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Limit laws of transient 1-d diffusions (11/2/91)

Suppose we were given a diffusion

$$dX_t = \sigma(X_t)\,dB_t + b(X_t)\,dt$$

which goes to $+\infty$ a.s... What form does the criterion of Hobson + Rogers take (for the a.s. convergence)? Let's make X reflect off 0,

$$s'(x) = \exp\left(-\int_0^x 2\sigma^{-2}b(z)\,dz\right),$$

and take

$$\varphi(x) \equiv \int_x^\infty s'(u)\,du = -s(x).$$

If $Y_t \equiv \varphi(X_t)$, we have that Y is a diffusion in natural scale with speed measure $m(dy) = (\varphi'\sigma\circ\varphi^{-1}(y))^{-2}\,dy$ and, if $\varphi(0) = a$,

$$m(y) = \int_y^a (\varphi'\sigma\circ\varphi^{-1}(z))^{-2}\,dz = \int_0^{\varphi^{-1}(y)} \frac{du}{|\varphi'\sigma^2(u)|},$$

whence,

$$c(y) = 2\int_0^{\varphi^{-1}(y)} \varphi'(z)\left(\int_0^z \frac{du}{\varphi'\sigma^2(u)}\right)dz.$$

(The condition for convergence in probability, that c is slowly varying, is equivalent to $x\,m(x)/c(x) \to 0$ $(x\downarrow 0)$.)

The condition for almost sure convergence now becomes

$$\boxed{\int^\infty s'(u)\,du \; \frac{\int_0^u s'(t)\left(\int_0^t \frac{dx}{s'\sigma^2(x)}\right)^2 dt}{\left(\int_0^u s'(t)\left(\int_0^t \frac{dx}{s'\sigma(x)^2}\right)dt\right)^2} < \infty}$$

Example An SDE which came up with Shi Zhan was

$$dX_t = dW_t + \alpha\,\frac{1+|X_t|^\beta}{X_t}\,dt, \qquad (0<\beta<1)$$

for which it is not too hard to check the boxed condition. Now the mean hitting time to level $x > 0$ grows like $x^{2-\beta}/\alpha(2-\beta)$, from which

$$\frac{X_t}{(\alpha(2-\beta)t)^{1/(2-\beta)}} \xrightarrow{a.s.} 1.$$

A question of D. Gough (16/2/91)

Take a finite sequence of harmonic motions with damping, and random forcing

$$\ddot{x}_i + \lambda_i \dot{x}_i + \omega_i^2 x_i = \sigma \dot{B}_i$$

where B_i is a Brownian motion. One observes only $X(t) = \sum_{i=1}^{N} x_i(t)$ and wants to make inferences about the unknown parameters λ_i, ω_i.

(i) If we assumed that $x_i(0) = \dot{x}_i(0) = 0$, then Laplace transforming gives

$$(s^2 + \lambda_i s + \omega_i^2)\, \tilde{x}_i(s) = \sigma \tilde{B}_i(s)$$

$$\Rightarrow \tilde{x}_i(s) = \frac{1}{(s-\alpha)(s-\beta)} \int_0^\infty e^{-st}\, dB_i(t)$$

where α, β are roots of the quadratic. Hence

$$x_i(t) = \frac{\sigma}{\alpha-\beta} \int_0^t \{ e^{\alpha(t-u)} - e^{\beta(t-u)} \}\, dB_i(u)$$

and so if we took instead

$$\boxed{x_i(t) = \sigma \int_{-\infty}^{t} f_i(t-u)\, dB_i(u)}$$

we would have the stationary form of x. $[f_i(t) \equiv (e^{\alpha t} - e^{\beta t})/(\alpha-\beta)]$

This puts us into time series technology.

(ii) The actual problem posed has $B_i = B$ for $i = 1, \dots, N$, so that

$$X(t) = \sigma \int_{-\infty}^{t} f(t-u)\, dB(u)$$

where $f = \Sigma f_i$, and so (if we define $f(t) = 0$ for $t < 0$) we can express the autocovariance of X as

$$E\, X(t)\, X(t+h) = \sigma^2 \int f(u) f(u+h)\, du,$$

whence the spectral function is

$$\int_{-\infty}^{\infty} e^{i\theta h}\, E[X(0)X(h)]\, dh = \sigma^2 |\hat{f}(\theta)|^2 = \sigma^2 \left| \sum_{j=1}^{N} \frac{1}{\theta^2 + i\theta\lambda_j - \omega_j^2} \right|^2.$$

(iii) The spectral density could be estimated from data, but since the form of f is so specific, it looks like time-domain methods are going to be more suitable. If we took some discrete approximation to the process x_i, say at integer time-steps, we'd have

$$x_{n+1} - 2x_n + x_{n-1} + \lambda_i (x_n - x_{n-1}) + w_i^2 x_n = \varepsilon_{n+1}$$

or $$\left(I - P_i(B)\right) x = \varepsilon$$

where $P_i(B) = (2 - \lambda_i - w_i^2) B + (\lambda_i - 1) B^2$, B the backward shift.
This implies

$$x = \left(I - P_i(B)\right)^{-1} \varepsilon$$

$$\therefore \quad X = \sum_{i=1}^{N} \left(I - P_i(B)\right)^{-1} \varepsilon$$

$$\therefore \quad \prod_{i=1}^{N} \left(I - P_i(B)\right) X = \left\{ \sum_{i=1}^{N} \prod_{j \neq i} \left(I - P_j(B)\right) \right\} \varepsilon$$

which is an ARMA(2N, 2N−2) process, and there are standard packages for estimating the parameters of such a thing. However, building in the relations between the coefficients could be untidy.

Pricing of an option when there is premature exercise (19/2/91)

If one considers an American call option, and assumes a non-negative interest rate, then no one should sell until the termination time T. But it could be that some individuals are forced by circumstances to sell early, so the writer of the option is actually doing a bit better than theory predicts. Take the simple model

$$dS_t = S_t \{ \sigma \, dW_t + \mu \, dt \}$$

where σ, μ are constants, and the exercise price is K, and the interest rate is $r > 0$, constant. Want to price $(S_\tau - K)^+$, where τ is the exercise time, perhaps $< T$. Thus conventional wisdom says price is

$$\tilde{E}\left[e^{-r\tau} (S_\tau - K)^+ \right] \equiv \tilde{E}\left[(\tilde{S}_\tau - K e^{-r\tau})^+ \right]$$

where $\tilde{S}_t = e^{-rt} S_t$ is the discounted price process, which is a martingale under $\tilde{P}$.
Suppose now that the option is exercised at rate ρ_t, so that we need to find

$$\tilde{E}\left[e^{-A_T}(\tilde{S}_T - Ke^{-rT})^+ + \int_0^T \rho_t e^{-A_t} (\tilde{S}_t - Ke^{-rt})^+ dt \right]$$

where $A_t \equiv \int_0^t \rho_s ds$. No real hope of this in general, but if we took $\rho_t = \rho(t, S_t)$ we can do F-K things; if we define

$$\psi(t,x) \equiv \tilde{E}\left[\int_t^T \rho_u e^{-(A_u - A_t)} (\tilde{S}_u - Ke^{-ru})^+ du + e^{-A_T + A_t} (\tilde{S}_T - Ke^{-rT})^+ \right]$$

when $S_t = x$ is the starting condition, then

$$\int_0^t \rho_u e^{-A_u} (\tilde{S}_u - Ke^{-ru})^+ du + \psi(t, \tilde{S}_t) e^{-A_t}$$ is a martingale, so

$$\boxed{\begin{aligned} &\frac{\partial \psi}{\partial t} + \tfrac{1}{2}\sigma^2 x^2 \frac{\partial^2 \psi}{\partial x^2} - \rho(t,x)\psi + \rho(x - Ke^{-rt})^+ = 0 \\ &\psi(T,x) \equiv (x - Ke^{-rT})^+ \end{aligned}}$$

It's unlikely that anything will work here, even if we took $\rho(t,x) = \rho(x)$. However, the rather trivial special case $\rho = $ const can be handled by the usual BS formula; if Y is the contingent claim

$$Y = e^{-\rho T}(\tilde{S}_T - Ke^{-rT})^+ + \int_0^T \rho e^{-\rho u} (\tilde{S}_u - Ke^{-ru})^+ du,$$

then

$$\boxed{\begin{aligned} \tilde{E}(Y) = &\left\{ S_0 \bar{\Phi}\left(\frac{\log(K/S_0) - rT - \sigma^2 T/2}{\sigma\sqrt{T}} \right) - Ke^{-rT} \bar{\Phi}\left(\frac{\log(K/S_0) - rT + \sigma^2 T/2}{\sigma\sqrt{T}} \right) \right\} e^{-\rho T} \\ &+ \int_0^T \rho e^{-\rho t} \left\{ S_0 \bar{\Phi}\left(\frac{\log(K/S_0) - rt - \sigma^2 t/2}{\sigma\sqrt{t}} \right) - Ke^{-rt} \bar{\Phi}\left(\frac{\log(K/S_0) - rt + \sigma^2 t/2}{\sigma\sqrt{t}} \right) \right\} dt. \end{aligned}}$$

Model for stock price with some memory (23/2/91)

(i) David Hobson suggests a model for log of a stock price in the form

$$\boxed{dY_t = \sqrt{S_t}\, dB_t + \mu_t\, dt}$$

where

$$S_t \equiv a \int_{-\infty}^{t} e^{\lambda(u-t)} (Y_u - Y_t)^2\, du .$$

Let

$$R_t \equiv \lambda \int_0^\infty e^{-\lambda u} (Y_t - Y_{t-u})\, du .$$

Always, we have $R_t^2 \leq \frac{\lambda}{a} S_t$. The drift μ will be made explicit later. It is not hard to prove that

$$\begin{cases} dR_t = dY_t - \lambda R_t\, dt \\ dS_t = \frac{2a}{\lambda} R_t\, dY_t + (\frac{a}{\lambda} - \lambda) S_t\, dt \end{cases}$$

(ii) Consider now $U_t \equiv R_t / \sqrt{S_t}$. We have from Itô's formula that

$$dU_t = \left(1 - \frac{aU^2}{\lambda}\right) dB + \frac{\mu}{\sqrt{S}} \left(1 - \frac{a}{\lambda} U^2\right) dt - \lambda U\, dt - \left(\frac{a}{\lambda} - \lambda\right) U \frac{dt}{2}$$

$$+ \frac{3a^2}{2\lambda^2} U^3\, dt - \frac{a}{\lambda} U\, dt$$

so if we take

$$\boxed{\mu = \gamma_1 R + \gamma_2 \sqrt{S}}$$

we get that U solves the autonomous SDE

$$\boxed{\begin{aligned} dU = \left(1 - \tfrac{a}{\lambda} U^2\right) dB - \left(\lambda + \tfrac{3a}{\lambda}\left(1 - \tfrac{a}{\lambda} U^2\right)\right) U \tfrac{dt}{2} \\ + \left(1 - \tfrac{a}{\lambda} U^2\right)(\gamma_1 U + \gamma_2)\, dt. \end{aligned}}$$

(iii) Let's now observe that $\log S$ is something we can characterise reasonably easily. Indeed, by Itô's formula once again

Notation is not very good. Define $\sigma^2 = a/\lambda$, $V = \sigma U$, $Z = \frac{V}{\sqrt{1-V^2}}$.

Then

$$\boxed{\begin{aligned} dZ &= \sigma\sqrt{1+Z^2}\, dB - \tfrac{\lambda}{2} Z(1+Z^2)\, dt + \gamma_1 Z\, dt + \sigma\gamma_2\sqrt{1+Z^2}\, dt, \\ A_t &= \lambda \int_0^t (Z_s^2 - 1)\, ds, \\ S_t &= \frac{S_0}{\lambda}(1+Z_t^2)\, e^{A_t} \\ R_t &= \sqrt{\frac{S_0}{a}}\, Z_t\, e^{\frac{1}{2}A_t}. \end{aligned}}$$

Now the case $\gamma_2 = 0$ seems to be most realistic in terms of the model.
Invariant density is proportional to

$$(1+x^2)^{-1+\gamma_1/\sigma^2}\, e^{-\lambda x^2/2\sigma^2}$$

in that case.

$$dV = \sigma(1-V^2)dB - \tfrac{1}{2}\left(\lambda + 3\sigma^2(1-V^2)\right)V\, dt + (1-V^2)(\gamma_1 V + \sigma\gamma_2)\, dt.$$

If we observed Y, we know $d\langle Y\rangle_t/dt = S_t$. So know $d\langle \log S_t\rangle/dt = \frac{4\sigma^2 Z^2}{1+Z^2}$
$= 4\sigma^2 V^2$

Thus we know $d\langle \sigma^2 V^2\rangle/dt = 4\sigma^6 V^2(1-V^2)^2$
$= 4\sigma^2 V^2 \cdot \sigma^4(1-V^2)^2$

$\therefore$ know $\sigma^4(1-V^2)^2$ $\therefore$ know $\sigma^2(1-V^2)$ $\therefore$ know $\frac{1}{V^2} - 1$ $\therefore$ know V^2, and hence know σ^2. Since we know $V^2 = \frac{Z^2}{1+Z^2}$, we know $1+Z^2$, and, from $\log S$, we can now deduce A_t, and hence λ.

Next, $R_t = e^{-\lambda t}\left(R_0 + e^{\lambda t}Y_t - Y_0 - \int_0^t \lambda e^{\lambda s} Y_s\, ds\right)$, so we can from this get (R_t).

$$d(\log S) = \frac{2a}{\lambda}\{U\,dB + (\gamma_1 U^2 + \gamma_2 U)\,dt\} + \left(\frac{a}{\lambda} - \lambda\right)dt - \frac{2a^2}{\lambda^2}U^2\,dt$$

and

$$d\log(\lambda - aU^2) = -\frac{2a}{\lambda}\{U\,dB + (\gamma_1 U^2 + \gamma_2 U)\,dt\} + \frac{aU^2}{\lambda - aU^2}\left\{\lambda + \frac{3a}{\lambda^2}(\lambda - aU^2)\right\}dt$$

$$- \left(\frac{a}{\lambda} + \frac{a^2U^2}{\lambda^2}\right)dt$$

which gives that

$$d\left[\log S + \log(\lambda - aU^2)\right] = \left\{\frac{\lambda a U^2}{\lambda - aU^2} - \lambda\right\}dt,$$

which we could just as well get from looking at

$$d(\lambda S - aR^2) = -\lambda(\lambda S - 2aR^2)\,dt.$$

Thus we can express S in terms of U: if $A_t \equiv \int_0^t \left\{\frac{\lambda a U_s^2}{\lambda - aU_s^2} - \lambda\right\}ds$,

then

$$\boxed{S_t = S_0(\lambda - aU_t^2)^{-1}\exp(A_t)}$$

Thus

$$\boxed{\begin{aligned} R_t &= S_0^{\frac{1}{2}}\frac{U_t}{(\lambda - aU_t^2)^{\frac{1}{2}}}\exp\left\{\frac{1}{2}\int_0^t\left(\frac{\lambda a U_s^2}{\lambda - aU_s^2} - \lambda\right)ds\right\} \\ \text{and}\quad Y_t &= R_t + \int_0^t \lambda R_s\,ds + Y_0 - R_0. \end{aligned}}$$

(iv) If $\theta = \sqrt{\lambda/a}$, I compute the invariant density of U to be proportional to

$$(\lambda - ax^2)^{-\frac{1}{2} - \frac{\lambda\gamma_1}{a}}\left(\frac{\theta + x}{\theta - x}\right)^{\lambda\gamma_2/a\theta}\exp\left\{-\frac{\lambda^3}{2a(\lambda - ax^2)}\right\}$$

Note that to preclude $R_t \to \infty$, or R_t exploding in an oscillatory way, must have $EA_t = 0$ under invariant law for U.

Convergence of certain measure-valued processes (27/2/91)

1) Consider the SDE

$$d\lambda_j = dB_j + \alpha\, dt \sum_{r\neq j} \frac{1+|\lambda_j-\lambda_r|^\beta}{\lambda_j-\lambda_r} \qquad (j=1,\dots,N)$$

where $0<\beta<1$. The case $\beta=0$ is the one considered by Terry Chan. For $\alpha \geq 1/2$, Shi Zhan has shown that the λ_i do not collide.

Set

$$v_j = \frac{\lambda_j}{(1+t)^\gamma}, \qquad \text{where } \gamma = \frac{1}{2-\beta} > \frac{1}{2}.$$

Then

$$dv_j = \frac{dB_j}{(1+t)^\gamma} + \left\{ -\gamma v_j + \alpha \sum_{r\neq j} \frac{|v_j-v_r|^\beta}{v_j-v_r} \right\} \frac{dt}{1+t}$$

$$+ \frac{\alpha\, dt}{(1+t)^{2\gamma}} \sum_{r\neq j} \frac{1}{v_j-v_r}.$$

If we time change by the deterministic additive functional $A_t = \int_0^t \frac{ds}{1+s} = \log(1+t)$, inverse $\tau_t = e^t-1$, with $\tilde v_j(t) \equiv v(\tau_t)$, we get

$$\boxed{d\tilde v_j = \left\{ -\gamma \tilde v_j + \alpha \sum_{r\neq j} \frac{|\tilde v_j-\tilde v_r|^\beta}{\tilde v_j-\tilde v_r} \right\} dt + dW_j\, e^{-\gamma\beta t/2} + \alpha e^{-\gamma\beta t}\left(\sum_{r\neq j} \frac{1}{\tilde v_j-\tilde v_r}\right) dt}$$

2) For large t, only the first term on the RHS remains important, and it looks like the $\tilde v_j(t)$ will converge to the solution of

$$\gamma x_j = \alpha \sum_{r\neq j} \frac{|x_j-x_r|^\beta}{x_j-x_r}.$$

This is proved elsewhere.

