other rate. More generally, we take the diffusion with generator

$$\tilde{\mathcal{L}} \equiv \tfrac{1}{2}\,\mathrm{div}\left((1+\eta)\,\mathrm{grad}\right)$$

where the (smooth) function η is a "small" perturbation in some sense. Then we have that the diffusion X obeys the SDE

(*) Can the expansion have any meaning? This would be OK if $\mathcal{A}G$ were a bounded operator on some Banach space, and $\mathcal{A}Gf = -\eta f + \frac{1}{2}\nabla\eta \cdot \nabla G f$, so the problem is all to do with ∇G. Brian Davies points out that if $f \in \mathcal{S}$, then

$$(\nabla_j G f)^\wedge(\omega) = i\omega_j \frac{2}{|\omega|^2}\hat{f}(\omega)$$

which can have arbitrarily large L^2 norm ...

Moreover, since for $k \in \mathbb{N}$

$$\int_{\mathbb{R}^d} f(x)^{2k}\,dx = \text{const.} \int_{\mathbb{R}^d} d\omega_2 \cdots \int_{\mathbb{R}^d} d\omega_{2k}\; \hat{f}(-\omega_2 - \cdots - \omega_{2k}) \prod_{r=2}^{2k} \hat{f}(\omega_r)$$

and taking $f(x) = (2\pi t)^{-d/2} e^{-|x|^2/2t}$, we can get that the power of t by which $\|D_j G f\|_{2k}$ scales is different from the power of t by which $\|f\|_{2k}$ scales, and that's fatal.

$$dX_t = \sigma(X_t)\, dW_t + \tfrac{1}{2} \nabla\eta(X_t)\, dt \qquad [\sigma^2 \equiv 1+\eta].$$

If we define $\boxed{X_t^{(n)} \equiv \frac{1}{n} X_{n^2 t}}$, then

we have that $X^{(n)}$ is a diffusion with generator

$$\tilde{\mathcal{L}}_n \equiv \tfrac{1}{2} \nabla\cdot(1+\eta_n)\nabla$$

where $\eta_n(x) \equiv \eta(nx)$.

2) Consider the Green's f^n $\tilde{G} \equiv (-\tilde{\mathcal{L}})^{-1}$ for $\mathcal{L}$. If $G \equiv (-\frac{1}{2}\Delta)^{-1}$ is the Green's fn for BM, we have

$$\tilde{G} = (-\tfrac{1}{2}\Delta - A)^{-1} \qquad \text{where } A \equiv \tfrac{1}{2}\nabla\cdot\eta\nabla,$$

$$= (G^{-1} - A)^{-1}$$

$$(*) \quad \boxed{\therefore\ \tilde{G} = \sum_{k\geq 0} (GA)^k G,}$$

at least formally. If the perturbation is "small", we hope to get an expansion of the Green f^n $\tilde{G}$ from this

3) The "effective diffusivity" σ_{eff}^2 of the perturbed diffusion can be defined by the recipe

$$\sigma_{\mathrm{eff}}^{-2} = \lim_{n\to\infty} E^0\left[\int_0^\infty I_{\{|X_s|\leq n\}}\, ds\right] / n^2$$

$$= \lim_{n\to\infty} E^0\left[\int_0^\infty I_{\{|X_s^{(n)}|\leq 1\}}\, ds\right]$$

$$= \lim_{n\to\infty} \int_{|y|\leq 1} \tilde{G}_n(0,y)\, dy.$$

(If the perturbation were $\equiv 0$, this would give an effective diffusivity $\sigma_{\mathrm{eff}}^2 = 1$).

Thus the aim is to get hold of the behaviour of

$$\int_{|y|\leq 1} (GA_n)^k G\,(0,y)\, dy.$$

4) <u>First term</u>. For standard BM, it is not hard to confirm that the expected time spent in ball radius R if started r from 0 is

$$g_R(r) = \begin{cases} R^2 - r^2/3 , & r \le R \\ 2R^3/3r , & r \ge R . \end{cases}$$

Hence

$$\int_{|y|\le 1} G(x,y)\,dy = g_1(|x|), \text{ and}$$

$$\int_{|y|\le 1} G\,A\,G(0,y)\,dy = \int dy \int_{|y|\le 1} G(0,x)\,dx \ \tfrac{1}{2}(\nabla\cdot\eta\nabla G)(x,y)$$

$$= \int dx\ G(0,x)\cdot \tfrac{1}{2}\nabla\cdot\big(\eta(x)\,\nabla g_1(|x|)\big)$$

$$= -\tfrac{1}{2}\int dx\ \nabla G(0,x)\ \eta(x)\ \nabla g_1(|x|)$$

$$= -\tfrac{1}{2}\int dx\ \eta(x)\ \frac{x}{2\pi|x|^3}\cdot\frac{x}{|x|}\left(\frac{2|x|}{3} I_{(|x|\le 1)} + \frac{2}{3|x|^2} I_{(|x|>1)}\right)$$

$$= -\int_0^\infty dr\ \bar\eta(r)\left[\frac{2r}{3} I_{\{r\le 1\}} + \frac{2}{3r^2} I_{\{r>1\}}\right]$$

where $\bar\eta(r)$ is the average of η over the ~~sphere~~ of radius r.

5) We could equally well define σ_{eff} by

$$\sigma_{\text{eff}}^{-2} = \lim_{n\to\infty} \frac{E^0\int_0^\infty g(\frac{1}{n}X_s)\,ds}{E^0\int_0^\infty g(\frac{1}{n}B_s)\,ds}$$

for some other $g \ge 0$, equal to 1 at 0 and cts. A natural choice would be $g(x) = e^{-|x|^2/2}/(2\pi)^{3/2}$

$$\sigma_{\text{eff}}^{-2} = \lim_{n\to\infty} \frac{E^0\int_0^\infty g(X_s^{(n)})\,ds}{E^0\int_0^\infty g(B_s)\,ds} = \tfrac{1}{2}\lim E^0\Big\{\int_0^\infty \exp(-\tfrac{1}{2}|X_s^{(n)}|^2)\,ds\Big\}.$$

6) The formal power-series $(*)$ looks like it would have problems. If we wanted it to converge, we'd try to get $\|GA\| < 1$ in some norm. For $f \in C_K^\infty$,

$$GAf(x) = \tfrac{1}{2}\int G(x,y)\ \nabla\eta\nabla f(y)\,dy$$

$$= \tfrac{1}{2}\int f(y)\ \nabla\eta\nabla G(x,y)\,dy$$

so to control the L^1 norm, we'd wish to prove that

$$\int |\nabla\eta\, \nabla G(x,y)|\, dx \leq C \quad \forall y.$$

Now if $\eta(x) = \sum_j \psi(x-\xi_j)$, where $\psi \in C_K^\infty$, supported in $B = \{x : |x| \leq 1\}$, it would be enough to prove

$$\sum \int |\nabla \eta_j\, \nabla G(x,y)|\, dx \leq C \quad \forall y$$

where $\eta_j \equiv \psi(\cdot - \xi_j)$. However, if $B + \xi_j$ doesn't overlap 0, $\nabla \eta_j \cdot \nabla G(x,0)$ $= \nabla\eta_j(x) \cdot \nabla G(x,0)$ and the contribution

$$\int |\nabla \eta_j\; \nabla G(x,0)|\, dx \quad \text{is} \quad \sim \frac{1}{dist(0,\xi_j)^2}$$

– and if the ξ_j are Poisson-distributed, this will not be cgt.

7) In § 12·4·1 of Crank, there is a discussion of this problem. Now he gives the result of Maxwell that to first order

$$\frac{D - D_b}{D + 2D_b} = \nu\, \frac{D_a - D_b}{D_a + 2D_b}$$

where D is the effective diffusivity, D_b the ambient, D_a the diffusivity in spheres, ν the volume fraction of the spheres, assumed small. Taking $D_b = 1$, $D_a = \sigma^2$, we get here to first order

$$D = 1 + \frac{\sigma^2 - 1}{\sigma^2 + 2}\,\nu$$

Whereas using the analysis of 4) above, $\bar{\eta} = (\sigma^2 - 1)\,\nu$, so we get from the first term in the expansion

$$D \equiv \sigma_{eff}^2 = 1 + (\sigma^2 - 1)\nu$$

which doesn't agree; possible causes (i) power series expansion is nonsense (ii) there are order ν terms further down the expansion than GAG – this does indeed seem to be the case.

