L.C.G. ROGERS.

Sudden mixing of diffusions? (19/3/92)

Peter Donnelly asks about the possibility of the "sudden mixing" of a diffusion like the sudden mixings which occur in various combinatorial Markov chains. As an example, let's take BM on the circle, to be thought of as $[0,1]$ with periodic boundary conditions. With

$$s_n(x) = \sqrt{2}\sin 2n\pi x \ , \ c_n(x) = \sqrt{2}\cos 2n\pi x \qquad (n\geq 1) \ , \ c_0(x) = 1,$$

we have a c.o.n.s. in $L^2[0,1]$, and can express the transition density as

$$p_t(x,y) = 1 + \sum_{n\geq 1} e^{-2n^2\pi^2 t}\{s_n(x)s_n(y) + c_n(x)c_n(y)\}$$

$$= 1 + \sum_{n\geq 1} e^{-2n^2\pi^2 t}\, 2\cos 2n\pi(x-y).$$

Thus

$$\int_0^1 |p_t(x,y) - 1|^2\,dy = 2\sum_{n\geq 1} e^{-4n^2\pi^2 t}$$

which drops to 0 as $t\to\infty$, quite rapidly. Indeed, this is actually <u>completely monotone</u> as a function of t, so cannot look like

So if this behaviour happens for this example, it must be with respect to some other notion of "distance".

Notice also that

$$\sup_y |p_t(x,y) - 1| = \sup_y \Big|2\sum_{n\geq 1} e^{-2n^2\pi^2 t}\cos 2n\pi y\Big|$$

$$= 2\sum_{n\geq 1} e^{-2n^2\pi^2 t},$$

so the sup-norm can't be the appropriate thing either!

Obtaining the law of X/Y from the law of (X,Y) (19/3/92)

If we are told

$$\varphi(\alpha,\beta) \equiv E\exp(-\alpha X-\beta Y)$$

where X, Y are positive random variables, how could we find

$$\psi(\lambda) \equiv E\exp(-\lambda X/Y) \ ?$$

Observe that

$$\int_0^\infty e^{-t\lambda}\psi(\lambda)\,d\lambda = E\,(t+X/Y)^{-1} = E\left(\frac{Y}{X+tY}\right)$$

and

$$-\frac{\partial\varphi}{\partial\beta}(\alpha,\beta) = E\left[Y e^{-\alpha X-\beta Y}\right],$$

so that

$$\boxed{\int_0^\infty e^{-t\lambda}\psi(\lambda)\,d\lambda = -\int_0^\infty d\alpha\ \frac{\partial\varphi}{\partial\beta}(\alpha,t\alpha).}$$

So we can get the LT of the LT of (X/Y) quite easily! This is certainly good enough to get moments of X/Y.

A treacherous pitfall in stable processes! (25/3/92)

(i) Discussion with Anton Thalmeier led me to the following. Suppose that μ is the Lévy measure on $\mathbb{R}^+\times\mathbb{R}^+$ of a stable (2-dimensional) subordinator;

$$\mu(cA) = c^{-\beta}\mu(A), \qquad A\in\mathcal{B}(\mathbb{R}^+\times\mathbb{R}^+)$$

for some $\beta>0$. Assuming that μ has a continuous density g, this can be expressed in terms of g as

$$g(cx,cy) = c^{-2-\beta}g(x,y)$$

whence

$$g(x,y) = x^{-2-\beta}g(1,y/x)$$

$$= y^{-2-\beta}g(x/y,1).$$

Now it is easy to calculate that

$$\mu\left[Y/X \in da \mid X=x \right] = g(1,a)\,da \Big/ \int_0^\infty g(1,y)\,dy$$

is the same for all $x>0$, and likewise

$$\mu\left[X/Y \in da \mid Y=y \right] = g(a,1)\,da \Big/ \int_0^\infty g(x,1)\,dx,$$

same for all y. <u>Thus</u> Y/X <u>is independent of</u> X, <u>and of</u> Y and yet

$$\mu\left[Y/X \in da \mid Y=y \right] = a^\beta g(1,a)\,da \Big/ \int_0^\infty g(x,1)\,dx$$

$$\neq \mu\left[Y/X \in da \mid X=x \right] \quad !!$$

I tried to apply this to the case where X is the lifetime of a Brownian excursion, Y is its squared maximum (so that $\beta = \frac{1}{2}$ here), and said that we know the law F of X given $Y=1$, so that $\check{F}$ should be the law of Y given $X=1$, where $\check{F}((0,a)) = F((a^{-1},\infty))$. This is <u>wrong</u>!

(ii) Notice that this similar treachery could not occur in the case of a probability measure - it's the fact that μ is σ-finite that screws it. If $X, Y, \ Z \equiv Y/X$ are r.v.s such that

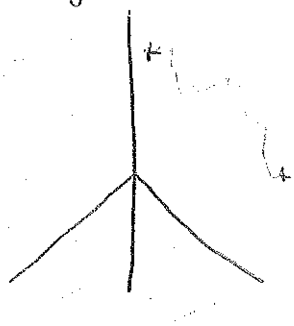
$$\begin{cases} X \text{ and } Z \text{ are independent} \\ Y \text{ and } Z \text{ are independent} \end{cases}$$

then the law of Z given $X=x$ <u>is</u> the unconditioned law of Z.

<u>Cats + dogs example - a dead end</u> (26/3/92) [see p47 of previous book]

I wondered whether there might be some simple expression for the law of $(X_1(t), \ldots, X_n(t), Y(t))$ under the constraint that by time t Y hasn't hit any X_j. This looks unlikely if we consider the case $n=2$, for then we have a $BM(\mathbb{R}^3)$ operating in the region $\{(x_1,x_2,x_3) : x_1 \geq x_2, x_3\} \equiv C$. Now C is the intersection of two halfspaces, perpendicular to $(\frac{1}{\sqrt2}, -\frac{1}{\sqrt2}, 0)$

and to $\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$, respectively. When we project perpendicular to $(1,1,1)$, which is the intersection line of the two half-spaces, we see a BM in a 120° cone in the plane



– can one now give a recipe for the transition density under the restriction of having stayed in the cone? Obvious reflection tricks fail because density vanishes along a line down the middle of the wedge.

<u>Resolvent for (A_t, B_t)</u> (27/3/92).

(i) If T is $\exp(\lambda)$ indept of B, and $(A_0, B_0) = (a, x)$, what is the law of (A_T, B_T)?

Define

$$\varphi(x) \equiv E^x\left[e^{i\theta B_T - \gamma A_T}\right] = E^x\left[\int_0^\infty \lambda e^{-\lambda t + i\theta B_t - \gamma A_t}\, dt\right] \equiv E^x Y.$$

Then

$$M_t \equiv E(Y \mid \mathcal{G}_t) = \int_0^t \lambda e^{-\lambda u + i\theta B_u - \gamma A_u}\, du + e^{-\lambda t - \gamma A_t}\, \varphi(B_t)$$

is a martingale. Define $\boxed{\alpha \equiv \sqrt{2\lambda},\ \beta \equiv \sqrt{2\lambda + 2\gamma}}$ so that, decomposing at H_0, we get

$$\left.\begin{aligned} &\text{for } x \geq 0, \quad \varphi(x) = e^{-\beta x}\varphi(0) + \left(e^{i\theta x} - e^{-\beta x}\right)\frac{\lambda}{\lambda + \gamma + \frac{1}{2}\theta^2} \\ &\text{for } x \leq 0, \quad \varphi(x) = e^{\alpha x}\varphi(0) + \left(e^{i\theta x} - e^{\alpha x}\right)\frac{\lambda}{\lambda + \frac{1}{2}\theta^2} \end{aligned}\right\}$$