The main point is that for large N, it looks like $\tilde v_j \sim N^\gamma x_j$ (very roughly) so let's define

$$\langle \mu_N(t), f\rangle = \frac{1}{N} \sum_{r=1}^{N} f(\tilde v_j(t)/N^\gamma)$$

for suitable test functions f (say, $f \in C_b^2(\mathbb{R})$). This defines the measure-valued process of interest. If we set

$$\xi_j = \tilde v_j / N^\gamma,$$

we have

$$d\xi_j = \left\{ -\gamma\xi_j + \frac{\alpha}{N}\sum_{r\neq j} \frac{|\xi_j-\xi_r|^\beta}{\xi_j-\xi_r} \right\}dt + \frac{e^{-\gamma\beta t/2}\,dW_j}{N^\gamma} + \frac{\alpha e^{-\gamma\beta t}}{N^{2\gamma}} \sum_{r\neq j}\left(\frac{1}{\xi_j-\xi_r}\right)dt$$

so that

$$d\left(\langle\mu_N, f\rangle\right) = b(\mu_N, f')\,dt + \frac{e^{-\gamma\beta t/2}}{N^{1+\gamma}}\sum_j f'(\xi_j)\,dW_j + \frac{\alpha e^{-\gamma\beta t}}{N^{2\gamma-1}}\, b_0(\mu_N, f')\,dt$$

$$+ \frac{e^{-\gamma\beta t}}{2N^{2\gamma}}(1-\alpha)\,\langle\mu_N, f''\rangle dt,$$

where

$$b(\mu, f) \equiv \frac{\alpha}{2}\iint \mu(dx)\,\mu(dy)\,\frac{f(x)-f(y)}{x-y}\,|x-y|^\beta - \int \mu(dx)\,\gamma x f(x),$$

$$b_0(\mu, f) \equiv \frac{1}{2}\iint \mu(dx)\,\mu(dy)\,\frac{f(x)-f(y)}{x-y}.$$

Thus we have for $f \in C_b^2(\mathbb{R})$

$$\langle\mu_N(t), f\rangle - \langle\mu_N(0) f\rangle = \int_0^t b(\mu_N(s), f')\,ds + \text{the rest},$$

where "the rest" is a semimg whose fv part has derivative $\leq$ const $/ N^{2\gamma-1}$, and whose martingale part has q.v. at worst $C\int_0^t N^{-2\gamma-1}ds$. Thus "the rest" converges to 0 in probability.

3) To get convergence of the measure-valued process, we can use Thm 3.9.1 of Ethier + Kurtz. This will give the relative compactness of the family $(\mu_N(t))_{t\geq0}$, $N=1,2,\ldots$ as a process <u>with values in</u> $\mathcal{P}([-\infty,\infty])$ provided we can prove the relative compactness of the real-valued processes

$$F(\langle\mu_N, g_1\rangle, \ldots, \langle\mu_N, g_k\rangle)$$

where $F \in C_b^2(\mathbb{R}^k)$, and the g_i are polynomials in $\tanh x$ (to be quite explicit!)

<u>Then</u> we have to prove that any possible limit has to take values in $\mathcal{P}(\mathbb{R})$.

The relative compactness of $F(\langle\mu_N, g_1\rangle, \ldots, \langle\mu_N, g_k\rangle)$ will be OK (If g is a polynomial in $\tanh x$, then $b(\mu, g)$ will be bounded — C_b^1 will do for this)

To prove that any possible limit must actually take values in $\mathcal{P}(\mathbb{R})$, let's consider $f(x) = (1+x^2)^{1/2}$, which has bounded first + second derivatives. We can easily bound the probability that "the rest" gets above some value, and only the integral of $b(\mu_N(s), f')$ needs any care. But

$$b(\mu, f') = \frac{\alpha}{2} \iint \mu(dx)\mu(dy) \underbrace{\frac{f'(x)-f'(y)}{x-y} |x-y|^\beta}_{\text{bounded}} - \int \mu(dx)\, \gamma \underbrace{x f'(x)}_{\text{non-negative}}$$

Thus given $\varepsilon > 0$, $T > 0$, can find K so large that $\forall N$

$$P\left[\sup_{s \le T} \langle \mu_N(s), f \rangle > K \right] \le \varepsilon$$

$\therefore \forall M$, $$P\left[\sup_{s\le T} \langle \mu_N(s), f \wedge M \rangle > K \right] \le \varepsilon$$

Thus if μ is any weak limit of the μ_N,

$$P\left[\sup_{s \le T} \langle \mu(s), f\wedge M \rangle > K \right] \le \varepsilon.$$

From this it is easy to conclude $P\left[\text{for all } s \le T,\ \mu(s)(\{-\infty, +\infty\}) = 0 \right] = 1.$

4) So we get that for all $f \in C_b^2(\mathbb{R})$

$$\boxed{\langle \mu(t), f \rangle = \langle \mu(0), f \rangle + \int_0^t b(\mu(s), f')\, ds}$$

Any limit μ should satisfy for all $f \in C_b^2(\mathbb{R})$

$$\boxed{0 = b(\mu, f') = \frac{\alpha}{2} \iint \mu(dx)\mu(dy) \frac{f'(x)-f'(y)}{x-y} |x-y|^\beta - \gamma \int x f'(x)\, \mu(dx)}$$

<u>ABRACADABRA</u> ! (1/3/91)

This nice observation on the ABRACADABRA example in DW's book Prob with Marts comes from Kalvis Jansons. If you have an alphabet of size K, then the mean hitting time to the word $a_1 \dots a_n$ is

$$\tau(a_1 \dots a_n) = \sum_{r=1}^{n} K^{n+1-r}\, I_{\{a_r \dots a_n = a_1 \dots a_{n-r+1}\}}.$$

[This is the generalisation of the story in DW, or in Sheldon Ross reference]

If now you consider what happens if you draw the letters one at a time i.i.d. from the alphabet, then

$$\tau(a_1 \cdots a_n) - K^n - n \quad \text{is a martingale !}$$

(now the a_i are random). The proof is based on the fact that for $r > 1$, $r \le n$,

$$P(a_r \cdots a_{n+1} = a_1 \cdots a_{n+2-r} \mid \mathcal{F}_n) = K^{-1} I_{\{a_r \cdots a_n = a_1 \cdots a_{n+1-r}\}}$$

and is immediate.

Hence in particular, the expected time to see a randomly-chosen sequence of length n is $K^n + n - 1$.

<u>Repelling particles with periodic boundary conditions</u> (4/3/91)

Fixing some $K > 0$, we could put particles down in $[0, K)$, then repeat positions periodically, and imagine these particles as diffusing with the $1/x$ repulsion. We would now have to compute

$$\sum_{n \in \mathbb{Z}} \frac{1}{x - (y + nK)} = \sum_{n \in \mathbb{Z}} \frac{1}{a - nK} \qquad (a \equiv x - y)$$

$$= \frac{1}{a} + \sum_{n \ge 1} \left(\frac{1}{a - nK} + \frac{1}{a + nK} \right)$$

$$= \frac{1}{a} + \sum_{n \ge 1} \frac{2a}{a^2 - n^2K^2}.$$

Now it can be shown that

$$\sum_{n \in \mathbb{Z}} \frac{1}{n^2 - a^2} = -\frac{\pi \cot \pi a}{2a},$$

so that the combined repulsion of the periodically-sited particles is none other than

$$\frac{\pi}{K} \cot \pi (x - y)/K,$$

and we have the SDE

$$\boxed{d\lambda_j = dB_j + \left\{ \alpha \sum_{r \ne j} \frac{\pi}{K} \cot \frac{\pi}{K}(\lambda_j - \lambda_r) \right\} dt.}$$

Note the identity $\displaystyle\sum_{n \ge 1} \frac{1}{n^2 + a^2} = \frac{1}{2a} \left\{ \pi \coth \pi a - \frac{1}{a} \right\}$ — excursion interpretation??

More on quadratic functionals (4/3/91)

1) Take BM $(B_t)_{0\le t\le 1}$, $B_0 = x$, a measure μ on $[0,1]$, $\mu(t) \equiv \mu([0,t])$, assume $\mu(0)=0$, $\mu(1)=1$, and μ is cts strictly increasing. Want to compute the joint LT

$$E^x \exp\left\{ -\tfrac12 \nu^2 \left(\int_0^1 B_s^2\, \mu(ds) - \rho\left(\int_0^1 B_s\, \mu(ds)\right)^2 \right) - \tfrac12 \theta^2 B_1^2 \right\} \equiv \varphi .$$

Abbreviate $\bar B \equiv \int_0^1 B_s\, \mu(ds)$, and assume $0 \le \rho \le 1$. Let W, Z be two indept Brownian motions, so that

$$\varphi = E^x \exp\left\{ -\tfrac12 \nu^2 \int_0^1 (B_s - \varepsilon \bar B)^2 \mu(ds) - \tfrac12 \theta^2 B_1^2 \right\}, \qquad 2\varepsilon - \varepsilon^2 \equiv \rho ;$$

$$= E^x \exp\left\{ -\tfrac12 \nu^2 \int_0^1 (B_{\tau_s} - \varepsilon \bar B)^2 ds - \tfrac12 \theta^2 B_1^2 \right\}, \qquad \text{where } \tau \text{ is inverse to } \mu ;$$

$$= E^x \exp\left\{ i\nu \int_0^1 (B_{\tau_s} - \varepsilon \bar B)\, dW_s + i\theta Z_1 B_1 \right\} .$$

Now we use the Yor trick on the interesting bit:

$$\int_0^1 (B_{\tau_s} - \varepsilon \bar B)\, dW_s = \int_0^1 (B_s - \varepsilon \bar B)\, d\tilde W_s \qquad \left(\tilde W_t \equiv W(\mu_t) \right)$$

$$= \int_0^1 \left\{ \tilde W_1 (1 - \varepsilon + \varepsilon \mu_s) - \tilde W_s \right\} dB_s + (1-\varepsilon) x \tilde W_1 ,$$

after a few calculations. Thus

$$\varphi = E \exp\left[i\nu \int_0^1 (\tilde W_1 (1-\varepsilon+\varepsilon\mu_s) - \tilde W_s)\, dB_s + i\nu (1-\varepsilon) x \tilde W_1 + i\theta Z_1 B_1 \right] .$$

We reduce this by firstly conditioning on W, Z:

$$\varphi = E \exp\Big[-\tfrac12 \nu^2 \int_0^1 (\tilde W_1 (1-\varepsilon+\varepsilon\mu_s) - \tilde W_s)^2 ds - \nu\theta \int_0^1 Z_1 (\tilde W_1 (1-\varepsilon+\varepsilon\mu_s) - \tilde W_s)\, ds$$
$$ - \tfrac12 \theta^2 Z_1^2 + i\nu(1-\varepsilon) x \tilde W_1 + i\theta x Z_1 \Big]$$

$$= E \exp\Big[-\tfrac12 \nu^2 \int_0^1 (W_1 (1-\varepsilon+\varepsilon s) - W_s)^2 \tau(ds) - \nu\theta Z_1 \int_0^1 (W_1 (1-\varepsilon+\varepsilon s) - W_s)\, \tau(ds)$$
$$ - \tfrac12 \theta^2 Z_1^2 + i\nu(1-\varepsilon) x \tilde W_1 + i\theta x Z_1 \Big] .$$

2) This is not much use except when $\boxed{x=0}$ which we now assume. Since

$$E \exp(-\tfrac{1}{2}\theta^2 Z^2 - a\theta Z) = (1+\theta^2)^{-1/2} \exp\left\{ \frac{a^2\theta^2}{2(1+\theta^2)} \right\},$$

we can condition on W, and get down to

$$E \exp\left\{ -\tfrac{1}{2}\nu^2 \int_0^1 Y_s^2 \,\pi(ds) + \frac{\nu^2\theta^2}{2(1+\theta^2)} \left(\int_0^1 Y_s \,\pi(ds) \right)^2 \right\} (1+\theta^2)^{-1/2}$$

where $Y_s \equiv W_1(1-\varepsilon+\varepsilon s) - W_s$.

But it is not hard to prove that $X_t \equiv Y_{1-t}$ has same law as $\beta_t - \frac{t}{T}\beta_T$, where $T = p^{-1}$, β a BM. Writing m for the reversal of π, $m(0,t] = 1 - \pi(1-t)$, we have

$$\varphi = E \exp\left(-\tfrac{1}{2}\nu^2 \left\{ \int_0^1 X_s^2 \, m(ds) - \frac{\theta^2}{1+\theta^2} \int_0^1 X_s \, m(ds) \right\} \right) (1+\theta^2)^{-1/2}.$$

Quadratic functionals of BM and the fundamental theorem of statistics (6/3/91)

1) The fundamental theorem of statistics is that if Q is pds, $X \sim N(0,V)$, $a \in \mathbb{R}^q$, then

$$(1) \quad E \exp -\tfrac{1}{2}(X+a)^T Q (X+a) = \det(I+QV)^{-1/2} \exp\left\{ -\tfrac{1}{2} a^T (I+QV)^{-1} Q a \right\}.$$

Let's apply (the extension to BM of) this result to work out the laws of some quadratic functionals of BM. So let (B_t) be throughout a BM($\mathbb{R}$), $B_0 = 0$. Let m be a measure on $[0,1]$ with a smooth uniformly positive density q, and let $a : [0,1] \to \mathbb{R}$ be continuous. We shall get an expression for

$$(2) \quad E \exp\left\{ -\tfrac{1}{2}\theta^2 \int_0^1 (B_s + a_s)^2 q_s \, ds \right\},$$

from which it will be simple to deduce closed-form expressions for things like

$$(3) \quad E^x \exp\left\{ -\tfrac{1}{2}\theta^2 \int_0^1 B_s^2 \, m_1(ds) + \rho \int_0^1 B_s \, m_2(ds) - \sigma(B_1 - b)^2 \right\},$$

and thence (by letting $\sigma \to \infty$) we can get things for Brownian bridge from $(0,x)$ to $(1,b)$, and so on.

2) The first thing to notice from FTS is that the first factor on the RHS can be calculated with $a \equiv 0$, and then we only need to worry about the second. In the case (2) of Brownian motion, with $a \equiv 0$, we can

easily calculate

$$\varphi = E \exp\left(-\tfrac{1}{2}\theta^2 \int_0^1 B_s^2 \, q_s \, ds\right)$$

by Ray-Knight:

$$\boxed{\varphi^2 = E^1 \exp(-\lambda H),}$$

where $\lambda \equiv \frac{1}{2}\theta^2$, and H is the hitting time to zero of the diffusion in $(0,1]$ with speed measure m, reflected at 1.

3) Let $\mathcal{G} \equiv \frac{1}{2}\frac{d^2}{dm\,dx}$ be the generator of the diffusion in $(0,1]$ with speed measure m, reflected at 1, let X denote this diffusion, and let $r_\lambda(\cdot,\cdot)$ be the λ-resolvent density of the process (w.r.to m, of course).

The first task now is to understand what $(I+QV)^{-1}$ is. If we have

$$y_t = ((I+QV)x)_t = x_t + \theta^2 q_t \int_0^1 (u\wedge t)\, x_u \, du ,$$

with y reasonably smooth, then with $\eta_t \equiv y_t/q_t$, $\xi_t \equiv x_t/q_t$, we get

$$\eta_t = \xi_t + \theta^2 \int_0^1 (u\wedge t)\, \xi_u \, q_u \, du$$

and we see that $\eta_0 = \xi_0$, $\eta_1' = \xi_1'$, and, differentiating twice,

$$\tfrac{1}{2}\eta'' = \tfrac{1}{2}\xi'' - \lambda \xi q$$

so that

$$\tfrac{1}{2}(\xi-\eta)'' - \lambda q(\xi-\eta) = \lambda q \eta$$

and thus (since $(\xi-\eta)_0 = 0$, $(\xi-\eta)_1' = 0$) we have the probabilistic solution

$$\boxed{\xi_x - \eta_x = -E^x \int_0^H \lambda e^{-\lambda t} \eta(X_t)\, dt = -\lambda R_\lambda \eta(x).}$$

Hence $\boxed{x = (I - \lambda Q R_\lambda Q^{-1})\, y}$ i.e. $\boxed{(I+QV)^{-1} = I - \lambda Q R_\lambda Q^{-1}}$

where Q is the operator $(Qf)_t \equiv \theta^2 q_t f_t$.

$$E \exp\left(-\lambda \int_0^1 B_s^2 \, m(ds) - 2 \int_0^1 B_s \, \nu(ds)\right)$$

$$= E \exp\left(-\lambda \int_0^1 B_s^2 \, m(ds)\right) \cdot \exp\left(\iint \nu(dx)\,\nu(dy)\, r_\lambda(x,y)\right)$$

Thus we have from (1) and (2)

$$\boxed{\begin{aligned} &E \exp\Big\{-\lambda\int_0^1 B_s^2\, m(ds) - 2\lambda\int_0^1 a_s B_s\, m(ds)\Big\} \\ &= \left(E^1 e^{-\lambda H}\right)^{\frac12} \exp\Big\{\tfrac12 \theta^2 \int_0^1 a_s\,(\lambda R_\lambda a)_s\, q_s\, ds\Big\} \\ &= \left(1-\lambda R_\lambda 1(1)\right)^{\frac12} \exp\Big\{\lambda\int_0^1 m(ds)\int_0^1 m(dt)\, a_s a_t\, \lambda r_\lambda(s,t)\Big\} \end{aligned}} \qquad (4)$$

At this point, we need explicit form of R_λ to be able to get much further.