8) This is apparently not what the physicists want in any case! What seems to be happening is that they want steady-state diffusivity, which I understand in the following way. Take a spherical shell $a \leq |x| \leq R$ of material of diffusivity σ_0^2, surrounded

by material of diffusivity σ_1^2. Keep ∞ at temperature 0, sphere radius a at temperature γ. What's the temperature of the outside of the sphere?

If φ is temperature, then

$$\mathrm{div}\,(\sigma^2 \nabla\varphi) = 0$$

$$\therefore \quad \varphi(r) = \begin{cases} A + B/r, & a \leq r \leq R \\ \dfrac{C}{r}, & r \geq R \end{cases}$$

and φ must be cts at R, the flux must be cts at R (so $\sigma_0^2 \frac{\partial\varphi}{\partial r}(R-) = \sigma_1^2 \frac{\partial\varphi}{\partial r}(R+)$)
Matching all the c/ts, we get

$$C = \gamma a R\, \sigma_0^2 \,/\, \{ a(\sigma_0^2 - \sigma_1^2) + R\sigma_1^2 \}.$$

So if we knew the harmonic f^{n} φ outside the sphere, we would be able to compute σ_0^2/σ_1^2.

9) One interesting remark; if we take a radially-symmetric divergence-form generator $\mathcal{L} = \frac{1}{2}\nabla\cdot(a(r)\nabla)$, then if $h(x) = E^x(\text{time to hit radius } R)$, we have h is obviously radially symmetric, and

$$-1 = \left(\frac{1}{2}\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r}\right)a(r) + \frac{1}{2}a'(r)\frac{\partial h}{\partial r} = \frac{1}{2}\,\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 a(r)\frac{\partial h}{\partial r}\right)$$

together with the bc $h'(0) = 0$.

Thus $$r^2 a(r)\frac{\partial h}{\partial r}(r) = A - 2r^3/3$$

$$\Rightarrow A = 0 \text{ (b.c.) and } \quad \boxed{h(r) = \int_r^R \frac{2x\,dx}{3\,a(x)}}$$

The interesting thing here is that if $a(r) \equiv 1$ for all $1 \leq r \leq R$, then the mean time starting from 1 to reach R is $(R^2-1)/3$, the value for BES(3) <u>irrespective of what a does inside the unit ball!!</u>

10) We could equally well consider for the effective diffusivity the mean time to exit from some big ball. The Green f^{n} for ball radius R is

$$\frac{1}{2\pi}\left\{ \frac{1}{|x-y|} - \frac{R^2}{|y|^2\,|x - yR^2/|y|^2|} \right\}$$

Self-financing portfolios (7/11/90)

(i) One frequently gets the situation where $(S^0, \dots, S^n) \equiv S$ is vector of prices and choose portfolio process θ so that $\theta^1, \dots, \theta^n$ are chosen at liberty, and θ^0_t is then chosen to make the portfolio self-financing

$$\theta_t \cdot S_t = \theta_0 \cdot S_0 + \int_0^t \theta_u \cdot dS_u .$$

This then becomes the problem of finding a real-valued process θ s.t.

$$\boxed{\theta_t S_t - \int_0^t \theta_u \, dS_u = \gamma_t}$$

where we have written S for S^0, and $\gamma_t = \sum_1^n \left(\int_0^t \theta^j_u dS^j_u - \theta^j_t S^j_t \right)$ is something adapted, but generally not much better.

(ii) Assume that S is a continuous strictly positive semimartingale, $\beta \equiv 1/S$.

Then if we define

$$\theta_t = \beta_t \gamma_t - \int_0^t \gamma_s \, d\beta_s$$

one can check easily with some Itô calculus that θ satisfies the boxed condition; so, provided S^0 is cts semimg, strictly positive, we can always chose the holding of θ^0 to make the portfolio self-financing.

Reversal of BES(3) from last hit on level (9/11/90)

Suppose that (R_t) is diffusion on $\mathbb{R}^+$ with generator

$$\frac{1}{2} \frac{d^2}{dr^2} + \mu \coth \mu r \, \frac{d}{dr}$$

where $\mu > 0$. (So this is BM with drift μ, conditioned not to hit 0, or alternatively, the modulus of BM in $\mathbb{R}^3$ with drift of magnitude μ).

If $\sigma \equiv \sup\{t : R_t = r\}$, and $X_t \equiv r - R_{\sigma - t}$, then X is a BM with drift μ run until it hits r (The case $\mu = 0$ of this result is in DW's path decomp. paper)

Explanation of monomer density for killed particles (21/11/90)

Take some $V : \mathbb{R}^d \to \mathbb{R}^+$, and kill Brownian motion at rate V. Thus if

$$A_t \equiv \int_0^t V(X_s)\,ds$$

then if we have natural killing at ζ, independent $\exp(\mu)$ r.v.,

$$\psi(x) \equiv P^x(\text{naturally killed}) = E^x\left[\exp - \int_0^\zeta V(X_s)\,ds\right] = E^x\left[\exp - A_\zeta\right]$$

Now fix some set $F \subseteq \mathbb{R}^d$, compact, and suppose we take lots of independent killed Brownian particles, with initial points distributed through $\mathbb{R}^d$ according to a Poisson process of rate m. For a single particle started at x, both naturally killed and V-killed,

$$h(x) \equiv E^x[\text{total time in } F \,;\, \text{not } V\text{-killed before natural killing}]$$

$$= E^x\left[\int_0^\zeta I_F(X_s)\,ds \;\; e^{-A(\zeta)}\right]$$

$$= E^x\left[\int_0^\infty I_{(\zeta>s)} I_F(X_s)\, e^{-A(\zeta)}\,ds\right]$$

$$= E^x\left[\int_0^\infty I_{(\zeta>s)}\, e^{-A(s)}\, I_F(X_s)\,\psi(X_s)\,ds\right]$$

$$= E^x\left[\int_0^\infty e^{-\mu s - A(s)}\, I_F(X_s)\,\psi(X_s)\,ds\right]$$

$$= R^V_\mu(I_F\psi)(x)$$ where R^V_λ is resolvent for V-killed process.

If now we consider the total time in F from all particles, with starting points $\xi_1, \xi_2, \dots$

$$E(\text{total time in } F) = E\left[\sum_j E^{\xi_j}(\text{time in } F;\ \text{not killed})\right]$$

$$= E\left[\sum_j h(\xi_j)\right]$$

$$= m\int_{\mathbb{R}^d} dx\;\; R^V_\mu(I_F\psi)(x)$$

$$= m\int_{\mathbb{R}^d} dx \int_F dy\;\; r^V_\mu(x,y)\,\psi(y)$$

where $r_\mu^V(x,y)$ is resolvent density;

$$= m \int_F dy\, \psi(y) \int_{\mathbb{R}^d} r_\mu^V(y,x)\, dx$$

(since $(\frac{1}{2}\Delta - \mu - V)$ is self adjoint w.r.t. Lebesgue measure, the resolvent density is symmetric);

$$= m \int_F \psi(y)\, dy\; E^y\left[\int_0^\infty e^{-\mu t - A_t}\, dt\right]$$

$$= \frac{m}{\mu} \int_F \psi(y)^2\, dy.$$

<u>An example in the continuous theory of trading</u> (22/11/90) (Thatcher Day!)

