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Another fundamental theorem of statistics (9/10/91)

Suppose $X_1, \dots, X_n$ are indept $N(0,V)$, where $V = S^2$ is $d \times d$, $n \geq d$. Let $\Sigma = n^{-1}\sum_{j=1}^{n}(X_j - \bar{X})(X_j-\bar{X})^T$ be the sample covariance matrix. How to characterise the law of Σ?

For any $x \in \mathbb{R}^d$,

$$E\exp(-\tfrac{1}{2}x^T \Sigma x) = E\exp\Big\{-\tfrac{1}{2}n^{-1}\sum_{j=1}^{n}\big(x^T(X_j-\bar{X})\big)^2\Big\}$$

Now if $\xi_j = X_j - \bar{X}$, then

$$\begin{pmatrix}\xi_1\\ \vdots \\ \xi_n\end{pmatrix} = \left(I_{nd} - \frac{1}{n}\begin{pmatrix} I_d & I_d & \cdots & I_d \\ \vdots & \vdots & & \vdots \\ I_d & I_d & \cdots & I_d\end{pmatrix}\right)\begin{pmatrix}X_1\\ \vdots \\ X_n\end{pmatrix} \equiv P\begin{pmatrix}X_1\\ \vdots \\ X_n\end{pmatrix}$$

and

$$\sum_{j=1}^{n}\big(x^T(X_j-\bar{X})\big)^2 = (\xi_1^T \cdots \xi_n^T)\begin{pmatrix} xx^T & & \\ & \ddots & \\ & & xx^T\end{pmatrix}\begin{pmatrix}\xi_1\\ \vdots \\ \xi_n\end{pmatrix}.$$

Now let's observe that $S^{-1}X_j$ are i.i.d. $N(0, I_d)$, so by replacing x by Sx, we may wlog assume $V = I$, and compute

$$E\exp(-\tfrac{1}{2}y^T\Sigma y), \qquad y \equiv Sx$$

$$= \det(I+Q)^{-\frac{1}{2}},$$

where

$$Q = \frac{1}{n}P^T\begin{pmatrix} yy^T & & \\ & \ddots & \\ & & yy^T\end{pmatrix}P$$

$$= \frac{1}{n}\left\{\begin{pmatrix} A & & \\ & \ddots & \\ & & A\end{pmatrix} - \frac{1}{n}\begin{pmatrix} A & \cdots & A \\ \vdots & & \\ A & \cdots & A\end{pmatrix}\right\} \qquad A \equiv yy^T$$

Thus

$$\det(I+Q) = \det\left(I + \frac{|y|^2}{n}\left\{\begin{pmatrix} J & & \\ & \ddots & \\ & & J\end{pmatrix} - \frac{1}{n}\begin{pmatrix} J & \cdots & J \\ \vdots & & \\ J & \cdots & J\end{pmatrix}\right\}\right)$$

where $J \equiv e_1 e_1^T$,

by applying a rotation. Next we should swap columns + rows of the matrix to reduce it to

$$\left(\begin{array}{cccc|c|c}
1+a-a/n & -a/n & -a/n \cdots & -a/n & & \\
-a/n & 1+a-a/n & -a/n \cdots & -a/n & & \\
\vdots & \vdots & \vdots & \vdots & \cdot & \\
-a/n & -a/n & -a/n \cdots & 1+a-a/n & & \cdot \\
\hline
 & \cdot & & & I_d & \cdot \\
\hline
 & & & & \cdot & I_d \ddots \\
 & \cdot & & & &
\end{array}\right) \qquad a \equiv |y|^2/n$$

so we just need

$$\det\left((1+a)I - \tfrac{a}{n}\,11^T\right) = \det\left((1+a)I - a\, e_1 e_1^T\right) \qquad \text{by rotating}$$

$$= (1+a)^{n-1}$$

Thus

$$\boxed{E\,\exp\left(-\tfrac{1}{2} x^T \Sigma x\right) = \left(1 + x^T V x/n\right)^{-(n-1)/2}}$$

Note that as $n \to \infty$, we get $\exp(-\frac{1}{2} x^T V x)$, as we should. The case $d=1$ just gives the well known result that

$$E\,\exp\left\{-\tfrac{1}{2}\lambda \sum_1^n (X_j - \bar{X})^2\right\} = \left(1 + \lambda\sigma^2\right)^{-(n-1)/2}.$$

<u>Another interacting particles example.</u> (9/10/91)

(i) Taking a one-dimensional example

$$dX_j = \sigma_N\, dB_j - D_j \varphi_N(X_1, \dots, X_N)\, dt$$

with

$$\varphi_N(x_1, \dots, x_N) = -\frac{\alpha}{2N} \sum_{i \neq j}\sum \log|x_i - x_j| + \theta \sum_i U(x_i),$$

we obtain the evolution of the m.v. limit process as

$$\frac{d}{dt}\langle \mu_t, f\rangle = \frac{\alpha}{2} \iint \frac{f'(x) - f'(y)}{x-y}\, \mu_t(dx)\, \mu_t(dy) - \theta\, \langle \mu_t, f' U' \rangle,$$

as before. One particularly interesting example is with $\boxed{U(x) = \frac{1}{4}(x^2-1)^2}$

Taking $f(x) = \frac{1}{x-z}$, $M_t(z) = \int f(x)\, \mu_t(dx)$, we obtain

$$\boxed{\frac{\partial}{\partial t} M_t(z) = \frac{\alpha}{2} \frac{\partial}{\partial z}\left(M_t(z)^2\right) + \theta \frac{\partial}{\partial z}\left((z^3 - z) M_t(z)\right) + 2\theta z + \theta \int x \mu_t(dx).}$$

The presence of $\int x \mu_t(dx)$ complicates matters; for example,

$$\frac{d}{dt}\left(\int x \mu_t(dx)\right) = -\theta \int (x^3 - x)\, \mu_t(dx)$$

so that the mean doesn't evolve autonomously...

(ii) <u>What can we say about the equilibria?</u> If we set $m \equiv \int x \mu(dx)$, we want to take M to solve

$$\frac{\partial}{\partial z}\left[\frac{\alpha}{2} M^2 + \theta(z^3 - z) M + \theta z^2 \right] + \theta m = 0$$

whence

$$(*) \qquad \frac{\alpha}{2} M^2 + \theta(z^3 - z) M + \theta(z^2 + mz - c) = 0$$

for some constant c. Let's consider the asymptotics as $z = ib \to \infty$. We have

$$M(z) + \frac{1}{z} + \frac{m}{z^2} = \int \mu(dx) \left[\frac{1}{x-z} + \frac{1}{z} + \frac{x}{z^2} \right]$$

$$= \frac{1}{z^2} \int \frac{x^2 \mu(dx)}{(x-z)} \sim -\frac{1}{z^3} \int x^2 \mu(dx)$$

and $M(z) \to 0$ as $z \to \infty$, $z M(z) \to -1$, so we obtain from $(*)$

$$0 = \lim_{z \to \infty} \left\{ z^3 \left(M + \frac{1}{z} + \frac{m}{z^2}\right) - zM - c \right\}$$

so that

$$\boxed{c = 1 - \int x^2 \mu(dx).}$$

If the measure μ has a density ρ, then $\pi \rho(x) = \operatorname{Im} M(x)$, $x \in \mathbb{R}$. So the interesting cases are when the quadratic $(*)$ has non-real roots.

We obtain

$$\boxed{\pi\rho(x) = \operatorname{Im} M(x) = \frac{1}{\alpha}\left[\left(2\alpha\theta(x^2+mx-c) - \theta^2(x^3-x)^2\right)^+\right]^{1/2}}$$

The constants m, c have to satisfy the consistency conditions

$$\boxed{\int \rho(x)\,dx = 1, \quad \int x\rho(x)\,dx = m, \quad \int x^2\rho(x)\,dx = 1-c}$$

Can these be achieved?

<u>Axiomatic approach to term structure of interest rates</u> (9/10/91)

There are certain natural properties which one might ask for for the bond prices $P(t,\tau)$. So if $P(t,\tau)$ is the price at time t of a sure £1 at time $\tau \geq t$, we will ask

(i) $P(t,t) = 1$ <u>for all</u> t, $P(t,\tau)$ <u>is continuous and strictly positive</u>;

(ii) <u>For some continuous increasing (adapted) process</u> R,

$$\left(e^{-R_t} P(t,\tau)\right)_{0\leq t\leq\tau} \quad \text{\underline{is a martingale}};$$

(iii) $\{P(t+s,\tau) : s\geq 0, \tau\geq t+s\}$ <u>should be conditionally independent</u>

<u>of</u> $\{P(u,\tau) : 0\leq u\leq t, \tau\geq u\}$ <u>given</u> $\{P(t,\tau) : \tau\geq t\}$.

Condition (i) is justified by the technical convenience of working with cts processes (and the obvious condition $P(t,t)=1$). Condition (ii) is simply a rephrasing of

$$P(t,\tau) = E_t \exp\{-(R_\tau - R_t)\}$$

where $R_t \equiv \int_0^t r_u\,du$ if there's an instantaneous interest rate process. The Markovian property (iii) says that the yield curve now is all you need to say what will happen in the future.

Let $S = \{$continuous decreasing $f : [0,\infty] \to [0,1]$, $f(0)=1$, $f(\infty)=0\}$. We can make S into an LCCB Hausdorff space by thinking of it as a subset of the compact metric space $\mathcal{P}([0,\infty])$. We define the S-valued process

$$\xi_t \equiv \left(u \mapsto P(t,t+u)\right).$$

The last assumption is reasonably innocent:

(iv) (ξ_t) is a time-homogeneous Markov process.

If we work on the canonical space $\Omega = C(\mathbb{R}^+, S)$ with canonical process ξ; (and any initial law for ξ)
then

$$\left(e^{-R_t(\omega)}\, \xi_t(\omega)(T-t)\right)_{0\le t\le T} \quad \text{is a martingale}$$

$$\therefore \quad \left(e^{-R_t(\theta_a\omega)}\, \xi_t(\theta_a\omega)(T-t)\right)_{0\le t\le T} \quad \text{is a martingale}$$

$$= \left(e^{-R_t(\theta_a\omega)}\, \xi_{a+t}(\omega)(T-t)\right)_{0\le t\le T}$$

But also

$$\left(e^{-R_{t+a}(\omega)}\, \xi_{t+a}(\omega)(T+a-t-a)\right)_{0\le t\le T} \quad \text{is a martingale}$$

and now we have a little result.

LEMMA. If ξ_t is a continuous positive semimartingale, and V_t, $\tilde{V}_t$ are two positive f.v. processes such that $V_0 = \tilde{V}_0$ and

$$V_t\xi_t, \quad \tilde{V}_t\xi_t \quad \text{are both continuous local martingales,}$$

then $V_t = \tilde{V}_t$.

Hence

$$R_{t+a}(\omega) = R_t(\theta_a\omega) + R_a(\omega)$$

so that R is an additive functional! (Some care needed over perfection etc, but who cares, really?)

Thus the only types of $P(t,\tau)$ satisfying the four natural properties above are of the form

$$\boxed{P(t,\tau) = E_t\left[\exp-(R_\tau - R_t)\right]}$$

where R is some cts incr additive functional of some time-homogeneous Markov process. It's trivial to verify that any such has properties (i)–(iv).

More about quadratic functionals (14/10/91)

(i) Let's suppose that A is the generator of a Markov chain on some finite state-space I, and suppose also that the chain dies out, and is symmetrisable with respect to the diagonal matrix D, with Green fn $G \equiv -A^{-1}$. Then

$$\boxed{E \exp\{-\tfrac{1}{2}\lambda\langle \xi+a, \xi+a\rangle\} = E\exp\{-\tfrac{\lambda}{2}\langle\xi,\xi\rangle\}.\exp\{-\tfrac{\lambda}{2}\langle a,a\rangle + \tfrac{\lambda^2}{2}\langle a, R_\lambda a\rangle\}}$$

where ξ is a zero-mean Gaussian process with covariance $-A^{-1}D^{-1}$, which is symmetric, and where $\langle f,g\rangle \equiv f^T Dg$ is the natural inner product.

Proof Using the FTS,

$$E\exp -\tfrac{1}{2}(X+a)^T Q(X+a) = E\exp(-\tfrac{1}{2}X^TQX).\ \exp\{-\tfrac{1}{2}a^T(I+QV)^{-1}Qa\},$$

we substitute λD for Q and $-A^{-1}D^{-1}$ for the covariance matrix V.

This generalises the result of pp 12-14 in book IV, in some sense.

However, the quadratic functional in that result didn't have to be exactly the same as the covariance of the Gaussian process. We had some pos-def diagonal Λ, and were looking at

$$\boxed{\begin{aligned} &E\exp\{-\tfrac{\lambda}{2}(X+a)^T D\Lambda(X+a) + \tfrac{\lambda}{2}a^TD\Lambda a\} \\ &= E\exp(-\tfrac{\lambda}{2}X^TD\Lambda X)\exp\{\tfrac{\lambda^2}{2}a^T\Lambda D(\tilde{R}_\lambda)a\}\end{aligned}}$$

where $\tilde{R}_\lambda \equiv (\lambda - \Lambda^{-1}A)^{-1}$, the resolvent of the underlying symmetrisable Markov chain time-changed by Λ. This is obtained in exactly the same way from FTS, and matches the result referred to above.

(ii) In order to model the instantaneous interest-rate process r_t, it is tempting to make it the squared modulus of $X_t + a_t$, where a is some trajectory in $\mathbb{R}^d$, and

$$dX_t = \sigma\, dW_t - K X_t\, dt$$

where σ, K are constant $d\times d$ matrices. A simple special case to do first is $d=1$, $\sigma=1$, $K = c \geq 0$. In this case, X is an OU diffusion with restoring constant c, and we want to find

$$E_c \exp - \lambda\int_0^T (X_s + a_s)^2\, ds = E_0 \exp\Big[-\int_0^T c X_s\, dX_s - \tfrac12 \int_0^T c^2 X_s^2\, ds - \int_0^T \lambda (X_s + a_s)^2\, ds \Big]$$

$$= e^{cT/2} E_0 \exp\Big[-\tfrac{c}{2} X_T^2 - \int_0^T \{ (\lambda + \tfrac12 c^2) X_s^2 + 2\lambda a_s X_s + \lambda a_s^2 \} ds \Big]$$

$$= e^{cT/2} E_0 \exp\Big[-\int_0^T X_s^2\, m(ds) - 2\int_0^T \lambda a_s X_s\, ds - \int_0^T \lambda a_s^2\, ds \Big]$$

$$= e^{cT/2} E_0 \exp\Big\{ -\int_0^T X_s^2\, m(ds) \Big\} \exp\Big[\int_0^T\!\!\int_0^T \lambda a_s \lambda a_t\, r_1(s,t)\, ds\, dt - \int_0^T \lambda a_s^2\, ds \Big]$$

where $m(dx) \equiv (\lambda + \tfrac12 c^2)\, dx + \tfrac{c}{2} \delta_T(dx)$, and r_1 is the resolvent density (wrt m) of the diffusion in $[0,T]$ with speed m, reflected at T, killed at 0.