4) As an example, let's take m to be Lebesgue measure, $a_s = x$ for all s. The resolvent density of BM reflected at 1, killed at 0, is

$$r_\lambda(x,y) = \frac{2\sinh\theta x\,\cosh\theta(1-y)}{\theta\cosh\theta} \qquad (0\le x\le y\le 1)$$

and $\int_0^1 ds\int_0^1 dt\ \lambda r_\lambda(s,t) = 1-\theta^{-1}\tanh\theta$. Thus from (4)

$$E\exp\left(-\lambda\int_0^1 (B_s+x)^2\,ds\right) = \left(\frac{1}{\cosh\theta}\right)^{\frac12}\exp\left(-\frac{x^2}{2}\theta\tanh\theta\right) \qquad (5)$$

as can be checked by excursion theory.

5) We can equally well do the calculations when we replace the Brownian motion B by the Brownian bridge Y. The only thing which changes is that the resolvent is now for speed measure m, with killing at 0 and 1. The quadratic part at the beginning also needs to be reinterpreted. We get

$$E\exp\Big\{-\lambda\int_0^1 Y_s^2\, m(ds) - 2\int_0^1 Y_s\, \nu(ds)\Big\} \qquad (6)$$

$$= E\exp\Big\{-\lambda\int_0^1 Y_s^2\, m(ds)\Big\}\cdot\exp\Big\{\int_0^1 \nu(ds)\int_0^1 \nu(dt)\, r_\lambda^\partial(s,t)\Big\}$$

When $m(dx) = \mathrm{Leb}(dx)$, we shall have $r_\lambda^\partial(x,y) = 2\sinh\theta x\,\sinh\theta(1-y)/\theta\sinh\theta$ for $0\le x\le y\le 1$.

Writing $\gamma^2 = 2\rho\lambda$ allows us to express (7) in the neater form

$$(\cosh\theta)^{-\frac{1}{2}} \exp\left\{ - \frac{\theta\tanh\theta}{2}(a+\rho)^2 + \rho^2\theta^2/2 \right\}$$

Can express RHS of (8) as

$$\frac{1}{\lambda}\left\{ 1 - \frac{2}{\theta}\tanh\frac{\theta}{2} \right\}.$$

6) We can extend the calculation of part 4) above to give

$$\boxed{\begin{aligned} &E \exp\left[-\tfrac{1}{2}\theta^2 \int_0^1 (B_s+a)^2 ds - \gamma \int_0^1 (B_s+a)\, ds\right] \\ &= (\cosh\theta)^{-\frac{1}{2}} \exp\left\{-\frac{a^2}{2}\theta\tanh\theta - \gamma a \frac{\tanh\theta}{\theta} + \frac{\gamma^2}{4\lambda}\left(1-\frac{\tanh\theta}{\theta}\right)\right\}. \end{aligned}} \quad (7)$$

For the bridge, it's not too difficult to compute

$$\int_0^1 dx \int_0^1 dy\, r_\lambda^\partial(x,y) = \frac{2}{\theta^3 \sinh\theta}\left\{\theta\sinh\theta + 2 - 2\cosh\theta\right\}. \quad (8)$$

To compute

$$E \exp\left\{-\tfrac{1}{2}\theta^2 \int_0^1 (Y_s + a + (b-a)s)^2 ds - \gamma \int_0^1 (Y_s+a+(b-a)s)\, ds\right\}$$

$$= \left(\frac{\theta}{\sinh\theta}\right)^{\frac{1}{2}} \exp\left\{-\tfrac{1}{2}k_1(a^2+b^2) + k_2 ab + k_3(a+b) + k_4\right\} \quad (9)$$

for functions k_1, k_2, k_3, k_4 which are quadratic in γ, and may depend on θ, but not a, b. Taking $a=b=0$, (6) and (8) give us

$$\boxed{k_4 = \frac{\gamma^2}{2\theta^3\sinh\theta}\left\{\theta\sinh\theta + 2 - 2\cosh\theta\right\}}$$

By mixing (9) over $b \sim N(a,1)$, we must recover (7), for all a. Hence

$$\boxed{k_1 = \theta\coth\theta - 1} \quad \boxed{k_2 = -1 + \theta\,\mathrm{cosech}\,\theta}, \quad \boxed{k_3 = -\theta\rho\tanh\tfrac{\theta}{2}},$$

$$\boxed{k_4 = \rho^2\lambda\left(1 - \tfrac{2}{\theta}\tanh\tfrac{\theta}{2}\right)}$$

with $\gamma = 2\rho\lambda$. Thus

$$\boxed{\begin{aligned} &E \exp\left\{-\lambda\int_0^1 (Y_s+a+(b-a)s)^2 ds - 2\rho\lambda\int_0^1 (Y_s+a+(b-a)s)\, ds\right\} \\ &= \left(\frac{\theta}{\sinh\theta}\right)^{\frac{1}{2}} \exp\Big[-\tfrac{1}{2}(\theta\coth\theta - 1)(a^2+b^2) - (1-\theta\,\mathrm{cosech}\,\theta)ab - \left(\theta\rho\tanh\tfrac{\theta}{2}\right)(a+b) \\ &\qquad + \rho^2\lambda\left(1-\tfrac{2}{\theta}\tanh\tfrac{\theta}{2}\right)\Big] \end{aligned}}$$

This simplifies a bit if we note that $\tanh\frac{\theta}{2} = \coth\theta - \operatorname{cosech}\theta$, so in fact everything can be deduced from the identity

$$E\exp\left\{-\lambda\int_0^1 (Y_s + a + (b-a)s)^2\,ds\right\}$$

$$= \left(\frac{\theta}{\sinh\theta}\right)^{1/2} \exp\left\{-\tfrac{1}{2}(a^2+b^2)(\theta\coth\theta - 1) - ab(1-\theta\operatorname{cosech}\theta)\right\}.$$

7) (21/3/91). If we have cts functions $f_j : [0, T_j] \to \mathbb{R}$, and set

$$\bar{f}_j \equiv \frac{1}{T_j}\int_0^{T_j} f_j(s)\,ds, \qquad \bar{f} \equiv \sum T_j \bar{f}_j \Big/ \sum T_j, \quad T = \sum T_j$$

then we can represent

$$\sum_{j=1}^n \int_0^{T_j} (f_j(s) - \bar{f})^2\,ds = \sum_{j=1}^n \left\{\int_0^{T_j}(f_j(s) - \bar{f}_j)^2\,ds + T_j(\bar{f}_j - \bar{f})^2\right\}.$$

By scaling, if $\tilde{Y}$ is a Brownian bridge from 0 to 0 of duration T, then

$$E\exp\left\{-\lambda\int_0^T \left(\tilde{Y}_s + a + (b-a)\tfrac{s}{T}\right)^2 ds\right\}$$

$$= E\exp\left\{-\lambda T^2\int_0^1\left(Y_s + \frac{a}{\sqrt{T}} + \frac{b-a}{\sqrt{T}}s\right)^2 ds\right\}$$

$$= \left(\frac{\theta T}{\sinh\theta T}\right)^{1/2}\exp\left[-\frac{a^2+b^2}{2T}(\theta T\coth\theta T - 1) - \frac{ab}{T}(1-\theta T\operatorname{cosech}\theta T)\right].$$

Thus if $Z \equiv \int_0^T(\tilde{Y}_s + a + (b-a)s/T)^2\,ds$, $\xi \equiv \int_0^T(\tilde{Y}_s + a + (b-a)s/T)\,ds$, then from the foot of previous page

$$E\exp(-\lambda Z - 2\rho\lambda\xi)$$

$$= \left(\frac{\theta T}{\sinh\theta T}\right)^{1/2}\exp\Big[-\frac{a^2+b^2}{2T}(\theta T\coth\theta T - 1) - \frac{ab}{T}(1-\theta T\operatorname{cosech}\theta T)$$

$$- \frac{\rho(a+b)}{T}(\theta T\coth\theta T - \theta T\operatorname{cosech}\theta T) + \rho^2\lambda T$$

$$- \rho^2\,\theta T(\coth\theta T - \operatorname{cosech}\theta T)\Big].$$

One could in principle use this to calculate the r.c.d. of Z given ξ, but it's a complete mess.

8) Can we model the time-dependent behaviour of a polymer's moment of inertia?

The obvious thing is to let $X_t(\cdot) \equiv X(t,\cdot)$ be the path at time t,

$$X(t,\tau) = e^{-\frac{1}{2}t} B(e^t, \tau)$$

where B is the standard Brownian sheet.

Then
$$X(t+h,\tau) \stackrel{\mathcal{D}}{=} e^{-\frac{1}{2}(t+h)}\left(B(e^t,\tau) + \tilde{B}(e^t(e^h-1),\tau)\right)$$
$$\stackrel{\mathcal{D}}{=} e^{-h/2} X(t,\tau) + e^{-h/2}\tilde{B}(e^h-1,\tau)$$

where $\tilde{B}$ is a Brownian sheet independent of B.

Thus to calculate the conditional distribution of X_{t+h} given X_t is in a sense quite easy. We want to compute the law of the MI of X_{t+h} given X_t, so want law of

$$\int_0^1 (W_s - \bar{W} + a_s)^2\, ds$$

where W is standard BM, a is any (cts) function (integrating to 0). We get

$$\boxed{\begin{aligned} &E \exp\left(-\tfrac{1}{2}\theta^2 \int_0^1 (W_s - \bar{W} + a_s)^2\, ds\right) \\ &\quad = \left(\frac{\theta}{\sinh\theta}\right)^{1/2} \exp\left[-\lambda\left(\int_0^1 a_s^2\, ds + \left(\int_0^1 a_s\, ds\right)^2 - \lambda\int_0^1 ds \int_0^1 dt\; a_s a_t\, r_\lambda(s,t)\right)\right]\end{aligned}}$$

where

$$\lambda r_\lambda(x,y) = \frac{\theta \cosh\theta x \cosh\theta(1-y)}{\sinh\theta}, \qquad 0 \le x \le y \le 1,$$

is resolvent density of BM in $[0,1]$ reflected at the ends.

Even if we use the fact that $\int a_t\, dt = 0$ in the application, this does not help very much; the conditional distribution depends on all of the behaviour of a, and not just the integral of the square.

A trading game (22/3/91)

1) Let the continuous non-negative semimartingale S model the price of a stock. Player 1 has an American call option on the stock, strike price K_1, to be exercised at or before time T. Player 2 has an American put option on the stock, strike price $K_2 \leq K_1$, to be exercised at or before time T.

Let τ_j be the stopping time at which player j exercises the option, with the convention $\tau_j = +\infty$ if he doesn't exercise at all.

$$\begin{cases} \text{If } \tau_1 < \tau_2, & \text{then player 1 buys from player 2 at price } K_1; \\ \text{If } \tau_2 < \tau_1, & \text{then player 1 buys from player 2 at price } K_2; \\ \text{If } \tau_1 = \tau_2 \leq T, & \text{then } \text{---------------------} \tfrac{1}{2}(K_1+K_2); \end{cases}$$

Otherwise, no trade takes place (i.e. when $\tau_1 = \tau_2 = +\infty$).

<u>What should be a fair price for player 1 to pay to enter this game?</u>

2) Let's assume that there exists an equivalent martingale measure $\tilde{P}$, and that the interest is <u>zero</u> (if we didn't have this, by letting $K_1 \to \infty$ we would get the intractable American put).

If the players choose stopping times τ_1, τ_2, then 1 receives

$$Y(\tau_1, \tau_2) = (S_{\tau_1} - K_1) I_{(\tau_1 < \tau_2)} + (S_{\tau_2} - K_2) I_{(\tau_2 < \tau_1)}$$

$$+ \left(S_{\tau_1} - \tfrac{1}{2}(K_1+K_2)\right) I_{\{\tau_1 = \tau_2 < \infty\}} .$$

The fair price for the contingent claim $Y(\tau_1, \tau_2)$ is just

$$v(\tau_1, \tau_2) \equiv \tilde{E}\, Y(\tau_1, \tau_2) .$$

3) PROPOSITION. <u>Define</u>

$$\boxed{\begin{aligned} \tau_1^* &= T \text{ if } S_T \geq K_1; \qquad & \tau_2^* &= T \text{ if } S_T \leq K_2; \\ &= +\infty \text{ else} & &= +\infty \text{ else}. \end{aligned}}$$

<u>Then the pair (τ_1^*, τ_2^*) is a saddle point for the game</u> —

$$r^* \equiv r(\tau_1^*, \tau_2^*) = \max_{\tau_1} r(\tau_1, \tau_2^*) = \min_{\tau_2} r(\tau_1^*, \tau_2).$$

Proof Let ρ be some stopping time and consider

$$r(\rho, \tau_2^*) = \tilde{E}\, Y(\rho, \tau_2^*)$$

$$= \tilde{E}[S_\rho - K_1 : \rho < T] + \tilde{E}[S_T - \tfrac{1}{2}(K_1+K_2) ; \rho = \tau_2^* = T]$$

$$+ \tilde{E}[S_T - K_1 : \rho = T < \tau_2] + \tilde{E}[S_T - K_2 : \tau_2^* = T < \rho]$$

$$\leq \tilde{E}[S_\rho - K_1 : \rho < T] + \tilde{E}[S_T - K_1 : \rho = T < \tau_2]$$

$$+ \tilde{E}[S_T - K_2 : \tau_2^* = T \leq \rho]$$

$$= \tilde{E}[S_T - K_1 : \rho < T] + \tilde{E}[S_T - K_1 : \rho = T < \tau_2] + \tilde{E}[S_T - K_2 : \tau_2^* = T \leq \rho]$$

since S is a $\tilde{P}$-martingale. The first term we break up and estimate as follows:

$$\tilde{E}[S_T - K_1 : \rho < T = \tau_1^*] + \tilde{E}[S_T - K_1 : \rho < T,\ S_T \in (K_2, K_1)]$$

$$+ \tilde{E}[S_T - K_1 : \rho < T,\ S_T \leq K_2]$$

$$\leq \tilde{E}[(S_T - K_1)^+ : \rho < T = \tau_1^*] + \tilde{E}[S_T - K_1 : \rho < T = \tau_2^*]$$

$$\leq \tilde{E}[(S_T - K_1)^+ : \rho < T] + \tilde{E}[S_T - K_2 : \rho < T = \tau_2^*].$$

Hence

$$r(\rho, \tau_2^*) \leq \tilde{E}[(S_T - K_1)^+ : \rho < T] + \tilde{E}[(S_T - K_1)^+ : \rho = T < \tau_2]$$

$$+ \tilde{E}[S_T - K_2 : T = \tau_2^*]$$

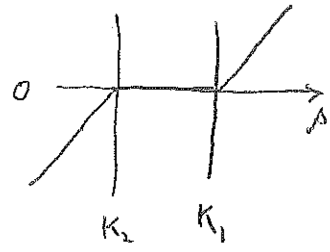
$$\leq \tilde{E}[(S_T - K_1)^+] - \tilde{E}[(K_2 - S_T)^+]$$

$$= r(\tau_1^*, \tau_2^*).$$

The proof in the other direction is wholly analogous.

4) If player 2 were to exercise at $\tau_2 < T$, then 1 would receive $S_{\tau_2} - K_2$ which he could turn into $S_T - K_2$ at time T. But

$$s - T \geq f(s) = (s-K_1)^+ - (K_2 - s)^+$$



and so $Y(\tau_1^*, \tau_2^*) = f(S_T) \leq S_T - K_2$.

Thus if player 1 enters the game, he is certain to be able to get at least $Y(\tau_1^*, \tau_2^*)$ by time T.

If he were allowed to enter the game for amount $r < r^*$, there would be an arbitrage opportunity. One way to think of it is as follows. If he had a fortune F at time 0, he'd take out r^*, and trade the fortune $F - r^*$ so as to get exactly $F - Y^*$ at time T. Of the r^* taken out at time 0, he'd set aside $r^* - r$, pay r to play the game, which would deliver at least Y^* at time T, so at time T he would surely have at least $(r^* - r) + (F - Y^*) + Y^*$, so an arbitrage opportunity exists. So no one would let 1 play the game for less than r^*. But similarly, 1 will never pay more than r^* to play, or his opponent has an arbitrage opportunity. So r^* is the fair price for 1 to pay to play this game.

A remarkable integral equation (23/3/91)

$$\int_{-1}^{1} (1-x^2)^{(1-\beta)/2} \frac{|a-x|^\beta}{a-x}\, dx = a \cdot \frac{\pi(1-\beta)}{\cos(\beta\pi/2)} \qquad \text{for } |a| < 1,$$

for each $\beta \in (0,1)$.

Proof

1) Let $f_1(z) = i/(-iz)^{1-\beta}$, $z \in \mathbb{H}$. Then $f_1 : \mathbb{H} \to \mathbb{H}$ is analytic, and

$$f_1(re^{i\theta}) = r^{\beta-1} \exp\left[-i\frac{\beta\pi}{2} - i\theta(1-\beta) + i\pi\right],$$

which has argument $\begin{cases} \pi - \beta\pi/2 & \text{on } \mathbb{R}^+ \\ \beta\pi/2 & \text{on } \mathbb{R}^-. \end{cases}$

Thus $\quad \operatorname{Re} f_1(x) = -\operatorname{sgn}(x) \cos(\beta\pi/2) \; |x|^{\beta-1} \quad$ for $x \in \mathbb{R}\setminus\{0\}$.