If we have the standard situation

$$dS_t^j = S_t^j \{ \sigma_{jk}(t)\, dW_t^k + b_j(t)\, dt \} \qquad j = 1, \dots, n$$

then Steve Satchell says that if the c/es are constant, the trading required to hit some function of S_T^1, $\varphi(S_T^1)$, say, will involve only S^1. This is evident, since S^1 is a Markov process, so

$$\tilde{E}[\varphi(S_T^1) \mid \mathcal{F}_t] \equiv \varphi(t, S_t^1)$$

and the martingale will be a s.i. w.r.t. dS^1. But if the c/es are <u>not</u> constant, then the same conclusion need not hold: for example

$$dS_2 = S_2\, dW_2$$

$$dS_1 = S_1 \frac{S_1 - S_2}{S_1 + S_2}\, dW_1 .$$

Here, (S_1, S_2) is Markovian, so $E[f(S_T^1)/\mathcal{F}_t] = \tilde{f}(t, S_t^1, S_t^2)$, and to have no trading in S^2, we'd need $\partial\tilde{f}/\partial s_2 = 0$. But the fact that $\tilde{f}$ is space-time harmonic implies

$$\frac{\partial \tilde{f}}{\partial t} + \frac{1}{2} s_1^2 \left(\frac{s_1 - s_2}{s_1 + s_2}\right)^2 \frac{\partial^2 \tilde{f}}{\partial s_1^2} = 0$$

so $\tilde{f}$ can't be independent of s_2 (unless constant!)

Equilibrium charge + capacity for RBM outside smooth compact body (22/11/90)

Take some compact body $K \subseteq \mathbb{R}^d$, $d \geq 3$, with a smooth (at least C) boundary. If we consider BM with normal reflection off ∂K, then this is a transient Markov process with Green's f^n G_K, and so one could represent the potential

$$h(x) \equiv P^x(\text{hit } K)$$

as either of

$$h(x) = \int_{\partial K} G(x,y)\,\mu(dy) = \int_{\partial K} G_K(x,y)\,\mu_K(dy)$$

where G is the Green's f^n of ordinary BM, μ the eq^m charge, and μ_K is the equilibrium charge for G_K.

Let σ denote surface measure on ∂K. Then I claim that

$$\boxed{\mu(dy) = \mu_K(dy) = \frac{1}{2}\frac{\partial h}{\partial n}(y)\,\sigma(dy)}$$

(where the derivative is along the normal pointing <u>into</u> K).

<u>Proof</u> (i) Let B_R be ball of radius R (always R big enough that $B_R \supseteq K$) and let $\Omega_R = B_R \setminus K$. Then

$$f_R(x) \equiv \int_{\partial\Omega_R} \frac{\partial h}{\partial n}(y)\,G(x,y)\,\sigma(dy)$$

$$= \int_{\Omega_R} \operatorname{div}(G(x,y)\nabla h(y))\,dy$$

$$= \int_{\Omega_R} \nabla G(x,y)\cdot\nabla h(y)\,dy.$$

But also

$$f_R(x) = \int_{\partial\Omega_R} \{\nabla(hG) - h\nabla G\}\cdot d\underline{n}$$

$$= \int_{\Omega_R} \Delta(hG)\,dy - \int_{\partial K} \nabla G\cdot dn + O(1/R)$$

$$= \int_{\Omega_R} 2\nabla h(y)\cdot\nabla G(x,y)\,dy - 2h(x) - \int_{\partial\Omega_R} \nabla G\cdot dn + \int_{\partial B_R} \nabla G\cdot dn + O(1/R)$$

$$= \int_{\Omega_R} 2\nabla h(y)\cdot\nabla G(x,y)\,dy - 2h(x) + O(1/R)$$

So letting $R \to \infty$, $f_R(x) \to \int_{K^c} \nabla h(y) \cdot \nabla G(x,y)\, dy$

$$= 2 \int_{K^c} \nabla h(y) \cdot \nabla G(x,y)\, dy \quad \cancel{2} h(x)$$

whence $\boxed{f(x) \equiv \int_{\partial K} \tfrac{1}{2} \frac{\partial h}{\partial n}(y)\, G(x,y)\, \sigma(dy) = h(x)}$ $\quad$ (using $\frac{\partial h}{\partial n}(x) = O(1/|x|^2)$)

(ii) If we write for y fixed

$$G_K(x,y) - G(x,y) \equiv \varphi_y(x) \qquad (x, y \notin K^\circ)$$

then φ_y is harmonic in K^c, and has normal derivative on ∂K equal to $-\frac{\partial G}{\partial n}(x,y)$.

<u>I need to assume</u> $G_K(x,y) = G_K(y,x)$, <u>so that</u> $\varphi_y(x) = \varphi_x(y)$.

Now

$$\int_{\partial K} \mu(dy)\, \varphi_y(x) = \int_{\partial K} \tfrac{1}{2} \frac{\partial h}{\partial n}(y)\, \varphi_y(x)\, \sigma(dy)$$

$$= \lim_R \int_{\partial \Omega_R} \tfrac{1}{2} \frac{\partial h}{\partial n}(y)\, \varphi_x(y)\, \sigma(dy)$$

$$= \lim_R \int_{\Omega_R} \tfrac{1}{2} \nabla \varphi_x \cdot \nabla h\, dy$$

$$= \int_{K^c} \tfrac{1}{2} \nabla \varphi_x \cdot \nabla h\, dy$$

But $\displaystyle\int_{\partial \Omega_R} \tfrac{1}{2} \varphi_x(y) \nabla h(y) \cdot d\underline{n}(y) = \int_{\partial \Omega_R} \left\{ \tfrac{1}{2} \nabla(\varphi_x h) - h \nabla \varphi_x \right\} d\underline{n}(y)$

$$= \int_{\Omega_R} \nabla \varphi_x \cdot \nabla h\, dy - \int_{\partial \Omega_R} h \nabla \varphi_x \cdot dn$$

But $\displaystyle\int_{\partial \Omega_R} h \nabla \varphi_x \cdot dn = \int_{\partial K} \nabla \varphi_x \cdot dn + \int_{\partial B_R} h \nabla \varphi_x \cdot dn$

$$= -\int_{\partial K} \nabla G(x,y) \cdot d\underline{n}(y) + O(1/R) = O(1/R) \text{ as before,}$$

assuming $\nabla\varphi_x(y) = O(|y|^{-2})$.

(iii) If one starts the RBM on the surface of the body K with law $C(K)^{-1}\mu(dy)$, then if L is local time on the surface (occupation density of ε-nbhd renormalised) we get

$$E L_\infty = \lim \int_{\partial K} \frac{\mu(dy)}{C(K)} \int G_K(y,x)\, \frac{1}{\varepsilon} I_{K_\varepsilon}(x)\, dx$$

$$= \lim_{\varepsilon\downarrow 0} \frac{1}{C(K)} \int \frac{1}{\varepsilon} I_{K_\varepsilon}(x)\, h(x)\, dx$$

$$= (\text{Surface area of } K)/C(K).$$

"Purity" laws for OU processes (3/1/91)

1) This arises from a question of J. Zabczyk (well, a result, in fact) who considers a family of stationary OU processes

$$dX^i = dW^i - \alpha_i X^i\, dt, \qquad 0\le t\le 1,$$

$$d\tilde X^i = d\tilde W^i - \tilde\alpha_i \tilde X^i\, dt, \qquad 0\le t\le 1,$$

where the $W^i, \tilde W^i$ are all indept standard BMs, $\alpha_i, \tilde\alpha_i > 0\ \forall i$, and the processes start with invariant law. The question is to consider $(X_1, X_2, \dots)$ and $(\tilde X_1, \tilde X_2, \dots)$ as random elements of $(C[0,1])^{\mathbb N}$, and to decide when the two laws are equivalent. His result is that this is iff

$$0 < c \le \alpha_i/\tilde\alpha_i \le C < \infty \quad \underline{\text{and}} \quad \sum (\alpha_i - \tilde\alpha_i)^2/\alpha_i < \infty$$

2) If we just took two sequences of independent normals, zero mean, variances $(\alpha_i), (\tilde\alpha_i)$, it's clearly necessary for the laws of the sequences to be equivalent that $\alpha_i/\tilde\alpha_i \to 1$ (If not, look down a subsequence + use strong law...) The test is

$$\prod E^{P_i} \sqrt{\frac{d\tilde P_i}{dP_i}} > 0$$

by usual Kakutani thing, and in this case with $P_i = N(0,\alpha_i)$ it's easy to

Calculate directly that $\beta_i \equiv E^{P_i}(d\tilde{P}_i/dP_i)^{\frac{1}{2}} = \frac{\sqrt{2}}{\sqrt{\alpha_i+\tilde{\alpha}_i}}(\alpha_i\tilde{\alpha}_i)^{\frac{1}{4}}$, so the condition for equivalence reduces after a few calculations to

$$\sum \frac{(\alpha_i-\tilde{\alpha}_i)^2}{(\alpha_i+\tilde{\alpha}_i)^2} < \infty \qquad \text{i.e.} \qquad \sum \frac{(\alpha_i-\tilde{\alpha}_i)^2}{\alpha_i^2} < \infty$$

which is different.