Write $\theta \equiv \sqrt{c^2 + 2\lambda}$. The resolvent density has the form $(0 \le x \le y \le T)$

$$r_1(x,y) = 2 f_+(x) f_-(y) / \{ -f_+(x) Df_-(x) + f_-(x) Df_+(x) \}$$

where

$$f_+(x) = \sinh \theta x,$$

$$f_-(x) = \cosh \theta (T-x) + \frac{c}{\theta} \sinh \theta (T-x).$$

Hence for $0 \le x \le y \le T$,

$$\boxed{ r_1(x,y) = \frac{2 \sinh \theta x \{ \cosh \theta (T-y) + \frac{c}{\theta} \sinh \theta (T-y) \}}{\theta \cosh \theta T + c \sinh \theta T} }.$$

Next,
$$\Big(E_0 \exp\Big\{ -\int_0^T X_s^2\, m(ds) \Big\} \Big)^2 = E^T (e^{-H_0})$$

$$= f_-(T)/f_-(0)$$

$$= \Big\{ \cosh \theta T + \frac{c}{\theta} \sinh \theta T \Big\}^{-1},$$

so

$$\boxed{ E_c \exp - \lambda \int_0^T (X_s + a_s)^2\, ds = \frac{\exp\Big[\tfrac12 cT + \int_0^T\!\int_0^T \lambda^2 a_s a_t r_1(s,t)\, ds\, dt - \lambda \int_0^T a_s^2\, ds \Big]}{\{ \cosh \theta T + \frac{c}{\theta} \sinh \theta T \}^{1/2}} }$$

A Lemma on Uniform integrability (16/10/91)

1) LEMMA. Let M be a continuous local martingale, $M_0 = 0$, $\langle M \rangle_\infty < \infty$ a.s., and let $\bar{M}_t \equiv \sup\{M_u : u \le t\}$. Then M is a UI martingale if and only if the following three conditions hold

$$\boxed{(i)\ M_\infty \in L^1 \quad (ii)\ EM_\infty = 0 \quad (iii)\ \lim_{\lambda \uparrow \infty} \lambda P(\bar{M}_\infty > \lambda) = 0.}$$

Proof. The conditions are clearly necessary. To show sufficiency, define

$$H_a \equiv \inf\{t : M_t = a\}, \quad T_a \equiv H_a \wedge H_{-a}.$$

Then

$$0 = E\, M(T_a) = E\big[M_\infty : T_a = \infty\big] + a\, P\big[H_a < H_{-a}\big] - a\, P\big[H_{-a} < H_a\big];$$

from (iii),

$$a\, P[H_a < H_{-a}] \le a\, P[H_a < \infty] \le a\, P[\bar{M}_\infty \ge a] \to 0,$$

and by (i) and (ii) the first term goes to zero, so we conclude that

$$a\, P\big[H_{-a} < H_a\big] \to 0 \qquad (a \uparrow \infty).$$

But

$$a\, P\big[\, H_a < H_{-a} < \infty \,\big] \le a\, P\big[H_a < \infty\big] \to 0, \text{ so we}$$

deduce that $a\, P[H_{-a} < \infty] \to 0$, hence $aP[T_a < \infty] \to 0$. We can now appeal to the result of Azéma-Gundy-Yor, Sem Prob XIV, provided we can check the condition

$$\sup_t E\, |M_t| < \infty.$$

But

$$E\, |M_{t \wedge T_n}| \le E\, |M_{T_n}| = n\, P\big[T_n < \infty\big] + E\big[|M_\infty| : T_n = \infty\big]$$

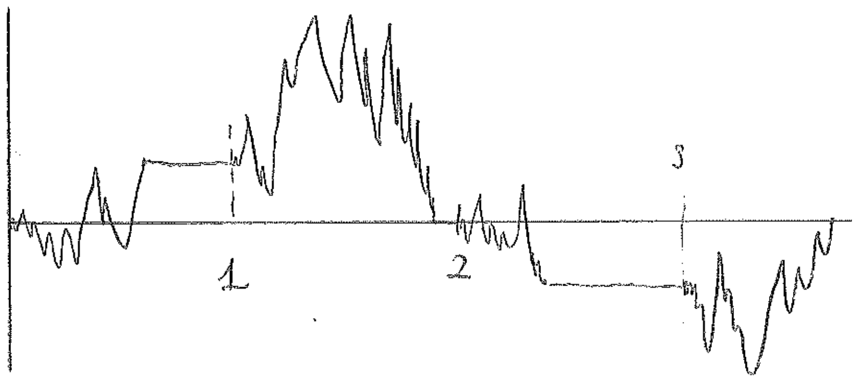
$$\le E\, |M_\infty| + 1 \qquad \text{for large } n.$$

Hence by Fatou, $E|M_t| \le E|M_\infty| + 1$, and that's it.

The interest of this would be if you could characterise all laws of $(B_T, \bar{B}_T)$ with T a finite stopping time, then you could immediately identify the law which corresponded to T for which $B(\cdot \wedge T)$ was UI.

2) Is it possible similarly to identify the T for which a cts local mg is L^1-bdd? This seems a wrong question, for consider the following cts local mg.

Take BM, $B_0 = 0$, and by using $\int_0^t \frac{dB_s}{1-s}$, make a cts local mg which gets out to $\{\pm 1\}$ by time 1. Then similarly speed up so as to get back to 0 by time 2. Go on repeating this. You get a cts local mg M with the property that $E|M_t| \le 1$ for all t, yet then it is most certainly not cgt!

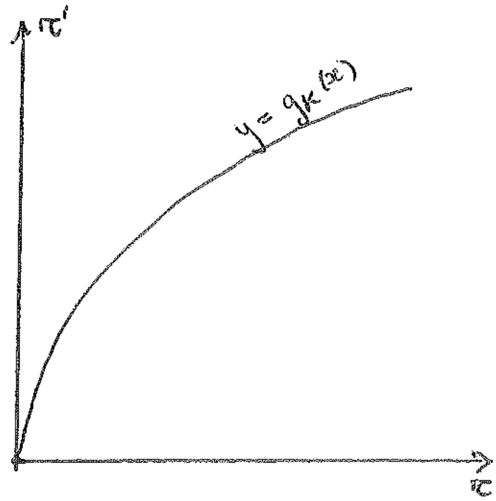


Behaviour of $\int_0^t I_{\{B_u > 0\}}\, du$ for small t (28/10/91)

(i) Write $A_t = \int_0^t I_{\{B_u > 0\}}\, du$. Suppose that $f : \mathbb{R}^+ \to \mathbb{R}^+$ is increasing continuous, with $f(0) = 0$.

When can we say
$$\liminf_{t \downarrow 0} A_t / f(t) > 0 \quad \text{a.s. ?}$$

$\liminf A_t/f(t) > K \Rightarrow \liminf \tau_t / f(\tau_t + \tau'_t) > K$, where $\tau_t = A(\gamma_t)$, $\tau'_t \equiv \gamma_t - \tau_t$ are indept stable($\frac{1}{2}$) subordinators, γ the inverse to L, so let's address the question



<u>When is</u> $$\liminf_{t \downarrow 0} \frac{\tau_t}{f(\tau_t + \tau'_t)} > K \,?$$

(ii) Consider the curve
$$\frac{x}{f(x+y)} = K,$$
equivalently, $y = g_K(x) \equiv f^{-1}\left(\frac{x}{K}\right) - x$

We naturally want g_K to be increasing, so assume $\boxed{f(t)/t \downarrow 0 \text{ as } t \downarrow 0}$

which ensures $f^{-1}(s)/s \uparrow \infty$ as $s \downarrow 0$, and so g_K is positive in a nbd of 0.

The bivariate subordinator (τ_t, τ_t') has Green function proportional to $(x+y)^{-3/2}$, since

$$\int_0^\infty dt \; \frac{t e^{-t^2/2x}}{\sqrt{2\pi x^3}} \cdot \frac{t e^{-t^2/2y}}{\sqrt{2\pi y^3}} = \text{const.} \; (x+y)^{-3/2}$$

What's the expected number of crossings of $y = g_K(x)$ by the time τ hits 1? It's

$$\propto \int_0^1 dx \int_0^{g_K(x)} dy \; (x+y)^{-3/2} \cdot (g_K(x) - y)^{-1/2}$$

$y = xu$

$$= \int_0^1 \frac{dx}{x} \int_0^{g_K(x)/x} \frac{du}{(1+u)^{3/2}} \left(\frac{g_K(x)}{x} - u \right)^{-\frac{1}{2}}$$

Now observe that $g_K(x)/x = \dfrac{f^{-1}(x/K)}{x} - 1 \uparrow \infty$ as $x \downarrow 0$, so we need to understand the asymptotics as $a \to \infty$ of

$$\int_0^a \frac{du}{(1+u)^{3/2}} (a-u)^{-\frac{1}{2}} = \int_0^1 \frac{a\,dv}{(1+av)^{3/2} \sqrt{a} \, (1-v)^{\frac{1}{2}}}$$

$$= \frac{1}{a} \int_0^1 \frac{dv}{(a^{-1}+v)^{3/2} (1-v)^{\frac{1}{2}}}$$

$$\sim \frac{1}{a} \int_0^1 \frac{dv}{(a^{-1}+v)^{3/2}} \sim \frac{1}{\sqrt{a}}$$

Thus the expected number of crossings of g_K is finite iff

$$\int_0^1 \frac{dx}{x} \sqrt{\frac{x}{f^{-1}(x/K)}} < \infty$$

$$\Leftrightarrow \int_{0+} \frac{dy}{y^{3/2}} \sqrt{f(y)} < \infty.$$

Thus $\boxed{\int_{0+} \frac{dy}{y^{3/2}} \sqrt{f(y)} < \infty \Rightarrow \liminf_{t \downarrow 0} \frac{\tau_t}{f(\tau_t + \tau_t')} = +\infty \text{ a.s.}}$

Some heuristics on how BM($\mathbb{C}$) approaches $\mathbb{R}$ (4/11/91)

Best to take X_t a BM($\mathbb{R}$), Y_t a BES(3) both started at 0, + see how the process leaves 0. I want to know when the trajectory of (X,Y) lies above some curve for a positive amount of time.

If $Z_t \equiv X_t + iY_t$, then

$$d \log Z_t = \frac{1}{Z_t}\, dZ_t = \frac{X_t - iY_t}{R_t^2}(dX_t + i\,dY_t) \qquad \left(dY = dW + \frac{dt}{Y}\right)$$

$$= \frac{XdX + YdW}{R^2} + i\,\frac{XdW - YdX}{R^2} + i\,\frac{X\,dt}{R^2\,Y} + \frac{dt}{R^2}$$

Perform a time change so that the generator gets multiplied by R^2. If $\log \tilde{Z} \equiv \rho + i\theta$, then

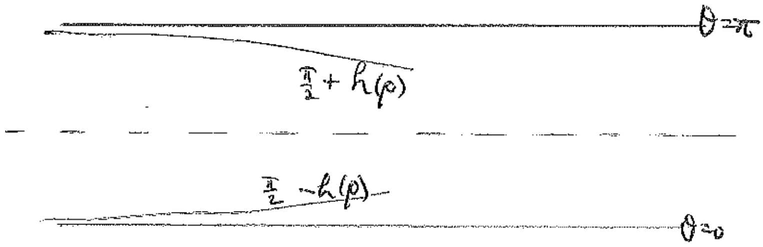
$$\boxed{\begin{aligned} d\rho &= dW_1 + dt \\ d\theta &= dW_2 + \cot\theta\, dt \end{aligned}}$$

— independent diffusions.

The scale function for θ is $s(\theta) \equiv \eta = -\cot\theta$, and

$$d\eta = (1+\eta^2)\, dW_2,$$

so that the invariant density is $\boxed{m(d\eta) \equiv \dfrac{2\,d\eta}{\pi(1+\eta^2)^2}}$



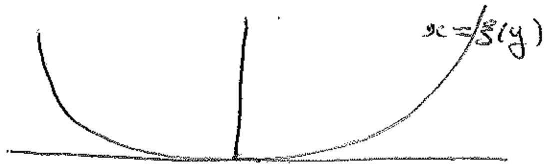
Rate at which local time at 0 grows $= \frac{2}{\pi}$, so roughly, $L_t \doteq 2t/\pi$.

Since the ρ-coordinate is drifting BM, this is growing more or less like t. The rate of excursions away from $\theta = \pi/2$ which get out to $|\theta - \pi/2| = \xi$ is $1/s(\pi/2 + \xi) = 1/\tan(\xi)$

So if we started (ρ,θ) at $(-K, 0)$ and sought the probability that it doesn't hit $\frac{\pi}{2} \pm h(\rho)$ before ρ hits 0, it must be approximately

$$\exp\left\{ -\int_K^0 \frac{2\,dt/\pi}{\tan h(t)} \right\}$$

so looks like $\boxed{\displaystyle\int_{-\infty}^0 \frac{dt}{\tan h(t)} < \infty \iff \int_{-\infty}^0 \frac{dt}{\frac{\pi}{2} - h(t)} < \infty}$ is the condition.



If we take this in the Cartesian form with $(\xi(y), y)$ as the boundary, and assume y/ξ_y^2 remains bounded

then $y = o(\xi_y)$ and the integral criterion converts to

$$\int_{0+} dy/\xi_y^2 \quad < \infty$$

as n+s condition not to cross the curve for a while. Is this correct?

Tauberian theory + divisor functions (5/11/91)

(i) There's a variant of the Karamata Tauberian theorem with exponential growth as well.

PROPOSITION. Suppose $a > 0$, $\rho \geq 0$, L is slowly varying, and U is increasing right continuous, $U(0) = 0$. If $\hat{U}$ is the Laplace transform of U, then

$$\boxed{U(x) \sim e^{ax} x^{\rho} L(x), \quad x \to \infty \iff \hat{U}(\lambda) \sim \frac{a}{(\lambda - a)^{1+\rho}} L\left(\frac{1}{\lambda - a}\right) \Gamma(1+\rho), \quad \lambda \downarrow a}$$

Proof Suppose the LHS, and write

$$U_0(x) \equiv e^{-ax} U(x) \sim x^{\rho} L(x).$$

Then

$$\begin{aligned}
\hat{U}(\lambda) &\equiv \int_0^\infty e^{-\lambda x} U(dx) \\
&= \int_0^\infty \lambda e^{-\lambda x} U(x)\, dx \\
&= \int_0^\infty \lambda e^{-(\lambda - a)x} U_0(x)\, dx \\
&= \frac{\lambda}{\lambda - a} \int_0^\infty e^{-(\lambda - a)} U_0(dx) \\
&\sim \frac{\lambda}{\lambda - a} (\lambda - a)^{-\rho} L\left(\frac{1}{\lambda - a}\right) \Gamma(1+\rho)
\end{aligned}$$

by the Karamata Tauberian Theorem.

Conversely,

$$\hat{u}(\lambda) \equiv \int_0^\infty e^{-x(\lambda-a)}\, e^{-ax}\, U(dx) \sim \frac{a}{(\lambda-a)^{\rho+1}} L\left(\frac{1}{\lambda-a}\right)\Gamma(1+\rho)$$

$$\Rightarrow \int_0^x e^{-ay}\, U(dy) \sim a\, x^{\rho+1} L(x)\, \Gamma(1+\rho)/\Gamma(2+\rho)$$

by the classical Karamata thm. Now

$$\int_0^x e^{-ay}\, U(dy) = e^{-ax} U(x) + \int_0^x a e^{-ay} U(y)\,dy$$

$$\equiv U_0(x) + \int_0^x a\, U_0(y)\,dy$$

$$\sim a \frac{x^{\rho+1}}{\rho+1} L(x) .$$

Multiply both sides by e^{ax}, and see

$$\frac{d}{dx}\left[e^{ax}\int_0^x U_0(y)\,dy\right] \sim a\, e^{ax}\frac{x^{\rho+1}}{\rho+1} L(x),$$

implying

$$e^{ax}\int_0^x U_0(y)\,dy \sim e^{ax}\frac{x^{\rho+1}}{\rho+1} L(x).$$

Now by the Lemma on p 446 of Feller, we conclude

$$U_0(x) \sim x^\rho L(x)$$

as required. $\square$

(ii) <u>On to divisor functions.</u> For $n = p_1^{a_1}\cdots p_k^{a_k} \in \mathbb{N}$, write $\omega(n) = k$, $\Omega(n) = a_1 + \cdots + a_k$, so that $\omega(n)$ = no. of distinct prime divisors of n, $\Omega(n)$ = total no of prime divisors of n.