Next,

$$f_2(z) \equiv \sqrt{1-z^2}$$

maps $\mathbb{H}$ to $\mathbb{H}$, and is real for $z \in [-1,1]$ and

has argument $\begin{cases} -\frac{\pi}{2} & \text{on } (1,\infty) \\ \frac{\pi}{2} & \text{on } (-\infty,-1). \end{cases}$

Thus if we consider $f_3(z) \equiv f_2(z)^{1-\beta} f_1(z-a)$, this is analytic in $\mathbb{H}$, and has real part on the real axis (with $|a|<1$)

$$\operatorname{Re} f_3(x) = -\operatorname{sgn}(x-a) \cos\left(\beta\tfrac{\pi}{2}\right) |x-a|^{\beta-1} (1-x^2)^{(1-\beta)/2}$$

for $|x| \le 1$;

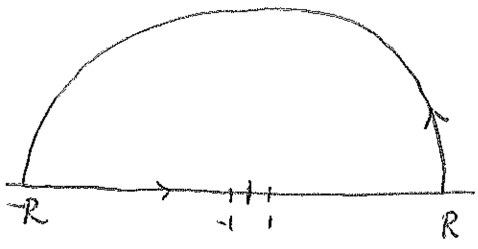
$$= 0$$

for $|x| > 1$.

Hence

$$\boxed{\cos(\beta\pi/2) \int_{-1}^{1} (1-x^2)^{(1-\beta)/2} \frac{|a-x|^\beta}{a-x} dx = \operatorname{Re} \int_{-\infty}^{\infty} f_3(x)\, dx}$$

2) We evaluate the right-hand side using Cauchy's theorem on the big semicircular contour. We just need to know what happens on the curved piece, so want



$$\int_0^\pi iRe^{i\theta} f_3(Re^{i\theta})\, d\theta .$$

For very large R, $\quad f_1(Re^{i\theta}-a) \simeq \dfrac{iR^{\beta-1}}{(-ie^{i\theta})^{-\beta+1}} \left(1 - \dfrac{a}{Re^{i\theta}}\right)^{\beta-1}$

$$= R^{\beta-1} \frac{i}{(-ie^{i\theta})^{1-\beta}} \left\{1 - (\beta-1)\frac{a}{Re^{i\theta}} + O(R^{-2})\right\},$$

and
$$f_2(z)^{1-\beta} = \left(1 - R^2 e^{2i\theta}\right)^{(1-\beta)/2}$$
$$= R^{1-\beta} e^{i(1-\beta)\theta - i\pi(1-\beta)/2}\left(1 - R^{-2}e^{-2i\theta}\right)^{(1-\beta)/2}$$
$$= R^{1-\beta} e^{i(\theta-\pi/2)(1-\beta)}\left\{1 - \frac{1-\beta}{2} e^{-2i\theta} R^{-2} + O(R^{-4})\right\}$$

Thus we obtain
$$iR\int_0^\pi e^{i\theta} e^{i\pi/2}\left\{1 - (\beta-1)\frac{a}{Re^{i\theta}} + O(R^{-2})\right\}\left\{1 - \frac{1-\beta}{2} e^{-2i\theta} R^{-2} + O(R^{-4})\right\} d\theta$$
$$= -R\left[\int_0^\pi e^{i\theta}\left(1 - (\beta-1)\frac{a}{Re^{i\theta}}\right) d\theta + O(R^{-2})\right]$$
$$= -2iR + (\beta-1)a\pi + O(R^{-1})$$

Hence
$$\mathrm{Re}\left(\int_0^\pi i Re^{i\theta} f_3(Re^{i\theta})\, d\theta\right) = (\beta-1)a\pi + O(R^{-1}),$$
from which the stated result follows.

3) We can let $\beta \uparrow 1$ and easily deduce the limit result for $\beta = 1$. The case $\beta = 0$ looks more delicate. Rather than attempt to tighten up the limiting behaviour, notice that the contour integration argument above is unchanged in the case $\beta = 0$, except that there is a residue at the pole of $f_1(z-a)$ at a. But the residue contribution is pure imaginary, so does not contribute when we look at real parts.

A question of David Williams (27/3/91)

1) David wants to know the law of
$$Z = \int_0^\infty e^{-\lambda t} f(B_t)\,dt$$

in the special case $\lambda=1$, $f(x) = I_{(x>0)}$, $B_0=0$. Let's notice that
$$E^x Z^n = n!\,E^x \int_0^\infty dt_1\, e^{-\lambda t_1} f(B_{t_1}) \int_{t_1}^\infty dt_2\, e^{-\lambda t_2} f(B_{t_2}) \cdots \int_{t_{n-1}}^\infty dt_n\, e^{-\lambda t_n} f(B_{t_n})$$
$$= n!\, R_{n\lambda} f R_{(n-1)\lambda} f \cdots f R_\lambda f\,(x).$$

Let's therefore define $h_0(x) \equiv 1$, and
$$h_{n+1}(x) = (R_{(n+1)\lambda} f h_n)(x) \qquad (n\geq 0)$$
and for $\theta \in \mathbb{R}$
$$\psi(\theta,x) \equiv \sum_{n\geq 0} \theta^n h_n(x) = \sum_{n\geq 0} \frac{\theta^n}{n!} E^x(Z^n) = E^x(e^{\theta Z}).$$
Using the resolvent identity $(\lambda - \mathcal{G})^{-1} = R_\lambda$ gives us that
$$\mathcal{G} h_n = n\lambda h_n - f h_{n-1} \qquad (n\geq 1).$$
Hence
$$\boxed{\frac{1}{2}\frac{\partial^2\psi}{\partial x^2} = \lambda\theta\frac{\partial\psi}{\partial\theta} - \theta f\psi}$$
with
$$\psi(0,x)\equiv 1.$$

2) Let's now specialise to $f = I_{(0,\infty)}$, $\lambda = 1$, and observe that this gives
$$\boxed{\lim_{x\to-\infty}\psi(\theta,x) = 1, \quad \lim_{x\to\infty}\psi(\theta,x) = e^\theta.}$$

Since the P^x-law of $1-Z$ is the same as the P^{-x}-law of Z, we get
$$\boxed{\psi(\theta,-x) = e^\theta\,\psi(-\theta,x)}$$

3) It's neater to take $f(x) = \mathrm{sgn}(x)$, so that $\psi(\theta, x) = \psi(-\theta, -x)$, and we can restrict attention to $\theta > 0$. Writing

$$\psi(\theta, x) \equiv \varphi(-\log\theta, x) \equiv \varphi(t, x)$$

we obtain

$$\frac{1}{2}\frac{\partial^2\varphi}{\partial x^2} = -\frac{\partial\varphi}{\partial t} - e^{-t}\,\mathrm{sgn}(\cdot)\,\varphi$$

with $\lim_{t\to\infty} \varphi(t,x) = 1$. Thus

$$\boxed{\begin{aligned}\varphi(t,x) &= E^x \exp\left(e^{-t}\int_0^\infty e^{-u}\,\mathrm{sgn}(B_u)\,du\right)\\ &\equiv E^{(t,x)} \exp\left(\int_t^\infty e^{-u}\,\mathrm{sgn}(B_u)\,du\right).\end{aligned}}$$

This is really no improvement!

<u>Lagrange inversion theorem</u> (30/3/91)

This is taken from "Generatingfunctionology" by Herbert S. Wilf, Academic Press, 90.

Let f, φ be analytic in some neighbourhood of 0, $\varphi(0) = 1$. Then in some nbd of 0, the implicit equation

$$u = t\,\varphi(u)$$

has a unique root $u(t)$ which is analytic in t, and when expanded as a power series in t gives

$$\underline{t^n}\rfloor\left(f(u(t))\right) = \underline{u^{n-1}}\rfloor\left(\frac{1}{n} f'(u)\,\varphi(u)^n\right)$$

<u>Proof</u>. To get the c/e of u^{n-1} in $f'\varphi^n$, we form

$$\oint \frac{du}{2\pi i}\,\frac{f'(u)\,\varphi(u)^n}{u^n} = \oint \frac{du}{2\pi i}\,\frac{f'(u)}{t(u)^n}$$

$$= \oint \frac{dt}{2\pi i}\,u'(t)\,\frac{f'(u(t))}{t^n}$$

$$= \underline{t^{n-1}}\rfloor\left(\frac{d}{dt} f(u(t))\right)$$

$$= n \cdot \underline{t^n}\rfloor\left(f(u(t))\right).$$

Wilf applies this to computing the number of rooted labelled trees on n vertices.

(PTO)

Other nice/useful things include

- Dirichlet series, $\sum_{n\geq 1} a_n/n^s$, of which the ζ-f^n is an example. There is the Möbius inversion result, + examples which show how one can derive number-theoretic results this way
- Exponential formula. This allows one to take the index set $\{1,\dots,n\}$ split it into k subsets of weights (= sizes) $j_1,\dots,j_k$, and to each index subset of weight w assign one of a finite set D_w of possible "pictures". How is the number $h(n,k)$ of such splittings related to $|D_j| = d_j$? There's a formula, and applications include counting no. of partitions of a set, no of connected graphs ...
- Generalisation of inclusion - exclusion
- asymptotics of (a_n) from generating f^n.

Neat formula for the moments of Gaussian variables (3/4/91)

According to Kahn's, the following is well known. If $(X_1, \dots, X_{2n})$ is zero mean Gaussian vector, then

$$E[X_1 X_2 \cdots X_{2n}] = \sum{}' E X_{i_1} X_{i_2} \, E X_{i_3} X_{i_4} \cdots E X_{i_{2n-1}} X_{i_{2n}}$$

where the sum is taken over all partitions of $\{1, 2, \dots, 2n\}$ into n pairs. There are $(2n)!\, 2^{-n}/n! = (2n-1)(2n-3)\cdots 3.1$ of these. Since both sides are linear in each X_j, it's enough to do when $X_1 = \dots = X_{2j} = \xi_1$, $X_{2j+1} = \dots = X_{2k} = \xi_2, \dots$ $X_{2r+1}, \dots, X_{2n} = \xi_q$, with ξ_q i.i.d. $N(0,1)$'s. But then it is easy to see that it will be enough just to take all the ξ_i the same, and then it's just a statement of the $2k^{th}$ moment of a Gaussian variable.

Endogenous stochastic volatility (19/4/91)

(i) A simpler model for a stock price with memory (see p 5) would be (with Y the log price as before)

$$dY = \sigma(\Delta)\, dB + \mu_t\, dt$$

where

$$\Delta_t \equiv Y_t - A_t \equiv Y_t - \int_{-\infty}^{t} \lambda e^{-\lambda(t-u)} Y_u\, du$$

because

$$d\Delta = dY - \lambda\Delta\, dt = \sigma(\Delta)\, dB + \mu\, dt - \lambda\Delta\, dt.$$

Thus quite a neat and general model would be to take Δ to be an autonomous diffusion, and then define Y by

$$\boxed{Y_t - Y_0 = \Delta_t - \Delta_0 + \int_0^t \lambda \Delta_s\, ds.}$$

This has the virtue that one can get a lot more control on the moments of Y.

(ii) But is there any situation where one can compute the law of Y_1 under some EMM? The EMM changes to Δ solving

$$d\Delta = \sigma(\Delta)\, d\tilde{B} - \lambda\Delta\, dt - \tfrac{1}{2}\sigma(\Delta)^2\, dt$$

and $$dY = d\Delta + \lambda\Delta\, dt = \sigma(\Delta)\, d\tilde{B} - \tfrac{1}{2}\sigma(\Delta)^2\, dt.$$

It looks unlikely that this could be solved to give the law of Y_1 explicitly...

More on the random motion of eigenvalues (25/4/91)

1) Let's consider a very general problem in $\mathbb{R}^d$, where we have

$$dX_j = \sigma_N dB_j - \nabla_j \varphi_N(X_1, \dots, X_N)\, dt$$

and the functional φ_N takes the form

$$\varphi_N(x_1, \dots, x_N) = \alpha_N \sum_{i \neq j}\sum V(x_i - x_j) + \theta_N \sum_j U(x_j).$$

The generator is thus

$$\mathcal{G}_N = \sum_j \left(\tfrac{1}{2}\sigma_N^2 \Delta_j^2 - \nabla_j \varphi_N \cdot \nabla_j\right)$$

$$= \sum_j \tfrac{1}{2}\sigma_N^2 \exp\{2\sigma_N^{-2}\varphi_N\} \nabla_j \cdot \left(\exp\{-2\sigma_N^{-2}\varphi_N\} \nabla_j\right)$$

which gives as invariant density $\exp\{-2\sigma_N^{-2}\varphi_N\}$.

2) Now we take the m-v description:

$$d\langle \mu_t^N, f\rangle = \frac{1}{N}\sum_j \left(\nabla f(X_j^N)\, dX_j^N + \tfrac{1}{2}\Delta f(X_j^N)\,\sigma_N^2\, dt\right)$$

$$= \frac{\sigma_N}{N}\sum_j \nabla f(X_j^N)\, dB_j + \frac{\sigma_N^2}{N}\sum_j \tfrac{1}{2}\Delta f(X_j^N)\, dt$$

$$- \frac{1}{N}\sum_j \nabla f(X_j^N)\left\{2\alpha_N \sum_{r \neq j} \nabla V(X_j^N - X_r^N) + \theta_N \nabla U(X_j^N)\right\} dt$$

assuming $\boxed{V(-x) = V(x).}$ In order that the second term remains decent, must have $\sigma_N = O(1)$, which will mean that the first term is going to be small; so we get

$$= \Big\{\sigma_N^2 \langle \mu_t^N, \tfrac{1}{2}\Delta f\rangle - 2N\alpha_N \iint \mu_t(dx)\,\mu_t(dy)\, \nabla f(x)\, \nabla V(x-y)$$

$$- \theta_N \int \mu_t(dx)\, \nabla f(x)\, \nabla U(x) \Big\}\, dt + o(1)$$

Let's now take $\boxed{\theta_N = \theta, \quad \alpha_N = \alpha/2N}$ so that in the limit we shall get

$$\frac{d}{dt}\langle \mu_t, f\rangle = -\frac{\alpha}{2}\iint_{\{x\neq y\}} \mu_t(dx)\mu_t(dy)\big(\nabla f(x)-\nabla f(y)\big)\cdot \nabla V(x-y)$$
$$- \theta\langle \mu_t, \nabla f\cdot\nabla U\rangle + \delta\langle \mu_t, \tfrac{1}{2}\Delta f\rangle$$

where $\delta = 1$ if $\sigma_N^2 \equiv 1$, zero else.

3) Let's define the energy of μ by

$$\mathcal{E}(\mu) \equiv \tfrac{1}{2}\alpha \iint_{\{x\neq y\}} \mu(dx)\mu(dy)\, V_n(x-y) + \theta\int \mu(dx)\, U_n(x).$$

Now consider

$$d(\mathcal{E}(\mu^N)) \equiv d\Big(\frac{\alpha}{2N^2}\sum_{i\neq j}\sum V_n(X_i^N - X_j^N) + \frac{\theta}{N}\sum_j U_n(X_j^N)\Big)$$

$$= \frac{\alpha}{2N^2}\sum_{i\neq j}\sum{}' \Big(\nabla V_n\cdot(dX_i - dX_j) + \tfrac{1}{2}\Delta V_n\, 2\sigma_N^2\, dt\Big)$$
$$+ \frac{\theta}{N}\sum_j \Big(\nabla U_n\cdot dX_j + \tfrac{1}{2}\sigma_N^2\,\Delta U_n\, dt\Big)$$

$$= \frac{\alpha\sigma_N}{2N^2}\sum_{i\neq j}\sum \nabla V_n(X_i - X_j)(dB_i - dB_j) + \frac{\theta\sigma_N}{N}\sum_j \nabla U_n(X_j)\, dB_j$$

$$- \frac{\alpha}{2N^2}\sum_{i\neq j}\sum \nabla V_n(X_i - X_j)\Big\{\sum_{r\neq i}\frac{\alpha}{N}\nabla V(X_i - X_r) - \sum_{r\neq j}\frac{\alpha}{N}\nabla V(X_j - X_r)\Big\}dt$$

$$- \frac{\alpha}{2N^2}\sum_{j\neq i}\sum \nabla V_n(X_i - X_j)\big(\theta\nabla U(X_i) - \theta\nabla U(X_j)\big)dt$$

$$- \frac{\theta}{N}\sum_j \nabla U_n(X_j)\cdot\sum_{r\neq j}\frac{\alpha}{N}\nabla V(X_j - X_r)dt - \frac{\theta^2}{N}\sum_j \nabla U_n(X_j)\cdot\nabla U(X_j)dt$$

$$+ \frac{\alpha\sigma_N^2}{N^2}\sum_{i\neq j}\sum \tfrac{1}{2}\Delta V_n(X_i - X_j)\, dt + \frac{\theta\sigma_N^2}{N}\sum_j \tfrac{1}{2}\Delta U_n(X_j)\, dt$$

$$= d(\text{little martingale})$$

$$- dt \cdot \int \mu_t^N(dx) \left| \theta \nabla U(x) + \alpha \int_{\{y \neq x\}} \mu_t^N(dy) \nabla V(x-y) \right|^2$$

$$+ \frac{\alpha \sigma_N^2}{N^2} \sum_{i \neq j}\sum \tfrac{1}{2} \Delta V(X_i - X_j)\, dt + \theta \sigma_N^2 \langle \mu_t^N, \tfrac{1}{2}\Delta U \rangle\, dt$$

Case 1: $\delta = 0$ (i.e. $\sigma_N^2 = o(1)$). In this case, we shall have in the limit

$$\boxed{\frac{d}{dt} \mathcal{E}(\mu_t) = - \int \mu_t(dx) \left| \theta \nabla U(x) + \alpha \int_{\{y \neq x\}} \mu_t(dy) \nabla V(x-y) \right|^2}$$