3) I followed the calculations through part of the way. If $\mathbb{P}^x$ is law (on $C[0,1]$) of BM started at x, $m \equiv \int dx\, \mathbb{P}^x$ then the OU process $dX = dW - \alpha X dt$ has law Q described by

$$\frac{dQ}{dm} = \frac{e^{-\alpha x^2}}{\sqrt{\pi/\alpha}}\left[\exp - \alpha\int_0^1 X_u\, dX_u - \tfrac{1}{2}\alpha^2\int_0^1 X_u^2\, du\right]$$

$$= \sqrt{\frac{\alpha}{\pi}}\exp\left\{-\tfrac{1}{2}\alpha(X_1^2+x^2) + \tfrac{1}{2}\alpha - \tfrac{1}{2}\alpha^2\int_0^1 X_u^2\, du\right\}$$

So the Kakutani criterion we need to calculate $\int\left(\frac{dQ}{dm}\frac{d\tilde{Q}}{dm}\right)^{\frac{1}{2}} dm$

$$= \left(\frac{\alpha\tilde{\alpha}}{\pi^2}\right)^{\frac{1}{4}}\int dm\, \exp\left\{-\tfrac{1}{4}(\alpha+\tilde{\alpha})(x^2+X_1^2) + \tfrac{1}{4}(\alpha+\tilde{\alpha}) - \tfrac{1}{4}(\alpha^2+\tilde{\alpha}^2)\int_0^1 X_u^2\, du\right\}.$$

By a bit of judicious excursion theory, I got for B a $BM(\mathbb{R})$ started at 0

$$E\exp\left\{-\tfrac{1}{2}\lambda(x+B_1)^2 - \gamma\int_0^1(x+B_u)^2\, du\right\}$$

$$= \left(\frac{\theta}{\theta\cosh\theta + \lambda\sinh\theta}\right)^{\frac{1}{2}}\exp\left\{-x^2\left(\frac{\lambda\theta\cosh\theta + \theta^2\sinh\theta}{2(\theta\cosh\theta + \lambda\sinh\theta)}\right)\right\} \qquad \theta \equiv \sqrt{2\gamma}$$

so that

$$\boxed{\int\left(\frac{dQ}{dm}\frac{d\tilde{Q}}{dm}\right)^{\frac{1}{2}} dm = (\alpha\tilde{\alpha})^{\frac{1}{4}}\, e^{\lambda/2}\,\beta^{\frac{1}{2}}\left(\frac{2}{\lambda+\gamma}\right)^{\frac{1}{2}}}$$

$\beta \equiv \theta(\theta\cosh\theta + \lambda\sinh\theta)^{-1}$, $\gamma \equiv (\lambda\cosh\theta + \theta\sinh\theta)\beta$, $\lambda \equiv \frac{1}{2}(\alpha+\tilde{\alpha})$, $\theta = \left(\frac{\alpha^2+\tilde{\alpha}^2}{2}\right)^{\frac{1}{2}}$

The asymptotic analysis got glutinous here, but in principle we could go further ... but how did

Zabczyk do it? I'd be interested to know.

4) We can persevere with the analysis to obtain (writing $\alpha = \lambda+\delta$, $\tilde\alpha = \lambda-\delta$) for the square of $E^{Q}(d\tilde Q/dQ)^{\frac12}$

$$r \equiv (\lambda^2-\delta^2)^{\frac12}\ e^{\lambda}\cdot \frac{2\theta}{(\lambda^2+\theta^2)\sinh\theta + 2\lambda\theta\cosh\theta}$$

$$= \frac{2(1-\rho^2)^{\frac12}\, e^{\lambda}}{(2+\rho)\sinh\sqrt{\lambda^2+\delta^2} + 2(1+\rho)^{\frac12}\cosh\sqrt{\lambda^2+\delta^2}} \qquad \rho \equiv \delta^2/\lambda^2$$

$$= \frac{2(1-\rho^2)^{\frac12}}{(2+\rho)\, e^{\sqrt{\lambda^2+\delta^2}-\lambda} + \left(2(1+\rho)^{\frac12} - 2-\rho\right) e^{-\lambda}\cosh\sqrt{\lambda^2+\delta^2}}$$

and since $\rho \to 0$, the second part of the denominator is o(first part), so this is all

$$\sim \frac{2(1-\rho^2)^{\frac12}}{2+\rho}\ e^{\lambda-\sqrt{\lambda^2+\delta^2}},$$

but this is misleading. In order that the product be positive, we must have $(\lambda^2+\delta^2)^{\frac12} - \lambda \to 0$, and, indeed, $\sum\{(\lambda^2+\delta^2)^{\frac12}-\lambda\} < \infty$, since $r \le \exp(\lambda - \sqrt{\lambda^2+\delta^2})$

But $$\sum\left\{(\lambda^2+\delta^2)^{\frac12}-\lambda\right\} = \sum \frac{\delta^2}{\lambda+(\lambda^2+\delta^2)^{\frac12}}$$

$$\propto \sum \frac{(\alpha-\tilde\alpha)^2}{\alpha+\tilde\alpha + \left((\alpha+\tilde\alpha)^2+(\alpha-\tilde\alpha)^2\right)^{\frac12}}$$

$$= \sum \frac{(\alpha-\tilde\alpha)^2}{\alpha+\tilde\alpha+(2\alpha^2+2\tilde\alpha^2)^{1/2}}$$

$$\sim \sum \frac{(\alpha-\tilde\alpha)^2}{\alpha}$$

So a <u>necessary</u> condition for absolute continuity is that $\boxed{\sum \frac{(\alpha-\tilde\alpha)^2}{\alpha} < \infty.}$

Now we show that this condition is also sufficient. Assume ↑, and notice that

$$r \ge \frac{(1-\rho^2)^{\frac12}}{(1+\rho)^{\frac12}}\cdot e^{\lambda-\sqrt{\lambda^2+\delta^2}} = \sqrt{1-\rho}\ e^{\lambda-\sqrt{\lambda^2+\delta^2}}.$$

Since the product of $e^{\lambda - \sqrt{\lambda^2 + \delta^2}}$ is > 0, we have only got to prove $\prod (1-p) > 0$, i.e. $\sum p = \sum \delta/\lambda < \infty$ i.e. $\sum \left| \frac{\alpha - \tilde{\alpha}}{\alpha} \right| < \infty$, which is immediate.

<u>Some results on Brownian winding.</u> (16/1/91)

1) Take BM inside the unit disc, normally reflected off the circle, started at 1. Let θ_t be the winding by time t: what's the asymptotics of this for large time? If R_t is a BES(2) process in $(0,1]$ reflected at 1, it's all down to the asymptotic behaviour of $\int_0^t R_s^{-2}\, ds$.