For $s > 1$, let P_s be the law on $\mathbb{N}$ given by

$$P_s(X=n) \equiv n^{-s}/\zeta(s)$$

Let N_r be the number of times p_r is a factor of X. Then under P_s, the N_r are <u>independent geometrics</u>, $P(N_r \geq k) = p_r^{-ks}$.

Now let $U(x) \equiv \sum_{n \le x} \omega(n)$, and consider

$$E_s[\omega(X)] = \sum_{n \ge 1} \omega(n) n^{-s} / \zeta(s)$$

$$= \int x^{-s} U(dx) / \zeta(s)$$

(*)

$$= E_s\left[\sum_{r \ge 1} I_{\{N_r > 0\}}\right]$$

$$= \sum_{r \ge 1} p_r^{-s}$$

(a) $\sum_{r \ge 1} p_r^{-s} = \int_0^\infty x^{-s} \pi(dx)$ where π is counting measure on the primes

$$= \int_0^\infty e^{-sy} \pi_\ell(dy) \quad \text{where } \pi_\ell(a) \equiv \pi(e^a)$$

The Prime Number Theorem says that $\pi(x) \sim x/\log x \therefore \pi_\ell(y) \sim e^y/y$, implying

$$\sum_{r \ge 1} p_r^{-s} \sim \int_1^\infty s\, e^{-sv} \frac{e^v dv}{v}$$

But if $\psi(\theta) = \int_1^\infty e^{-\theta v} \frac{dv}{v}$, then $\frac{\partial \psi}{\partial \theta}(\theta) = -\frac{1}{\theta} e^{-\theta}$, so for small θ, $\psi(\theta) \sim \log(1/\theta)$ which implies that

$$\boxed{\sum_{r \ge 1} p_r^{-s} \sim \log\left(\frac{1}{s-1}\right), \quad s \downarrow 1}$$

(b) It is quite easy to prove that $\zeta(s) \sim \frac{1}{s-1}$ $(s \downarrow 1)$

(c) $\int_0^\infty x^{-s} U(dx) = \int_0^\infty e^{-sy} U_\ell(dy)$, where $U_\ell(a) = U(e^a)$

so we conclude from (*) that

$$\int_0^\infty x^{-s}\, U(dx) = \int_0^\infty e^{-sy}\, U_\ell(dy)$$

$$= \zeta(s) \sum_{r\geq 1} p_r^{-s}$$

$$\sim \frac{1}{s-1} \log\left(\frac{1}{s-1}\right).$$

Applying the extension of Karamata's Theorem from (i), we conclude that

$$U_\ell(x) \sim e^x \log x$$

$$\boxed{\therefore \quad U(x) \equiv \sum_{n\leq x} \omega(n) \sim x \log\log x}$$

(iii) <u>The Prime Number Theorem itself?</u> We can actually prove this too by applying the correct Tauberian results, together with the Euler product:

$$P_s(X=1) = \frac{1}{\zeta(s)} = \prod_{r\geq 1}\left(1-p_r^{-s}\right)$$

which is probabilistically obvious. Now it's easy to show $\zeta(s) \sim \frac{1}{s-1}$ $(s\downarrow 1)$, so

$$-\log\zeta(s) = \sum_{r\geq 1} \log(1-p_r^{-s}) \sim -\log\left(\frac{1}{s-1}\right) \qquad (s\downarrow 1)$$

$$\sim -\sum_{r\geq 1} p_r^{-s}$$

so we have

$$\sum_{r\geq 1} p_r^{-s} = \int_0^\infty x^{-s}\, \pi(dx) \sim \log\left(\frac{1}{s-1}\right)$$

$$= \int_0^\infty x^{-(s-1)}\, \frac{\pi(dx)}{x}$$

Set $\nu(dx) = \pi(dx)/x$;

$$= \int_0^\infty e^{-(s-1)y}\, \nu_\ell(dy)$$

$$\Rightarrow \nu_\ell(y) \sim \log(y) \Rightarrow \nu(y) \sim \log\log y.$$

But $\nu_\ell(y) = \sum_{p_r\leq y} \frac{1}{p_r}$, which <u>suggests</u> $p_r \sim r\log r$, but this actually seems to be on the edge of lots of Tauberian results.

Liptser + Shiryaev "Statistics of Random Processes I" p280 give an account of the relationship which is ultimately the same (using Riccatti equation) but which is presented from a different stochastic calculus viewpoint: assuming $a=0$,

$$d\left(\tfrac{1}{2} X_t^T Q_t X_t\right) = X_t^T Q_t \, dX_t + \tfrac{1}{2} X_t^T \dot{Q}_t X_t \, dt + \tfrac{1}{2} \operatorname{tr} Q_t \, dt$$

$$= \left\{ X_t^T Q_t \, dX_t + \tfrac{1}{2} X_t^T Q_t^2 X_t \, dt \right\} + \tfrac{1}{2} \operatorname{tr} Q_t \, dt - \tfrac{1}{2} X_t^T K_t X_t \, dt$$

Since $Q_1 = 0$, we get $-\int_0^1 X_t^T Q_t \, dX_t - \frac{1}{2} \int_0^1 X_t Q_t^2 X_t dt$

$$= -\tfrac{1}{2} \int_0^1 X_t^T K_t X_t \, dt + \tfrac{1}{2} \int_0^1 \operatorname{tr} Q_t \, dt$$

and that's it.

Another route to quadratic functionals (11/11/91)

(i) Suppose X is a BM in $\mathbb{R}^d$ and that K_t is a $d\times d$-pd-matrix-valued $f^{\underline{n}}$. To calculate

$$\varphi(t,x) \equiv E\left[\exp\left(-\tfrac{1}{2}\int_t^1 X_u^T K_u X_u\, du\right) \Big| X_t = x\right]$$

we could speculate that φ has the form

$$\varphi(t,x) = \exp\left[-\tfrac{1}{2}(x-a_t)^T Q_t (x-a_t) + \gamma_t\right],$$

so that the Itô-formula PDE for φ [i.e. $\frac{1}{2}\Delta\varphi + \dot{\varphi} - \frac{1}{2}x^T K x\,\varphi = 0$] translates into

$$0 = \frac{1}{2}(x-a_t)^T Q_t^2 (x-a_t) - \frac{1}{2}\operatorname{tr} Q_t + \dot{\gamma} - \tfrac{1}{2}(x-a_t)^T \dot{Q}_t (x-a_t) + \dot{a}_t^T Q_t (x-a_t) - \tfrac{1}{2}x_t^T K_t x_t .$$

Thus we would get

$$\boxed{\begin{aligned} &Q_t^2 - \dot{Q}_t - K_t = 0, \qquad Q_1 = 0 = a_1 = \gamma_1 \\ &Q_t(-\dot{a}_t + Q_t a_t) = \dot{Q}_t a_t \\ &\tfrac{1}{2}\{a_t^T Q_t^2 a_t - \operatorname{tr} Q_t - a_t^T \dot{Q}_t a_t - 2\dot{a}_t^T Q_t a_t\} + \dot{\gamma} = 0 \end{aligned}}$$

(ii) Example : $d=1$, $K=\theta^2$. This is easily solved to

$$Q(t) = \theta \tanh \theta(1-t), \quad a_t \equiv 0, \quad \gamma_t = -\tfrac{1}{2}\log\cosh \theta(1-t)$$

as we knew already.

(iii) Remark Clearly if we changed x to $-x$, $\varphi(t,x)$ should not be altered, hence $a=0$. This then simplifies the above equation to

$$\boxed{\begin{aligned} &Q_t^2 - \dot{Q}_t - K_t = 0 \\ &\dot{\gamma}_t = \tfrac{1}{2}\operatorname{tr} Q_t \end{aligned}}$$

Notice that $\exp(\frac{1}{2}\operatorname{tr} Q) = \sqrt{\det \exp(Q)}$.

However, it would be possible to approach the more general problem

$$\varphi(t,x) \equiv E\left[\exp\left(-\tfrac{1}{2}\int_t^1 (X_u+b_u)^T K_u (X_u+b_u)\,du\right) \Big| X_t = x\right]$$

in the same manner to yield

$$\boxed{\begin{aligned} & Q_t^2 - \mathring{Q}_t - K_t = 0 \\ & Q_t^2 a_t - Q_t \mathring{a}_t + K_t b_t = \mathring{Q}_t a_t \qquad \text{ie} \quad K(a+b) = Q\mathring{a} \\ & \tfrac{1}{2} a_t^T Q_t^2 a_t - \tfrac{1}{2}\operatorname{tr} Q_t + \mathring{\gamma}_t - \tfrac{1}{2} a_t^T \mathring{Q}_t a_t - \mathring{a}_t^T Q_t a_t - \tfrac{1}{2} b_t^T K_t b_t = 0 \end{aligned}}$$

If we set $\eta_t \equiv \gamma_t - \frac{1}{2} a_t^T Q_t a_t$, the last equation simplifies to

$$\boxed{\mathring{\eta}_t = \tfrac{1}{2}\left\{\operatorname{tr} Q_t + b_t^T K_t b_t - a_t^T Q_t^2 a_t\right\}}$$

(iv) <u>An example</u> Take a two-dimensional example, where $K_t = \theta^2 \begin{pmatrix}\cos wt \\ \sin wt\end{pmatrix}(\cos wt \;\; \sin wt)$.
It makes more sense to work in backward time, so we wish to solve the ODE

$$\ddot{Q}_t = K_{1-t} - Q_t^2 = \theta^2 \begin{pmatrix}\cos w(1-t) \\ \sin w(1-t)\end{pmatrix}(\cos w(1-t) \; \sin w(1-t)) - Q_t^2, \quad Q_0 = 0,$$

and set $A \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}$. Let's set $R_t \equiv \begin{pmatrix} \cos w(1-t) & -\sin w(1-t) \\ \sin w(1-t) & \cos w(1-t)\end{pmatrix} = \exp w(1-t)A$

and consider

$$V_t \equiv R_t^T Q_t R_t.$$

Then

$$\begin{aligned}\mathring{V}_t &= \mathring{R}^T Q R + R^T Q \mathring{R} + R^T \mathring{Q} R \\ &= wAV - wVA + \theta^2 \begin{pmatrix}1 & 0 \\ 0 & 0\end{pmatrix} - V^2\end{aligned}$$

Suppose now we express V in diagonal form:

$$V_t = S_t^T \Lambda_t S_t \quad , \text{ with } \quad S_t = \begin{pmatrix} \cos\varphi_t & -\sin\varphi_t \\ \sin\varphi_t & \cos\varphi_t \end{pmatrix} .$$

Then we obtain

$$\dot{V} = \dot{\varphi}\, VA - \dot{\varphi} A V + S^T \dot{\Lambda} S$$

More usefully,

$$\dot{\Lambda} = -\Lambda^2 + \theta^2 \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} (\cos\varphi \;\; \sin\varphi) + (\dot{\varphi}A + \omega SAS^T)\Lambda - \Lambda(\dot{\varphi}A + \omega SAS^T)$$

from which

$$\boxed{\begin{aligned} \dot{\lambda}_1 &= -\lambda_1^2 + \theta^2 \cos^2\varphi \\ \dot{\lambda}_2 &= -\lambda_2^2 + \theta^2 \sin^2\varphi \\ 0 &= \theta^2 \cos\varphi\sin\varphi + (\lambda_1 - \lambda_2)(\dot{\varphi} + \omega) \end{aligned}}$$

Solution?

<u>Invariant measures via excursion theory</u> (14/11/91)

We use the notation of the paper ZW 63 237–255, 1983.

(i) Suppose we have a Mkv process with distinguished state a, which is recurrent, so $\delta \equiv$ killing rate at $a = 0$, and $n(\xi = \infty) = 0$, where ξ is the excursion lifetime. So we have the representation

$$R_\lambda f(x) = R_\lambda^\partial f(x) + \psi_\lambda(x)\, R_\lambda f(a),$$

$$R_\lambda f(a) = \frac{n_\lambda f + \gamma f(a)}{\lambda n_\lambda 1 + \lambda\gamma} .$$

Here, $n_\lambda f \equiv \int_U n(dp) \int_0^\infty e^{-\lambda t} f(p_t)$ is the LT of the excursion entrance law.

I claim that

$$\boxed{n_0 + \gamma \delta_a \text{ is an invariant measure}}.$$

The proof is quite straightforward: if $\mu_\varepsilon \equiv n_\varepsilon + \gamma \delta_a$, then

$$\mu_\varepsilon \lambda R_\lambda f = n_\varepsilon \left(\lambda R^\partial_\lambda f + \psi_\lambda \lambda R_\lambda f(a) \right) + \gamma \lambda R_\lambda f(a)$$

$$= \lambda \frac{n_\varepsilon - n_\lambda}{\lambda - \varepsilon} f + n_\varepsilon \{ 1 - \lambda R^\partial_\lambda 1 \} \lambda R_\lambda f(a) + \gamma \lambda R_\lambda f(a)$$

$$= \frac{\lambda}{\lambda-\varepsilon} (n_\varepsilon - n_\lambda) f + \left(n_\varepsilon 1 - \frac{\lambda}{\lambda - \varepsilon} (n_\varepsilon - n_\lambda) 1 \right) \lambda R_\lambda f(a) + \gamma \lambda R_\lambda f(a)$$

$$= \frac{\lambda}{\lambda-\varepsilon} (n_\varepsilon - n_\lambda) f + \left\{ \frac{-\varepsilon n_\varepsilon 1}{\lambda - \varepsilon} + \frac{\lambda}{\lambda-\varepsilon} n_\lambda 1 \right\} \lambda R_\lambda f(a) + \gamma \lambda R_\lambda f(a)$$

$$\xrightarrow[(\varepsilon \downarrow 0)]{} n_0 f - n_\lambda f + (\lambda n_\lambda 1 + \gamma \lambda) R_\lambda f(a)$$

since $\varepsilon n_\varepsilon 1 = \int n(dp) \left(1 - e^{-\varepsilon \zeta} \right) \to 0$ as $n(\zeta = \infty) = 0$;

$$= (n_0 + \gamma \delta_a) f ,$$

as required.

(ii) It is easy to see that

$$\lambda n_0 \psi_\lambda \equiv \lambda n_0 \{ 1 - \lambda R^\partial_\lambda 1 \}$$

$$= \lambda n_\lambda 1 \qquad \text{by resolvent identity.}$$

This then gives

$$\int \pi(dx) \, \lambda \psi_\lambda(x) = \text{rate of } \lambda\text{-killed excursions},$$

at least if the normalisation of the invariant measure π is chosen correctly.