It's necessary that φ should be C^1 at zero, so this implies

$$\boxed{\varphi(0) = \frac{\alpha^2}{\alpha + \beta}\left\{\frac{1}{\beta - i\theta} + \frac{1}{\alpha + i\theta}\right\}}.$$

The Fourier transform is easy to undo :

$$E^0\left[e^{-\gamma A_T} : B_T \in dx\right]/dx = \frac{\alpha^2}{\alpha+\beta}\left[e^{-\beta x} I_{(x>0)} + e^{\alpha x} I_{(x<0)}\right].$$

The Laplace transform is not, though ; if $g_t \equiv \sup(u<t : B_u = 0)$, then

$$E\, e^{-\gamma A(g_T)} = \int_0^1 du\, E\, e^{-\gamma u g_T}$$

$$= \int_0^1 du \int_0^1 \frac{ds}{\pi\sqrt{s(1-s)}} \int_0^\infty \lambda e^{-\lambda t}\, e^{-\gamma u s t}\, dt$$

$$= \int_0^1 du \int_0^1 \frac{ds}{\pi\sqrt{s(1-s)}} \int_0^\infty \frac{dv}{us}\, \lambda e^{-\lambda v/us}\, e^{-\gamma v}.$$

This is going to be too complicated to be useable, especially when we start away from 0.

(ii) Let's just record

$$P^0\left[g_t \in ds,\ B_t \in da\right]/ds\,da = \frac{a\, e^{-a^2/2(t-s)}}{\sqrt{2\pi(t-s)^3}} \cdot \frac{1}{\sqrt{2\pi s}} \qquad (a>0)$$

$$\equiv \frac{a}{t-s}\, p_{t-s}(a)\, \frac{1}{\sqrt{2\pi s}} = \frac{a}{t-s}\, p_{t-s}(a)\, p_s(0)$$

(iii) Take $x \geq 0$ and compute for $y \geq 0$, $v < t$

$$P^x\left[B_t \in dy,\ A_t \leq v\right] = \int_0^v \frac{x e^{-x^2/2u}}{\sqrt{2\pi u^3}}\, du\; P^0\left[B_{t-u} \in dy,\ A_{t-u} \leq v\right]\, dy$$

$$= \int_0^v \underbrace{q(x,u)}_{\text{first passage density}}\, du \int_{t-v}^{t-u} ds\; q(y, t-u-s)\, \frac{1}{\sqrt{2\pi s}} \cdot \frac{v-(t-s)}{s} \cdot dy$$

from which

$$\boxed{P^x\left[B_t \in dy,\ A_t \in dv\right]/dy\,dv = \int_0^v q(x,u)\, du \int_{t-v}^{t-u} ds\; q(y, t-u-s)\, \frac{1}{\sqrt{2\pi s^3}}}$$

This gives us easily

$$P^x(B_t\in dy,\ A_t\in dv)/dy\,dv = \int_0^v q(x,u)\,du \int_0^{v-u} ds\ q(y,s)\ \left(2\pi(t-u-s)^3\right)^{-1/2}.$$

Now always $t-v \le t-u-s \le t$, so if h is the harmonic function,

$$E^x\left[h(A_t,B_t) : \begin{array}{l} A_t \le c, \\ B_t \ge 0 \end{array}\right] = \int_0^\infty dy \int_0^c dv\ h(y,c) \int_0^v q(x,u)\,du \int_0^{v-u} \frac{ds}{\sqrt{2\pi}\ (t-u-s)^{3/2}}\, q(y,s)$$

$$\ge \int_0^\infty dy \int_0^c dv \int_0^v du \int_0^{v-u} ds\ \ h(y,c)\,q(x,u)\,q(y,s)\ /\sqrt{2\pi t^3}$$

and also $\le$ .... $/\sqrt{2\pi(t-c)^3}$

So from the first inequality we learn that the integral is finite, from the second we learn that the expectation goes to 0 as $t\to\infty$.

(iv) Taking the starting point $x=-a<0$, we now get

$$P^x\left[B_t\in dy,\ A_t\in dv\right]/dy\,dv = \int_0^t q(a,w)\,dw\ P^0\left[B_{t-w}\in dy,\ A_{t-w}\in dv\right]/dy\,dv$$

and $E^x\left[h(A_t,B_t) : H_0 > \tfrac12 t\right] \to 0$, so

$$E^x\left[\,h(A_t,B_t) : B_t\ge 0,\ A_t\le c,\ H_0 \le \tfrac12 t\right]$$

$$= \int_0^{t/2} q(a,w)\,dw\ E^0\left[h(A_{t-w},B_{t-w}) : B_{t-w}\ge 0,\ A_{t-w}\le c\right]$$

$$\to 0 \qquad (t\uparrow\infty),$$

since always $t-w \ge t/2$. Thus we have now taken care of the case $B_t\ge 0$.

(v) For $x\ge 0 > y$, we have for $0\le v\le t$

$$P^x\left[B_t \in dy,\ A_t \le v\right]/dy = \int_0^v q(x,u)\,du \int_0^{t-u} ds\ q(|y|, t-u-s)\ \frac{1}{\sqrt{2\pi s}} \cdot \left[\frac{v-u}{s} \wedge 1\right]$$

so that

$$P^x\left[B_t \in dy,\ A_t \in dv\right]/dy\,dv = \int_0^v q(x,u)\,du \int_{v-u}^{t-u} ds\ q(|y|, t-u-s)\ \frac{1}{\sqrt{2\pi s^3}}$$

$$= \int_0^v q(x,u)\,du \int_0^{t-v} ds\ q(|y|, s)\ \frac{1}{\sqrt{2\pi\,(t-u-s)^3}}$$

and

$$P^x\left[B_t < 0,\ A_t \in dv\right]/dv = \int_0^v q(x,u)\,\frac{du}{2\pi}\int_0^{t-v}\left(s(t-u-s)^3\right)^{-1/2} ds$$

$$= \int_0^v q(x,u)\,\frac{du}{2\pi}\cdot\frac{2}{t-u}\sqrt{\frac{t-v}{v-u}}$$

from which

$$E^x\left[h(A_t, B_t) : B_t \le 0,\ A_t \le c\right] = \int_0^c dv \int_0^v q(x,u)\ \frac{du}{\pi(t-u)}\sqrt{\frac{t-v}{v-u}}\ h(v,0).$$

Since $t-c \le t-u \le t$, we conclude as before that this goes to 0 as $t \to \infty$. The case of $x<0$ is deduced in a similar manner from this result, as we had in (iv) above (except with negligible probability, $H_0 \le t/2$, ...)

(vi) If we now go back to the situation at the foot of p 50 in the previous book, we see that we really can time change by the inverse to A_t:

$$\tilde{h}(0,x) = E^{(0,x)}\left[\tilde{h}(A(t\wedge\tau_c),\ B(t\wedge\tau_c))\right]$$

$$= E^{(0,x)}\left[\ \tilde{h}(c,\ B(t\wedge\tau_c))\ ;\ \tau_c \le t\right] + E^{(0,x)}\left[\tilde{h}(A_t, B_t) : \tau_c > t\right]$$

and we have just proved that this last piece goes to zero.

Thus $\tilde{h}$ is just a space-time harmonic function for Brownian motion, and as such is well understood.

(vii) It now remains to understand the harmonic minorant h_x.