Put R into natural scale: $Y_t \equiv \log R_t$, $dY_t = e^{-Y_t} dW_t - dL_t$

so we can realise Y from a standard BM B by time change:

$A_t \equiv \int_0^t e^{2B_s} I_{(B_s < 0)} ds$, $\tau_t = \inf\{u : A_u > t\}$, $Y_t = B(\tau_t)$ at least in distribution,

and we want to know asymptotics of

$$\int_0^t ds\ e^{-2Y_s} = \int_0^t ds\ e^{-2B(\tau_s)} = \langle Y \rangle_t = \tau_t .$$

So if we knew asymptotics of A_t, we'd have a good chance of asymptotics of τ_t.

$$A_t = \int_{-\infty}^0 e^{2y} L(t,y)\, dy \overset{\mathcal{D}}{=} \int_{-\infty}^0 e^{2y} \sqrt{t}\, L(1, y/\sqrt{t})\, dy$$

$$= \int_{-\infty}^0 e^{2v\sqrt{t}} L(1,v)\, dv\ t$$

$$\sim \tfrac{1}{2}\sqrt{t}\ L(1,0)$$

so that $\dfrac{2A_t}{\sqrt{t}} \xrightarrow{\mathcal{D}} L(1,0)$

Hence $P[\tau_t > at^2] = P[t > A_{at^2}] = P\left[\dfrac{2A_{at^2}}{\sqrt{a}\, t} < \dfrac{2}{\sqrt{a}} \right]$

$$\to P\left[L(1,0) > \frac{2}{\sqrt{a}}\right] = P[H_2 > a]$$

Hence $\boxed{\dfrac{\tau_t}{t^2} \xrightarrow{\mathcal{D}} H_2}$ which implies $\boxed{\dfrac{\theta_t}{2t} \xrightarrow{\mathcal{D}} C_1}$

2) Can we work out the asymptotic windings if we reflect BM($\mathbb{C}$) off a growing disc of radius $g(t)$ at time t? This looks hard, but as a first stab, what if we just use the clock

$$C_t \equiv \int_0^t R_s^{-2} I_{(R_s > g_s)} \, ds \; ?$$

We can at least compute some moments. The BES(2) transition density is

$$p_t(x,y) = \frac{y}{t} e^{-(x^2+y^2)/2t} I_0(xy/t)$$

So

$$E^x\left[R_t^{-2} I_{(R_t > g_t)} \right] = \int_{g_t}^\infty t^{-1} y \, e^{-(x^2+y^2)/2t} I_0(xy/t) \frac{dy}{y^2}$$

$$= \frac{1}{t} e^{-x^2/2t} \int_{g_t/\sqrt{t}}^\infty e^{-v^2/2} I_0\left(\frac{xv}{\sqrt{t}}\right) \frac{dv}{v}$$

If $x>0$, we only need worry about this for large t.

<u>Case 1</u>: $g_t/\sqrt{t} \to 0$. In this case

$$E^x\left[R_t^{-2} ; R_t > g_t\right] \sim -\log(g_t/\sqrt{t}) \frac{1}{t}$$

and

$$E \, C_t \sim \int^t -\log(g_s/\sqrt{s}) \frac{ds}{s}$$

In particular, if $g_s = s^{\alpha/2}$, where $0<\alpha<1$, we have

$$E \, C_t \sim \int^t \frac{1-\alpha}{2} \log s \, \frac{ds}{s} \sim \frac{1-\alpha}{4} (\log t)^2 .$$

<u>Case 2</u>: $g_t = k\sqrt{t}$. In this case $E\left[R_t^{-2} ; R_t > g_t\right] \sim \frac{1}{t}$, so that

$$E \, C_t \sim c . \log t$$

<u>Case 3</u>: $g_t/\sqrt{t} \to \infty$. In this case, assuming $g_t/t \to 0$, we get

$$E^x\left[R_t^{-2} ; R_t > g_t\right] \sim \frac{1}{g_t^2} e^{-g_t^2/2t} .$$

3) It is of related interest to know when you can start at 0 and not immediately have infinite winding. Restricting attention to $g(t) = t^{\alpha/2}$, we get explosion

of the mean iff $\alpha \geq 1$.

Notice that if we take R_t to be the BES(2) process reflected off $g(t)$, started at $x \geq 0$, then $\tilde{R}_t \equiv c R_{t/c^2}$ is BES(2) reflected off $\tilde{g}(t) = c\, g(t/c^2)$ started at cx. In particular, if $x = 0$, $g(t) = t^{\alpha/2}$, then

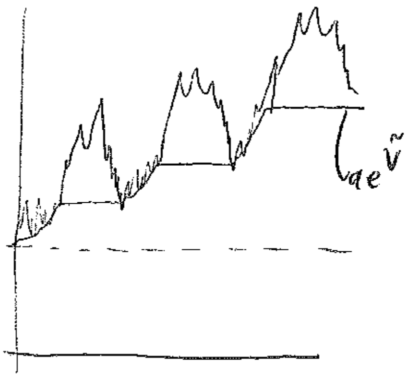
$\tilde{R}_t$ is BES(2) reflected off $c^{1-\alpha} t^{\alpha}$, started at 0.

In the particular case $\alpha = 1$, this makes law of $\tilde{\Theta}_1$ = law of Θ_{c^2} which could only be possible if Θ explodes immediately.

4) Let's now imagine a polymer winding around a cylinder of initial radius a. As the polymer wraps around, the radius grows. How to model this? Want to have the radius growing only when the local time on the surface is growing.

If we take a BM($\mathbb{R}$) and let $L_t = \sup_{s \leq t} B_s^-$, then $\beta_t \equiv B_t + L_t$ is RBM in $\mathbb{R}^+$. Now let's have some incr. process V which grows only when L grows and set

$$\eta_t = \alpha + \beta_t + V_t \qquad (\alpha \equiv \log a)$$



We now time-change this using

$$A_t \equiv \int_0^t e^{2\eta_s}\, ds, \qquad \tau \equiv A^{-1}$$

to get

$$\tilde{\eta}_t \equiv \eta(\tau_t) = \alpha + \tilde{\beta}_t + \tilde{V}_t .$$

Thus if we define

$$\rho_t \equiv \exp \tilde{\eta}_t$$

we see that ρ bounces off the increasing process $a \exp(\tilde{V}_t)$. Now

$$\begin{aligned} d\left(\rho_t - a e^{\tilde{V}_t}\right) &= d\left((e^{\tilde{\beta}} - 1)\, a e^{\tilde{V}} \right) \\ &= a\, e^{\tilde{V}} e^{\tilde{\beta}} \left(d\tilde{\beta} + \tfrac{1}{2} d\langle \tilde{\beta} \rangle \right) \\ &= \rho \left(d\tilde{B} + d\tilde{L} + \tfrac{1}{2} e^{-2\tilde{\eta}}\, dt \right) \\ &= dW + \frac{dt}{2\rho} + \rho\, d\tilde{L} \end{aligned}$$

so that if ℓ is the local time at zero of $\rho - a e^{\tilde{V}}$, then $d\ell = \rho\, d\tilde{L}$.

One case of considerable interest is when the barrier $ae^{\tilde{V}}$ grows like $\rho^{-1}\,d\ell$, since this corresponds to the "thickening" happening uniformly around the circumference. This is easily solved by

$$a e^{\tilde{V}_t} - a = \tilde{L}_t \qquad \text{so} \qquad \boxed{V_t = \log(1 + a^{-1} L_t)}$$

Hence in this case

$$\boxed{\eta_t = \alpha + B_t + L_t + \log(1+L_t)} \qquad \text{taking } a = 1 \text{ wlog}$$

To know about windings, we need to know about asymptotics $\tau_t = \int_0^t \rho_s^{-2}\,ds$, for which knowing asymptotics of A_t would be useful:

$$A_t = \int_0^t \exp(2\eta_s)\,ds.$$

We should be able to get asymptotics of EA_t, which will be a start. Assume that $B_0 = 0$. If $Y_t \equiv B_t + L_t$, then the joint law of (L_T, Y_T), where T is indept $\exp(\frac{1}{2}\theta^2)$, is easy, so a few calculations give

$$E\, e^{2\eta(T)} = e^{2\alpha}\, E\, e^{2Y_T}(1+L_T)^2$$

$$= e^{2\alpha}\, \frac{\theta}{\theta - 2}\left(1 + \frac{2}{\theta} + \frac{2}{\theta^2}\right)$$

So if we look instead at

$$\int_0^\infty \gamma e^{-\gamma t}\, E\left(e^{2\eta_t - 2t}\right) dt = \frac{2\gamma}{\theta(\theta-2)}\left\{1 + \frac{2}{\theta} + \frac{2}{\theta^2}\right\} \qquad (\theta^2 \equiv 4 + 2\gamma)$$

$$\sim 5 \quad \text{as } \gamma \downarrow 0$$

$$\Rightarrow \quad \boxed{\int_0^T E\left(e^{2\eta_t - 2t}\right) dt \sim 5T \quad (T \to \infty)}$$

Hence $\boxed{E A_t \sim \int^t 5 e^{2s}\, ds \sim \frac{5}{2} e^{2t},}$ $(t \to \infty)$?? To justify this, would need ultimate monotonicity of $E(e^{2\eta_t - 2t})$,

(~~Note that this is the same as we'd get if the $\log(1+L_t)$ term was not present in η.~~)
which can be readily confirmed.