Case 2: $\delta = 1$, so $\sigma_N^2 = 1$

$$\boxed{\begin{aligned}\frac{d}{dt} \mathcal{E}(\mu_t) = &- \int \mu_t(dx) \left| \theta \nabla U(x) + \alpha \int_{\{y \neq x\}} \mu_t(dy) \nabla V(x-y) \right|^2 \\ &+ \theta \langle \mu_t, \tfrac{1}{2}\Delta U \rangle + \alpha \iint_{\{x \neq y\}} \mu_t(dx)\mu_t(dy)\, \tfrac{1}{2}\Delta V(x-y)\end{aligned}}$$

Another quadratic functional calculation (30/4/91)

Let Y denote a Brownian bridge from 0 to 0 (duration 1) and let

$$X_t \equiv Y_t + x + t(y-x)$$

be the Brownian bridge from x to y. The aim here is to compute

$$J \equiv E \exp\left\{ - \int_0^1 X_t^2 \mu(dt) \right\}.$$

We take the basic result (6) on p. 14:

$$E \exp\left(-\lambda \int_0^1 Y_s^2 m(ds) - 2\int_0^1 Y_s \nu(ds) \right)$$

$$= E \exp\left(-\lambda \int_0^1 Y_s^2 m(ds) \right) \exp \int_0^1 \nu(ds) \int_0^1 \nu(dt)\, r_\lambda^{\partial}(s,t)$$

and apply it with $\lambda m = \mu$, $\nu(ds) = (x + s(y-x))\mu$, to obtain

$$J = \exp\left\{-\int_0^1 \lambda (x+\delta s)^2 m(ds)\right\} E \exp\left\{-\lambda \int_0^1 Y_s^2 m(ds)\right\} \exp \int_0^1 \nu(ds) \int_0^1 \nu(dt) r_\lambda^\partial(s,t),$$

where $\delta \equiv y - x$. All that's now needed is to understand the double integral

$$I \equiv \lambda \int_0^1 m(ds)\,((1-s)x+sy) \int_0^1 m(dt)((1-t)x+ty)\lambda r_\lambda^\partial(s,t).$$

Let x be diffusion in nat-scale, speed m in $[0,1]$, killed at $H_0 \wedge H_1 \equiv \zeta$.
Let

$$\varphi_\lambda^+(x) \equiv E^x\left[e^{-\lambda\zeta} : x_\zeta = 1\right], \quad \varphi_\lambda^-(x) = E^x\left[e^{-\lambda\zeta} : x_\zeta = 0\right]$$

which both satisfy

$$\frac{1}{2} \frac{d^2}{dm\,dx} \varphi_\lambda^\pm = \lambda \varphi_\lambda^\pm .$$

<u>C/e of y^2 in I</u> :

$$\lambda \int_0^1 m(ds) \,.\, s \,.\, \underbrace{\int_0^1 m(dt)\, t \,.\, \lambda r_\lambda^\partial(s,t)}$$

$$E^s\left[x(T_\lambda) : T_\lambda < \zeta\right] = s - P^s(\zeta < T_\lambda,\ x_\zeta = 1)$$

$$= s - \varphi_\lambda^+(s)$$

$$= \lambda \int_0^1 s^2 m(ds) - \int_0^1 s \underbrace{m(ds)\ \lambda \varphi_\lambda^+(s)}_{d\left(\frac{1}{2} D\varphi_\lambda^+(s)\right)}$$

$$= \lambda \int_0^1 s^2 m(ds) - \frac{1}{2} D\varphi_\lambda^+(1) + \underbrace{\int_0^1 \tfrac{1}{2} D\varphi_\lambda^+(s)\, ds}_{=\frac{1}{2}}$$

$$= \lambda \int_0^1 s^2 m(ds) - \frac{1}{2}\left(D\varphi_\lambda^+(1) - 1\right).$$

Notice the excursion interpretation:

$$\frac{1}{2}\left(D\varphi_\lambda^+(1) - 1\right) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon}\left\{1 - \varphi_\lambda^+(1-\varepsilon) - \varepsilon\right\}$$

$$= \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} E^{1-\varepsilon}\left(1-e^{-\lambda \zeta} : x_\zeta = 1\right)$$

= rate of excursions down from 1 which are marked but don't reach 0

$\equiv n_{11}(\lambda)$, say

By similar simple calculations, c/e of x^2 is

$$\lambda \int_0^1 (1-s)^2 m(ds) - n_{00}(\lambda)$$

where

$n_{00}(\lambda)$ = rate of excursions up from 0 which are marked but don't reach 1,

and the c/e of xy is

$$2\lambda \int_0^1 (1-s)s \, m(ds) - 2n_{01}(\lambda)$$

where

$n_{01}(\lambda) \equiv$ rate of excursions up from 0 which get marked and cross to 1

$= n_{10}(\lambda)$

by symmetry.

Excursion argument shows that

$$\left[E \exp\left(-\lambda \int_0^1 Y_s^2 m(ds)\right)\right]^2 = 1 - 2n_{01}(\lambda),$$

so assembling:

$$\boxed{E \exp\left(-\int_0^1 X_t^2 dt\right) = \left(1-2n_{01}(\lambda)\right)^{\frac{1}{2}} \exp\left\{-x^2 n_{00}(\lambda) - 2xy\, n_{01}(\lambda) - y^2 n_{11}(\lambda)\right\}}$$

Eigenvalue movement again (9/5/91)

Return to the story on p.28 for the behaviour of the energy of μ. I reckon that under some suitable assumptions, will be able to prove that $\mathcal{E}$ is strictly convex, and has a unique minimum μ^*, and we would like to be able to prove that $\mu_t \Rightarrow \mu^*$. One possible approach to this is to consider

$$\frac{d}{dt}\mathcal{E}(\mu_t) = -\int \mu_t(dx)\left|\theta\nabla U(x) + \alpha\int_{\{y\neq x\}} \nabla V(x-y)\,\mu_t(dy)\right|^2 \equiv -\mathcal{G}(\mu_t),$$

say, and to argue that (under some conditions on U, V) there will be a cgt s/seq μ_{t_n} convergent to μ_∞ and (if $\mathcal{G}$ is cts) $\mathcal{G}(\mu_{t_n}) \to \mathcal{G}(\mu_\infty)$, so that we might be able to get $\mathcal{G}(\mu_\infty)=0$ as a result. However, $\mathcal{G}$ is not continuous. Consider the case of $\mathbb{R}^1$, with $V(x) = -\log|x|$, $U(x) = x^{2k}$, $\theta=1$, $\alpha=\frac{1}{2}$, $k\in\mathbb{N}, k\geq 2$, which has an energy-minimising measure μ^* which in general we don't know, but seems likely to be of compact support. If we took η_n to be uniform on $(a_n, 1+a_n)$, and $\mu_n = (1-\frac{1}{n})\mu^* + \frac{1}{n}\eta_n$, then $\mathcal{E}(\mu_n) \to \mathcal{E}(\mu^*)$ provided $n^{-1}a_n^{2k} \to 0$, and yet

$$\mathcal{G}(\mu_n) = (1-\tfrac{1}{n})\int \mu^*(dx)\left|2kx^{2k-1} + \tfrac{1}{2}\int_{\{y\neq x\}} \frac{\eta_n(dy)}{y-x}\right|^2$$

$$+\ \frac{1}{n}\int \eta_n(dx)\left|2kx^{2k-1} + \tfrac{1}{2}\int_{\{y\neq x\}}\frac{\mu_n(dy)}{y-x}\right|^2$$

$$\to \infty \quad \text{if } n^{-1}a_n^{4k-2} \to \infty,$$

which we can arrange. Notice also that $\mu_n \Rightarrow \mu^*$, so $\mathcal{G}$ is not a weakly cts fn.

Recurrent potential theory for real Lévy processes. (13/5/91)

Suppose we have a recurrent Lévy process on $\mathbb{R}$ which hits points. The aim would be to consider $\int \mu(dx)\, r_\lambda(a-x)$ as $\lambda\downarrow 0$, but this would diverge, so we need take instead

$$\lim_{\lambda\downarrow 0}\int \mu(dx)\{r_\lambda(a-x) - r_\lambda(a)\},$$

for example. Now we have to understand

$$\lim_{\lambda\downarrow 0} r_\lambda(0) - r_\lambda(x) = \lim_{\lambda\downarrow 0} E\left[\int_0^\infty e^{-\lambda t}\,dL(t,0) - \int_0^\infty e^{-\lambda t}\,dL(t,x)\right]$$

assuming that there is a local time process. Assume also that the Lebesgue measure of time spent in a state is zero. Let $q_{0\partial}$ be rate of excursions from 0 which are λ-killed before they reach a (if at all), q_{0a} be rate of excns from 0 to a which are not λ-killed before they reach a. Then if we time change by $L(\cdot,0)+L(\cdot,a)$ we get a Mkv chain with two states,

$$Q = \begin{pmatrix} -q_{0a}-q_{0\partial} & q_{0a} \\ q_{a0} & -q_{a0}-q_{a\partial} \end{pmatrix}$$

and $$R_\lambda = (-Q)^{-1} = |\Delta|^{-1} \begin{pmatrix} q_{a0}+q_{a\partial} & q_{0a} \\ q_{a0} & q_{0a}+q_{0\partial} \end{pmatrix}.$$

from which easily we have $\boxed{q_{a0}+q_{a\partial} = q_{0a}+q_{0\partial}}$ and

$$r_\lambda(0) - r_\lambda(a) = \frac{q_{a0}+q_{a\partial}-q_{0a}}{q_{a0}q_{0\partial}+q_{0a}q_{a\partial}+q_{0\partial}q_{a\partial}} = \frac{q_{0\partial}}{q_{0\partial}q_{a0}+q_{a\partial}q_{0a}+q_{0\partial}q_{a\partial}}$$

$$= \{ q_{a0} + q_{a\partial} + q_{a\partial}q_{0a}/q_{0\partial} \}^{-1}.$$

In the special case of a recurrent symmetric Lévy process, we have $q_{a0} = q_{0a} \to$ rate of excns from 0 to a as $\lambda \downarrow 0$, and

$$r_\lambda(0) - r_\lambda(a) = \int \frac{1-e^{-i\theta a}}{\lambda+\psi(\theta)} \frac{d\theta}{2\pi}$$

$$\to \int \frac{1-\cos\theta a}{\psi(\theta)} \frac{d\theta}{2\pi} \quad \text{as } \lambda\downarrow 0.$$

The special case of $\psi(\theta) = |\theta|^\alpha$, $1<\alpha<2$, the symmetric stable process, gives

$$\lim_{\lambda\downarrow 0} \{ r_\lambda(0) - r_\lambda(a) \} = \text{const.}\ |a|^{\alpha-1}, \quad \text{with positive const.}$$

Hence if ν is difference of two prob^{ys},

$$\boxed{-\iint \nu(dx)\,\nu(dy)\,|x-y|^{\alpha-1} = \lim_{\lambda\downarrow 0} \iint \nu(dx)\,\nu(dy)\,r_\lambda(x-y) \geq 0.}$$

Some examples of pricing in incomplete market (14/6/91)

Let's consider the situation $U(x) = -e^{-\gamma x}$, and the stock price is a log BM, $dS = (\sigma dB + \mu dt) S$, and we wish to price some smooth f^n f of W_T, where W is an independent BM which we can't trade on, but can observe

$$\psi(x,s) \equiv \sup_\theta EU\left(x + \int_0^T \theta_u dS_u + sf(W_T)\right)$$

$$= e^{-\gamma x} \sup_\theta E - \exp\left\{-\gamma \int_0^T \theta_u dS_u - \gamma s f(W_T)\right\}.$$

Now suppose $E\left[\exp(-\gamma s f(W_T)) \mid \mathcal{F}_t\right] = g_s(t, W_t)$, a martingale. We shall have (writing $\theta dS \equiv H(\sigma dB + \mu dt)$) that

$$E \exp\left\{-\gamma \int_0^T H(\sigma dB + \mu dt) - \gamma f(W_T)\right\}$$

$$a \equiv \mu/\sigma$$

$$= \tilde{E} \exp\left[a\tilde{B}_T - \tfrac{1}{2}a^2 T - \gamma\sigma \int_0^T H \, d\tilde{B} - s\gamma f(W_T)\right], \qquad \tilde{B}_t \equiv B_t + at$$

$$= \tilde{E} \exp\left[-\tfrac{1}{2}a^2 T + \int_0^T (a - \gamma\sigma H) \, d\tilde{B} - s\gamma f(W_T)\right]$$

$$\geq \tilde{E} \exp\left[-\tfrac{1}{2}a^2 T + \int_0^T (a - \gamma\sigma H) \, d\tilde{B} - \tfrac{1}{2}\int_0^T (a - \gamma\sigma H)^2 dt - s\gamma f(W_T)\right]$$

$$= e^{-a^2 T/2} g_s(0, W_0) \qquad \text{since } d\langle \tilde{B}, W \rangle = 0.$$

This lower bound is attained when $H = a/\gamma\sigma$, so this tells us the optimal policy. Note that it is constant, and, in particular, completely independent of W. This is slightly remarkable – one might have thought that being able to see W could be exploited...

We have explicitly

$$\boxed{\psi(x,s) = -e^{-\gamma x - a^2 T/2} g_s(0,0).}$$

Eg1: $f(x) = x$. We have $g_s(0,0) = \exp\left(\tfrac{1}{2}\gamma^2 s^2 T\right)$ so that $\psi(x,s)$ is $\downarrow$ in s; the contingent claim isn't ≥ 0, and isn't desirable!

Eg2: $f(x) = x^2$. Here, $g_s(0,0) = (1 + 2\gamma s T)^{-1/2}$.

Can the assumption of independence of B, W be dropped? It appears so. Suppose $W = \rho B + \upsilon\beta$, $\upsilon^2+\rho^2=1$, B, β indpt BM, then

$$\min_H E \exp - \gamma\sigma\int_0^T H\, d\tilde{B} - s\gamma f(\rho B_T + \upsilon \beta_T)$$

$$= \min_H \tilde{E} \exp\{ a\tilde{B}_T - a^2T/2 - \gamma\sigma \int_0^T H\, d\tilde{B} - s\gamma f(\rho\tilde{B}_T - \rho a T + \upsilon\beta_T)\}.$$

Now if $E \exp - s\gamma f(x - \rho a T + \upsilon\beta_T) \equiv \exp - V(x)$, then the problem is

$$\min_H \tilde{E} \exp\{ a B_T - a^2T/2 - \gamma\sigma\int_0^T H\, d\tilde{B} - V(\rho\tilde{B}_T)\}$$

FALSE!! H will in general depend on β!

and if we make the integral representation $V(\rho\tilde{B}_T) = c + \int_0^T K_s\, d\tilde{B}_s$, then we have as before

$$\tilde{E}\exp(a\tilde{B}_T - a^2T/2 - \gamma\sigma\int_0^T H\, d\tilde{B} - c - \int_0^T K_s\, d\tilde{B}_s)$$

$$\geq \tilde{E}\exp\{ -a^2T/2 - c + \int_0^T (a - \gamma\sigma H_s - K_s)\, d\tilde{B}_s - \tfrac{1}{2}\int_0^T (a - \gamma\sigma H_s - K_s)^2 ds\}$$

$$= \exp(-c - a^2T/2)$$

with equality if $a = \gamma\sigma H_s + K_s$. The optimal control is a function only of $\tilde{B}$, so we do not use the additional information implicit in β.

<u>Eg</u>: $\underline{f(x) = x^2}$. Here we can easily prove that

$$C \equiv E\, V(\rho\tilde{B}_T) = \frac{s\gamma\rho^2}{(1+2s\gamma\upsilon T)}(a^2T^2+T) - \tfrac{1}{2}\log(1+2s\gamma\upsilon T),$$

whence

$$\boxed{\psi(x,s) = (1+2s\gamma\upsilon T)^{-\frac{1}{2}}\exp\left\{-\gamma x - a^2T/2 - \frac{s\gamma\rho^2(a^2T^2+T)}{1+2s\gamma\upsilon T}\right\}},$$

and

$$\boxed{\theta_t^* = \xi^{-1}\left\{\frac{\mu}{\gamma\sigma^2} - \frac{2s\rho^2(\tilde{B}_t - aT)}{\sigma(1+2s\gamma\upsilon T)}\right\}}.$$

<u>Pricing of a futures contract</u> (24/6/91) (cf book VII, p41)

Suppose that at time 0 Short agrees to sell Long at time T one share at strike price K. In a futures contract of this type, what happens is that Short sells Long one share at time T at the market price S_T, but that there are payments going on in the meantime to offset the movement in price.

At time t, Short has paid to Long an amount $S_t - K$, so that the overall payment of Long to Short is exactly K. If (β_t) is the stochastic discount process, then the value to Short of the cashflow received is

$$\tilde{E}_0\Big[K - S_0 - \int_0^T \beta_u \, dS_u \Big]$$

(at time T, must pay S_T to get a share which is immediately sold for S_T)

$$= \tilde{E}_0\Big[\int_0^T S_u \, d\beta_u + K - \beta_T S_T \Big]$$

so that the fair strike price to pay at time 0 is

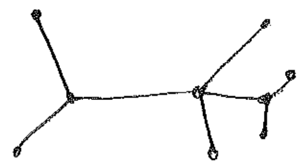
$$K = \tilde{E}_0 \Big[-\int_0^T S_u \, d\beta_u + \beta_T S_T \Big]$$

Compare with a <u>forward</u> contract (Long pays Short an amount K_1 at time 0, and Short delivers one share at time T), where fair strike price is

$$K_1 = \tilde{E}_0 \big[\beta_T S_T \big] \leq K \quad .$$

<u>Interpretation of a formula of David Dean</u> (24/6/91)

Take some tree structure, put a measure μ onto it and try to get the law of the moment of inertia about the centroid:

$G =$ 

$$\varphi = E \exp\Big\{ - \int (X - \bar{X})^2 d\mu \Big\} \qquad \text{where } \bar{X} = \rho \int X d\mu, \quad \rho = \Big(\int d\mu\Big)^{-1}.$$

$$= E \exp \Big\{ - \Big(\int X^2 d\mu - \rho \Big(\int X d\mu\Big) \Big) \Big\}.$$

Here, X is a Brownian motion whose time-parameter set is the tree (the lengths of the edges give the duration of the pieces of Brownian path), with X started at 0 at some distinguished vertex. So X is a Branching BM with the named tree shape.