Quadratic functionals and optimal control of linear systems (19/11/91)

(i) If we go back to FTS

$$E \exp\{-\tfrac{1}{2}(X+a)^T Q (X+a)\} = \det(I+QV)^{-1/2} \exp\left[-\tfrac{1}{2} a^T (I+QV)^{-1} Q a\right]$$

let's observe that

$$\min_y \left[\tfrac{1}{2}(y+a)^T Q (y+a) + \tfrac{1}{2} y^T V^{-1} y \right]$$

$$= \tfrac{1}{2} a^T (I+QV)^{-1} Q a$$

(ii) This suggests that we should be considering the problem

$$\boxed{\min_y \left\{ \tfrac{1}{2} \int_t^1 (y_s+a)^T K_s (y_s+a)\, ds + \int_t^1 \tfrac{1}{2} |\dot{y}_s|^2\, ds \right\} \equiv V(t,a),}$$

which is the Brownian analogue of FTS. Then we should find that whatever the function y,

$$\tfrac{1}{2}\int_0^t \{ y_s^T K_s y_s + |\dot{y}_s|^2 \}\, ds + V(t, y_t) \quad \underline{\text{is increasing}}, \quad \underline{\text{and}}$$

<u>constant if y is optimal</u>. Thus

$$\tfrac{1}{2} y_t^T K_t y_t + \tfrac{1}{2}|\dot{y}_t|^2 + \frac{\partial V}{\partial t}(t,y_t) + \dot{y}_t \nabla V(t,y_t) \geqslant 0.$$

If we try

$$\boxed{V(t,x) \equiv \tfrac{1}{2} x^T Q_t x}$$

where $Q_1 = 0$, $Q_t^2 - \dot{Q}_t - K_t = 0$, then

$$\min_{x,v} \left[\tfrac{1}{2} x^T K_t x + \tfrac{1}{2}|v|^2 + \tfrac{1}{2} x^T \dot{Q}_t x + v^T Q_t x \right]$$

$$= \min_x \tfrac{1}{2} x^T (K_t + \dot{Q}_t - Q_t^2) x \qquad = 0$$

Thus

$$\tfrac{1}{2}\int_0^t \{ y_s^T K_s y_s + |\dot{y}_s|^2 \}\, ds + \tfrac{1}{2} y_s^T Q_s y_s \quad \text{is increasing, and}$$

is constant if $\boxed{\dot{y}_t = -Q_t y_t}$

That links the quadratic-functional story to deterministic optimal control.

<u>Moment-of-inertia of tree-indexed BM (2/12/91)</u>

Suppose we have a tree $\mathcal{T}$ as the index set of a BM X, and want to know about the law of

$$\int_{\mathcal{T}} (X_t - \bar{X})^2 \, ds$$

where $\bar{X} \equiv |\mathcal{T}|^{-1} \int_{\mathcal{T}} X_s \, ds$ is the centroid of the polymer. We want expressions which don't depend on a special choice of a root.

Let's consider $\xi_t = X_t - \bar{X}$ as a Gaussian process in its own right, and get the covariance. The neat way to do this is to replace X by a stationary OU process with restoring parameter ε, so that

$$E \, X_s X_t = \frac{1}{2\varepsilon} \exp\left[-\varepsilon \, d(s,t)\right]$$

where $d(s,t)$ is the distance from s to t through the tree. Thus

$$E \, \xi_t \xi_s = E\left[X_s X_t - (X_s + X_t)\bar{X} + \bar{X}^2\right]$$

$$= \frac{1}{2\varepsilon}\left\{ e^{-\varepsilon d(s,t)} - \int_{\mathcal{T}} \frac{du}{|\mathcal{T}|}\left\{ e^{-\varepsilon d(u,t)} + e^{-\varepsilon d(u,s)} \right\} + \int_{\mathcal{T}}\int_{\mathcal{T}} \frac{du}{|\mathcal{T}|}\frac{dv}{|\mathcal{T}|} e^{-\varepsilon d(u,v)} \right\}$$

$$\to -\frac{1}{2}\left[d(s,t) - \int_{\mathcal{T}} \{ d(u,t) + d(u,s) \} \frac{du}{|\mathcal{T}|} + \int_{\mathcal{T}}\int_{\mathcal{T}} \frac{du}{|\mathcal{T}|}\frac{dv}{|\mathcal{T}|} d(u,v) \right]$$

as $\varepsilon \downarrow 0$.

Hence

$$\boxed{E \, \xi_t \xi_s = \frac{1}{2} |\mathcal{T}|^{-2} \int_{\mathcal{T}}\int_{\mathcal{T}} du \, dv \left\{ d(u,t) + d(v,s) - d(u,v) - d(s,t) \right\}}$$

Some comments on strongly stochastically monotone processes (30/12/91)

(i) If μ_1, μ_2 are two measures on $\mathbb{R}$, we say

$$\mu_1 \leq^{sst} \mu_2 \Leftrightarrow \frac{d\mu_2}{d\mu_1}(\cdot) \text{ is increasing (or has an increasing version).}$$

A Markov process X is said to be strongly stochastically monotone (SSM) if for $s<t$, $x_1 < x_2$

$$\mathcal{L}(X_t \mid X_s = x_1) \leq^{sst} \mathcal{L}(X_t \mid X_s = x_2).$$

Gareth Roberts (JAP 28, 74–83, 1991) proves that if X is SSM, and $f, g : [0,T] \to \mathbb{R}$ are R-functions, $f \leq g$, then

$$\boxed{\begin{array}{l}[X \mid \tau_f > T] \text{ is SSM, and} \\ \qquad [X \mid \tau_f > T] \overset{sst}{\leq} [X \mid \tau_g > T],\end{array}}$$

this last meaning that for any $0 \leq s \leq t \leq T$, $x \in \mathbb{R}$,

$$\mathcal{L}(X_t \mid X_s = x, \tau_f > T) \leq^{sst} \mathcal{L}(X_t \mid X_s = x, \tau_g > T).$$

Of course, $\tau_f \equiv \inf\{t : X_t > f(t)\}$, and we assume X has R-paths.

(ii) The way to prove these results is to look at a discrete skeleton $(X_{k\delta})$ and prove them there. A little bit of weak convergence gets us through.

So let's suppose

$$(X_n)_{n=0}^N \text{ is a SSM, real-valued, time-inhomogeneous Markov process,}$$

with transition kernels $P_k(x, dy) \equiv P(X_{k+1} \in dy \mid X_k = x)$, which have densities $p_k(\cdot,\cdot)$ with respect to some reference measure dy.

We now generalise slightly to take in the possibility of killing into an additive functional, and reweight the law of the process using

$$Z \equiv \prod_{i=1}^N \varphi_i(X_i)$$

where the φ_i are non-negative, but otherwise not yet restricted. This transforms the transition kernels to

$$\tilde{P}_k f \equiv g_k^{-1} P_k(\varphi_{k+1} g_{k+1} f),$$

where $g_N \equiv 1$, $g_k = P_k(\varphi_{k+1} g_{k+1})$ for $k<N$.

(iii)

PROPOSITION. The process with transition kernels $\tilde{P}_k$ is again SSM.

Proof The proof is quite elementary and uses

① If $\psi(\cdot,\cdot)$ has the property that for $x_1 \le x_2$

$$\frac{\psi(x_2,y)}{\psi(x_1,y)} \text{ is increasing in } y$$

then for $y_1 \le y_2$,

$$\frac{\psi(x,y_2)}{\psi(x,y_1)} \text{ is increasing in } x. \qquad \text{(This is trivial)}$$

② If $f_2(x)/f_1(x) \uparrow$, and $\mu_1 \le^{sst} \mu_2$, then

$$\boxed{\int d\mu_2(x) f_2(x) \int d\mu_1(x) f_1(x) \ge \int d\mu_2(x) f_1(x) \int d\mu_1(x) f_2(x)}$$

(this is (a form of) the FKG inequality; see Lemma 2.1 of Robert's article, where he refers to Wijsman JASA 80, 472-475, 1985.)

As a piece of notation, let $\tilde{p}_{s,t}(x,y) = \tilde{P}(X_t \in dy \mid X_s = x)/dy$, for $s<t$.

We have to prove that for $x_1 \le x_2$, and $0 \le s < t \le N$

$$(*) \qquad \frac{\tilde{p}_{s,t}(x_2,y)}{\tilde{p}_{s,t}(x_1,y)} \text{ is incr in } y.$$

The proof is by induction on $|t-s|$. For $|t-s|=1$, the result is immediate from the SSM property of X. For the inductive step,

$$\tilde{p}_{s,t}(x,y) = \int \tilde{p}_s(x,z)\ \tilde{p}_{s+1,t}(z,y)\,dz$$

and we apply ② with

$$f_j(z) \equiv \tilde{p}_s(x_j,z) \qquad \mu_j(dz) = \tilde{p}_{s+1,t}(z,y_j)\,dz$$

By the inductive hypothesis and ①, $\mu_1 \leq^{sst} \mu_2$; (*) follows immediately from ②.

(iv) Now we consider the result for different boundaries. In the more general context considered here, we take functions φ_j and φ_j^*, and assume

$$\underline{\text{for each } j},\quad \varphi_j^*/\varphi_j \quad \underline{\text{is increasing}}$$

Notice that this implies

$$\underline{\text{for each } j},\quad g_j^*/g_j \quad \underline{\text{is increasing}}.$$

[Indeed, if $x_1 \leq x_2$,

$$g_j^*(x_i) = \int p_j(x_i,y)\ \varphi_{j+1}^*\ g_{j+1}^*(y)\,dy$$

$$= \int f_i(y)\ \mu^*(dy)$$

where $f_2/f_1 = p_j(x_2,y)/p_j(x_1,y)$ is increasing, and $\mu \leq^{ssc} \mu^*$

(at least, when $j=N-1$, this is clear; and then it follows by induction). The statement $g_j^*/g_j \uparrow$ now comes immediately from ②.]

We can simplify notation by assuming $\varphi_j \equiv 1$, $g_j \equiv 1$, which corresponds to taking the law $\tilde{P}$ as reference measure. Let's now do that.

In order to prove $[X] \leq^{sst} [X^*]$, we must prove that for $s \leq t$, and any x,

$$\frac{p^*_{s,t}(x,y)}{p_{s,t}(x,y)} \quad \text{is increasing in } y.$$

We shall in fact prove the stronger statement

$$\boxed{\text{for any } s<t,\ x\in\mathbb{R}, \quad \frac{p^*_{s,t}(x,y)}{g^*_t(y)\ p_{s,t}(x,y)} \quad \text{is increasing in } y}$$

The proof is by induction on $t-s$; for $t-s=1$, we get the answer trivially, since φ^* is increasing, and X is SSM.

For the inductive step

$$\frac{p^*_{s,t+1}(x,y)}{g^*_{t+1}(y)\ p_{s,t+1}(x,y)} = \frac{\int p^*_{s,t}(x,z)\ g_t(z)^{-1}\ p_t(z,y)\ \varphi^*_{t+1}(y)\ dz}{\int p_{s,t}(x,z)\ p_t(z,y)\ dz}$$

For $y_1<y_2$, let's define $f_j(z) \equiv p_t(z,y_j)$, so that, by ①, $f_2(z)/f_1(z)$ is increasing.

Next define

$$\mu_1(dz) \equiv p_{s,t}(x,z)\,dz$$

$$\mu_2(dz) = p^*_{s,t}(x,z)\ g_t(z)^{-1}\ dz\ ;$$

by inductive hypothesis, $\mu_1 \leq^{sst} \mu_2$.

Hence

$$\frac{p^*_{s,t+1}(x,y_2)}{g^*_{t+1}(y_2)\,p_{s,t+1}(x,y_2)} \Big/ \frac{p^*_{s,t+1}(x,y_1)}{g^*_{t+1}(y_1)\,p_{s,t+1}(x,y_1)} = \frac{\varphi^*_{t+1}(y_2)}{\varphi^*_{t+1}(y_1)} \cdot \frac{\int f_2(z)\,\mu_2(dz)}{\int f_2(z)\,\mu_1(dz)} \cdot \frac{\int f_1(z)\,\mu_1(dz)}{\int f_1(z)\,\mu_2(dz)}$$

$$\geq 1$$

by ② and monotonicity of φ^*_{t+1}.

The inequality ② is very useful!

A Wiener-Hopf example (31/12/91)

(i) Let's consider the additive functional

$$A_t \equiv \int_0^t \varphi(B_s)\,dB_s = \Phi(B_t) - \Phi(B_0) - \tfrac{1}{2}\int_0^t \varphi'(B_s)\,ds$$

of Brownian motion and see whether we can identify $Y_t^+ = B(\tau_t^+)$, where, as usual,

$$\tau_t^+ = \inf\{u : A_u > t\}.$$

Let's assume

$\underline{\Phi \text{ is convex increasing}}$, $\Phi(-\infty) = 0$.

(ii) Try the old martingale trick: can we find g_λ so that

$$\boxed{(*) \quad g_\lambda(B_t)\, e^{\lambda A_t} \text{ is a loc. martingale?}}$$

In fact, if we take

$$\boxed{g_\lambda(x) = \psi_\lambda(x)\, e^{-\lambda \Phi(x)}}$$

where

$$\boxed{\psi_\lambda'' = \lambda \varphi' \psi_\lambda, \quad \psi_\lambda \text{ increasing}, \quad \psi_\lambda(-\infty) = 1}$$

then (*) holds, and g_λ is __bounded__. It is simple to verify via Itô's formula that (*) holds, but to see how g_λ was arrived at, we try to deduce g_λ from

$$g_\lambda(x) = E^x \exp \lambda A(H_\alpha) \qquad \text{for large } \alpha$$

$$= E^x \exp\{\lambda \Phi(\alpha) - \lambda \Phi(x) - \tfrac{1}{2}\lambda \int_0^{H_\alpha} \varphi'(B_s)\,ds\}$$

$$= \text{const.}\ e^{-\lambda \Phi(x)}\, \psi_\lambda(x).$$

Now, as in Dym-McKean,

$$\psi_\lambda(x) = \sum_{n \geq 0} p_n(x), \qquad \text{where } p_0 \equiv 1,$$

$$p_{n+1}(x) = \int_{-\infty}^x dy \int_{-\infty}^y dv\ \lambda \varphi'(v)\, p_n(v)\, dv$$

and one proves easily by induction that

$$p_n(x) \leq \lambda^n \Phi(x)^n / n! \qquad \forall n, \ \forall x.$$

This now allows us to say

$$\boxed{g_\lambda(x) = E^x\left[g_\lambda(Y_t^+)\, e^{\lambda t} \right]} \qquad \forall x, \forall \lambda, \forall t$$

Can this transform be unravelled?

<u>First passage to a barrier for Brownian motion (31/12/91)</u>

Suppose we're given some smooth $f: \mathbb{R}^+ \to \mathbb{R}$, $f(0) = a > 0$, and we want to know about the density of the first passage time to f by Brownian motion B. This is same as first time τ

$$X_t \equiv B_t - (f(t) - f(0)) \quad \text{reaches } a$$

If we use

$$\left.\frac{d\tilde{P}}{dP}\right|_{\mathcal{F}_t} = Z_t, \quad \text{where} \quad dZ_t = -f'(t) Z_t \, dX_t$$

then under $\tilde{P}$ the canonical process is $B_t - f_t + a$. Thus for any test f^n φ supported in some compact,

$$E\,\varphi(\tau) = \tilde{E}\,\varphi(H_a)$$

$$= E\left[\varphi(H_a)\, \exp\left\{ -\int_0^{H_a} f_s'\, dX_s - \tfrac{1}{2}\int_0^{H_a} f_s'^2\, ds \right\}\right]$$

Thus the $\tilde{P}$-dist^n of H_a $(= \mathcal{L}(\tau))$ has density

$$\rho(t) \equiv E\left[\exp\left(-\int_0^t f_s'\, dX_s - \tfrac{1}{2}\int_0^t f_s'^2\, ds\right) \Big| H_a = t \right]$$

with respect to the standard Brownian first-passage density. Now we can use the idea of BLMS **17**, 157–161, extended by Eric Pauwels (JAP **24**, 370–371), which is to use the fact that

$$\boxed{\text{given } H_a = t, \quad (X_s)_{0 \le s \le t} \overset{\mathcal{D}}{=} \left(a - \frac{t-s}{t} R_a\left(\frac{ts}{t-s}\right)\right)_{0 \le s \le t}}$$

where $R_a(\cdot)$ is a BES(3) process started at a. There are other ways of specifying the conditional law of X, for example

$$dX_s = dW_s - \frac{ds}{a - X_s} + \frac{a - X_s}{t-s}\, ds.$$

We can also re express $R_a(\cdot)$ by time inversion:

$$\left(t R_a(1/t) \right) \overset{\mathcal{D}}{=} \left(|\beta_t + at| \right)$$

where β is a 3-d BM, a stands for $(a,0,0)$ also.