5) As another excursion exercise on Brownian winding, let's take a disc of radius a, and reflect BM($\mathbb{C}$) off the disc, and see how the winding goes, when we do the following:

(i) kill on the circle at rate γ in local time;
(ii) make the boundary sticky, with parameter $\delta \geq 0$;
(iii) kill at constant rate $\lambda \equiv \frac{1}{2}\theta^2$.

Let $A_t = \int_0^t R_s^{-2}\,ds$, and let T be the time of λ-or-γ killing. We want to compute

$$E \exp(-\tfrac{1}{2}\nu^2 A_T) = E \exp i\Theta(T)$$

when the starting point is on the circle. If we introduce ν-killing at rate $\nu^2/2r^2$, we want to know

$$P(\text{ T comes before any } \nu\text{-killing}).$$

Let c_1 = rate in loc. time of λ-or-ν killed excursions
c_2 = ————— excns which are λ-killed before ν-killed.

Then

$$\boxed{E \exp(-\tfrac{1}{2}\nu^2 A_T) = \frac{c_2 + \gamma + \frac{1}{2}\delta\theta^2}{c_1 + \gamma + \frac{1}{2}\delta\left(\theta^2 + \nu^2/a^2\right)}}$$

and it just remains to get c_1, c_2. Firstly, c_1. If $dR = dB + \frac{dt}{2R} + dL$ is the reflecting BES(2) process, seek f such that

$$f(R_t)\exp(-\lambda t - \tfrac{1}{2}\nu^2 A_t) \quad \text{is a mg until hit } a$$

so

$$\tfrac{1}{2}f'' + \tfrac{1}{2r}f' - (\lambda + \tfrac{\nu^2}{2r^2})f = 0 \quad \Rightarrow \quad f(r) = K_\nu(\theta r) \qquad (\text{or } I_\nu(\theta r))$$

and so

$$e^{c_1 L_t} f(R_t)\exp(-\lambda t - \tfrac{1}{2}\nu^2 A_t) \quad \text{is a mg}$$

and Itô calculus gives us that

$$\boxed{c_1 = -\theta K_\nu'(\theta a)/K_\nu(\theta a).}$$

For c_2, want to consider

$$g(x) = E^x\left(\int_0^{H_a} \lambda e^{-\lambda t - \frac{1}{2}\nu^2 A_t}\,dt\right) \equiv P^x(\text{get } \lambda \text{ mark before reach } a, \text{ or get } \nu)$$

which solves $\boxed{\mathcal{G}g - (\lambda + \tfrac{\nu^2}{2r^2})g + \lambda = 0, \quad 0 = g(a) \leq g \leq 1}$ hence $\boxed{c_2 = g'(a+)}$

Estimating the Green's function for a Lévy process (30/1/91)

1) If (X_t) is a Lévy process on $\mathbb{R}$, $\tau \equiv \inf\{t>0 : X_t > 0\}$, we wish to estimate

$$E^x\left[\int_0^\tau I_{[a,0]}(X_s)\,ds\right] \equiv \int_a^0 G^\partial(x,dy)$$

for $x<0$, $a<0$. We shall show that for some const C

$$E^x\left[\int_0^\tau I_{[a,0]}(X_s)\,ds\right] \leq C(1+a^2) \qquad \forall\, a,x \leq 0.$$

By replacing X by $X+\varepsilon W$ if need be, we may without loss of generality assume 0 is regular for $(-\infty,0)$ and $(0,\infty)$. Then if $\bar{X}_t \equiv \sup\{X_s : s \leq t\}$, the process $\bar{X}-X$ has a local time L at 0, with inverse A. Let $Y_t = \bar{X}(A_t)$, so that (A_t, Y_t) is a bivariate subordinator with Laplace exponent $\varphi_+(\cdot,\cdot)$.

Then

$$E^x\left[\int_0^\tau I_{[a,0]}(X_s)\,ds\right]$$

$$= E^x\left\{\int_0^\infty I_{[a,0]}(Y_s)\;g([a-Y_s,0])\,ds\right\}$$

where g is the Green function for excursions down from 0 of $\bar{X}-X$;

$$\int_{-\infty}^0 g(dy)\,h(y) = \int \nu(d\rho)\int_0^\zeta h(\rho_s)\,ds$$

in the notation of "A new identity for real Lévy processes" AIHP $\underset{\sim}{20}$ 1984.

$$\leq E^x\left(\int_0^\infty I_{[a,0]}(Y_s)\,ds\right)\; g([a,0]).$$

Thus the task is to estimate the Green function g of the excursions, and also the Green function of the ladder process Y.

2) Here is an auxiliary result which makes short work of the problem.

LEMMA. If $U : \mathbb{R}^+ \to \mathbb{R}^+$ is increasing right continuous, $U(0)=0$, and

$$\limsup_{\lambda \downarrow 0} \lambda^\gamma \int_0^\infty e^{-\lambda x}\,U(dx) \leq c \Rightarrow \limsup_{x\to\infty} \frac{U(x)}{x^\gamma} \leq ec.$$

Here, $\gamma > 0$ is fixed. The proof is very simple. Suppose that

(*) $\limsup_{\lambda \downarrow 0} \lambda^\gamma \int_0^\infty e^{-\lambda x} U(dx) \le C$ and $\exists x_n \uparrow \infty$ such that $\frac{U(x_n)}{x_n^\gamma} \ge (C+\varepsilon)e$

Then

$$\lambda^\gamma \int_0^\infty e^{-\lambda x} U(dx) = \left[\lambda^\gamma e^{-\lambda x} U(x)\right]_0^\infty + \int_0^\infty \lambda^{\gamma+1} e^{-\lambda x} U(x)\,dx$$

$$= \int_0^\infty \lambda^{\gamma+1} e^{-\lambda x} U(x)\,dx$$

(if for some $\lambda > 0$ $e^{-\lambda x} U(x) \not\to 0$ as $x \uparrow \infty$, then $e^{-\lambda x/2} U(x)$ is unbounded as $x \to \infty$, so the integral $\int_0^\infty e^{-\lambda x/2} U(dx)$ explodes, in contradiction of (*))

$$\ge \lambda^\gamma (C+\varepsilon) e \, x_n^\gamma e^{-\lambda x_n}$$

for each n. Taking $\lambda = x_n^{-1}$ gives a contradiction.

3) If now G_+ is Green function of Y, we have

$$\int_0^\infty G_+(0,dy)\, e^{-\lambda y} = E^0 \int_0^\infty e^{-\lambda Y_s}\, ds = 1/\varphi_+(0,\lambda).$$

Now since $\varphi_+(0,\lambda)$ is a Laplace exponent, $\lambda^{-1}\varphi_+(0,\lambda) \uparrow$ as $\lambda \downarrow 0$, whence

$$\limsup_{\lambda \downarrow 0} \lambda \int_0^\infty G_+(0,dy)\, e^{-\lambda y} \le C \quad \therefore \quad \limsup_{x\to\infty} \frac{G_+(0,x)}{x} \le eC,$$

$\therefore$ $G_+(0,x) \le C(1+x)$ for all x. Now we want to estimate for $x<0$

$$E^x\left(\int I_{[a,0]}(Y_s)\,ds\right) = E^0 \int I_{[a-x,-x]}(Y_s)\,ds$$

$$= \int_{a-x}^{-x} G_+(0,dy)$$

$$\le G_+(0,|a|) \le C(1+|a|)$$

by obvious probabilistic reasoning. The estimation of $g([a,0])$ follows similar lines, but is a little bit easier – we need to use Silverstein's