The particular vertex chosen to make the starting point is immaterial. If one chooses vertex k, then

$$\varphi^2 = \Big(\prod_{\sigma\in G} p_{\sigma,k}\Big) / (\varphi\, d_k),$$

where $d_k \equiv$ rate of μ-killed excursions from k, and

$p_{\sigma,k} = P^i$ (reach j without μ-killing)

where $\sigma = (i,j)$ is the edge from i to j (j nearest to k), and we take a BM with statespace G, picking excursions from the various vertices at equal rate.

Now if we just look at this BM when it's at the vertices of G, we see a Markov chain on the vertices of G, with a Q-matrix Q. Moreover,

$$q_{ij} \equiv n_i(\quad \text{excns which get to } j \text{ without killing})$$

$$= q_{ji},$$

$$q_i = \sum_{j\neq i} n_i(\text{excns going along } (i,j) \text{ which reach } j \text{ or are killed}).$$

Then

$$(*) \qquad \boxed{\Big(\prod_{\sigma\in G} p_{\sigma,k}\Big) / d_k = \prod_{\sigma} q_{ij} \Big/ |\det Q|}$$

The proof is by induction on the number of vertices. With two vertices, one edge σ,

$$p_{\sigma,1} \equiv P^2[\text{reach } 1 \text{ with no killing}] = q_{21}/q_2$$

$$d_1 = q_1 - q_{12} + q_{12}(1 - q_{21}/q_2) = (q_1 q_2 - q_{12} q_{21})/q_2$$

and the result is immediate.

Suppose next that we adjoin one more vertex 0 to our graph, linked to vertex 1. The original Q-matrix Q gets modified to

$$\bar{Q} = \left(\begin{array}{c|ccc} -q_0 & q_{01} & 0 \cdots & 0 \\ \hline q_{01} & & & \\ 0 & & Q - q_0 e_1 e_1^T & \\ \vdots & & & \\ 0 & & & \end{array}\right)$$

and we now have

$$\bar{d}_1 = d_1 + (q_0 - q_{01}) + q_{01}(1 - q_{01}/q_0)$$

$$= d_1 + q_0 - q_{01}^2/q_0 \,,$$

$$P_{\sigma,1} = q_{01}/q_0 \qquad \text{for } \sigma = (0,1)\,,$$

$$\det \bar{Q} = -q_0 \left| Q - q_0 e_1 e_1^T \right| - q_{01}^2 \cdot \det Q \cdot (Q^{-1})_{11}$$

$$= \det Q \left\{ -q_0 \det\left(I - q_0 Q^{-1} e_1 e_1^T\right) + q_{01}^2 \, g_{11} \right\}$$

where $(g_{ij}) = (-Q)^{-1}$ is the Green function, $g_{11} \equiv 1/d_1$;

$$= \det Q \left\{ q_{01}^2 \, g_{11} - q_0 \left(1 - q_0 Q^{-1}_{11}\right) \right\}$$

$$= \det Q \left\{ q_{01}^2 \, g_{11} - q_0 + q_0^2 (-g_{11}) \right\}.$$

To get from the LHS of (*) for the old graph to the LHS for the new, we multiply by

$$\frac{q_{01}/q_0}{d_1 + q_0 - q_{01}^2/q_0} \cdot d_1 = \frac{q_{01}}{q_0 + (q_0^2 - q_{01}^2) g_{11}}$$

and on the RHS the new factor is

$$q_{01} \frac{|\det Q|}{|\det \bar{Q}|} = \frac{q_{01}}{|q_0 + (q_0^2 - q_{01}^2) g_{11}|}\,,$$

which is the same, completing the proof.

Relation between Schrödinger-type operators and divergence-form operators (26/6/91)

If we had $\mathcal{G} = \frac{1}{2}\Delta - V$, and we could find some positive φ such that $\mathcal{G}\varphi = 0$, then

$$Z_t \equiv \varphi(X_t)\exp\left(-\int_0^t V(X_s)\,ds\right) \text{ is a martingale,}$$

And we have the prospect of changing measure using Z. If $\tilde{\mathbb{P}}$ is the new measure and P_t^V is the semigroup of $\mathcal{G}$, we have

$$\tilde{\mathbb{E}}^x f(X_t) = \varphi(x)^{-1}\,\mathbb{E}^x\,\varphi f(X_t)\,e^{-A_t} = \varphi(x)^{-1} P_t^V \varphi f(x)$$

$$\Rightarrow \tilde{\mathcal{G}} f(x) = \varphi(x)^{-1}\,\mathcal{G}(\varphi f) = \varphi^{-1}\left\{\varphi\cdot\tfrac{1}{2}\Delta f + \nabla\varphi\,\nabla f\right\}$$

$$= \left(\tfrac{1}{2}\Delta + \nabla\log\varphi\,\nabla\right) f.$$

Thus the semigroup of $\mathcal{G}$ is unitarily equivalent to the semigroup of $\tilde{\mathcal{G}}$, and this allows us to pass from one to the other. It is essentially all that's going on in the 'Prob Thy + Polymer physics' paper!

A question of Peter Donnelly (27/6/91).

Suppose that $(P_t)_{t\geq 0}$ is the semigp of a Feller process with bounded generator $\mathcal{G}$ on some ordered ~~state space~~ I. There is a result of Harris which says that if $P_t : \mathcal{M} \to \mathcal{M} \equiv \{\text{increasing } f^{ns} \text{ on } I\}$ then

$$P_t(fg) \geq P_t f\cdot P_t g \quad \forall f,g\in\mathcal{M} \Leftrightarrow \Gamma(f,g) \equiv \mathcal{G}(fg) - f\mathcal{G}g - g\mathcal{G}f \geq 0 \quad \forall f,g\in\mathcal{M}.$$

Peter wants to get a similar negative correlation result, assuming that $I = I_1\times\cdots\times I_N$, and

$$-g,\ f\in\mathcal{M}_A = \{\text{functions on } I \text{ which are } \uparrow \text{ in the } A\text{-variables, decr in } A^c \text{ variables}\}$$

where $A \subseteq \{1,\dots,N\}$.

If we simply assume $\boxed{P_t : \mathcal{M}_A \to \mathcal{M}_A}$, then the same proof as before works and we get

$$\boxed{P_t fg \leq P_t f\, P_t g \quad \forall f\in\mathcal{M}_A,\ g\in\mathcal{M}_{A^c} \equiv -\mathcal{M}_A \Leftrightarrow \Gamma(f,g)\leq 0 \quad \forall f\in\mathcal{M}_A,\ g\in\mathcal{M}_{A^c}}.$$

Pricing in an incomplete market: example (28/6/91)

1) We return to the situation on p. 35 and analyse it correctly for one special case. If B, W are independent standard Brownian motions, the goal is to

$$\min_H E \exp\left[-\int_0^T H_s\, dB_s - \alpha\, (B_T + \varepsilon W_T + c)^2 \right]$$

If we set

$$\varphi(t,c) \equiv \min_H E \exp\left[-\int_0^t H_s\, dB_s - \alpha\, (B_t + \varepsilon W_t + c)^2 \right],$$

then the process

$$Y_t \equiv \varphi(T-t,\, B_t + \varepsilon W_t) \exp\left[-\int_0^t H_u\, dB_u \right]$$

is a submartingale, and a martingale under optimal control. Thus

$$-\dot\varphi + \tfrac{1}{2}(1+\varepsilon^2)\varphi'' + \tfrac{1}{2} H^2 \varphi - H\varphi' \geq 0$$

with equality at optimum. Also, $\varphi(0,c) = \exp(-\alpha c^2)$. Minimising over H gives

$$\boxed{H^* = \varphi'/\varphi}$$

for optimal strategy, and the pde for φ is

$$\tfrac{1}{2}(1+\varepsilon^2)\varphi'' - \dot\varphi - \tfrac{1}{2}\varphi'^2/\varphi = 0.$$

Writing $u(t,x) \equiv \log \varphi(t,x)$, we see that the pde is

$$\boxed{\tfrac{1}{2}(1+\varepsilon^2) u'' + \tfrac{1}{2}\varepsilon^2 u'^2 - \dot u = 0, \qquad u(0,x) \equiv -\alpha x^2}$$

which is solved exactly by

$$\boxed{u(t,x) = -\frac{\alpha x^2}{1+2\alpha\varepsilon^2 t} - \frac{1+\varepsilon^2}{2\varepsilon^2}\log\left(1+2\alpha\varepsilon^2 t\right),}$$

and

$$\boxed{H^* = -\frac{2\alpha x}{1+2\alpha\varepsilon^2(T-t)}.}$$

2) We can use this to give an expression for the solution to the original problem,

which was

$$\max_{\theta} E - \exp\{-\gamma x - \gamma \int_0^T \theta_u \, dS_u - \gamma \rho (B_T + \varepsilon W_T)^2\}$$

$$= -\exp\left[-\gamma x - \frac{a^2 T}{2} - \frac{\gamma\rho\, a^2 T^2}{1 + 2\gamma\rho\,\varepsilon^2 T}\right]\left(1 + 2\gamma\rho\,\varepsilon^2 T\right)^{-(1+\varepsilon^2)/2\varepsilon^2}.$$

We can check that as $\rho \downarrow 0$, $\varepsilon \uparrow \infty$ so that $\rho\varepsilon^2 \to 1$, we get the answer on p. 34 to the simpler problem when the contingent claim is simply W_T^2.
We can also read off the optimal portfolio:

$$\sigma\gamma\theta^*_t \equiv H^*_t S_t^{-1} + a S_t^{-1} = \frac{-2\gamma\rho\,(\tilde{B}_t + \varepsilon W_t - aT)}{1 + 2\gamma\rho\,\varepsilon^2(T-t)} S_t^{-1} + a S_t^{-1}$$

which again checks out OK when $\varepsilon \uparrow \infty$, $\rho\varepsilon^2 \to 1$.

Convexity of the energy functional (2/7/91)

The use of recurrent potential theory for Lévy processes on p 31-32 is rather an elaborate way to prove what is essentially a simpler result.

(i) If ν is the difference of two prob'y measures on $\mathbb{R}^d$, and $0 < \beta \le 2$, then

$$-\iint_{\mathbb{R}^d \times \mathbb{R}^d} \nu(dx)\,\nu(dy)\, |x-y|^\beta \ge 0. \qquad \text{(assuming the integral absolutely cgt)}$$

Proof. It's enough to do for compactly supported ν. If (X_t) is symmetric stable (β) process, then

$$E\, e^{i\theta\cdot X_t} = \exp(-t|\theta|^\beta)$$

so that

$$\iint \nu(dx)\,\nu(dy)\, e^{-\varepsilon|x-y|^\beta} = \iint \nu(dx)\,\nu(dy) \int p_\varepsilon(a)\, e^{i(x-y)a}\, da$$

$$= \int p_\varepsilon(a)\, da\, |\hat{\nu}(a)|^2$$

$$\ge 0$$

Thus
$$\iint \nu(dx)\,\nu(dy)\left(e^{-\varepsilon|x-y|^\beta}-1\right) \geq 0$$
and the result follows when we divide by ε, let $\varepsilon \downarrow 0$.

(ii) If ν is the difference of two prob^y measures, then
$$-\iint_{(\mathbb{R}^d)^2} \nu(dx)\,\nu(dy)\,\log|x-y| \geq 0$$
assuming the integral absolutely convergent.

Proof. Use
$$-\log|x| = \text{const.}\ \lim_n \int_0^n \left\{ e^{-|x|^2/2t} - e^{-1/2t} \right\} \frac{dt}{2\pi t} \ ,$$

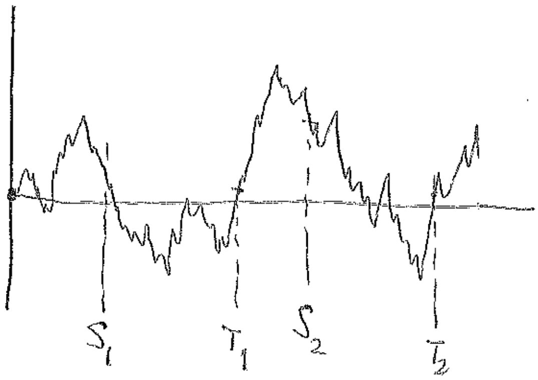
Note: the method of (i) works whenever $V(x) = \log E(e^{i x\cdot X})$, with X a symmetric infinitely divisible law on $\mathbb{R}^d$.

Mechanical trading rules on foreign exchanges (9/7/91)

A paper presented by Richard M. Levich (with Lee R. Thomas) at Gerzensee entitled "The significance of technical trading-rule profits in the foreign exchange market: a bootstrap approach" discusses certain "x%-filter" rules. These can be described as follows.

Let log price $X_t \equiv B_t + \mu t$, $T_0 \equiv 0$,
$$S_{n+1} = \inf\{t > T_n : X_t < \sup_{[T_n,t]} X_u - \varepsilon\} \qquad (n \geq 0)$$
$$T_n = \inf\{t > S_n : X_t > \inf_{[S_n,t]} X_u + \varepsilon\} \qquad (n \geq 1),$$

where $\varepsilon > 0$ is some fixed parameter.



At time 0, invest \$1 in the foreign currency, to buy FC$(1/P_0)$ (so $P_t \equiv e^{X_t}$ is no of \$/FC) and at time S_1 when the foreign currency has weakened, you convert all back into \$, when you get \$$(P_{S_1}/P_0)$. At time T_1, when the foreign currency looks better, you put \$ back into FC, obtaining
$$FC\left(\frac{P_{S_1}}{P_0}\cdot\frac{1}{P_{T_1}}\right)$$
and at S_2 you sell back, to get \$ $\exp\left[-X_0 + X_{S_1} - X_{T_1} + X_{S_2}\right]$.

Now the r.v.'s $X(S_j) - X(T_{j-1})$ are i.i.d., as are the r.v.'s $X(T_j) - X(S_j)$. The laws are

$$X(S_j) - X(T_{j-1}) \sim \exp(\rho) - \varepsilon$$

$$X(T_j) - X(S_j) \sim -\exp(\rho') + \varepsilon$$

where ρ is the rate of downward xans $\lim_{h\downarrow 0} \frac{1}{h} \frac{s(0)-s(-h)}{s(0)-s(-\varepsilon)} = \lim_{h\downarrow 0} \frac{1}{h} \frac{-1+e^{2\mu h}}{-1+e^{2\mu\varepsilon}}$, so

$$\rho = \frac{2\mu}{e^{2\mu\varepsilon}-1}, \quad \rho' = \frac{-2\mu}{e^{-2\mu\varepsilon}-1}.$$

Now $E\{B(S_j) - B(T_{j-1})\} = E\left[X(S_j) - X(T_{j-1}) - \mu(S_j - T_{j-1})\right] = 0$

$$\Rightarrow E(S_j - T_{j-1}) = \frac{1}{\mu}\left\{\frac{1}{\rho} - \varepsilon\right\} = \frac{e^{2\mu\varepsilon}-1-2\mu\varepsilon}{2\mu^2}$$

which is $\sim \varepsilon^2$ as $\varepsilon \downarrow 0$; and $E(T_j - S_j) \sim \varepsilon^2$ likewise.
Thus there are essentially $t/2\varepsilon^2$ spells during which you hold FC before time t, each of which gets you on average

$$E\{X(S_j) - X(T_{j-1})\} = \frac{1}{\rho} - \varepsilon = \frac{e^{2\mu\varepsilon}-1-2\mu\varepsilon}{2\mu} \sim \mu\varepsilon^2$$

[Note $-E(X(T_j) - X(S_j)) = \varepsilon - \frac{1}{\rho'} = \varepsilon + \frac{e^{-2\mu\varepsilon}-1}{2\mu} = \frac{2\mu\varepsilon - 1 + e^{-2\mu\varepsilon}}{2\mu} \sim \mu\varepsilon^2$ also]

Thus by time t, the holding in \$ is worth essentially $e^{\mu t}$!!

Trading with transaction costs (13/7/91)

Suppose we have a bond with zero interest rate, and a stock whose price at time t is S_t, a semimg. If now we consider the situation where we have to pay βS_t to buy 1 unit of stock, receive ρS_t if we sell one unit of stock $(0<\rho<1<\beta)$, then only f.v. trading strategies may be considered. Let the holding of stock at time t be

$$H_t = H_0 + H_t^+ - H_t^- \qquad 0 = H_0^+ = H_0^-, \quad H^{\pm} \text{ incr adapted R-processes,}$$

and let the holding of cash at time t be C_t. Self-financing $\Rightarrow$

$$\boxed{C_t = C_0 - (\beta-\rho)\int_{(0,t]} S_u \, dH_u^+ - \rho S_t H_t + \rho \int_{(0,t]} H_{u-} \, dS_u}$$

and so if $V_t = C_t + \rho S_t H_t$ is the cash value of the holding at time t, then

$$V_t = C_0 - (\beta - \rho) \int_{(0,t]} S_u \, dH_u^+ + \rho \int_{(0,t]} H_{u-} \, dS_u \quad .$$

$$\equiv C_0 + V_t(H), \quad \text{say}.$$

There are now two questions of great interest; can we replicate a c.c. Y by some H, and, even more importantly, can we dominate Y?