All depends on getting a useable form for

$$\int_0^t f'(s)\, dX_s = f'(t) X_t - \int_0^t f''(s)\, X_s\, ds.$$

<u>Possible further correction for variance estimation</u> (24/1/92).

1) In one days trading, observe opening Y, close X, sup S and inf I in the log-price, but only observe with limited accuracy (1p in case of UK shares!) We always have

$$I \leq X, Y \leq S,$$

but the <u>true</u> values of the log prices do not have to be in the set $\{ \log k\delta : k \in \mathbb{N} \}$, where δ is the price-spacing.

Let's consider the possible range for the <u>true</u> estimator $\hat{\sigma}^2_{RS}$ (or $\hat{\sigma}^2_{RS,k}$) based on the <u>true</u> values $I' \leq X', Y' \leq S'$.

Holding X', Y' fixed, the biggest possible value for $\hat{\sigma}^2_{RS}$ would be

$$(S^* - Y')(S^* - X') + (I_* - Y')(I_* - X')$$

where S^* is the biggest possible value of S' (so if $S = \log k\delta$, we'd have $S^* = \log((k+\frac{1}{2})\delta)$) and I_* is smallest possible value of I' (thus $I_* = \log(j-\frac{1}{2})\delta$ if $I = \log j\delta$). This is

$$S^{*2} + I_*^2 - (S^* + I_*)(X' + Y') + 2X'Y'$$

$$a = \tfrac{1}{2}(S^* + I_*)$$

$$= S^{*2} + I_*^2 - 2(X' - a)(Y' - a) + 2a^2$$

and thus we can now maximise over choice of X', Y' in the permitted interval.

The analysis of the maximal value of $\hat{\sigma}^2_{RS,h}$ is quite simple, as

$$\hat{\sigma}^2_{RS,h} = \frac{a\sqrt{h}(S-I) + \{a^2h(S-I)^2 + (1-2bh)\hat{\sigma}^2_{RS}\}^{\frac{1}{2}}}{(1-2bh)}$$

and the same maximisation procedure will work just as well.

2) As for minimisation, if we see

$$I < X, Y < S,$$

then the best thing to do is to pull S down to S_*, push I up to I_* and proceed then to minimise over permitted values of X, Y just as with the maximisation.

If we see

$$I < X < Y = S$$

then with I, X held fixed we minimise $\hat{\sigma}^2_{RS}$ by $Y' = S' = S_*$, and then with X held fixed we minimise using $I' = I^*$, and finally we minimise over X' at $X = X_*$. This also minimises $\hat{\sigma}^2_{RS,h}$.

If we see

$$I = X < Y = S$$

we minimise $\hat{\sigma}^2_{RS}$ by taking $I' = I^* = X'$, $S' = S_* = Y'$, which also minimises $\hat{\sigma}^2_{RS,h}$.

If we see

$$I < X = Y = S$$

then minimise $\hat{\sigma}^2_{RS}$, $\hat{\sigma}^2_{RS,h}$ by $X' = Y' = S' = S_*$, $I' = I^*$.

Whatever we do, we end up with a possible interval $[\sigma_*^2, \sigma^{*2}]$ in which our estimator $\hat{\sigma}^2$ would lie if we had been able to measure with total precision. I suggest that we could <u>either</u> reject days where $|\sigma_*^2 - \sigma^{*2}|$ is too large a fraction of $\sigma_*^2 + \sigma^{*2}$, <u>or else</u> pool such days with others where trading was better.

3) How would we combine estimators on several days trading in order to get

a (hopefully) more reliable estimate? If we accept the "constant-variance-per-trade" hypothesis, then on a day when there are N trades, we form an estimate of $N\sigma^2$ (as if we saw the BM for N units of time). The variance of this estimator (for $c=0$) is known to be const. $\sigma^4 N^2$, so that if we divide our estimator by N to form an estimator of σ^2, the variance of this is the same however busy the day was! Thus (apart from the bumpy bits when trading is thin) there is no virtue in weighting estimators differently on busier days.

<u>Calculations on an example of Bolthausen, Deuschel + Schmock</u> (9/2/92)

1) BDS consider reweighting Wiener measure on $C(\mathbb{R}^+, \mathbb{R})$ by

$$Z_T^{-1} \exp\left\{ -\frac{\tau^2}{4T} \int_0^T ds \int_0^T du \, (X_s - X_u)^2 \right\} \equiv Z_T^{-1} \exp\{-H_T\}$$

where

$$Z_T = E\exp\left\{ -\frac{\tau^2}{4T} \int_0^T ds \int_0^T du \, (X_s - X_u)^2 \right\}$$

is the appropriate normalisation. <u>Is there some limiting measure as</u> $T \to \infty$, <u>and</u>, <u>if so</u>, <u>what is it</u>?

2) Firstly, let's notice that

$$Z_T = E\exp\left[-\frac{\tau^2}{4T} \left\{ 2T \int_0^T X_s^2 \, ds - 2\left(\int_0^T X_s \, ds\right)^2 \right\} \right]$$

$$= E\exp\left[-\frac{\tau^2 T^2}{2} \left\{ \int_0^1 X_s^2 \, ds - \left(\int_0^1 X_s \, ds\right)^2 \right\} \right] \qquad \text{by scaling}$$

$$= \left(\frac{\tau T}{\sinh \tau T} \right)^{1/2} \qquad \text{by the moment-of-inertia of Brownian motion result.}$$

3) More generally, if we can compute

$$Z_T^{-1} E_t \exp\{-H_T\}$$

then letting $T \to \infty$ we should obtain the change-of-measure martingale.

$$e^{-H_T} = \exp -\frac{\tau^2}{2T}\left\{ \int_0^t ds \int_s^t du\,(X_s-X_u)^2 + \int_0^t ds \int_t^T du\,(X_s-X_u)^2 + \int_t^T ds \int_s^T du\,(X_s-X_u)^2 \right\}$$

So if we set $X_{t+u} = X_t + \beta_u$, then β is a Brownian motion independent of $\mathcal{F}_t$, and

$$\int_0^t ds \int_t^T du\,(X_s-X_u)^2 = \int_0^t ds \int_0^{T-t} du\,(\beta_u + X_t - X_s)^2$$

$$= t\int_0^{T-t}\beta_u^2\,du + 2\int_0^t ds\,(X_t-X_s)\cdot\int_0^{T-t}\beta_u\,du + (T-t)\int_0^t (X_t-X_s)^2\,ds.$$

Let's abbreviate $Y_1 \equiv \int_0^{T-t}\beta_u\,du$, $Y_2 \equiv \int_0^{T-t}\beta_u^2\,du$,

$$a \equiv \int_0^t ds\int_s^t du\,(X_s-X_u)^2 + (T-t)\int_0^t (X_t-X_s)^2\,ds \qquad \in \mathcal{F}_t,$$

$$b \equiv \int_0^t ds\,(X_t - X_s) \qquad \in \mathcal{F}_t,$$

so that

$$E_t e^{-H_T} = e^{-a\tau^2/2T} E_t \exp -\frac{\tau^2}{2T}\left\{ tY_2 + 2bY_1 + (T-t)Y_2 - Y_1^2 \right\}$$

$$= e^{-a\tau^2/2T} E_t \exp -\frac{\tau^2}{2T}\left\{ TY_2 + 2bY_1 - Y_1^2 \right\}.$$

Now we recall the result

$$E\exp\left[-\frac{\theta^2}{2}\int_0^t (B_s+x)^2\,ds\right] = (\operatorname{sech}\theta t)^{1/2}\exp\left(-\frac{x^2}{2}\theta\tanh\theta t\right)$$

from which

$$E\exp\left[-\frac{\theta^2}{2}\int_0^t B_s^2\,ds - x\theta^2\int_0^t B_s\,ds\right] = (\operatorname{sech}\theta t)^{1/2}\exp\left(\frac{x^2\theta^2 t}{2}\left\{1 - \frac{\tanh\theta t}{\theta t}\right\}\right).$$

Thus

$$E\exp\left[-\frac{\tau^2}{2}Y_2 - \tau^2\gamma Y_1\right] = (\operatorname{sech}\tau(T-t))^{1/2}\exp\left[\frac{\gamma^2\tau^2(T-t)}{2}\left\{1 - \frac{\tanh\tau(T-t)}{\tau(T-t)}\right\}\right].$$

Taking $\gamma = \frac{b}{T} + \varepsilon$, and mixing over ε with a $N(0, \frac{1}{T\tau^2})$ law, the left-hand side becomes the thing we want

$$E \exp\{-\tfrac{1}{2}\tau^2 Y_2 - \tfrac{b\tau^2}{T} Y_1 + \tau^2 Y_1^2/2T\},$$

and the right-hand side becomes

$$\left(\operatorname{sech} \tau(T-t)\right)^{1/2} \left(\frac{T}{t + \frac{\tanh \tau(T-t)}{\tau}}\right)^{1/2} \exp \frac{b^2}{2T} \; \frac{\tau^2(T-t)\left\{1 - \frac{\tanh \tau(T-t)}{\tau(T-t)}\right\}}{t + \frac{\tanh \tau(T-t)}{\tau}}$$

If we now multiply by Z_T^{-1} and let $T \uparrow \infty$, we obtain

$$\boxed{\lim_{T\uparrow\infty} E_t\left(Z_T^{-1} e^{-H_T}\right) = (1+t\tau)^{-1/2} \exp\left[\frac{\tau t}{2} - \frac{\tau^2}{2}\int_0^t (X_t - X_s)^2 ds + \left(\int_0^t (X_t - X_s) ds\right)^2 \frac{\tau^3}{2(1+t\tau)}\right].}$$

4) BDS prove that the limiting measure is an OU law attracted to a randomly chosen centre $c \sim N(0, \tau^{-1})$, with restoring constant τ. If $\mathbb{Q}^{c,\tau}$ is the law of the OU process

$$dX = dB - \tau(X-c)\,dt$$

then we have

$$\left.\frac{d\mathbb{Q}^{c,\tau}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left[-\tau\int_0^t (X_s - c)\,dX_s - \tfrac{1}{2}\int_0^t (X_s-c)^2 ds \,.\, \tau^2\right]$$

$$= \exp\left[-\frac{\tau}{2}(X_t^2 - t) + c\tau X_t - \frac{\tau^2}{2}c^2 t + \tau^2 c\int_0^t X_s ds - \frac{\tau^2}{2}\int_0^t X_s^2 ds\right]$$

$$\equiv \exp\left[-c^2\tau^2 t/2 + cV_t + U_t\right]$$

$$\left(V_t \equiv \tau\left(X_t + \tau\int_0^t X_s ds\right),\; U_t \equiv -\frac{\tau^2}{2}\int_0^t X_s^2 ds - \frac{\tau}{2}(X_t^2 - t)\right)$$

$$\equiv \rho(c),$$

say. By Bayes Theorem, the posterior density of the law of

c given $\mathcal{F}_t$ is

$$\boxed{N\left(\frac{X_t + \tau\int_0^t X_s\,ds}{1+\tau t},\ \frac{1}{\varepsilon+\tau^2 t}\right)}.$$

The density w.r.t. $\mathbb{P}$ of the limit law on $\mathcal{F}_t$ is simply

$$\int \rho(c)\ \exp(-c^2\tau/2)\ \frac{dc}{\sqrt{2\pi/\tau}}$$

$$= (1+t\tau)^{-1/2}\exp\left[-\frac{\tau^2}{2}\int_0^t X_s^2\,ds - \frac{\tau}{2}(X_t^2 - t) + \frac{\tau\left(X_t + \tau\int_0^t X_s\,ds\right)^2}{2\,(1+\tau t)}\right],$$

which, it can be confirmed, is the same as we get above.

5) If we write $\hat{P}^\tau$ for the limiting measure, then under $\hat{P}^\tau$, $\hat{E}^\tau(c|\mathcal{F}_t)$ $= (1+\tau t)^{-1}\left(X_t + \tau\int_0^t X_s\,ds\right)$ must be a martingale, whose qv is identified, so we get

$$X_t + \tau\int_0^t X_s\,ds = (1+\tau t)\int_0^t \frac{dW_s}{1+\tau s},$$

whence

$$\boxed{dX_t = dW_t - \tau(X_t - c_t)\,dt}$$

with

$$c_t \equiv \hat{E}^\tau(c|\mathcal{F}_t) = \frac{X_t + \tau\int_0^t X_s\,ds}{1+\tau t}.$$

This is, of course, also obvious directly from filtering.

Another quadratic functional – more difficult (10/2/92)

1) Another qf calculation which BDS are interested in is

$$(*) \quad E \exp\left\{ -\frac{\theta^2}{2} \int_0^T \int_0^T (X_s - X_t)^2 \rho(s,t)\, ds\, dt \right\},$$

where ρ is some symmetric non-negative weight function.

Introduce a zero-mean Gaussian random measure ξ on the subsets of $[0,T]^2$, with

$$E\, \xi(A)\xi(B) = \iint_{A\cap B} \rho(s,t)\, ds\, dt.$$

Then

$$(*) = E \exp i\theta \iint (X_s - X_t)\, \xi(ds,dt)$$

$$= E \exp\left(i\theta \int_{s=0}^T \int_{t=0}^s \xi(ds,dt) \int_t^s dX_u - i\theta \int_{s=0}^T \int_{t=s}^T \xi(ds,dt) \int_s^t dX_u \right)$$

$$= E \exp i\theta \int_0^T dX_u \underbrace{\Big(\xi((u,T)\times(0,u)) - \xi((0,u)\times(u,T)) \Big)}_{\equiv Y_u}$$

$$= E \exp\left(-\theta^2/2 \int_0^T Y_u^2\, du \right).$$

Now Y is a zero-mean Gaussian process with

$$\boxed{E(Y_u Y_v) = 2\int_0^u ds \int_v^T dt\, \rho(s,t)} \qquad 0 \le u \le v \le T$$

(If ρ is bdd, then Y has a cts version)

2) An example of particular interest is where

$$\rho(s,t) \equiv \frac{\alpha}{2} \exp(-\alpha|s-t|)$$

and then

$$E\, Y_u Y_v = \frac{1}{\alpha}(e^{\alpha u} - 1)(e^{-\alpha v} - e^{-\alpha T}).$$

So if we set

$$f(u) \equiv (e^{\alpha u} - 1)/\sqrt{\alpha}, \quad g(v) = (e^{-\alpha v} - e^{-\alpha T})/\sqrt{\alpha},$$

we have the structural form

$$\boxed{E\, Y_u Y_v = f(u)\, g(v) \qquad \text{for } u \le v,}$$

strictly.

where $f(0) = 0 = g(T)$, f incr, g decr. Let's stick to this structural form for now, so that

$$2\rho(s,t) = -f'(s)\,g'(t) \quad \text{for} \quad 0 \le s \le t \le T,$$

and if f', g' are bdd, then there's a continuous version. Now observe that for $u < v$,

$$E\left[Y_u \mid Y_v\right] = \frac{f(u)}{f(v)}\, Y_v$$

and, remarkably,

$$E\left[\,Y_u \mid Y_s : s \ge v\right] = \frac{f(u)}{f(v)}\, Y_v$$

because for $s \ge v$

$$E\left(\frac{f(u)}{f(v)}\, Y_v Y_s\right) = f(u)\,g(s) = E\left(Y_u Y_s\right).$$

<u>Thus this Gaussian process is</u> (<u>inhomogeneous</u>) <u>Markov</u>, and in fact the given structural form is also <u>necessary</u> for the process to be Mkv.

Moreover, this process can be made by a deterministic transformation from the Brownian bridge!