We have for any $a>0$ that, starting from $(0,0)$,

$$P\left[A(H_{-a})\in du\right]/du = \int_0^\infty \frac{dx}{a}\, e^{-x/a}\, \frac{x e^{-x^2/2u}}{\sqrt{2\pi u^3}} \equiv \varphi_a(u).$$

This gives us

$$\infty > h(0,0) \geq h_*(0,0) = \lim_{a\to\infty} E^{(0,0)}\left[\, h(A(H_{-a}), -a)\,\right]$$

$$= \lim_a \int_0^\infty \varphi_a(u)\left\{ h(u,0) + a\rho(u)\right\} du.$$

Hence

$$h(0,0) \geq \lim_a \int_0^\infty a\varphi_a(u)\,\rho(u)\, du$$

$$= \lim_a \int_0^\infty \rho(u)\left\{\int_0^\infty e^{-x/a}\, dx.\, \frac{x e^{-x^2/2u}}{\sqrt{2\pi u^3}}\right\} du$$

$$= \int_0^\infty \frac{\rho(u)}{\sqrt{2\pi u}}\, du, \qquad \text{an increasing limit.}$$

Similarly, starting from $(t,0)$, we conclude that

$$\boxed{\int_0^\infty \rho(t+u)\,\frac{du}{\sqrt{2\pi u}} \leq h(t,0) < \infty \quad \text{for all } t\geq 0.}$$

Electrical flow in networks when the network changes (2/4/92)

(i) We shall consider electricity flow in a finite connected network, with two distinct nodes O (the sink) and N (the source). The conductivity of edge (i,j) is q_{ij}. The flow from i to j will be denoted φ_{ij}. If we put $1A$ into the network at N and take it out at O, the potential of node j will be denoted by V_j, so that

$$\varphi_{ij} = (V_i - V_j)\, q_{ij}$$

and Kirchoff's laws say

$$\sum_{j\neq i} q_{ij}(V_i - V_j) = 0 \quad, i\neq 0, N; \qquad \sum_{j\neq N}(V_N - V_j)\, q_{ij} = 1.$$

There's a way to describe this in ~~terms~~ of a Markov chain. If $q_{ii} \equiv -\sum_{j \neq i} q_{ij}$, then we let Q be the matrix $(q_{ij})_{i,j \neq 0}$. This will be the generator of a substochastic Mkv chain, which dies out eventually (= hits 0 eventually). Then we have

$$V = (-Q)^{-1} e_N$$

where $e_N(j) \equiv \delta_{jN}$.

(ii) Now what will happen if we change the network slightly, by changing the conductivity of edge (i,j) to $q_{ij} + w$? If we partition Q

$$Q = \begin{array}{cc} \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) & \begin{array}{l} \{i,j\} \\ \{i,j\}^c \end{array} \end{array}$$

we have $C = B^T$, $A = A^T$, $D = D^T$ and we can write $A = \begin{pmatrix} +q_{ii} & q_{ij} \\ q_{ji} & q_{jj} \end{pmatrix}$. Now

$$-Q^{-1} = \begin{pmatrix} -(A - BD^{-1}C)^{-1} & (A - BD^{-1}C)^{-1} BD^{-1} \\ (D - CA^{-1}B)^{-1} CA^{-1} & -(D - CA^{-1}B)^{-1} \end{pmatrix}$$

and we have

$$\begin{pmatrix} V_i \\ V_j \end{pmatrix} = (A - BD^{-1}C)^{-1} BD^{-1} e_N .$$

We write $BD^{-1} e_N = -(\rho_i \ \rho_j)^T$, with $\rho_i \geq 0$, $\rho_j \geq 0$. Now the only way the expression for (V_i, V_j) depends on q_{ij} is through A; <u>before</u> we change the conductivity of (i,j), we have

$$A - BD^{-1}C \equiv \begin{pmatrix} -\alpha & \gamma \\ \gamma & -\beta \end{pmatrix}; \quad \underline{\text{after}}, \text{ we have } A_w - BD^{-1}C \equiv \begin{pmatrix} -\alpha - w & \gamma + w \\ \gamma + w & -\beta - w \end{pmatrix}$$

$$\equiv \begin{pmatrix} -\alpha' & \gamma' \\ \gamma' & -\beta' \end{pmatrix}.$$

Thus

$$-(A - BD^{-1}C)^{-1} = \frac{1}{\Delta} \begin{pmatrix} \beta & \gamma \\ \gamma & \alpha \end{pmatrix}, \quad \text{with } \Delta \equiv \alpha\beta - \gamma^2 > 0.$$

We abbreviate $-(A - BD^{-1}C)^{-1} \equiv G$, $-(A_w - BD^{-1}C)^{-1} \equiv G_w$, and have

$$\begin{pmatrix} V_i \\ V_j \end{pmatrix} = G \begin{pmatrix} \rho_i \\ \rho_j \end{pmatrix}, \qquad G_w - G = \frac{w}{\Delta \Delta_w} \begin{pmatrix} \gamma - \beta \\ \alpha - \gamma \end{pmatrix} (\beta - \gamma,\ \gamma - \alpha).$$

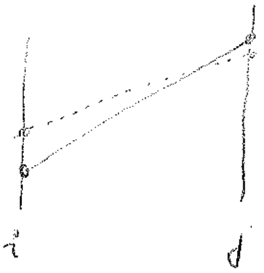
Thus we obtain

$$\boxed{\begin{pmatrix} V_w(i) - V(i) \\ V_w(j) - V(j) \end{pmatrix} = \frac{w}{\Delta_w} \begin{pmatrix} \beta - \gamma \\ -(\alpha - \gamma) \end{pmatrix} \{ V(j) - V(i) \}.}$$

Observe that if $V(j) > V(i)$, then $V_w(i) > V(i)$, $V_w(j) < V(j)$, which seems quite natural. Also,

$$\boxed{V_w(j) - V_w(i) = \frac{\Delta}{\Delta_w} \left(V(j) - V(i) \right).}$$

This gives us a picture of before and after:



(iii) In the light of one simple example, if we were to take a Mkv chain on the vertex set I of the network which increased the weight of an edge after it crossed it, and if we let $V(j, \xi)$ be the voltage at vertex j when we put 1A into the network at N, when the "environment" (= conductivities of edges) is ξ, then one may conjecture that

$V(X_n, \xi_n)$ is a supermartingale?

For this, we need to know about

$$V_w(j) - V(i) = \frac{\Delta + (\beta - \gamma) w}{\Delta_w} \{ V(j) - V(i) \},$$

and for $i \neq N$,

$$\sum_{j \neq i} q_{ij} \left[V(j, \xi^j) - V(i, \xi) \right] \qquad (\xi^j \text{ is environment with conductivity}(ij) \text{ raised by } w_j)$$

$$= \sum_{j \neq i} q_{ij} (V_j - V_i) \left(\frac{\Delta + (\beta - \gamma) w}{\Delta_w} \right)_j$$

$$= \sum_{j \neq i} q_{ij} \left(V_j - V_i\right) \left(\frac{-(\alpha-\gamma)w}{\Delta_w}\right)_j \qquad \text{since } V \text{ is harmonic for } Q.$$

Need this necessarily always be ≤ 0? No, in fact; if we were to choose $w_j = 0$ if $V_j - V_i > 0$, $= 1$ if $V_j - V_i < 0$, we get in general a positive sum. We would be all right if the condition

$$V_j < V_i \Rightarrow \alpha = \gamma$$

holds; this means that the graph has a tree structure, which explains why the method worked before.

(iv) Could this possibly work if we had always $w = 1$?

Another conjecture is that if we used the equilibrium potential instead of V, it might work. But the equilibrium potential is a multiple of V, and if we put in one more edge, $V(N, \xi)$ will drop. Thus

$$h(\cdot, \xi') \equiv V(\cdot, \xi') / V(N, \xi') \geq \frac{V(\cdot, \xi')}{V(N, \xi)}$$

Thus if the supermartingale property fails for V, it will certainly fail for h.