I get:

$$d\langle \Psi(\theta), \Psi(\eta)\rangle = \frac{\theta\eta}{(\theta+\eta)^3}\Big[(1+\theta)\big(\Psi(\eta) - S_0 - \eta S_1\big) - \eta(\theta+\eta)\big(\Psi'(\eta) - S_1\big) - (1+\eta)\big(\Psi(\theta) - S_0 - \theta S_1\big) + \theta(\eta+\theta)\big(\Psi'(\theta) - S_1\big) \Big]$$

$$d\langle \Psi(\theta), \Psi(\theta)\rangle = \frac{1}{6}\left\{ \theta^3 \Psi'''(\theta) + 9\,\theta^2 \Psi''(\theta) + 28\,\theta\,\Psi'(\theta) + 34 \right\}.$$

A sequence $(s_n)_{n\geq 0}$ is a moment sequence iff $\sum_{j=0}^{n}\sum_{k=0}^{n} a_j a_k s_{j+k} \geq 0$ $\forall$ real (a_j). [Akhiezer's book has this] identity ((15) on p 25 of ANIFRLP):

$$\int_{-\infty}^{0} e^{\lambda y} g(dy) = 1/\varphi_-(0,\lambda).$$

Random motion of eigenvalues of certain matrix diffusions (30/1/91)

We consider

$$d\lambda_j = dB_j + \alpha\Big(\sum_{r\neq j} \frac{1}{\lambda_j-\lambda_r}\Big)dt \qquad j=1,\dots,N$$

with initial conditions $\lambda_1(0)<\dots<\lambda_N(0)$. Can prove non-collision of eigenvalues if $\alpha\geq\frac{1}{2}$ by considering the process

$$U \equiv \sum\sum_{i<j} \log(\lambda_j-\lambda_i)$$

1) Let $S_m \equiv \sum_{j=1}^{N}(\lambda_j)^m$. Then we have

$$dS_m = m\sum_j \lambda_j^{m-1}dB_j + \tfrac{1}{2}\alpha m\Big(\sum_{r=0}^{m-2} S_r S_{m-2-r}\Big)dt$$

$$+ \frac{m(m-1)}{2} S_{m-2}\,(1-\alpha)\,dt$$

after some calculations.

Hence if $\Psi(\theta) \equiv \sum_{m\geq 0} \theta^m S_m$, we have
$d\Psi(\theta) = \sum_{m\geq 1}\sum_j m\theta^{m-1}x_j\,dB_j + \frac{\alpha\theta}{2}dt\,\frac{\partial}{\partial\theta}\big(\theta^2\Psi(\theta)^2\big) + \frac{1-\alpha}{2}\theta^2\Psi''(\theta)dt$

2) If we specialise to $m=2$, we obtain

$$dS_2 = 2\sum\lambda_j\,dB_j + (\alpha N^2+N-N\alpha)\,dt$$

$$= 2\sqrt{S_2}\,dW + (\alpha N^2+N-N\alpha)\,dt,$$

so S_2 is a BESQ process, dimension $(\alpha N+1-\alpha)N$ — in particular, no a.s. limit behaviour is going to happen, even if we scale the λ_j down.

$[\,d\langle S_m, S_k\rangle = km\,S_{m+k-2}\,dt\,]$

If $\Phi(z) \equiv \sum_j \frac{1}{x_j - z}$ $\qquad (x_j \equiv \lambda_j)$

we get

$$d\,\Phi(z) = -\sum_j \frac{dB_j}{(x_j - z)^2} + \left\{ \alpha\, \Phi\, \Phi'(z) + \tfrac{1}{2}\Phi''(z) - \tfrac{1}{2}\alpha\, \Phi''(z) \right\} dt$$

$$d\,\langle \Phi(z), \Phi(z) \rangle = \frac{1}{6}\, \Phi'''(z)\, dt$$

3) Shi Zhan has considered

$$\prod_{i<j}\prod (\lambda_j - \lambda_i) \equiv Z.$$

An Itô analysis yields

$$dZ = Z \left\{ \sum V_j \, dB_j + \alpha \sum V_j^2 \, dt \right\}$$

where

$$V_j = \sum_{r \neq j} \frac{1}{\lambda_j - \lambda_r}$$ — hence Z is a time-change of BES$(1+2\alpha)$.

4) Write $\mathcal{U}(\lambda_1, \dots, \lambda_N) = \frac{1}{2} \sum_{i \neq j}\sum \log |\lambda_i - \lambda_j|$

Then $\dfrac{\partial \mathcal{U}}{\partial \lambda_j} = \sum_{r \neq j} \dfrac{1}{\lambda_j - \lambda_r}$, so that we get the SDE in the form

$$d\lambda_j = dB_j + \alpha \, D_j \mathcal{U}(\lambda) \, dt$$

Now heuristics suggest that $\lambda_j(t)$ will grow like $\sqrt{t}$, so we consider $v_j(t) \equiv \dfrac{\lambda_j(t)}{\sqrt{1+t}}$ which satisfies

$$dv_j = \frac{dB_j}{\sqrt{1+t}} + \frac{\alpha D_j \mathcal{U}(\lambda)}{\sqrt{1+t}} dt - \frac{dt}{2(1+t)} v_j(t)$$

$$= \frac{dB_j}{\sqrt{1+t}} + \frac{\alpha \, dt}{1+t} \sum_{r \neq j} \frac{1}{v_j - v_r} - \frac{dt}{2} v_j /(1+t) .$$

Thus if we do a deterministic time-change

$$\tilde{v}_j(t) = v_j(e^t - 1)$$

we have

$$\boxed{d\tilde{v}_j = d\tilde{B}_j + \alpha \sum_{r \neq j} \frac{dt}{\tilde{v}_j - \tilde{v}_r} - \frac{1}{2} \tilde{v}_j \, dt.}$$

So if we were to define

$$V(\lambda) \equiv \alpha \, \mathcal{U}(\lambda) - \frac{1}{4} |\lambda|^2,$$

then the generator of $\tilde{v}$ is simply $\mathcal{G} = \frac{1}{2}\Delta + \nabla V \cdot \nabla$, which allows us to express

$$\mathcal{G} \equiv \frac{1}{2}\left(\Delta + \frac{\nabla\psi}{\psi}\cdot\nabla\right) \qquad \text{where} \qquad \psi = e^{2V}$$

from which we see that the process $\tilde{v}$ has generator

$$\mathcal{G} = \frac{1}{2\psi}\nabla\cdot(\psi\nabla)$$

and so has invariant measure with density proportional to

$$\boxed{e^{2V} = e^{-|\lambda|^2/2} \prod_{i \neq j}\prod |\lambda_i - \lambda_j|^{\alpha}.}$$

(This is all in the Dyson paper in some form or other.....)

Consumption / investment model considered by Karatzas (5/2/91).

There is an endowment process $(\varepsilon_t)_{0\le t\le T}$ and a (strictly concave, strictly increasing) utility function U given, and the aim is to choose a consumption process $(c_t)_{0\le t\le T}$ so that

$$E\int_0^T U(c_s)\,ds \text{ is maximal, subject to } \int_0^T \{c_s - \varepsilon_s\}\,ds = 0.$$

It appears that the solution must in general make use of the explicit form of U; there is no "general theory" quick fix!

To see this, consider a situation where $0 < a < b$, and

$$P(Y=a) = p = 1 - P(Y=b) = 1-q,$$

$\mathcal{F}_t = \{\emptyset, \Omega\}$ for $0 \le t < \frac{1}{2}$, $\mathcal{F}_t = \sigma(Y)$ for $t \ge \frac{1}{2}$, and Y is going to be the total endowment. The control problem is nearly trivial; at time 0, one chooses an amount c to be consumed between 0 and $\frac{1}{2}$ (which is then consumed at constant rate $2c$) and then once Y is revealed, the remaining $Y-c$ is consumed uniformly in $(\frac{1}{2},1)$. Then

$$E\left[\int_0^1 U(c_s)\,ds\right] = \int_0^{\frac{1}{2}} U(2c)\,ds + \int_{\frac{1}{2}}^1 EU(2Y-2c)\,ds$$

$$= \frac{1}{2}\left\{U(2c) + p\,U(2a-2c) + q\,U(2b-2c)\right\}$$

and for optimality

$$0 = U'(2c) - p\, U'(2a-2c) - q\, U'(2b-2c).$$

The optimal c will evidently depend on the explicit form of U.