Indeed, if we try to find some $\tau : [0,1] \to [0,T]$ strictly increasing cts, and cts $a : [0,1] \to \mathbb{R}^{++}$ such that

$$\tilde{Y}_s \equiv a(s)\, Y_{\tau(s)} \quad \text{is Brownian bridge,}$$

then this can be done iff for $0 \le s \le t \le 1$,

$$s(1-t) = E\,\tilde{Y}_s \tilde{Y}_t = a(s)\,a(t)\,f(\tau(s))\,g(\tau(t))$$

which will certainly happen if

$$\left.\begin{aligned} a(s)\,f(\tau(s)) &= s \\ a(t)\,g(\tau(t)) &= 1-t \end{aligned}\right\}$$

So if we pick τ so that

$$\frac{f(\tau(s))}{g(\tau(s))} = \frac{s}{1-s}$$

which we can do since $f(u)/g(u) \uparrow$ from 0 at 0 to ∞ at T,

and then set $a(s) = s / f(\tau(s))$, we have it.

3) Returning to our favoured example, $f(s) \equiv (e^{\alpha s}-1)/\sqrt{\alpha}$, $g(s) \equiv (e^{-\alpha s} - e^{-\alpha T})/\sqrt{\alpha}$, we can show that

$$\tau(s) = \frac{1}{\alpha} \log\left[\frac{1 - s(1+e^{-\alpha T}) + \{(1-s(1+e^{-\alpha T}))^2 + 4s(1-s)\}^{\frac{1}{2}}}{2(1-s)}\right]$$

by standard calculation. Hence

$$a(s) = \frac{2\sqrt{\alpha}\, s(1-s)}{\{(1-s(1+e^{-\alpha T}))^2 + 4s(1-s)\}^{\frac{1}{2}} - 1 + s - se^{-\alpha T}}.$$

The thing we want to calculate is the law of

$$\int_0^T Y_u^2\, du = \int_0^1 Y_{\tau(s)}^2 \; \tau'(s)\, ds$$

$$= \int_0^1 \left(\frac{\tilde{Y}_s}{a_s}\right)^2 \tau'(s)\, ds.$$

In principle, this can be computed?

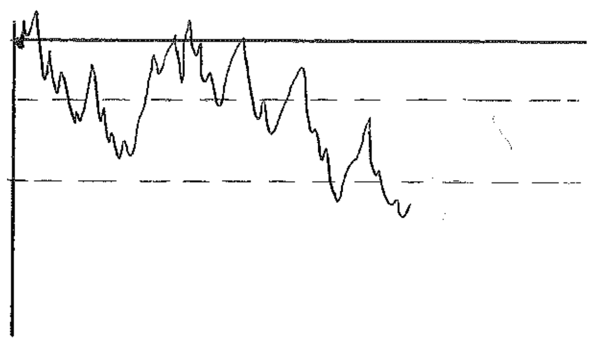
An example studied by Marc Yor + F. Petit (11/2/92)

(i) If B is BM, ℓ its local time at 0, $f: \mathbb{R}^+ \to \mathbb{R}^+$ cts strictly incr, $f(0)=0$, with inverse function φ, and

$$X_t \equiv |B_t| - f(\ell_t)$$

then the aim is to specify the law of $\{ L(\tau_s, -x) : x \geq 0 \}$, where L is the local time pr of X, and τ is the inverse to $L(\cdot, 0)$. (IT IS incidentally quite obvious that $\{L(\tau_s, x) : x \geq 0\}$ is a BESQ(0)).

(ii) Fix $0 < a < b \equiv a + \delta$, and consider what would be the law of $L(\tau_s, -b)$ given $L(\tau_s, -a) = \ell$ and all of $L(\tau_s, x)_{-a < x \leq 0}$.



The conditional law obviously only depends on $L(\tau_s, -a)$, by the independence of excursion processes above and below $-a$.

The calculation now splits into two:

(I) compute the law of $L(H_{-b}, -a) \equiv S$;

(II) Once we know the law of S, then if $S \geq \ell$ there can be no local time at $-b$ before τ_s. However, if $S = s < \ell$, there is a local time interval of length $\ell - s$ during which excursions (a $P((\ell-s)/2\delta)$ number of them) cross down from $-a$ to $-b$, each adding an indept $\exp(1/2\delta)$ to $L(\cdot, -b)$, in addition to the first $\exp(1/2\delta)$ which happened when X got to $-b$.

(iii) (I) While X is passing from $-a$ to $-b$, the local time of B at 0 will rise from $\varphi(a)$ to $\varphi(b)$. While $\ell_t = x$, excursions of X which reach a happen at rate $(-a + f(x))^{-1}$, and on each such excursion an indept $\exp(\frac{1}{2}(f(x)-a)^{-1})$ of local time is added at a before the excursion falls back down to the current minimum of X.

We therefore have a compound Poisson random measure indexed by $[\varphi(a), \varphi(b)]$, and if Y is the total contribution to $L(\cdot, a)$ while X passes from a down to b, we have

$$E \exp(-\lambda Y) = \exp - \int_{\varphi(a)}^{\varphi(b)} \left\{ 1 - \frac{1}{1+2\lambda(f(x)-a)} \right\} \cdot \frac{dx}{f(x)-a}$$

$$= \exp\left\{ - \int_{\varphi(a)}^{\varphi(b)} \frac{2\lambda \, dx}{1+2\lambda(f(x)-a)} \right\}. \qquad (*)$$

Hence if we carry out the step II, we shall have

$$E\left[e^{-\lambda L(\tau_\delta, -b)} \,\middle|\, L(\tau_\delta, -a) = \ell \right]$$

$$= \int_0^\ell P(Y \in dy) \frac{1}{1+2\delta\lambda} \exp\left\{ - \frac{\lambda(\ell - y)}{1+2\delta\lambda} \right\} + P(Y \geq \ell)$$

by the usual sort of Ray-Knight argument.

(iv) It seems unlikely that many examples will work out at all explicitly, but the special case $f(x) = \mu x$ $(\mu > 0)$ does simplify somewhat. We get

$$E \exp(-\lambda Y) = \left(\frac{1}{1+2\lambda\delta} \right)^{1/\mu}$$

so that

$$P(Y \in dy)/dy = \left(\frac{y}{2\delta} \right)^{\frac{1}{\mu}-1} e^{-y/2\delta} \cdot \frac{1}{2\delta},$$

a gamma distn.

(v) We can express the transition mechanism by the double LT:

$$\boxed{\begin{aligned} &\int_0^\infty \alpha e^{-\alpha \ell} E\left[e^{-\lambda L(\tau_\delta, -b)} \,\middle|\, L(\tau_\delta, -a) = \ell \right] d\ell \\ &= 1 - \frac{\lambda(1+2\delta\alpha)}{\alpha + \lambda(1+2\delta\alpha)} \exp\left\{ - \int_{\varphi(a)}^{\varphi(b)} \frac{2\alpha \, dx}{1+2\alpha(f(x)-a)} \right\} \end{aligned}}$$

(*) A nice feature is that $P(Y=0) > 0$ if $\int_{\varphi(a)+} (f(x)-a)^{-1} dx < \infty$!

Ray-Knight thm for cts time rw on $\mathbb{Z}$ (15/2/92)

Let $(Y_t)_{t\geq 0}$ be simple rw on $\mathbb{Z}$, $q_{i,i+1} = \lambda > \mu = q_{i,i-1}$ $\forall i \in \mathbb{Z}$. If we set $L(t,j) \equiv \int_0^t I_{\{Y_u = j\}}\,du$, then the claim is that

$$\{ L(\infty, j) \; ; \; j \geq 0 \} \text{ is a Markov process}$$

1) If we take $X_t = B_t + bt$, where $e^{2b} = \lambda/\mu$, then the local time process of X, $Z_x \equiv L(\infty, x)$ satisfies

$$dZ_x = 2\sqrt{Z_x}\, dW_x + 2(1 - bZ_x)\,dx \quad \text{in } x \geq 0,$$

with Z_0 distributed as $\exp(b)$ [This is one form of the Ray-Knight thm - see (2) below]. If we time-change X by the additive functional

$$A_t = c \sum_n L(t, n) \qquad (\text{where } c = b/(\lambda - \mu), \text{ in fact})$$

then we get a process equal in law to Y, with local time process just (a multiple of) the process $(Z_n)_{n\geq 0}$, which we know is Markovian by R.

2) Here is an analyst's proof of R-K in the situation considered here; it is a variant of the proof of McGill, Sem de Prob $\overline{XV}$.

Take some $g \geq 0$, $g \in C_K^\infty(\mathbb{R}^+)$, and let $A_t \equiv \int_0^t g(X_s)\,ds$. Define

$$f(x) \equiv E^x e^{-A_\infty}$$

so that $\boxed{\mathcal{G} f - g f = 0}$, $0 \leq f \leq 1$, $f \uparrow 1$ as $x \uparrow \infty$. $(\mathcal{G} \equiv \tfrac{1}{2}D^2 + bD)$

Next, let $\mathcal{L} \equiv 2x\,D^2 + 2(1 - bx)\,D$, the generator of the local time process, as we shall prove.

If we now let $\tilde{X}_t = B_t - bt$, with $\tilde{A}_t \equiv \int_0^t g(\tilde{X}_s)\,ds$, then the function $\tilde{f}(x) \equiv E^x \exp(-\tilde{A}_\infty)$ solves

$$\tilde{\mathcal{G}} \tilde{f} - g\tilde{f} = 0, \quad 0 \leq \tilde{f} \leq 1, \quad \tilde{f} \uparrow 1 \text{ as } x \downarrow -\infty \qquad (\tilde{\mathcal{G}} \equiv \tfrac{1}{2}D^2 - bD$$

and we can now take the martingale

$$M_x = (\tilde{f}(x))^{-1} \exp\left[- \int_{-\infty}^{x} g(y) Z_y \, dy - \left(\tilde{f}'(x) / 2 \tilde{f}(x) \right) Z_x \right]$$

where (Z_x) is a stationary diffusion with generator $\mathcal{L}$ (and invariant law $\exp(b)$). This gives us

$$1 = M_{-\infty} = E\, M_\infty = \frac{1}{\tilde{f}(\infty)} E \exp\left(- \int_{-\infty}^{\infty} g(y) Z_y \, dy \right)$$

$$\Rightarrow \quad E \exp\left[- \int g(y) Z_y \, dy \right] = \tilde{f}(\infty)$$

$= f(-\infty)$ by time-reversal of the downward-drifting BM;

$= f(0)$ since g is zero in $\mathbb{R}^-$;

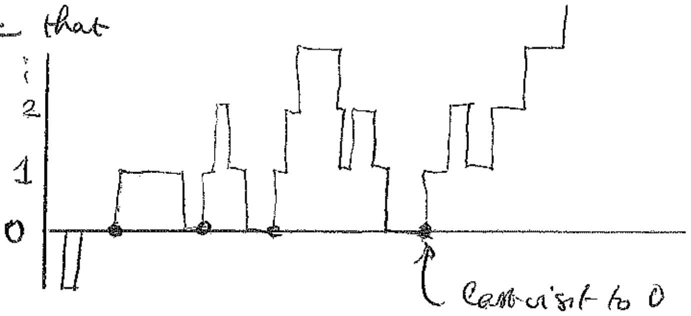
$= E^0 \exp\left(- \int_0^\infty g(y) L(\infty, y) \, dy \right)$.

Since g was arbitrary, that's it.

3) Now we see the excursion way to prove the result, which makes the transition mechanism explicit. Observe that

$P^1(\text{never reach } 0) = 1 - \mu/\lambda$,

and that the r.w. started at 1 conditioned to hit 0 is a rw with rates μ up, λ down.



The excursions up from 0 which return to 0 come at rate $\lambda \cdot \frac{\mu}{\lambda} = \mu$

——— down ——— " ——— " ——— μ,

and the excursions which escape to ∞ come at rate $(\lambda - \mu)$.

So if $L(\infty, 0) = \ell$, the no. of excursions up from 0 which return is $\sim \mathcal{P}(\mu \ell)$, and on each of these an $\exp(\lambda)$ amount of time is spent at 1. On the infinite excursion, an $\exp(\lambda)$ amount of time is also spent, as one sees by observing that the time must be exponential, and the mean is λ, because the mean time spent in 1 before last excursion

$$= \frac{1}{\lambda} \cdot E\, \mu L(\infty, 0) = \frac{\mu}{\lambda} \cdot \frac{1}{\lambda - \mu} = \frac{1}{\lambda - \mu} - \frac{1}{\lambda}$$

and the mean time spent in 1 is $(\lambda-\mu)^{-1}$.

Hence

$$E \exp\{-\alpha L(\infty,1) \mid L(\infty,0)=\ell\}$$

$$= \frac{\lambda}{\lambda+\alpha} \exp\{-\mu\ell\alpha/(\lambda+\alpha)\}.$$

The conditional independence of $(L(\infty,k))_{k\geq j}$ and $(L(\infty,k))_{k\leq j}$ given $L(\infty,j)$ is immediate from excn thy.

<u>Term structure in binary tree model</u> (17/2/92).

1) Let's suppose that $\Omega = \{-1,+1\}^{\mathbb{N}}$, with coordinate process X, and partial sum process

$$S_n = \sum_{j=1}^{n} X_j, \qquad S_0 = 0,$$

and $\mathcal{F}_n \equiv \sigma(\{X_j : j \leq n\})$. Let's also suppose that there is stochastic discounting; assume the interest rate on day j is r_j, so that \$1 at the beginning $(n=0)$ is worth

$$\$ \exp\left(\sum_{j=1}^{m} r_j\right) \equiv \$ \exp(R_m)$$

at time m. The r_j are non-negative and adapted. Accordingly, a bond which delivers \$1 at time τ is conventionally assumed to be worth

$$\boxed{V(t,\tau) \equiv E\left(\exp(-R_\tau + R_t) \mid \mathcal{F}_t\right)}$$

at time $t \leq \tau$. Here, E is expectation with respect to some given law on Ω. Define also

$$M(t,\tau) \equiv e^{-R_t} V(t,\tau) = E\left(e^{-R_\tau} / \mathcal{F}_t\right),$$

a martingale.

2) Now consider bonds of two different maturities $\tau < \tau'$, and consider forming a riskless hedge at time t of these two, if this

can be done. Abbreviate $V(t,\tau)$ to V_t, $V(t,\tau')$ to V'_t. Because of the special structure of the sample space, we have

$$V_t = V_t(X_1, \dots, X_t)$$

so let's write

$$\xi_+ = V_{t+1}(X_1, \dots, X_t, +1) - V_t(X_1, \dots, X_t)$$

$$\xi_- = V_{t+1}(X_1, \dots, X_t, -1) - V_t(X_1, \dots, X_t),$$

with $\xi'_\pm$ defined similarly in terms of the increments of V'.

If we make a portfolio of 1 unit of V_t and $-\lambda$ of V'_t, then at time $t+1$ we have gained

$$\xi_+ - \lambda \xi'_+ \quad \text{if } X_{t+1} = +1,$$

$$\xi_- - \lambda \xi'_- \quad \text{if } X_{t+1} = -1$$

and these two are the same iff

$$\boxed{\xi_+ - \xi_- = \lambda(\xi'_+ - \xi'_-)}$$

3) No arbitrage $\Rightarrow$ common value is zero, so we must have

NB: this isn't really right – see (5) below.

$$\lambda = \xi_+/\xi'_+ = \xi_-/\xi'_-$$

or in other words

$$\boxed{V_{t+1} - V_t \equiv \Delta V_{t+1} = \lambda \, \Delta V'_{t+1},}$$

where λ is $\mathcal{F}_t$-measurable.