A question of Nina Gantert (3/4/92)

(i) To keep things concrete, let's suppose that $\Omega = D([0,1], S)$, where S is some Polish space, with canonical process X, and shifts

$$(\theta_t w)(s) = w((t+s) \wedge 1) \quad , \qquad t, s \in [0,1].$$

If $\mathbb{P}$ is a law on Ω under which X is Markov, and if $\tilde{\mathbb{P}} \ll \mathbb{P}$ has the property that X is also Markov under $\tilde{\mathbb{P}}$, what can we say?

(ii) Assume for the moment that $\tilde{\mathbb{P}} \sim \mathbb{P}$, and let

$$Z_t = \left.\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}_t}.$$

Let $\mathcal{F}^t \equiv \theta_t^{-1} \mathcal{F}$ be the σ-field of events determined by the path after time t. We have that Z is a positive $\mathbb{P}$-martingale.

$$\boxed{\text{Claim: } Z_{t,1} \equiv Z_1 / Z_t \text{ is a function of } \theta_t w.}$$

For this, it is enough to prove that for any $\xi \in b\mathcal{F}^t$, $\eta \in b\mathcal{F}_t$,

$$E\left(Z_{t,1}\ \xi\eta \right) = E\left[Z_{t,1}\ E(\xi\eta / \mathcal{F}^t) \right].$$

But the RHS of this is

$$E\left[Z_{t,1}\ \xi\ E(\eta \mid \mathcal{F}^t) \right] = E\left[Z_{t,1}\ \xi\ E(\eta \mid X_t) \right]$$

$$= E\left[E(Z_{t,1}\ \xi / X_t) \cdot \eta \right]$$

$$= E\left[E(Z_{t,1}\ \xi \mid \mathcal{F}_t) \cdot \eta \right]$$

since by assumption

$$\tilde{E}\left[\xi \mid \mathcal{F}_t \right] \equiv E\left[Z_{t,1}\ \xi \mid \mathcal{F}_t \right] = \varphi(t, X_t),$$

and therefore $E\left[Z_{t,1}\ \xi / \mathcal{F}_t \right] = E\left[Z_{t,1}\ \xi / X_t \right]$;

$$= E\left[Z_{t,1}\ \xi\eta \right], \qquad \text{as required.}$$

(iii) One can similarly prove that for $0 \le s \le t$,

$$Z_t / Z_s \quad \text{is meas wrto} \quad \sigma\left(\{ X_u : s \le u \le t \} \right),$$

which is saying in some sense that Z must be a multiplicative functional of the process X (since X is not assumed time-homogeneous, this needs to be interpreted with a little care.) When one does an h-~~transform~~, Z_t is $h(t, X_t)$, but this can't hold in general (take OU process, Wiener process).

<u>A nice example of Hans Föllmer + Peter Imkeller</u> (4/4/92)

Take $\Omega = C([0,1], \mathbb{R})$, canonical process X, Wiener measure P, and let $\mathcal{G}_t \equiv \mathcal{F}_t \vee \sigma(X_1)$. Wrto $\mathcal{G}_t$, the semimg rep^n of X is

$$X_t = W_t + \int_0^t \frac{X_1 - X_s}{1-s}\, ds \equiv W_t + \int_0^t \alpha_s\, ds.$$

Now the martingale (local martingale, really)

$$Z_t = \exp\left[-\int_0^t \alpha_s\, dW_s - \tfrac{1}{2} \int_0^t \alpha_s^2\, ds \right]$$

converts X back into a Brownian motion (at least on each $\mathcal{F}_{1-\varepsilon}$ if we stop at ε!) so shouldn't it be giving us Wiener measure in the limit $\varepsilon \downarrow 0$?

A few lines of calculus reduce the expression for Z to

$$Z_t = \exp\left[\frac{(X_1 - X_t)^2}{2(1-t)} + \tfrac{1}{2}\log(1-t) - \tfrac{1}{2}X_1^2 \right] = p_1(0,X_1) / p_{1-t}(X_t, X_1)$$

so for $t<1$, $Y \in b\mathcal{F}_t$,

$$\tilde{E}\left[Y e^{i\theta X_1} \right] \equiv E\left[Y Z_t e^{i\theta X_1} \right] = E\left[Y e^{i\theta X_t} \cdot Z_t e^{i\theta(X_1 - X_t)} \right] = E\left[Y e^{-\theta^2/2} \right]$$

after some calculations.

Thus using $Z_{1-\varepsilon}$ as the density, the canonical process X is like BM on $[0, 1-\varepsilon]$, and X_1 is an <u>independent</u> $N(0,1)$!!

<u>An example to do with EMM (7/4/92)</u>.

(i) Let's consider how we might make an approximate arbitrage opportunity when the likelihood-ratio martingale Z is a Brownian motion started at 1, then stopped at rate $\lambda(S_t)$ ($S_t = \sup(Z_u : u \le t)$) or stopped when it first reaches $\delta \in (0,1)$. The idea is to make a wealth process

$$\xi_t = \int_0^t \varphi(S_u)\, d\left(\frac{1}{Z_u}\right)$$

$$= \frac{\varphi(S_t)}{Z_t} - g(S_t),$$

where $\varphi'(x) = x g'(x)$, $\varphi(1) = g(1) = 0$, φ is non-negative increasing, and g remains bounded. To be quite explicit, $g(\infty) = 1$.

(ii) Suppose we take Brownian excursions up from 0, with killing at rate $\lambda = \frac{1}{2}\theta^2$, and stopped when the xcn reaches $a > 0$. Then

$$n(\text{xcns which reaches } a \text{ or gets killed before reaching } a) = \theta \coth \theta a,$$

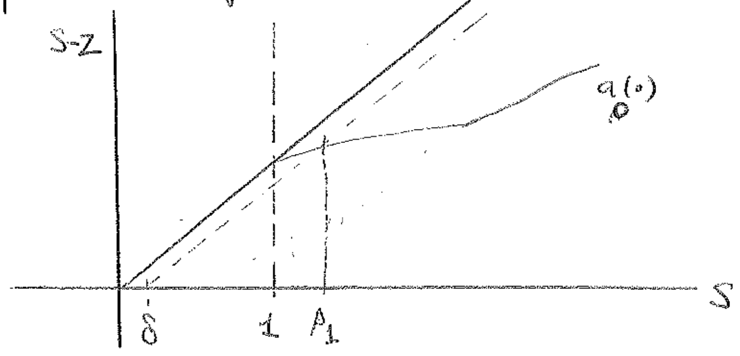
$$n(\text{xcns which reach } a \text{ before killing}) = \theta \operatorname{cosech} \theta a,$$

$$n(\text{xcns which die at } dx, \text{ having killed before reach } a) = \frac{\theta^2 \sinh \theta(a-x)\, dx}{\sinh \theta a}.$$

(iii) Now we choose a stopping boundary, stopping at

$$T_0 \equiv \inf\{t : Z_t < S_t - a_0(S_t)\}$$

if we haven't been λ-stopped, or stopped at δ before then.



We shall insist that for all $s \geq 1$

$$a_0(s) = s - \frac{\varphi(s)}{b + g(s)}$$

$$= \frac{sb + \int_1^s g(x)\,dx}{b + g(s)},$$

where $b > 0$ is some fixed parameter. The effect of this is that

$$\boxed{\xi_{T_0} = b \quad \text{when } T_0 < H_\delta \wedge \zeta.}$$

Thus if $T_0 < H_\delta \wedge \zeta$, we "win" – the trading strategy makes $b > 0$. If on the other hand we get killed before T_0, the bad situation is to be killed at a time t when $Z_t > \varphi(S_t)/g(S_t)$ i.e., $S_t - Z_t < \int_1^S g(x)\,dx \,/ g(s) \equiv \alpha(s)$.