PS (6/7/93) Suppose we could find a mg M such that

$$\boxed{\int_0^T I(M_s)\,ds = \int_0^T \varepsilon_s\,ds \quad \text{a.s.},}$$

where $I \equiv (U')^{-1}$.

Then c^* would be $c^*_t \equiv I(M_t)$. This is because if c is any other feasible consumption plan,

$$E\left[\int_0^T \{U(c_s) - U(c^*_s)\}\,ds\right] \leq E\left[\int_0^T (c_s - c^*_s)\,U'(c^*_s)\,ds\right]$$

$$= E\left[\int_0^T (c_s - c^*_s)\,ds\; M_T\right]$$

$$= 0.$$

(ii) Taking RBM in wedge, what's the easiest way to prove that the process is a semimg $\Leftrightarrow \alpha = (\theta_1+\theta_2)/\xi < 1$?

(iii) Some of the h-transformed things are interesting; eg with Lipschitz domain, can one show that $\frac{1}{t}\log P_x^h(\tau_D > t) \to -\lambda_D$, where λ_D is lowest e-value of $-\frac{1}{2}\Delta$?

10) Can one do Skorohod embedding à la Azéma-Yor, but with the barrier set depending on L_t?

One can embed a random walk with steps of finite variance in a BM; can we get Donsker's thm from this? (The problem is to show that if T_n are the embedding times, $ET_n \equiv \rho_n^2$, then $(T_n - \rho_n^2)/\rho_n \xrightarrow{L^1} 0$)

11) Suppose we take isotropic diff^n in $\mathbb{R}^d$, $d \geq 3$, with generator

$$\mathcal{L} = \frac{1}{2} D_i a D_i$$

with a being the diffusivity. The effective diffusivity of this is (in $d<3$)

$$a_{eff} = \lim_{R\to\infty} \frac{1}{R^2} E^0[\text{time spent in ball of radius } R],$$

assuming this limit exists. Can we get bounds

$$\lim_R \left(\int_{|x|\leq R} a(x)^{-1} \frac{dx}{V_R}\right)^{-1} \leq a_{eff} \leq \lim_R \int a(x) \frac{dx}{V_R}, \quad V_R = \text{vol of ball radius } R \text{ ?}$$

12) Take two i.i.d one-dimensional diffusions with different starting laws - can one get the distribution of the time til they meet?

$$= -2T\beta \int_0^1 \frac{\varphi_t-\varphi_u}{|\varphi_t-\varphi_u|} \frac{du}{(1+T^\gamma|\varphi_t-\varphi_u|)^{1+\beta}}$$

If now we supposed that φ lies always along a straight line, we would get from this

$$T^{\gamma-2}\,\ddot\varphi(t) \doteq -2T\beta \int_0^1 \mathrm{sgn}(t-u)\,\frac{du}{|\varphi_t-\varphi_u|^{1+\beta}}\; T^{-\gamma(1+\beta)},$$

the integral existing as a Cauchy principal value. This suggests that the powers of T on each side must match, so $\gamma = 3/(2+\beta)$ (the Flory exponent in dimension β!)

6) RBM in 2D – I wondered whether there might be some "reversibility" of the form

$$\mathrm{Re}\left(\frac{1}{\psi(z)}\right) G(z,z_0) = \mathrm{Re}\,\psi(z_0)\; G^*(z_0,z)$$

where G^* is got as for G, but with θ replaced by $-\theta$. Notice that this is crazy; if θ is held away from $-\frac{\pi}{2}, \frac{\pi}{2}$, then $\mathrm{Re}\,\psi(z) \sim \mathrm{Im}\,\psi(z) \sim |\psi(z)|$, and if you get transience for θ, you certainly won't for $-\theta$!! So G^* is not defined!

7) Some outstanding questions on the polymer models: firstly in $d=1$,
  (i) does one get linear growth for $f \in (C_K^1)^+$? $f \in (L^1)^+ \cap C^1$?
  (ii) What happens for $f =$  ?
  (iii) Can we compute the prob'y that the process goes on the upper route when $f(x) \sim x^{-\beta}$?
then in $d>1$,
  (iv) for the slow tail do we have asymptotics like in $d=1$?
  (v) Compactly supported $f \Rightarrow X$ behaves like B?

8) James Taylor poses the question "If Z is $BM(\mathbb{C})$, can the path $\{Z_t : 0 \le t \le 1\}$ be cut by a straight line?" Chris Burdzy has shown that this can be done with a Lipschitz curve.

9) Reviewing Dante De Blassie's work, there are a number of interesting questions which feel like they should have an easier answer:
  (i) Take some cone $C \subseteq \mathbb{R}^d$ (not necessarily convex), let τ_C be the exit time, and consider for $x \in C$, $P_x(\tau_C > t)$. What are the asymptotics as $t \to \infty$? Feels like skew product + Pitman-Yor things + Tauberian should be a viable approach.

QUESTIONS, CONJECTURES, etc...

1) Mike Gage, University of Rochester, asks the following question. Let K be a compact subset of $\mathbb{R}^d$, let K_n be the compact $\frac{1}{n}$-nbd of K, and let τ be the first exit time from K (τ_n the 1st exit time from K_n). The lowest e-value of $-\frac{1}{2}\Delta$ on D with Dirichlet b.c.s satisfies

$$\alpha = \sup\{\lambda > 0 : E^x \exp(\lambda\tau) < \infty\}$$

and clearly, since $\tau_n \geq \tau$, $\alpha_n \uparrow$. Is $\lim \alpha_n = \alpha$?

2) Ruth Williams + Wei An Zheng have a construction of reflecting BM in bad domains by Dirichlet process techniques. Is there some general Mkv-process construction of a "reflecting" process, using some sort of conditioning/killing?

3) Let X^1, X^2 be two indep. Mkv processes with invariant σ-fields $\mathcal{I}^1, \mathcal{I}^2$. If $\mathcal{I}$ is the invariant σ-field of (X^1, X^2) then $\mathcal{I}^1 \vee \mathcal{I}^2 \subseteq \mathcal{I}$. How about the other way? Not true - see p6.

4) Jean-Dominique Deuschel tells me that the following inequality is known. If $X_1, \ldots, X_n$ are i.i.d, mean 0, say, and variance σ^2, and f is C^1, then

$$\operatorname{var}(f(X_1, \ldots, X_n)) \leq \sigma^2 \sum_{k=1}^{n} (\|D_k f\|_\infty)^2$$

The case $n=1$ is easy, but apparently the others are toughies...

5) The physicists' model of polymers leads one quite naturally to study the problem of

$$\min \int_0^T \dot{x}_s^2\, ds + \int_0^T ds \int_0^T du\ \psi(x_s - x_u),$$

particularly to give info on end-to-end distance. If we suppose $\psi(x) = (1+|x|)^{-\beta}$, for $0<\beta<1$, we conjecture that best x looks like $x_t \sim T^\gamma \varphi(t/T)$ for some $\gamma > 0$, and "limit-shape" φ. The variational form of the problem is

$$\ddot{x}_s = 2\int_0^T du\ \nabla\psi(x_s - x_u)$$

i.e. $$T^{\gamma-2}\ddot{\varphi}(\tfrac{t}{T}) \doteq 2\int_0^T du\ \nabla\psi\left(T^\gamma(\varphi_{t/T} - \varphi_{u/T})\right)$$

$$\Rightarrow T^{\gamma-2}\ddot{\varphi}(t) = 2T\int_0^1 du\ \nabla\psi\left(T^\gamma(\varphi_t - \varphi_u)\right)$$

$\sqrt{a+ib} = \alpha + i\beta$, where $a+ib \notin (-\infty, 0)$,

and $\alpha = \left(\frac{\sqrt{a^2+b^2}+a}{2}\right)^{\frac{1}{2}}$, $\beta = \operatorname{sgn}(b)\left(\frac{\sqrt{a^2+b^2}-a}{2}\right)^{\frac{1}{2}}$