To simplify things technically, let us assume that

$$\boxed{\text{for all } t < \tau, \quad \xi_+ \neq \xi_-, \quad \text{and that } R \text{ is strictly increasing.}}$$

Then expressing the condition on ΔV, $\Delta V'$ in terms of the martingales $M'_t \equiv M(t,\tau')$, $M_t \equiv M(t,\tau)$, we get

$$e^{R_{t+1}}(M_{t+1} - \lambda M'_{t+1}) = e^{R_t}(M_t - \lambda M'_t)$$

implying that

$$M_{t+1} - \lambda M'_{t+1} = e^{-R_{t+1}+R_t} (M_t - \lambda M'_t)$$

Taking conditional expectation w.r.to $\mathcal{F}_t$ gives

$$M_t - \lambda M'_t = E(e^{-R_{t+1}}/\mathcal{F}_t)\, e^{R_t} (M_t - \lambda M'_t)$$

which can only happen if

$$M_t = \lambda M'_t$$

implying $M_{t+1} = \lambda M'_{t+1}$.

Thus $\dfrac{M_{t+1}}{M'_{t+1}} = \lambda$, an $\mathcal{F}_t$ - meas random variable;

$$= \frac{M_t}{M'_t}.$$

Hence for all $t \leq \tau$, the ratio

$\dfrac{M_t}{M'_t}$ <u>takes the same value</u>, <u>which must therefore be $\mathcal{F}_0$-meas, or constant</u>!

Thus

$$E_\tau \exp -(R_{\tau'} - R_\tau) = \text{const}.$$

4) Dropping the assumption of the binary tree, if I have \$1 at $t=0$, I can <u>either</u> invest it in the bank, which will give me \$ e^{R_t} at time t,

<u>or</u> buy $\dfrac{1}{V(0,t)}$ of bonds maturing at time t, which gives me

$1/V(0,t)$ at time t

$$E\left[\frac{1}{V(0,t)}\right] = E\left[\frac{1}{E_0(e^{-R_t})}\right] < E\, E_0\, e^{R_t} = E e^{R_t}$$

– so on average I do better by investing at a stochastic interest rate ...

5) It would however not be impossible for a risk-free hedge to make a positive gain, the effect of this being to define a risk-free interest rate. If we can make a portfolio $Y = \sum_{j=1}^{n} \lambda_j \, V(\cdot, \tau_j)$ which is riskless from time 0 to time 1, then we may suppose that

$$Y_0 = 1, \quad Y_1 = c.$$

Since $e^{-R_t} Y_t$ is a P-mg, we conclude that

$$\boxed{c = 1/E_0 e^{-R_1}}$$

which defines the riskless rate of interest R_1^* via

$$c \equiv e^{R_1^*} = (E_0 e^{-R_1})^{-1} \leq E_0 e^{R_1}$$

and more generally

$$\exp(R_t^* - R_{t-1}^*) = \{E_{t-1} \exp(-R_t + R_{t-1})\}^{-1}$$

Thus under the (a?) EMM P^*, we must have that the asset price processes $V(\cdot, \tau)$ discounted by the riskless interest rate form a martingale, so that

$$e^{-R_t^*} V(t, \tau) \text{ is a } P^*\text{-martingale}.$$

Hence $$\boxed{\left.\frac{dP^*}{dP}\right|_{\mathcal{F}_t} = e^{R_t^* - R_t}}.$$

[This is, of course, only going to work if we can hedge perfectly ...].

6) Compare with the cts case where hedging is possible:

$$e^{-R_t} V(t, \tau) = M(t, \tau) \text{ is a P-mg,}$$

so if one can form a riskless hedge $Y_t = \sum_j \alpha_j(t) V(t, \tau_j)$, where the α_j are (self-financing portfolio and) f.v., we get

$$dY_t = \sum \alpha_j \, dV(t, \tau_j) = \sum \alpha_j \, V(t, \tau_j) \, dR_t \quad \text{(because of riskless assumption)}$$

$= Y_t \, dR_t$. Thus in this case R_t __is__ the riskless interest process.

Perhaps useful:

$$\left(\frac{x}{3\sqrt{3}}\right)^{1/2} K_{1/3}\left(\tfrac{2}{3}\, x^{3/2}\right) \equiv f(x)$$

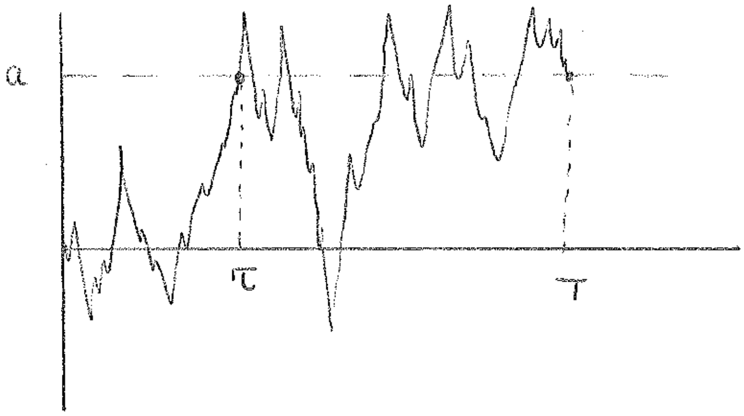
solves the Airy equation $f''(x) = x\, f(x)$.

Polymer measure in 1 dimension? (21/2/92)

1) Take

$$\frac{dP_T}{dP} \equiv Z_T^{-1} \exp\left\{ -c \int L(T,x)^2 \, dx \right\}.$$

The first problem is to find what the partition f^n does as $T \to \infty$, for which we look at $T \sim \exp(\lambda)$, $\theta \equiv \sqrt{2\lambda}$ and try to understand the process then. The law of $|X_T|$ is $\exp(\theta)$, and given $X_T = a$, the law of $L(T,a)$ is independent $\exp(\theta)$ also.



2) If we condition on $X_T = a > 0$, $L(T,a) = \ell$, we shall obtain that the process

$$Z_x^{(1)} \equiv L(T, a+x)$$

solves

$$\boxed{dZ_x^{(1)} = 2\sqrt{Z_x^{(1)}}\, dW_x^{(1)} - 2\theta Z_x^{(1)} \, dx, \quad Z_0^{(1)} = \ell,}$$

and the process

$$Z_x^{(2)} \equiv L(T, a-x)$$

solves

$$\boxed{dZ_x^{(2)} = 2\sqrt{Z_x^{(2)}}\, dW_x^{(2)} + 2\left(I_{[0,a]}(x) - \theta Z_x\right) dx, \quad Z_0^{(2)} = \ell.}$$

3) If we set

$$\psi_j(x) \equiv E\left[\exp\left(-c \int_0^\infty (Z_x^{(j)})^2 \, dx \right) \,\Big|\, Z_0^{(j)} = x \right] \qquad j=1,2,$$

then we find that

$$\boxed{\begin{aligned} &\psi_1'' - \theta \psi_1' - \tfrac{c}{2} x \psi_1 = 0, \qquad \psi_1(0) = 1 \geq \psi_1 \geq 0 \\ &2x\psi_2'' + 2\left(I_{[0,a]} - \theta x\right)\psi_1' - c x^2 \psi_2 = 0, \qquad 0 \leq \psi_2 \leq 1. \end{aligned}}$$

Solution??

4) It may be useful to record the joint law of $(L(T,a), L(T,0))$. When at 0, the rate of excursions getting over to a will be

$$\gamma_+ \equiv \frac{\theta e^{2\theta a}}{e^{2\theta a}-1}, \quad \equiv \frac{\theta\xi}{\xi-1} \quad \text{for short.}$$

When at a, the rate of excursions getting across to 0 will be significantly less, namely

$$\gamma_- = \frac{\theta}{e^{2\theta a}-1} \equiv \frac{\theta}{\xi-1} \equiv \gamma_+/\xi .$$

Thus if the local time at a takes value l, there will be $1+P(l\gamma_-)$ no of contributions to the local time at 0, each with $\exp(\gamma_+)$ dist^{n}. Now if we mix over the law of $L(T,a)$ (which is simply $\exp(\theta)$), we obtain

$$\boxed{E \exp\left[-\alpha L(T,0) - \beta L(T,a)\right] = \theta^2 / \left\{(\alpha+\theta)(\beta+\theta) - \alpha\beta e^{-2\theta a}\right\}.}$$

<u>Exit from a cone in $\mathbb{R}^n$</u> (26/2/92)

1) Brownian motion on S^{n-1} satisfies

$$dU = (I - UU^T)\, dX - \tfrac{n-1}{2} U\, dt,$$

from which the component $Z_t \equiv (e_1, U_t)$ satisfies

$$dZ_t = \sqrt{1-Z_t^2}\, dW_t - \tfrac{n-1}{2} Z_t\, dt .$$

Hence if we take a BM in $\mathbb{R}^n$, let $R_t \equiv |X_t|$, $U_t \equiv X_t/R_t$, $Z_t \equiv (e_1, U_t)$, we find that (R_t, Z_t) is a diffusion with generator

$$\mathcal{G} = \frac{1}{2}\frac{\partial^2}{\partial r^2} + \frac{n-1}{2r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left\{\tfrac{1}{2}(1-z^2)\frac{\partial^2}{\partial z^2} - \frac{n-1}{2} z \frac{\partial}{\partial z}\right\}$$

[The martingale parts of R, Z are uncorrelated.]

2) <u>For what circular cones in $\mathbb{R}^n$ is the mean exit time finite?</u> We take the circular cone to be defined by $C \equiv \{(r,z) : r \geq 0,\ z > \alpha\}$, which is a cone of half-angle $\theta \equiv \cos^{-1}(\alpha)$.

By scaling and symmetry, if T is the exit time

$$E^x[T] = r^2 g(z) \equiv \psi(x),$$

and so the PDE $\frac{1}{2}\Delta\psi = -1$, $\psi = 0$ on ∂C simplifies to

$$\boxed{n g(z) + \tfrac{1}{2}(1-z^2) g''(z) - \tfrac{1}{2}(n-1) z g'(z) = -1 \quad , \quad g(\alpha) = 0}$$

If $\alpha < 1/\sqrt{n}$, this is solved by

$$g(z) = \frac{nz^2 - 1}{n(n\alpha^2 - 1)} - \frac{1}{n}$$

and hence

$$E^{e_1}(T) = \frac{1}{n}\left(\frac{n-1}{(n\alpha^2-1)} - 1 \right) = \frac{1-\alpha^2}{(n\alpha^2 - 1)}.$$

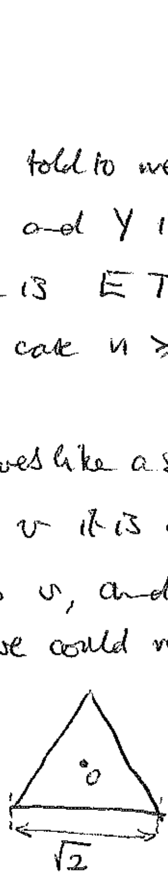
So the mean exit time from the cone is finite iff

$$\boxed{\alpha \equiv \cos\theta < n^{-\frac{1}{2}}}$$

3) The interest in this springs from a question of Dan Stroock, told to me by Ago Pistone. If $X_1, \dots, X_n$ are indept BM($\mathbb{R}$) started at 0, and Y is a BM started at 1, and $T = \inf\{u : Y_u = X_j(u) \text{ for some } j\}$, when is $ET < \infty$? The answer apparently is that $ET < \infty \Leftrightarrow n \geq 3$, and the case $n \geq 3$ is quite difficult (solved by Kesten?)

If we set $\xi_j \equiv Y - X_j$, then in $n=3$, we see that ξ moves like a standard 2dim BM in the plane $\perp v \equiv (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$, and along v it is a BM with q.v. $4\,dt$. Thus if $\rho_t \equiv v \cdot \xi(t)$ is the projection onto v, and $Z_t \equiv (I - vv^T)\xi(t)$ is the projection onto the plane $\perp v$, we could rephrase the problem as the question

"If Z is BM($\mathbb{C}$), $Z_0 = 0$, and B is indept BM, $B_0 = 1$, is the mean exit time from the equilateral triangle for Z_t / B_{4t} finite or infinite?"



Another way to view this would be to transform $\mathbb{R}^3$ by $x \mapsto (I - vv^T)x + \frac{1}{2} vv^T x$ which converts ξ into standard BM, but opens out the orthant into a "wider" convex cone. The orthant is contained in a cone of half angle $\cos^{-1}(1/\sqrt{3})$, so might just be handled by comparison with the conical problem, but the "wider" orthant is not, so the attempted comparison appears useless.

A neat inequality of Jean-Dominique Deuschel (27/2/92)

Suppose that $X \sim N(0, G)$, $X \in \mathbb{R}^n$, G is nnd symmetric and suppose that Φ is a Lipschitz function on $\mathbb{R}^n$. Then

$$\boxed{E \exp \Phi(X) \leq \exp\Big[E\Phi(X) + \sup_x \tfrac{1}{2} \sum_i \sum_j D_i\Phi(x)\, G_{ij}\, D_j\Phi(x) \Big]}$$

Proof If $G^{\frac{1}{2}} = A$, we write $X = AB_1$, where B is standard $BM(\mathbb{R}^n)$, and set $\Psi(B_1) \equiv \Phi(AB_1) = \Phi(X)$. Now by the integral representation,

$$\boxed{\Psi(B_1) = E\Psi(B_1) + \int_0^1 \nabla P_{1-t}\Psi(B_t)\, dB_t}$$

and so

$$1 = E\exp\Big\{ \int_0^1 \nabla P_{1-t}\Psi(B_t)\, dB_t - \tfrac{1}{2}\int_0^1 |\nabla P_{1-t}\Psi(B_t)|^2 dt \Big\}$$

$$= E\exp\Big[\Psi(B_1) - E\Psi(B_1) - \tfrac{1}{2}\int_0^1 \sum_j |P_{1-t} D_j \Psi(B_t)|^2 dt \Big]$$

$$\geq E \exp\Big[\Psi(B_1) - E\Psi(B_1) - \tfrac{1}{2}\int_0^1 P_{1-t} |\nabla\Psi|^2 (B_t)\, dt \Big]$$

$$\geq E \exp\Big[\Psi(B_1) - E\Psi(B_1) - \tfrac{1}{2}\sup_x |\nabla\Psi(x)|^2 \Big].$$

Using now $\nabla\Psi = A^T \nabla\Phi = A\nabla\Phi$, rearrangement now yields the inequality. [The article by Ledoux, SP XXII, 249–259, uses this integral representation idea extensively to prove estimates. MY gave me this reference.]

Observation on the problem posed by Paul Embrechts (7/3/92) (See last page of this book, no. 1) Transforming gives

$$\int_0^\infty dx\, \alpha e^{-\alpha x} \int_0^x \frac{\mu(dy)}{\mu(x-y)} = \int_0^\infty e^{-\alpha y}\mu(dy) \int_0^\infty \alpha e^{-\alpha x} \frac{dx}{\mu(x)} = c$$

so that if ν is Lévy measure, $\bar\nu(x) = 1/\mu(x)$, we have, with $\psi(\alpha) = \int_0^\infty (1-e^{-\alpha x})\nu(dx)$

$$\int_0^\infty e^{-\alpha y} \frac{\mu(dy)}{c} = \frac{1}{\psi(\alpha)} = E^0 \int_0^\infty e^{-\alpha X_t}\, dt$$

so that $\boxed{\frac{\mu}{c} \text{ is Green } f^n \text{ for subordinator.}}$

Harmonic functions for (A_t, B_t) (17/3/92)

(i) Lets define
$$A_t \equiv \int_0^t I_{\{B_u > 0\}}\, du,$$

where B is a $BM(\mathbb{R})$, and set $X_t \equiv (A_t, B_t)$, a Markov process in $S \equiv \mathbb{R}^+ \times \mathbb{R}$. Notice that while $B_t < 0$, A_t is not changing. Erwin asks what are the harmonic functions h for X, that is

(1) $\quad 0 \leq h(t,x) = E^{(t,x)} h(X_u) \qquad \forall\ (t,x) \in S, \ \forall u \geq 0.$

(ii) Suppose that h satisfies (1). Since $h(X_t)$ is always a martingale, we have

(2) $\quad \infty > h(t,x) \geq E^{(t,x)} h(H_0, 0) = \displaystyle\int_0^\infty \frac{x e^{-x^2/2u}}{\sqrt{2\pi u^3}}\, h(t+u, 0)$

for all $x > 0$, $t \geq 0$. A necessary condition for this $E^{(t,x)} h(H_0, 0) < \infty$ is that

(3) $$\boxed{\int_0^\infty \frac{e^{-\varepsilon/2u}}{u^{3/2}}\, h(u,0)\, du < \infty \qquad \forall \varepsilon > 0}$$

and it's not hard to see that this is also sufficient for (2).