Note that $a_0(s) > \alpha(s)$. Setting $a(s) \equiv a_0(s) \wedge (s-\delta)$, the process stops by the first time $a(S_t) = S_t - Z_t$. Let $T \equiv T_0 \wedge H_\delta \wedge \zeta$. Then

$$P[S_T > t] = \exp\left\{-\int_1^t \theta_u \coth \theta_u a_u \,du\right\} \equiv \exp(-R_t),$$

and if $s_1 = \inf\{u : a_0(u) < u - \delta\} = \inf\{u : \varphi(u) > \delta(b + g(u))\}$, then

$$P[\xi_T = b] = \int_{s_1}^\infty dt\; e^{-R_t}\; \theta_t \operatorname{cosech} \theta_t a_t,$$

$$P[\xi_T < 0] = \int_1^\infty dt\; e^{-R_t} \int_0^{\alpha(t)} \theta_t^2 \sinh \theta_t (a_t - x) \operatorname{cosech} \theta_t a_t \,dx,$$

$$E[\xi_T^-] = \int_1^\infty dt\, e^{-R_t} \int_0^{\alpha(t)} \frac{\theta_t^2 \sinh \theta_t (a_t - x)}{\sinh \theta_t a_t} \left\{ g(t) - \frac{\varphi(t)}{t - x} \right\} dx.$$

The condition for non UI is $\int^{\infty} v\,\theta(v)^2\,dv < \infty$.

What is the joint excursion law of (variation, overshoot) for symmetric stable $(\frac{1}{2})$? (5/5/92)

Let X, Y be independent stable $(\frac{1}{2})$ subordinators, $E e^{-\lambda X_t} = e^{-t\sqrt{2\lambda}}$, with $Z \equiv X - Y$, $V \equiv X + Y$. If T is an indept $\exp(\lambda)$ r.v., and if $\sigma_t \equiv \sup\{u < t : Z_u = \bar{Z}_u\}$, can we obtain

$$E \exp\{ s \bar{Z}_T - \alpha V(\sigma_T) \} \equiv \psi(s, \alpha)? \qquad (\mathrm{Re}(s) \le 0).$$

By the old WH factorisation business, $\psi(s,\alpha) \cdot \psi(-s,\alpha) \equiv |\psi(s,\alpha)|^2 = E\left(e^{sZ_T - \alpha V_T}\right)$

$= \lambda\{\lambda + 2\,\mathrm{Re}\sqrt{2\alpha + 2s}\}^{-1}$.

Now if we consider excursions down from max of Z, and think of killing such excursions at rate α in total variation, rate γ in the overshoot, as well as the λ-killing, then

$$\boxed{\psi(-\gamma, \alpha) = \frac{\text{rate of } \lambda\text{-killed excursions}}{\text{rate of excursions which are } \alpha, \gamma \text{ or } \lambda\text{-killed}}}.$$

Now if (τ_t) is the inverse to local time at the maximum for Z, the old Fristedt identity gives

$$E \exp\{-\eta \tau_t - \gamma Z(\tau_t)\} = \exp -t\,\varphi(\eta, \gamma),$$

where $\varphi(\eta,\gamma) = \exp \int_0^\infty \frac{dt}{t} \int_0^\infty (e^{-t} - e^{-\eta t - \gamma x})\,P(Z_t \in dx)$.

Thus with $\gamma = 0$, using symmetry of Z we learn that

$$E \exp\{-\eta \tau_t\} = \exp(-t\sqrt{\eta}),$$

so that $\boxed{\text{rate of } \lambda\text{-killed excursions} = \sqrt{\lambda}.}$

Now we divide both sides of the WH factorisation by λ, let $\lambda \downarrow 0$, and get

$$\boxed{\left| \int_0^\infty\!\!\int_0^\infty \mu(dx, dv)\{1 - e^{-sx - \alpha v}\} \right|^2 = 2\,\mathrm{Re}\sqrt{2\alpha + 2s},} \qquad \mathrm{Re}(s) = 0,$$

where μ is the Lévy measure of (overshoot, total variation) for excursions down from max of Z. This is in principal useable, because

$$\int_0^\infty\int_0^\infty \mu(dx, dv)\, \{1 - e^{-sx - \alpha v}\} \equiv h(s,\alpha)$$

is analytic in $Re(s) > 0$, and has positive real part there, so that $\log h(s,\alpha)$ is well defined and analytic. The last boxed equation gives the boundary values on $Re(s) = 0$ of $\log h(s,\alpha)$, so we could in principle now extend into the right half plane by analytic extension, but the Poisson integral involved looks unassailable.

If we knew h, we could compute $(H_a \equiv \inf\{t: Z_t > a\})$

$$\int_0^\infty \rho e^{-\rho a}\, da\; E\, e^{-\theta(Z(H_a) - a) - \alpha V(H_a)}$$

$$= \int_0^\infty \rho e^{-\rho a}\, da \int_0^a G_\alpha(dy) \int_{a-y}^\infty \nu_\alpha(dx)\; e^{-\theta(x+y-a)}$$

where G_α is the Green f^n, ν_α the Lévy measure of the subordinator $Z(\tau_t)$ killed at rate αV. Now a few lines of calculus reduce this to

$$\frac{\rho}{\rho - \theta}\{\psi(\rho) - \psi(\theta)\}\cdot\frac{1}{\psi(\rho)}, \qquad \psi(\rho) \equiv h(\rho, \alpha).$$

Thus knowing h also in principle tells us the joint law of (overshoot, variation) at first passage across a – but there's nothing in practice we can do.

Excursions via resolvents again (13/5/92)

Assume we are given (R_λ), (R^∂_λ) and are told the finite boundary B. The aim is to recover the representation (assume $\psi^a_\lambda(b) = \delta_{ab}$)

$$R_\lambda f(a) = \sum_b M_\lambda(a,b)\; \eta^b_\lambda f,$$

where all notation is as in ZW $\underset{\sim}{67}$, 473–476. Now since η^b_λ is an entrance law, we have

$$R_\lambda R^\partial_\beta f(a) = \sum_b M_\lambda(a,b)\,(\eta^b_\lambda f - \eta^b_\beta f)\,/(\beta - \lambda)$$

so that

$$\boxed{\sum_b M_\lambda(a,b)\,\eta^b_\beta f = R_\lambda f(a) + (\lambda-\beta) R_\lambda R^\partial_\beta f(a).}$$

From
$$M_\lambda(a,b)^{-1} = \delta^a + \lambda\gamma^a + \lambda n^a_\lambda 1 + \sum_{c\neq a} \nu^{ac}_\lambda \qquad (a=b)$$
$$= -\nu^{ab}_\lambda \qquad (a\neq b)$$
we see that the offdiagonal terms drop to zero, but the on diagonals remain significant:
$$M_\lambda(a,a) \sim (\delta^a + \lambda\gamma^a + \lambda n^a_\lambda 1)^{-1}.$$

Hence we could define
$$\eta^a_\beta f = \lim_{\lambda\to\infty} \frac{R_\lambda f(a) + (\lambda-\beta) R_\lambda R^\partial_\beta f(a)}{R_\lambda 1(a) + (\lambda-1) R_\lambda R^\partial_1 1(a)} \quad ;$$

we may have to pick out some subsequence to achieve this.

Optimal consumption problem (15/5/92)

Suppose that $U : [0,\infty) \to \mathbb{R}$ is C^1, with U' strictly decreasing, $U' > 0$ everywhere, and set
$$c \equiv \sup_x U'(x) = \lim_{x\downarrow 0} U'(x) \leq \infty.$$

Consider the problem
$$\max E \sum_1^N U(a_j)$$

where a_j is the amount consumed on day j in the following set-up; at the beginning of day j, you receive a random amount $Y_j \geq 0$, with the Y_j i.i.d.