EMM once again (13/8/91)

(i) One possible definition of arbitrage would be the following (let's just assume that S is a cts semimartingale for now)

(A') "There exists previsible θ s.t. $\int_0^T \theta_s^2 \, d\langle S\rangle_s < \infty$, $(\theta \cdot S)_t \geq -1 \;\; \forall t$, and $(\theta \cdot S)_1 \geq 0$, $P[(\theta \cdot S)_1 > 0] > 0$."

The conjecture would be $\exists$ EMM $\Leftrightarrow$ not(A'). Taking the simple case $dX_t = dB_t + f(t)\,dt$, where f is deterministic, $\int_0^1 f^2(t)\,dt = +\infty > \int_0^1 f(t)\,dt$, with $dS_t = S_t \, dX_t$, there is no EMM in this situation; is there an arbitrage opportunity in the sense of (A')? Indeed there is: solving

$$dJ_t = J_t \, f(t) \, dX_t \equiv J_t \, f(t) \, S_t^{-1} \, dS_t \qquad , \; J_0 = 1,$$

gives the solution

$$J_t = \exp\left(\int_0^t f_s \, dB_s + \tfrac{1}{2} \int_0^t f_s^2 \, ds \right)$$

which is always ≥ 0, but tends to $+\infty$. If now we simply stop when J reaches 2, we've got an (A') arbitrage opportunity.

(ii) The price process in general will be a cts semi mg

$$S_t = S_0 + M_t + A_t = S_0 + M_t + \int_0^t \alpha_u \, d\langle M\rangle_u$$

for there to be any prospect of an EMM. If there is going to be an EMM, it has to be given by the process

$$Z_t \equiv \exp\left\{ -\int_0^t \alpha_s \, dM_s - \tfrac{1}{2} \int_0^t \alpha_s^2 \, d[M]_s \right\}, \qquad 0 \leq t \leq T,$$

and the only problem is that this exponential local mg may fail to be UI. The paper by Azéma, Gundy + Yor in SP $\underline{\text{XIV}}$ gives us that

$$Z \text{ is UI} \iff \lambda P(Z^* > \lambda) \to 0 \quad (\lambda \uparrow \infty)$$

$$\iff \lambda P([Z]_\infty^{1/2} > \lambda) \to 0 \quad (\lambda \uparrow \infty)$$

$$\iff E Z_\infty = 1.$$

Time-changing by $\tau_t = \inf\{u : [Z]_u > t\}$ gives $\tilde{Z}_t = Z(\tau_t) \equiv B(t \wedge [Z]_\infty) \equiv B(t \wedge T)$, where B is an $\mathcal{F}(\tau_t)$-BM. Thus the UI of Z is the same as the UI of B^T.

(iii) This leads us to consider the UI of $B(t \wedge T)$, where B is a BM started at 1, T is a stopping time, $T \leq H_0$. In particular,

Can we characterise all possible laws of $(\bar{B}_T, B_T)$, $T \leq H_0$?

It appears that we can. Let $\tau_a \equiv \inf\{u > 0 : B_u > 1+a\}$, $a \geq 0$, and consider the $\tilde{\mathcal{F}}_a \equiv \mathcal{F}(\tau_a)$-martingale

$$\tilde{B}_a \equiv B(\tau_a) = \begin{cases} 1+a & \text{for } a < \bar{B}_\infty - 1 \equiv \zeta = \bar{B}_T - 1 \\ B_T & \text{for } a \geq \zeta. \end{cases}$$

Then

$$\tilde{B}_a = 1 + a \wedge \zeta + (B_T - \bar{B}_T)\, I_{(a \geq \zeta)}$$

$$= 1 + a \wedge \zeta - J_a$$

where J is the jump process

$$J_a \equiv (\bar{B}_T - B_T)\, I(a \geq \zeta).$$

We let $\nu(dt, dx)$ be the random measure on $(\mathbb{R}^+)^2$ corresponding to J, with previsible compensating measure $\tilde{\nu}(dt, dx)$. The aim is to characterise

$$E\, e^{-\lambda\zeta - \alpha J_\infty} = E \iint e^{-\lambda t - \alpha x}\, \nu(dt, dx)$$

$$= E \iint e^{-\lambda t - \alpha x}\, \tilde{\nu}(dt, dx).$$

Let's write

$$\mu(dt, dx) \equiv E\, \tilde{\nu}(dt, dx), \ \equiv E\, \nu(dt, dx)$$

and observe that μ is a probability, since $\iint \nu(dt, dx) = 1$.

Next,

$$0 = E\int_0^\infty e^{-\lambda s}\, d\tilde{B}_s = E\left[\int_0^\zeta e^{-\lambda s} ds - \int_0^\infty\int_0^\infty e^{-\lambda t}\, x\, \nu(dt,dx)\right]$$

implying $\boxed{\int x\,\mu(dt,dx) = dt\; P(\zeta\ge t)}$

Also, $\int_{[0,\infty)} \mu(dt,dx) = P(\zeta\in dt)$. Let's note that if $P(\zeta\in\cdot)$ has an atom at some point t_0, then the restriction of μ to $\{t_0\}\times[0,\infty)$ must be concentrated on $(t_0,0)$, since t_0 is a previsible time, and so the jump of the mg $\tilde{B}$ at t_0 has zero expectation. Let $F(t,dx)$ be a r.c.d. for x given $\zeta=t$.

The next thing to notice is that the rate of killing is

$$\boxed{\rho(dt) = \frac{P(\zeta\in dt)}{P(\zeta\ge t)} = \frac{dt}{\int_0^\infty x\,F(t,dx)}} \qquad (*)$$

This is actually a characterisation; more precisely, given a dist^n $P(\zeta\in\cdot)$ on $[0,\infty)$ and for each $t\ge 0$ a prob $F(t,dx)$ on $[0,t+1]$ with the property that $F(t,\cdot)$ is concentrated on 0 if t is an atom of the law of ζ, and such that for times which are not atoms of $P(\zeta\in\cdot)$ the relation $(*)$ holds, we can construct a BM started at 1 and a stopping time T such that the law of $(\bar{B}_T-1,\ \bar{B}_T-B_T)$ is given by $P(\zeta\in dt)\,F(t,dx)$.

To sketch out the way this happens, imagine that at level t, we pick a barrier according to the dist^n

$$\frac{x\,F(t,dx)}{\int y\,F(t,dy)} \equiv G(t,dx)$$

and we choose such a barrier independently for each possible level. When an excursion goes down from the maximum, it is stopped at the barrier if it should reach that far. Thus the rate of excursions which get stopped is

$$\int \frac{1}{x}\,G(t,dx) = \left(\int y\,F(t,dy)\right)^{-1}$$

and, given that the excursion does get stopped, the dist^n of the drop below the maximum is just $F(t,\cdot)$. The relation $(*)$ now just says that the law of ζ is consistent with this stopping rule (of course, the atoms have ~~to~~ be included, but that is not such a big

<u>Note</u> the condition for non UI is

$$\int_1^\infty \frac{dv}{v} \left\{ v\theta(v) \coth v\theta(v) - 1 \right\} < \infty$$

and since $x \coth x$ is strictly incr from 1, and goes like const x^2 for small x, the condition is equivalent to $\int_1^\infty v\,\lambda(v)\,dv < \infty$

Now if we think of the killing functional $K_t \equiv \int_0^t \lambda(\bar{B}_s)\,ds$ as being the thing which triggers the bomb when it reaches exp r.v., the non UI condition is equivalent to

$$E K_t < \infty$$

So non UI $\Rightarrow$ pos prob of no killing (even if we don't stop the BM at 0).

issue.)

(iv) If we set

$$\gamma_t \equiv 1+t - \int x\, F(t,dx) \equiv E\big[B_T \mid \bar{B}_T = 1+t\big],$$

then (ignoring atoms)

$$P(J \geq t) = \exp\Big[-\int_0^t \frac{ds}{1+s-\gamma_s}\Big]$$

so that

$$(1+t)P(J\geq t) = \exp\Big\{-\int_0^t ds\Big(\frac{1}{1+s-\gamma_s} - \frac{1}{1+s}\Big)\Big\}$$

$$= \exp\Big[-\int_0^t \frac{\gamma_s}{(1+s)(1+s-\gamma_s)}\,ds\Big]$$

so the non UI condition is exactly

$$\boxed{\int_0^\infty \frac{\gamma_s\, ds}{(1+s)(1+s-\gamma_s)} < \infty}$$

(v) Now let's consider the following recipe for stopping a BM B which starts at 1: we kill independently at rate $\lambda(\bar{B}_T)$. If $\theta(y) \equiv \sqrt{2\lambda(y)}$, then

Rate of killing when max is y = $\theta \coth \theta y$

(We also kill immediately the process reaches zero, of course.) Thus if $c_t \equiv \gamma_{t-1}$ $(t\geq 1)$ we obtain

$$P[\bar{B}_\infty \geq a] = \exp\Big(-\int_1^a \frac{dv}{v-c_v}\Big) = \exp\Big(-\int_1^a \theta(y)\coth y\theta(y)\, dy\Big)$$

implying

$$y - c_y = \big(\theta(y)\coth y\theta(y)\big)^{-1}$$

and

$$c_y = y - \big(\theta(y)\coth y\theta(y)\big)^{-1}.$$

The function $\theta(y)$ can be chosen so that B^T is not UI. The point of this example is the following. If we think we are going to make an arbitrage opportunity as a stochastic integral wrt $1/Z$ (equivalently, wrt S), then the time at which B gets stopped is <u>totally inaccessible</u>, and indeed

$$I_{[T,\infty)}(t) - \int_0^{t\wedge T} \lambda(\bar{B}_s)\, ds$$

is a martingale: the killing can come at <u>absolutely any time</u> !! Thus if one makes up $H \cdot S$, if this <u>ever</u> went negative, you <u>could</u> get caught out then! Thus this is an example where <u>there is no EMM, yet no arbitrage opportunity of the (A') kind exists</u> !

So we need a more relaxed notion of EMM.

<u>A variant of the arcsine law</u> (27/8/91)

David Hobson asks the following. If $A_t \equiv \int_0^t I_{\{B_s > 0\}}\, ds$, for what functions $g : \mathbb{R}^+ \to \mathbb{R}^+$ do we have

$$\liminf_{t\to\infty} \frac{A_t}{g(t)} = 0 \quad ?$$

Assume that $\boxed{g(t)/t \ \downarrow}$. If $\inf g(t)/t > 0$, then $\liminf A_t/g(t) = 0$ a.s., so wlog we may assume $g(t)/t \downarrow 0$. Assume also that $\boxed{g(t)\uparrow\infty}$. Then

$$\boxed{\liminf_{t\to\infty} \frac{A_t}{g(t)} = 0 \iff \int^\infty \sqrt{g(t)}\, \frac{dt}{t^{3/2}} = +\infty}$$

David's proof that $\int^\infty \sqrt{g(t)}\, dt/t^{3/2} < \infty \Rightarrow A_t/g(t) \to \infty$ (by looking at the times 2^n + using Borel-Cantelli) is about as simple as it could be.

The converse is more involved. We shall say that an excursion is <u>long</u> if its lifetime is at least $\varphi(L_t)$, where φ is increasing, and, to be quite explicit, we choose

$$\varphi(\ell) = g^{-1}(\ell^2/\varepsilon)$$

where $\varepsilon > 0$ is fixed but small. Let N_t be the number of long negative excursions before $\gamma_t = L^{-1}(t)$ which start at a time which is $\le L^2$. Thus

$$N_t - \int_0^t I_{\{\gamma_\ell \le \ell^2\}} \frac{\text{const}}{\sqrt{\varphi(\ell)}}\, d\ell \quad \text{is an } \mathcal{F}(\gamma_t)\text{-martingale},$$

and it is not hard to prove that

$$N_\infty = \infty \iff \int_0^\infty I_{\{\gamma_\ell \le \ell^2\}}\, \varphi(\ell)^{-1/2}\, d\ell = \infty.$$

So the aim now is to prove that

$$\int_0^\infty I_{\{\gamma_t \le t^2\}} \, \varphi(t)^{-\frac{1}{2}} \, dt = +\infty \text{ a.s.} \iff \int_0^\infty \sqrt{g(t)} \, t^{-3/2} \, dt = +\infty .$$

Since γ is a subordinator, the event $\{ \int_0^\infty I_{\{\gamma_t \le t^2\}} \varphi(t)^{-\frac{1}{2}} dt = +\infty \}$ is exchangeable, and so will have probability 0 or 1.

Define

$$V_t \equiv \int_0^t I_{\{\gamma_s \le s^2\}} \, \varphi(s)^{-\frac{1}{2}} \, ds ,$$

and $P(\gamma_s \le s^2) \equiv c_*$, same for all s. Denote $J_t \equiv I_{\{\gamma_t \le t^2\}}$. Let's now notice that for $s \le t$,

$$P\left[\gamma_t \le t^2 \,\middle|\, \gamma_s \le s^2\right] = \int_0^{s^2} P(\gamma_s \in du) \, \frac{P(\gamma_{t-s} \le t^2 - u)}{P(\gamma_s \le s^2)}$$

so that (taking $s = 1$ wlog)

$$0 \le P\left[\gamma_t \le t^2 \mid \gamma_1 \le 1\right] - P\left[\gamma_t \le t^2\right]$$

$$= \int_0^1 \frac{P(\gamma_1 \in du)}{P(\gamma_1 \le 1)} \left\{ P\left[HN > \frac{t-1}{\sqrt{t^2 - u}} \right] - P\left[HN > 1 \right] \right\}$$

$$= \int_0^1 \frac{P(\gamma_1 \in du)}{P(\gamma_1 \le 1)} \left\{ P \ HN \in \left(\frac{t-1}{\sqrt{t^2-u}} , 1 \right) \right\}$$

$$\le \text{const} \cdot 1/t$$

Thus

$$\operatorname{var}(V_t) = 2 \int_0^t \varphi(s)^{-\frac{1}{2}} ds \int_0^s \varphi(u)^{-\frac{1}{2}} du \ E\left(J_s J_u - c_*^2 \right)$$

$$\le 2 \int_0^t \varphi(s)^{-\frac{1}{2}} ds \int_0^s \varphi(u)^{-\frac{1}{2}} du \cdot \text{const} \cdot u/s$$

$$\propto 2 \int_0^t \varphi(s)^{-\frac{1}{2}} ds \left(\frac{1}{s} \int_0^s \frac{u}{\sqrt{\varphi(u)}} du \right)$$

But $u/\sqrt{\varphi(u)} = \sqrt{\varepsilon} \left(\dfrac{u^2/\varepsilon}{g^{-1}(u^2/\varepsilon)} \right)^{\frac{1}{2}} \downarrow 0$ by assumption and so

$$\operatorname{var}(V_t) = o\left((EV_t)^2 \right)$$

(We know that

$$EV_t = C_* \int_0^t \frac{ds}{\sqrt{\varphi(s)}} \propto \int^{g(t^2/\varepsilon)} \frac{g'(v)\,dv}{\sqrt{v g(v)}} \uparrow \infty \text{ by hypothesis)}$$

Thus we have

$$\frac{V_t}{EV_t} \xrightarrow{L^2} 1 \quad , \quad EV_t \uparrow \infty$$

which, together with the fact that $P(V_\infty = \infty) = 0$ or 1, implies that $P(V_\infty = \infty) = 1$.

A little example on characteristic functions (29/8/91)

Suppose $\varphi_n(\theta) \to \varphi(\theta)$ for all θ in some dense set in $\mathbb{R}$; can we conclude weak cgce of the μ_n? (here, $\varphi_n = \hat{\mu}_n$, of course).

As an example, consider $\mu_n = \delta_{2\pi \cdot 2^n}$. Then $\varphi_n(\theta) = \exp(2\pi i\, 2^n \theta)$ so that $\varphi_n(\theta) \to 1$ for all dyadic rational θ.

However, for almost every $\theta \in (0,1)$, the fractional parts of $2^n\theta$ will be dense in the unit circle, so for such θ, $\varphi_n(\theta)$ doesn't converge.

If we knew that the measures (μ_n) were tight, then the convergence of the cfs on a dense set does give convergence.

Stable processes (30/8/91)

Having worked out some of this stuff for the book, it seems worth recording.