(iii) Notice that since A does not change while $B < 0$, we have to have

(4) $$\boxed{h(t,x) = h(t,0) - \rho(t) x \qquad \forall x \leq 0, \ t \geq 0.}$$

(iv) Now let's define
$$h_*(t,x) \equiv \downarrow \lim_{n \to \infty} E^{(t,x)} h(A(H_{-n}), -n) \leq h(t,x).$$

I claim that this is a harmonic function for X. Indeed, for $n \in \mathbb{N}$,
$$E^{(t,x)} h_*(X_u) = E^{(t,x)}\left[h_*(X_u) : u < H_{-n}\right]$$
$$+ E^{(t,x)}\left[h_*(X_u) : H_{-n} \leq u\right].$$

Now the second term is dominated by $E^{(t,x)}\left[h(X_u) : H_{-n} \leq u\right] \to 0 \quad (n \to \infty)$, and the first is handled by the obvious estimation

$$E^{(t,x)}\left[h_*(X_u) \;:\; u < H_{-n} \right]$$

$$= E^{(t,x)}\left[\downarrow \lim_{n \le k \to \infty} E^{(t+A_u, B_u)}\left[h(A(H_{-k}), -k) \right] \;:\; u < H_{-n} \right]$$

$$= \downarrow \lim_{n \le k \to \infty} E^{(t,x)}\left[E^{(t+A_u, B_u)}\left[h(A(H_{-k}), -k) \right] \;:\; u < H_{-n} \right]$$

$$= \downarrow \lim_{k} E^{(t,x)}\left[h(A(H_{-k}), -k) \;:\; u < H_{-n} \right]$$

$$= h_*(t,x) - \lim_{k} \underbrace{E^{(t,x)}\left[h(A(H_{-k}), -k) \;:\; H_{-n} \le u \right]}$$

$$\le E^{(t,x)}\left[h(A(H_{-n}), -n) \;:\; H_{-n} \le u \right]$$

$$= E^{(t,x)}\left[h(A_u, B_u) \;:\; H_{-n} \le u \right] \to 0 \quad \text{as } n \to \infty.$$

(v) Now if we consider what h_* looks like when $x<0$, we have

$$E^{(t,x)}\left\{ h(A(H_{-n}), -n) \right\}$$

$$= \frac{|x|}{n} \left\{ h(t,0) + n\rho(t) \right\} + \left(1 - \frac{|x|}{n}\right) E^{(t,0)}\left[h(A(H_{-n}), -n) \right]$$

$$\to -x\rho(t) + h_*(t,0) \qquad \text{as } n \to \infty$$

Thus we have

$$\boxed{h_*(t,x) = h_*(t,0) - x\rho(t) \quad \forall\, x \le 0,\ t \ge 0.}$$

The conclusion is that

$h - h_* = \tilde{h} \ge 0$ <u>is harmonic,</u> <u>and</u> $\tilde{h}(t,x) = \tilde{h}(t,0) \;\; \forall\, x \le 0.$

Thus $\boxed{\tilde{h}(A_t, B_t) = \tilde{h}(A_t, B_t^+) \text{ is a martingale.}}$

Obviously we would now like to time-change by A_t, but <u>can</u> we . . . ?

(vi) One example we need to understand is

$$h(t,x) \equiv I_{(t<t_0)} \, (t_0-t)^{-\frac{1}{2}} \exp\{-(x^+)^2/2(t_0-t)\}.$$

This cannot have the property that $h(t, |B_t|)$ is a martingale, but could it be that $h(A_t, B_t)$ is a martingale? We have

$$E^{(s,x)} \, h(A(H_0), 0) = h(s,x) \qquad \forall (s,x),$$

but do we have $\boxed{E^{(0,0)} \, h(A_t, B_t) = h(0,0) \quad \forall t>0?}$ One way to check this is to write

$$h(t_0; t, x) \equiv I_{(t<t_0)} (2\pi(t_0-t))^{-\frac{1}{2}} \exp\left[-(x^+)^2/2(t_0-t)\right]$$

and then mix over $t_0 \sim \exp(\lambda)$. If the conjecture is true, we should have $\forall t>0$

$$E^{(0,0)}\left[\frac{\theta}{2} \exp(-\lambda A_t - \theta B_t^+)\right] = \frac{\theta}{2}$$

and this is clearly <u>not</u> true.

(vii) While trying to prove this by clumsy methods, I did some excursion calculations the results of which are worth recording.

Rate of α-marked excns which contain no λ-mark $\equiv \rho(\alpha,\lambda)$

$$= \frac{1}{2}\left[\sqrt{2\alpha+2\lambda} - \sqrt{2\lambda}\right]$$

Rate of λ-marked excursions containing no γ-mark before the λ-mark

$$= \sqrt{2\lambda}\left(\frac{\lambda}{\lambda+\gamma}\right)^{1/2}$$

Hence if $g_T \equiv \sup\{u<t : B_u = 0\}$, we shall find with $A_t^+ \equiv A_t$, $A_t^- \equiv t - A_t^+$,

$$E \exp\left[-\alpha A^+_{g_T} - \beta A^-_{g_T} - \gamma(T-g_T)\right] = \sqrt{2\lambda}\left(\frac{\lambda}{\lambda+\gamma}\right)^{\frac{1}{2}} \left\{\sqrt{2\lambda} + \rho(\alpha,\lambda) + \rho(\beta,\lambda)\right\}^{-1}$$

$$= 2\theta \sqrt{\frac{\lambda}{\lambda+\gamma}} \left\{\sqrt{2\alpha+2\lambda} + \sqrt{2\beta+2\lambda}\right\}^{-1}.$$

It is now quite easy to confirm that if $U \sim U[0,1]$ indept of B, T,

then
$$E \exp\{-\alpha(1-U)g_T - \beta U g_T\} = E\exp\{-\alpha A^+(g_T) - \beta A^-(g_T)\}$$

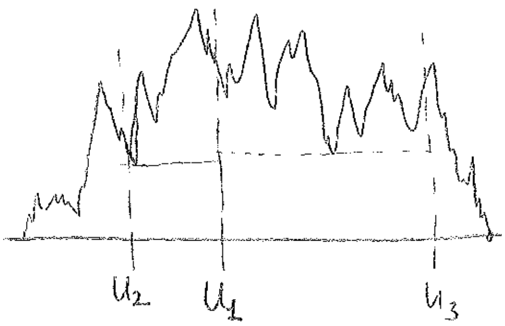
so that $\boxed{A^+(g_t)/g_t \text{ is } U[0,1] \text{ for any } t>0,}$ which I knew before...

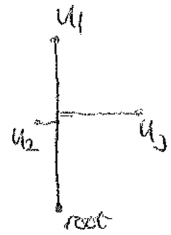
Another thing is to take $\sigma_t = \inf\{u>t : B_u = 0\}$, and then observe that the law of a λ-marked excursion lifetime has LT
$$\frac{1}{\sqrt{2\lambda}}\int_0^\infty (1-e^{-\lambda t})\frac{dt}{\sqrt{2\pi t^3}}\, e^{-\alpha t} = \frac{\sqrt{2\lambda+2\alpha} - \sqrt{2\alpha}}{\sqrt{2\lambda}}$$

so that
$$\boxed{E\exp\{-\alpha A^+(\sigma_T)\} = \frac{1}{2}\,\frac{\sqrt{2\lambda+2\alpha} - \sqrt{2\alpha} + \sqrt{2\lambda}}{\sqrt{2\lambda+2\alpha}}}$$

<u>A question arising from a talk of David Aldous</u> (10/3/92)

(i) Dave gives many recipes for a continuum random tree, one of which is to take $U_1, U_2, \dots$ i.i.d. $U(0,1)$ indept of a scaled Brownian excursion ξ, and then to draw the tree





by taking a line from "root" to "U_1" of length $\xi(U_1)$, then take a branch of length $\xi(U_2) - \min_{[U_2,U_1]}\xi$ sticking out at height $\min_{[U_2,U_1]}\xi$, then of length $\xi(U_3) - \min_{[U_1,U_3]}(\xi)$ at height $\min_{[U_1,U_3]}(\xi)$ in the diagram.

Another description is that we take points of a Poisson pr rate $r(t) = t$ on $\mathbb{R}^+$, and cut $\mathbb{R}^+$ at these points. Each new piece is attached at a randomly-chosen point of the tree so far constructed. These two recipes give the same distn.

From this, we conclude that if we took a BM started at 0 and run until H_a $(a>0)$, take U indept $U[0,1]$, then $S(UH_a) \sim U[0,a]$, where S is the

Supremum process.

(ii) Here is a direct proof of this fact. First we compute

$$E\left[\int_0^{H_a} e^{-\alpha S_t}\,dt\; e^{-\gamma H_a}\right]$$

$$= \int_0^\infty dt\; E\left[e^{-\alpha S_t - \gamma H_a} : S_t < a\right]$$

$$= \int_0^\infty dt \int_0^a dy \int_0^\infty dv\; \frac{2(y+v)\,e^{-(y^2+v^2)/2t}}{\sqrt{2\pi t^3}}\; e^{-\alpha y - \gamma t - \theta(a-y+v)}$$

$$(\theta = \sqrt{2\gamma})$$

$$= \int_0^a dy \int_0^\infty dv\; 2\,e^{-\alpha y - \theta(y+v) - \theta(a-y+v)}$$

$$= 2e^{-\theta a}\int_0^a dy \int_0^\infty dv\; e^{-\alpha y - 2\theta v}$$

$$= \frac{e^{-\theta a}}{\theta}\;\frac{1-e^{-\alpha a}}{\alpha}.$$

Now integrate wrto γ on both sides to get

$$E\left[\int_0^{H_a} e^{-\alpha S_t}\,\frac{dt}{H_a}\right] = \frac{1-e^{-\alpha a}}{\alpha a}.$$

12) Von Weizsäcker (Kaiserslautern) tells a question of Harry Reuter. If $f: \mathbb{R}^+ \to \mathbb{R}^+$ is meas, then $\{\delta>0 : \sum_{n\geq 1} f(n\delta) < \infty\} \equiv C$ is either Lebesgue null, or C^c is Lebesgue null.

He remarks that if $f \in L^1$, then $\mathrm{Leb}(\mathbb{R}^+ \setminus C) = 0$, so could always replace given f by bounded f (trivially), then the bdd f by C^∞ bdd f which is close in L^1. Hence can assume $f \in C^\infty$ if it helps.

Could one now use Fourier techniques?

6). From talking to Hanspeter Schmidli, the following question arises. When can one decide from the convergence (weak) of Markov processes whether the "asymptotics converge" in some sense...

7) The old chestnut; if $\varphi(B) = |B| - L$, then what can we say about $\bigcap_n \sigma(\varphi^n(B))$?

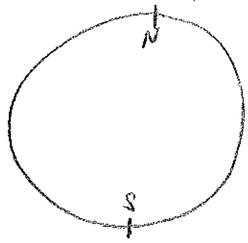
8) Is there some large deviations story for stopped BM?

9) The Dirac polaron is what you get when you reweight Wiener measure by

$$Z_T^{-1} \exp\{ + T^{\alpha} \int L_T(x)^2 \, dx \}$$

where the power α is chosen correctly ($\alpha \approx -1$?). EB+JDD are very keen to know what the limiting path measure is, and Agi Pisztora is working on some of this.

10) Götz Kersting asks the following: if $L = \frac{1}{2} a_{ij} D_i D_j + b_i D_i$, with a, b smooth on nbd of closed unit disc, a uniformly elliptic, and if f is the solution to Dirichlet problem in disc $D \equiv \{z : |z|<1\}$
with a bdy function with unique max at N,
min at S, monotone in between,
can one prove ∇f non-vanishing?



11) Michael Scheutzow asks: if (X_n) is a mg difference sequence, and the X_n are identical in law, does $\frac{1}{n}\sum_{1}^{n} X_j \to 0$ a.s.?

Next, if $\{Y_{ij} : 1 \le i < j \le n\}$ are i.i.d. mean zero, and $Y_{ij} = -Y_{ji}$ for $1 \le j < i \le n$, $Y_{ii} = 0$, and if the law of Y_{ij} is symmetric, and

$$\xi_j = \sum_{k=1}^{n} Y_{jk}$$

is the limit behaviour of $\max_{1 \le j \le n} \xi_j$ the same as if the Y_{ij} were all independent with the same law?

Book on large deviations by James A. Bucklew is supposed to be quite a good account.

Interesting questions.

1) Paul Embrechts asks the following. If μ is a (σ-finite) measure on $\mathbb{R}^+$ such that

$$\int_0^x \frac{\mu(dy)}{\mu(x-y)} = c \quad \forall x > 0,$$

then is $\mu(dy) = c' \cdot y^\alpha \, dy$ for some α? Certainly if μ is a solution, then so also is $\mu(ay) \equiv \mu_a(y)$.

2) The old conjecture of Terry Lyons that the exchangeable σ-field of a transient Markov chain is generated by $(L_j)_{j \in I}$, $L_j = \sum_{n \geq 0} I(X_n = j)$.

3) Erwin Bolthausen has looked at a random environment problem of the following form. Take i.i.d. $X(n,x)$, $n \in \mathbb{N}$, $x \in \mathbb{Z}^d$, $X(n,x) > 0$, $EX(n,x) = 1$, and for each $T \in \mathbb{N}$, define the weight of the r.w. path $(\omega_1, \dots, \omega_T)$ started at 0 as

$$W_T(\omega) = \prod_{j=1}^{T} X(j, \omega_j).$$

Now set $V_T \equiv (2d)^{-T} \sum_{\omega \in \Omega_T} W_T(\omega)$, and pick a path $\omega \in \Omega_T$ with prob^y $W_T(\omega)/V_T$. What can one say about the asymptotic behaviour of $|\omega_T|$? [In $d \geq 3$, V_T is a mg on the filtration of the environment $\mathbb{X}$, and is UI, at least when the variance of X is small enough. If $X = \exp(\beta Y)/E\exp\beta Y$, then the guess is that there is a "phase transition" for some critical value of β, $\beta_c(d)$; in dimensions 1+2, it seems that little is yet understood.]

4) The example reported on p 47 begs the question whether there might not be some "reflection" proof of the case $n = 3$, like the Itô McKean proof of the transition density for n Brownian particles, conditioned on no collisions...

5) Another (impossible?) question from Jean-Dominique Deuschel; if $\varphi \in C_K$, $\int_{-\infty}^{\infty} \varphi(x)dx = 0$, and

$$v_T(t,x) \equiv E^x \exp T^\alpha \int_0^t \varphi(B_{Tu})\,du, \quad \Lambda_\alpha(T) \equiv T^{-\alpha} \int (v(1,x) - 1)\,dx,$$

for what α if any does $\lim_{T\to\infty} \Lambda_\alpha(T)$ exist in $(0,\infty)$?