The value functions for this DP problem (in terms of time-to-go) satisfy

$$\boxed{\begin{array}{l} V_{n+1}(x) = \max_{0\leq a\leq x} \{U(a) + \tilde V_n(x-a)\}, \qquad \tilde V_n(x) \equiv E\, V_n(x+Y), \\ V_0(x) \equiv U(x). \end{array}}$$

Let $a_{n+1}(x)$ be the value of $a \in [0,x]$ at which the max is achieved:

$$V_{n+1}(x) = U(a_{n+1}(x)) + \tilde{V}_n(x - a_{n+1}(x)).$$

<u>Claim</u>: For all $n \in \mathbb{Z}^+$,

(i) $0 < a_{n+1}(x) \le a_n(x) \le x$ for $x > 0$;

(ii) $a_{n+1}(\cdot)$ is continuous, strictly increasing;

(iii) $V'_{n+1}(x) = U'(a_{n+1}(x))$.

Here, of course, $a_0(x) \equiv x$.

<u>Proof</u>. Firstly, the case $n=0$. Observe that

$$\tilde{V}_0 = \tilde{U} \ge U, \quad \text{and } \tilde{V}_0' = \tilde{U}' \le U',$$

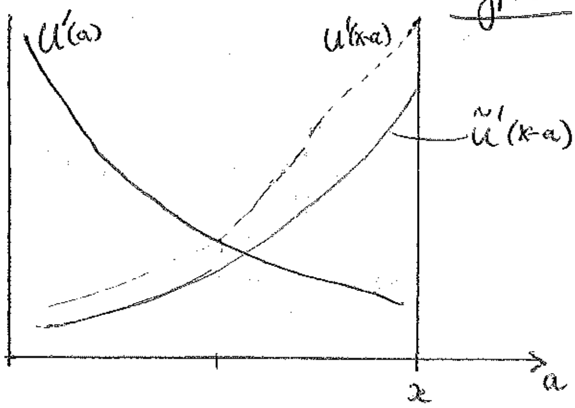
since $U \uparrow$, $U' \downarrow$.

Thus the slope of $U(a) + \tilde{V}_0(x-a)$ will vanish where

$$U'(a) = \tilde{V}_0'(x-a) \le U'(x-a),$$

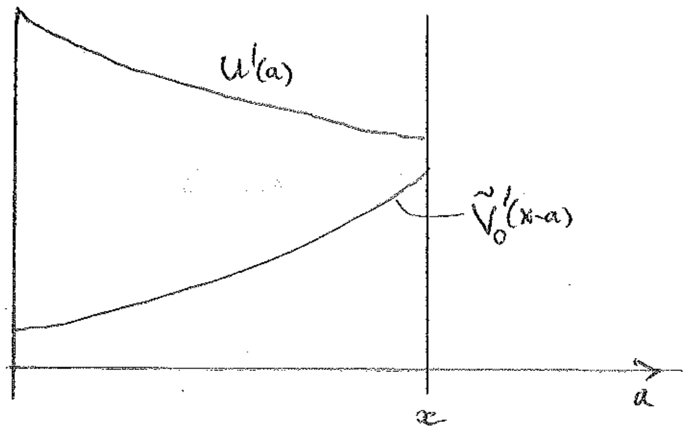
so certainly at some point $a \ge x/2$.

<u>typical</u>



One other possibility could arise for small x:

<u>Small x</u>:



In this situation, the best a to choose is $a = x$.

The possibility $\tilde{V}_0'(x) > U'(0)$ is ruled out by the fact that $\tilde{V}_0' \le U'$, so the third eventuality can be safely ignored.

Whichever the situation is, $x/2 \le a_1(x) \le a_0(x) = x$, so (i) is valid, and (ii) follows by inspection. As for (iii), for some small x we may have

$$V_1(x) = U(x) + \tilde{U}(0)$$

Notice that if $U^*(\lambda) \equiv \inf_x \{\lambda x - U(x)\}$ is the concave conjugate, we have

$$\boxed{V_{n+1}^*(\lambda) = U^*(\lambda) + \tilde{V}_n^*(\lambda)}$$

Can one relate $\tilde{V}^*(\cdot)$ to anything we know?

Convexity of V_n implies $V_n(x) \leq \tilde{V}_n(x) \leq V_n(x+b)$ where $b \equiv EY$.

in which case (iii) certainly holds; in the other typical situation,

$$V_1(x) = U(a_1(x)) + \tilde{U}(x - a_1(x))$$

so that
$$V_1'(x) = a_1'(x)\{U'(a_1(x)) - \tilde{U}'(x-a_1(x))\} + \tilde{U}'(x - a_1(x))$$
$$= \tilde{U}'(x - a_1(x))$$
$$= U'(a_1(x)),$$

establishing the inductive statement for $n=0$.

Suppose now that it's true up to and including $n-1$. We know that for $1 \le k \le n$,

$$V_k'(x) = U'(a_k(x)) \ge U'(a_{k-1}(x)) = V_{k-1}'(x)$$

by (i) and (iii) of the inductive hypothesis. Therefore for $1 \le k \le n$

$$c \ge \tilde{V}_k'(x) \equiv E\, V_k'(x+Y) \ge \tilde{V}_{k-1}'(x),$$

and so when we compute

$$V_{n+1}(x) = \max_{0 \le a \le x} \{U(a) + \tilde{V}_n(x-a)\}$$

we look for a to solve $U'(a) = \tilde{V}_n'(x-a) \ge \tilde{V}_{n-1}'(x-a)$, and hence

$$a_{n+1}(x) \le a_n(x),$$

and

$$V_{n+1}'(x) = U'(a_{n+1}(x))$$

as above. Property (ii) is easy to prove.

Notice that

$$\boxed{V_n'(x) \ge \tilde{V}_n'(x).}$$ Thus $\tilde{V}_n - V_n \ge 0$ is decreasing.

Let us also record that

$$\boxed{U(x) \le \tilde{U}(x) \le U(x) + U(EY)}$$

since $\tilde{U}(x) - U(x) = E\{U(x+Y) - U(x)\} \le U(x+EY) - U(x) \le U(EY)$.

Remarks on WH factorisation (5/6/92)

In the WH factorisation

$$V^{-1}Q\begin{pmatrix} I & \Pi^- \\ \Pi^+ & I \end{pmatrix} = \begin{pmatrix} I & \Pi^- \\ \Pi^+ & I \end{pmatrix}\begin{pmatrix} G^+ & \cdot \\ \cdot & -G^- \end{pmatrix},$$

in the exactly balanced case $\sum m_j v_j = 0$ both of the matrices $\Pi^\pm$ are stochastic so that $S = \begin{pmatrix} I & \Pi^- \\ \Pi^+ & I \end{pmatrix}$ is singular. What is the eigenvalue/eigenvector structure here?

Notice that G^+ has one e-value 0, no other e-value on $i\mathbb{R}$, and no Jordan vectors of e-value 0, cos if $G^+h = 1$, we left-multiply by invariant meas of G^+ and get $1=0$. Thus if $|E^+| = n$, $|E^-| = m$, G^+ has eigenspaces of dimension $n-1$ in $\{\operatorname{Re} z < 0\}$, G^- has eigenspaces of dimension $m-1$ in $\{\operatorname{Re} z < 0\}$.