Case 1 : $0<\alpha<1$

For $\theta > 0$,
$$\int_0^\infty \frac{dx}{x^{1+\alpha}}(e^{i\theta x}-1) = \int_0^\infty i\theta e^{i\theta t}\frac{dt}{\alpha t^\alpha}$$

$$= i\theta \int_0^\infty i\,dy \,\frac{e^{-\theta y}}{\alpha y^\alpha}\, e^{-i\alpha\pi/2}$$

$$= -\theta^\alpha e^{-i\alpha\pi/2} \int_0^\infty y^{-\alpha} e^{-y}\,\frac{dy}{\alpha}$$

$$= -\theta^\alpha e^{-i\alpha\pi/2}\,\Gamma(1-\alpha)/\alpha$$

Hence for $\theta \in \mathbb{R}$, we have

$$\boxed{\int_0^\infty \frac{dx}{x^{1+\alpha}}\left(e^{i\theta x}-1\right) = -\frac{|\theta|^\alpha\,\Gamma(1-\alpha)}{\alpha}\left\{\cos\left(\frac{\alpha\pi}{2}\right) - i\,\mathrm{sgn}(\theta)\,\sin(\alpha\pi/2)\right\}.}$$

Case 2: $\alpha=1$. For $\theta\in\mathbb{R}$, define

$$h(\theta) \equiv \int_0^\infty \frac{dx}{x^2}\left\{e^{i\theta x}-1 - i\theta(1-e^{-x})\right\},$$

and note $h(-\theta) = \overline{h(\theta)}$. If $f(z) = z^{-2}\left\{e^{i\theta z}-1 - i\theta(1-e^{-z})\right\}$, then

$$h(\theta) = \int_0^\infty f(x)\,dx = i\int_0^\infty dy\; f(iy)$$

$$= -i\int_0^\infty \frac{dy}{y^2}\left\{e^{-\theta y}-1 - i\theta\left(1-e^{-iy}\right)\right\}$$

$y = v/\theta$

$$= \int_0^\infty \theta\,\frac{dv}{v^2}\left\{-\theta\left(1-e^{-iv/\theta}\right) - i\left(e^{-v}-1\right)\right\}$$

$$= \theta^2\int_0^\infty \frac{dv}{v^2}\left\{e^{-iv/\theta}-1 - \frac{i}{\theta}\left(1-e^{-v}\right)\right\}$$

$$= \theta^2\, h(-1/\theta).$$

Next, observe that, for $\theta>0$,

$$\int_0^\infty \frac{dx}{x^2}\left\{e^{i\theta x}-1 - i\theta\left(1-e^{-x}\right)\right\}$$

$$= \int_0^\infty \frac{dx}{x^2}\left\{e^{i\theta x}-1 - i\left(1-e^{-\theta x}\right)\right\} + i\underbrace{\int_0^\infty \frac{dx}{x^2}\left\{1-e^{-\theta x} - \theta\left(1-e^{-x}\right)\right\}}_{\int_0^\infty \frac{dx}{x^2}\int_0^x \theta\left(e^{-\theta t} - e^{-t}\right)dt}$$

$$= \int_0^\infty \theta\left(e^{-\theta t}-e^{-t}\right)\frac{dt}{t}$$

$$= \theta\int_0^\infty \frac{dt}{t}\int_{\theta t}^{t} e^{-x}\,dx$$

$$= \theta\int_0^\infty e^{-x}\,dx\int_x^{x/\theta}\frac{dt}{t} = -\theta\log\theta$$

$$= \theta\,h(1) - i\theta\log\theta$$

But $h(1) = h(-1) = \overline{h(1)}$, so $h(1)$ is real, and thus

$$h(1) = \int_0^\infty \frac{dx}{x^2}(\cos x - 1) = -\pi/2$$

$$\therefore\quad \boxed{h(\theta) = -\pi|\theta|/2 - i\theta\log|\theta|}$$

Case 3 : $1<\alpha<2$. Here we have for $\theta>0$

$$\int_0^\infty \frac{dx}{x^{1+\alpha}}\left\{e^{i\theta x}-1-i\theta x I_{(x<1)}\right\} = \int_0^\infty d\left(\frac{-1}{\alpha x^\alpha}\right)\left(e^{i\theta x}-1-i\theta x I_{(x<1)}\right)$$

$$= \frac{i\theta}{\alpha}\int_0^\infty \frac{dx}{x^\alpha}\left(e^{i\theta x}-1\right) + \frac{i\theta}{\alpha-1}$$

$$= \frac{i\theta}{\alpha}\left(-|\theta|^{\alpha-1}\frac{\Gamma(2-\alpha)}{\alpha-1}e^{-i(\alpha-1)\pi/2}\right) + \frac{i\theta}{\alpha-1}$$

using result of case 1;

$$= \frac{|\theta|^\alpha}{\alpha(\alpha-1)}\Gamma(2-\alpha)\, e^{-i\alpha\pi/2} + \frac{i\theta}{\alpha-1}$$

Hence for $\theta\in\mathbb{R}$,

$$\boxed{\int_0^\infty \frac{dx}{x^{1+\alpha}}\left\{e^{i\theta x}-1-i\theta x I_{(x<1)}\right\} = \frac{|\theta|^\alpha}{\alpha(\alpha-1)}\Gamma(2-\alpha)\left\{\cos\frac{\alpha\pi}{2} - i\,\mathrm{sgn}(\theta)\sin\frac{\alpha\pi}{2}\right\} + \frac{i\theta}{\alpha-1}}$$

Bretagnolle's Lemma (9/9/91)

If (X,Y) is bivariate Gaussian, and $\varphi(X),\psi(Y)\in L^2$, then

$$|\rho(\varphi(X),\psi(Y))| \le |\rho(X,Y)|$$

and if $E(X-EX)\varphi(X)=0$, then

$$|\rho(\varphi(X),\psi(Y))| \le \rho(X,Y)^2 .$$

Proof. Without loss of generality $EX=EY=0$, $EX^2=EY^2=1$. Let's suppose $X=X_1$, $Y=Y_1$, where (X_t), (Y_t) are correlated standard Brownian motions. Then assuming wlog that $E\varphi(X)=E\psi(Y)=0$, we have the integral representation

$$\varphi(X) = \int_0^1 \theta_s\, dX_s = \int_0^1 (P_{1-s}\varphi)'(X_s)\,dX_s \qquad \text{in fact}$$

$$\psi(Y) = \int_0^1 \tilde\theta_s\, dY_s = \int_0^1 (P_{1-s}\psi)'(Y_s)\,dY_s .$$

Hence $E\varphi(X)\psi(Y) = \rho\, E\int_0^1 \theta_s\tilde\theta_s\,ds \le \rho\left(E\int_0^1\theta_s^2 ds\right)^{\frac12}\left(E\int_0^1\tilde\theta_s^2 ds\right)^{\frac12}$,

proving the first assertion. For the second, note that

$$E\,(P_{1-t}\varphi)'(X_t) \quad \text{is the same for all} \quad 0<t<1$$

so the statement $0 = E\,X\varphi(X) = E\int_0^1 (P_{1-t}\varphi)'(X_t)\,dt$ implies $E(P_{1-t}\varphi)'(X_t)=0$ for all t.

Hence

$$|E\,\varphi(X)\psi(Y)| = |\rho\, E \int_0^1 (P_{1-t}\varphi)'(X_t)\,(P_{1-t}\psi)'(Y_t)\,dt\,|$$

$$\leq \rho^2$$

by applying the first inequality to $(P_{1-t}\varphi)'(X_t)$, $(P_{1-t}\psi)'(Y_t)$.

<u>Excursions via resolvents again</u> (30/9/91).

1) The trick in the two papers [1] = ZW <u>63</u> 237-255, [2] = ZW <u>67</u> 473-476 was to fix $\beta>0$ and consider

$$R_\beta g(x) + (\beta-\lambda) R_\beta R_\lambda^\partial g(x)$$
$$= E^x\left[\int_0^\infty e^{-\beta t} g(X_t)\,dt\right] + (\beta-\lambda)\, E^x\left[\int_0^\infty e^{-\beta t}\,dt \int_t^{H\circ\theta_t} g(X_s)\,e^{-\lambda(s-t)}\,ds\right],$$

where $H \equiv \inf\{t>0:\ X_t \text{ or } X_{t-} \in B\}$. The second term is

$$(\beta-\lambda)\ E^x\left[\int_0^\infty g(X_s)\,e^{-\lambda s}\,ds \int_{\sigma_s}^{s} e^{-\beta t+\lambda t}\,dt\right], \qquad \text{where } \sigma_s \equiv \sup\{u<s:\ X_u \text{ or } X_{u-}\in B\};$$

$$= E^x \int_0^\infty g(X_s)\,e^{-\lambda s}\left\{e^{-(\beta-\lambda)\sigma_s} - e^{-(\beta-\lambda)s}\right\}ds$$

so that

$$\boxed{R_\beta g(x) + (\beta-\lambda) R_\beta R_\lambda^\partial g(x) = E^x \int_0^\infty g(X_s)\,e^{-\lambda s-(\beta-\lambda)\sigma_s}\,ds.}$$

Let's re-express the RHS in terms of excursions. Let $L(t,b)$ be the local time process at b, with $L(t) = \sum_{b\in B} L(t,b)$. So the RHS is

$$E^x\left[\sum_{(t,\rho_t)} e^{-\beta L^{-1}(t-)} \int_0^{\zeta(\rho_t)} g(\rho_t(s))\,e^{-\lambda s}\,ds\right]$$

Observe the following: If we kill X at rate λ, and consider the Markov chain $X(L_t^{-1})$ with values in $B \cup \{\partial\}$, then

$$\boxed{M_\lambda \text{ is the Green function of } X(L_t^{-1})}$$

where M_λ is as defined in [2]. Thus $E^b\left[\int_0^\infty e^{-\beta t} L(dt, b)\right]$

$$= E^b\left[\int_0^\infty e^{-\beta L_t^{-1}} I_{\{X_t = b\}} dt\right] = M_\lambda(b,b).$$

$$= E^x\left[\int_0^\infty e^{-\beta L^{-1}(t)} \sum_b n_\lambda^b g \, I_{\{p_t(0)=b\}} \, dt\right]$$

where $\{(t, p_t)\}$ is the point process of excursions. The formalism in the last statement is not perfect; but $\mathbb{R}^+$ is split up according to which point was the starting point of the current excursion, and this is what is intended. Any case, the above is equal to:

$$\sum_b n_\lambda^b g \quad E^x \int_0^\infty e^{-\beta t} L(dt, b)$$

$$= \sum_b n_\lambda^b g \quad E^x e^{-\beta H_b} \, . \quad E^b\left\{\int_0^\infty e^{-\beta t} L(dt, b)\right\},$$

and the constants $E^b \int_0^\infty e^{-\beta t} L(dt, b)$ are really only normalising constants for $L(\cdot, b)$. So we can take them all to be 1, yielding

$$\boxed{R_\beta g(x) + (\beta-\lambda) R_\beta R_\lambda^\partial g(x) = \sum_b n_\lambda^b g \quad E^x e^{-\beta H_b}.}$$

2) Now the puzzle is this. Suppose one were given some resolvent (R_λ) which extended the killed resolvent (R_λ^∂), which we are also told. Supposing, if necessary, that we are also told the size of the finite exit boundary B and the functions ψ_λ^a, $a \in B$, $\lambda > 0$, could we arrive at the expression (2) of [2], and discover what the n_λ^b, γ_b, δ_b must be?

The above analysis converts this into the question:

" Can we work out what $E^x(e^{-\beta H_b})$ is for $b \in B$? "

An optimal consumption problem (3/10/91)

Let $S_t = \sup\{B_u : u \le t\}$ and suppose we wish to

$$\max \; E \int_0^1 \log(C_s)\, ds \qquad \text{subject to} \quad \int_0^1 C_s \, ds = S_1$$

How can this be done? If

$$g(\tau, y, x) = \max\left\{E\int_0^\tau \log(C_s)\, ds \; ; \; \int_0^\tau C_s \, ds = (S_\tau - y)^+ + x\right\},$$

then

$$\int_0^t \log(C_s)\,ds + g(1-t,\ S_t - B_t,\ S_t - \int_0^t C_s\,ds)$$

is a supermg, and a mg under optimal control. This yields

$$\begin{cases} \log c - g_\tau + \tfrac{1}{2} g_{yy} - c\, g_x \le 0 , & g(0,y,x) = 0 \\ g_y + g_x = 0 \quad \text{when } y = 0 \end{cases}$$

where $Y_t \equiv S_t - B_t$. Maximising over C gives $C^* = 1/g_x$, and the PDE

(1) $$\boxed{\tfrac{1}{2} g_{yy} - g_\tau - \log g_x - 1 = 0 , \quad g(0,y,x) = 0 , \quad g_y + g_x = 0 \text{ on } y = 0.}$$

Now there's some scaling:

$$\int_0^\tau C_s\,ds = (S_\tau - y)^+ + x$$

$$= \int_0^{\lambda^2 \tau} \lambda^{-2} C_{v/\lambda^2}\,dv$$

$$\Rightarrow \int_0^{\lambda^2\tau} \underbrace{\lambda^{-1} C_{v/\lambda^2}}_{\equiv C^\lambda_v \text{, say}}\,dv = (\lambda S_\tau - \lambda y)^+ + \lambda x \overset{d}{=} (S_{\lambda^2 \tau} - y)^+ + \lambda x$$

and so

$$\int_0^{\lambda^2 \tau} \log(C^\lambda_v)\,dv = \lambda^2 \int_0^\tau \log C_u\,du - \lambda^2 \tau \log \lambda$$

whence

$$\boxed{g(\lambda^2 \tau, \lambda y, \lambda x) = \lambda^2 g(\tau, y, x) - \lambda^2 \tau \log \lambda} \qquad (2)$$

Thus if $h(x,y) \equiv g(1,x,y)$, we get, with $\xi \equiv x/\sqrt{\tau}$, $\eta \equiv y/\sqrt{\tau}$,

$$\boxed{g(\tau, y, x) = \tau h(\eta, \xi) - \tau . \tfrac{1}{2} \log \tau} \qquad (3)$$

Now a few calculations allow us to express the PDE in terms of h:

$$\boxed{\tfrac{1}{2} h_{\eta\eta} - h + \tfrac{1}{2}\eta h_\eta + \tfrac{1}{2}\xi h_\xi - \tfrac{1}{2} - \log h_\xi = 0 \quad , \quad h_\xi + h_\eta = 0 \quad \text{on } \eta = 0} \qquad (4)$$

Notice some easy asymptotics: as $y \uparrow \infty$, $g(\tau, y, x) \to \tau \log(x/\tau)$, since as $y \uparrow \infty$, you have essentially no chance of getting any more wealth, and the best thing is to consume what you have at a uniform rate. This gives the bc

$$\boxed{\lim_{\eta \uparrow \infty} h(\eta, \xi) = \log \xi} \qquad (5)$$

This PDE looks hard; no separable solution could satisfy the bc $h_\xi + h_\eta = 0$ on $\eta = 0$ and the asymptotic above; there's no solution of the form $\varphi_1(\xi) + \varphi_2(\eta)$, and it's a non-linear PDE, so ...

We also expect h is increasing in ξ, decreasing in η.

Taking $x = y = 0$ in (2), we find that $\varphi(t) \equiv g(t, 0, 0)$ solves

$$\varphi(at) = a\varphi(t) - \tfrac{1}{2} at \log a \qquad \text{for all } a, t > 0.$$

The general solution to this is

$$\boxed{\varphi(t) \equiv g(t, 0, 0) = At - \tfrac{1}{2} t \log t} \qquad \text{where } A = h(0, 0),$$

as we see from (3).

Another piece of information; if $t > 0$, $y > 0$ and $x = 0$, $\boxed{g(t, y, 0) = -\infty}$

so that $\boxed{h(\eta, 0) = -\infty}$

<u>Postscript</u>: the Chavel conjecture is FALSE: Burdzy (et al?) gave a counterexample in 1992.

Interesting questions + conjectures.

1) In the work on limit theorems with David Hobson, we got a sufficient condition for $H_x / c(x) \xrightarrow{a.s.} 1$, but Hsu Pei wants to know if one can find an integral test for $X_t \le f_1(t)$ eventually, or for $\sigma_x \le f_2(x)$ eventually.

2) Take two nice convex domains $D \subset \Omega \subset \mathbb{R}^d$

and consider the transition f^{ns} of reflecting BM in the domains. Isaac Chavel has made the conjecture that

$$p_t^D(x,y) \ge p_t^\Omega(x,y) \qquad \forall x,y \in D, \ \forall t>0.$$

One possible approach: fix $a \in D$, and consider for $T>0$ fixed

$$h(t,x) \equiv p_{T-t}^D(x,a) - p_{T-t}^\Omega(x,a) \qquad (0 \le t < T)$$

This extends to a bdd cts f^n on $[0,T] \times \bar{D}$, and $h(T,x) \equiv 0$.

By Itô's formula, we get $\qquad (dX = dB - n(X_t)\,dL_t)$

$$dh(t,X_t) = d(mg) - \nabla h(t,X_t)\, n(X_t)\, dL_t$$

$$= dmg + \frac{\partial p_{T-t}^\Omega}{\partial n}(X_t, a)\, dL_t$$

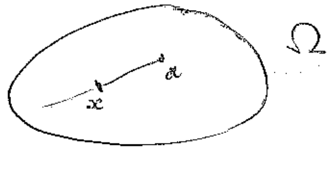
so that

$$h(0,x) = -E^x\left[\int_0^T \frac{\partial}{\partial n} p_{T-s}^\Omega(X_s,a)\, dL_s\right]$$

$$= -\int_0^T ds\left(\int_{\partial D} p_s^D(x,z)\, \frac{\partial}{\partial n} p_{T-s}^\Omega(z,a)\, dz\right).$$

We don't have $\frac{\partial}{\partial n} p_t^\Omega(x,a) \le 0 \quad \forall x \in \partial D$ as a rule, so that won't work...

because by varying the domain D, we could only have this if $\nabla p_t^\Omega(x,a)$ pointed along $x-a$, which would imply that $p_t^\Omega(\cdot,a)$ would have to be spherically symmetric...



Bessel process in dimension d has transition density

$$\frac{\exp-(x^2+y^2)/2t}{t\,(xy)^{\frac{d}{2}-1}}\, y^{d-1}\, I_{\frac{d}{2}-1}\left(\frac{xy}{t}\right)$$

f is lsc $\Leftrightarrow$ $f(x) \le \liminf\limits_{y\to x} f(y) \Leftrightarrow f^{-1}((-\infty,a])$ is closed $\Leftarrow$ f is sup of family of cts f^{ns}