Each Jordan vector of $G^\pm$ gives rise to a Jordan vector of $V^{-1}Q$, so the eigenspaces of $G^\pm$ away from $i\mathbb{R}$ account for $m+n-2$ of the dimensions of $V^{-1}Q$. There's also the e-space of the zero e-value, but, I claim, the Jordan decomposition of $V^{-1}Q$ looks like

$$V^{-1}Q = S\left(\begin{array}{c|c} \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix} & 0 \\ \hline 0 & \begin{matrix}\text{non-zero} \\ \text{eigenspaces}\end{matrix} \end{array}\right) S^{-1}$$

The only other possibility is that the first row + col of the Jordan form are zero. If this the case, the first column of S is a right-evector of $V^{-1}Q$ with e-value 0, and the first row of S^{-1} is a left-evector of $V^{-1}Q$ with e-value 0. Thus the first column of S is constant, the first row of S^{-1} is a multiple of mV. If we then multiply the first row of S^{-1} into the first column of S, we get

$$mV1 = 0 \neq 1, \text{ which it should be !!}$$

Invariant distⁿ for buffered fluid models. (6/6/92)

(i) Water flows into a reservoir of capacity a at rate $v(X_t)$, where X is a finite irreducible Markov chain. Outflow (inflow) ceases when the reservoir is empty (full). If φ_t is the content of the reservoir at time t, what is

$$\Pi(j,x) \equiv \lim_{t\to\infty} P(X_t = j,\ \varphi_t \le x) \ ?$$

(ii) The generator of the bivariate process (X_t, φ_t) is

$$(1) \qquad \mathcal{G}f = Qf + V\frac{\partial f}{\partial\varphi}$$

applied to functions for which $\frac{\partial f}{\partial\varphi}$ vanishes at $\varphi = 0, a$. Let's firstly remark that the measure $\Pi(j,dx)$ will have an atom at 0 if $j \in E^-$, and at a if $j \in E^+$. Let's assume that in $(0,a)$ $\pi(j,x) = \Pi(j,dx)/dx$ exists and is continuous. The adjoint equation for π says that for $f \in \mathcal{D}(\mathcal{G})$,

$$0 = \sum_j \int_{[0,a]} \Pi(j,dx)\ \mathcal{G}f(j,x)$$

$$= \sum_{j\in E^-} \Pi(j,0)\,(Qf)_j + \sum_{j\in E^+} \Delta\Pi(j,a)\,(Qf)_j$$

$$+ \ \pi.Vf(a) - \pi.Vf(0) + \int_0^a (\pi Q - \dot{\pi} V) f(x)\,dx,$$

where we take π, Π to be row vectors. So if $\Delta\Pi(0) = p_-$, $\Delta\Pi(a) = p_+$, we get

$$(2) \qquad \boxed{\begin{aligned} \dot{\pi}(x)V &= \pi(x)Q, \qquad x\in(0,a) \\ p_- Q &= \pi(0)V, \\ p_+ Q &= -\pi(a)V \end{aligned}}$$

(iii) We conclude from this that $p_- Q = p_-(C\ D)$, $p_+ Q = p_+(A, B)$, and

$$(3) \qquad \boxed{\begin{aligned} \pi(x)V &= p_-(C\ D)\, e^{xV^{-1}Q} \\ &= -p_+(A\ B)\, e^{(x-a)V^{-1}Q}, \end{aligned}}$$

[Notice $\pi(x)V\mathbf{1} = 0 \ \forall x$, which has a simple probabilistic interpretation

so that

(4) $$\boxed{p_-(C\ D) = -p_+(A\ B)\, e^{-aV^{-1}Q}}$$

Right multiplying by $\begin{pmatrix}\Pi_-\\ I\end{pmatrix}$ yields

(5) $$\boxed{p_- G_- = -p_+(A\ B)\begin{pmatrix}\Pi_-\\ I\end{pmatrix} e^{aG_-} = p_+\Pi_-\, e^{aG_-}\, G_-}$$

and similarly right-multiplying by $\begin{pmatrix}I\\ \Pi_+\end{pmatrix}$ yields

(6) $$\boxed{p_+ G_+ = p_-\Pi_+\, e^{aG_+}\, G_+ .}$$

(iv) <u>Assuming that G_- is invertible</u> (so that the additive f^{nl} drifts to too), we have

$$p_- = p_+\,\Pi_-\, e^{aG_-}$$

and

$$p_+\left(I - \Pi_- e^{aG_-}\,\Pi_+\, e^{aG_+}\right) G_+ = 0,$$

so that

(7) $$\boxed{p_+ = \nu_+\left(I - \Pi_- e^{aG_-}\,\Pi_+\, e^{aG_+}\right)^{-1} . c}$$

for some constant c, where ν_+ is the invariant measure of G_+, a multiple of $m_+(I - \Pi_-\Pi_+)$.

(v) <u>Assuming that G_+ is invertible</u>, we obtain similarly ($\nu_- \propto m_-(I - \Pi_+\Pi_-)$)

$$p_- \propto \nu_-\, e^{aG_-}\left(I - \Pi_+ e^{aG_+}\,\Pi_- e^{aG_-}\right)^{-1} = c.\, m_-(I-\Pi_+\Pi_-)(I-\Pi_+ e^{aG_+}\Pi_- e^{aG_-})^{-1}$$

In the limit as $a \to \infty$, which is probabilistically meaningful, we get

$$p_- \propto m_-(I - \Pi_+\Pi_-) \propto \nu_-$$

(vi) If we set $K_+(a) \equiv \left(I - \Pi_- e^{aG_-}\,\Pi_+\, e^{aG_+}\right)^{-1}$, with

$K_-(a)$ defined analogously, then one can show that when we assume

G_+ is invertible ($\Rightarrow m_+ = m_- \Pi_+$) then

(8) $$\boxed{\begin{aligned} p_- &= m_-(I - \Pi_+\Pi_-)\, K_-(a) \\ &\equiv m_-(I-\Pi_+\Pi_-)(I - \Pi_+ e^{aG_+}\Pi_- e^{aG_-})^{-1}, \end{aligned}}$$

by integrating the density π and adding in the masses at $0, a$, which must of course give the invariant distn m for the original chain. Using the simple fact that

$$K_-(a)\, \Pi_+ e^{aG_+} = \Pi_+ e^{aG_+} K_+(a),$$

we can express

(9) $$\boxed{p_+ = p_- \Pi_+ e^{aG_+} = m_+(I - \Pi_-\Pi_+)\, e^{aG_+} K_+(a).}$$

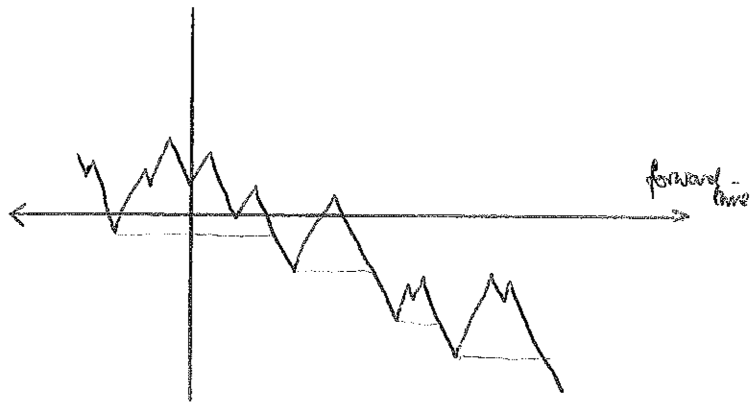
The formulae assuming G_- is invertible are analogous.

Notice the limiting form as $a \to \infty$ for p_- when G_+ is invertible;

(10) $$\boxed{p_- = m_-(I - \Pi_+\Pi_-).}$$

This has a simple and natural probabilistic interpretation! Indeed, in the reversed situation with $\hat{Q} \equiv M^{-1} Q^T M$, $\hat{V} \equiv -V$, $\hat{E}_\pm = E_\mp$ we see that for $i \in E_- \equiv \hat{E}_+$

$$p_-(i) = \hat{P}(X_t = i,\ \hat{\varphi} \text{ never falls below } \hat{\varphi}_t)$$



$$= m_i\,(1 - \hat{\Pi}_- 1(i))$$

$$= m_i - m_i \sum_{j \in E_+} \hat{\Pi}_-(i,j)$$

$$= m_i - m_i \sum_{j \in E_+} m_j\, \Pi_-(j,i)/m_i$$

so that $$\boxed{p_- = m_- - m_+\Pi_- = m_-(I - \Pi_+\Pi_-).}$$

(vii) <u>The balanced case</u>, <u>$m_+ 1 = m_- 1$</u> is, as one would expect, considerably more delicate. The fundamental boundary condition

(11) $$p_-(C\ D)\, e^{aV^{-1}Q} = -p_+(A\ B)$$

is again the starting point, but $S = \begin{pmatrix} I & \Pi_- \\ \Pi_+ & I \end{pmatrix}$ is singular, so we lose information when we multiply by S. From the WH factorisation, we have here

$$\left.\begin{array}{ll} m_+ - m_-\Pi_+ & \text{is a multiple of invariant law of } G_+ \\ m_- - m_+\Pi_- & \text{———————————————— } G_- \end{array}\right\} \text{of total mass } 0 \,: \quad \boxed{\begin{array}{l} m_+ = m_-\Pi_+ \\ m_- = m_+\Pi_- \end{array}}$$

The missing piece of information is supplied by taking z such that $V^{-1}Qz = 1$ (such a z exists - see p. 20) and right multiplying the bc (11) above by it:

$$p_-(C\,D)(I + aV^{-1}Q)z = -p_+(A\,B)z = -p_+ 1$$

$$= -p_- 1,$$

so that $\boxed{p_+ 1 = p_- 1.}$

[This is in any case probabilistically obvious!] So taking the bcs (11), multiplying by S, we get as before

$$(p_- - p_+\Pi_- e^{aG_-})\, G_- = 0, \quad (p_+ - p_-\Pi_+ e^{aG_+})\, G_+ = 0,$$

so each of the row vectors is a multiple of the invariant law of G_+, G_-. But the row vectors have total mass 0, whence

$$\boxed{p_- = p_+\Pi_- e^{aG_-}, \quad p_+ = p_-\Pi_+ e^{aG_+}.}$$

Thus p_+ is a multiple of the invariant law of $\Pi_- e^{aG_-}\Pi_+ e^{aG_+}$, which we would have guessed in any case from (8)!

If $p_- 1 = \theta$, we have

$$1 - 2\theta = \int_0^a \pi(x)\, 1\, dx = (0\ p_-)\,(e^{aQV^{-1}} - I)\, 1$$

$$= (0\ p_-)\, e^{aQV^{-1}} 1 - \theta$$

So if ν_- is the invariant law of $\Pi_+ e^{aG_+}\Pi_- e^{aG_-}$, we determine θ by

$$\boxed{\theta = \left(1 + (0, \nu_-)\, e^{aQV^{-1}} 1\right)^{-1}}$$

It's not clear whether this will simplify.

(viii) Since it seems not to have been recorded elsewhere so far, let's just take the reversal of the WH decomposition: if $Q^* = M^{-1} Q^T M$, then

$$\boxed{V^{-1} Q^* S^* = S^* \quad M^{-1} K^T \begin{pmatrix} G_+ & \cdot \\ \cdot & -G_- \end{pmatrix}^T (K^T)^{-1} M}$$

where $K^{-1} = \begin{pmatrix} I - \Pi_- \Pi_+ & \cdot \\ \cdot & I - \Pi_+ \Pi_- \end{pmatrix}$.

<u>A question raised by Ago Pistova</u> (24/6/12)

Ago asks the following. Take a Brownian bridge on $[0,1]$ from a to $a > 0$, conditioned not to hit 0. Can one prove that as $a \downarrow 0$ the law of this thing converges to the law of a scaled Brownian excursion?

Yes, one can, as follows. Let $(B_t)_{t \geq 0}$ be a Brownian motion in $\mathbb{R}^3$, and let $u \in \mathbb{R}^3$ be some fixed unit vector. Let

$$\tau_a = \inf\{ t : |B_t + at\, u| > a \}.$$

Then the process

$$\boxed{X^a_t \equiv (1-t) \left| B\left(\tau_a + \frac{t}{1-t}\right) + a\left(\tau_a + \frac{t}{1-t}\right) u \right|}$$

<u>is the Brownian bridge from a to a on $[0,1]$, conditioned not to hit 0</u>, and as $a \downarrow 0$ this converges a.s. uniformly to

$$X_t \equiv (1-t) \left| B\left(\frac{t}{1-t}\right) \right|$$

which is a well-known representation of Brownian excursion.

In a little more detail, if $p_t(x,y) \equiv (2\pi t)^{-1/2} \exp(-(x-y)^2/2t)$ is the standard 1 dimensional Brownian ~~transition~~ density,

$$\bar{p}_t(x,y) \equiv p_t(x,y) - p_t(x,-y) = (2\pi t)^{-1/2} e^{-(x^2+y^2)/2t} \; 2\sinh\left(\frac{xy}{t}\right)$$

then the transition density of BES(3) is

$$r_t(x,y) \equiv \bar{p}_t(x,y)\, y/x$$

and the transition density of $|B_t + at\, u|$ is

$$r_t^a(x,y) = e^{-a^2t/2}\sinh(ay)\,{}_0p_t(x,y)/\sinh(ax)$$

as is proved, for example, in Rogers + Pitman, Ann Prob $\underset{\sim}{9}$, 573–582. It is not at first sight obvious that $|B_t + atu|$ is a Markov process!

Now one can verify by direct calculation that the transition density of the inhomogeneous Markov process X^a is in fact

$$p_{s,t}(x,y) = {}_0p_{t-s}(x,y)\,{}_0p_{1-t}(y,a)\,/\,{}_0p_{1-s}(x,a),$$

confirming the fact that X^a is the Brownian bridge $a \to a$, conditioned to stay positive.

Bounds on the price of an Asian option (26/6/92)

(i) Suppose that we have an asset with price process

$$S_t = S_0 \exp\{\sigma B_t - \tfrac{1}{2}\sigma^2 t + rt\}$$

under the martingale measure. To price the Asian call option with strike price K, we have to compute,

$$\text{price} = E\, e^{-rT}\left(T^{-1}\int_0^T S_u\,du - K\right)^+$$

$$= e^{-rT}\, E\, E\left[\left(T^{-1}\int_0^T S_u\,du - K\right)^+ \Big/ X_T = a\right]$$

where

$$X_t = B_t + \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)t,$$

so that $S_t = S_0\exp(\sigma X_t)$. The law of X_T is $N(\mu T, T)$, where

$$\mu = \sigma^{-1}(r - \tfrac{1}{2}\sigma^2).$$

(ii) Firstly, let's get a lower bound.

$$E\left[\left(T^{-1}\int_0^T (S_u - K)\,du\right)^+ \Big| X_T = a\right]$$

$$\geq \left(E\left[T^{-1}\int_0^T (S_u - K)\,du \Big/ X_T = a\right]\right)^+$$

$$= \left(E\left[T^{-1}\int_{-\infty}^{\infty}(S_0 e^{\sigma x} - K)\,L(T,x)\,dx \,\Big|\, X_T = a\right]\right)^+.$$

Now we need to compute

$$E\left[L(T,x) \mid X_T = a\right] = \int_0^T p_t(0,x)\, p_{T-t}(x,a)\, dt \Big/ p_T(0,a)$$

$$= \int_0^T \exp\left\{-\frac{x^2}{2t} - \frac{(x-a)^2}{2(T-t)}\right\} \frac{dt}{2\pi\sqrt{t(T-t)}} \Big/ p_T(0,a).$$

If now we set

$$\varphi(x,y) \equiv \int_0^T e^{-x^2/2t - y^2/2(T-t)} \frac{dt}{2\pi\sqrt{t(T-t)}}$$

then φ is symmetric in x, y, and for $x, y > 0$,

$$\frac{\partial^2 \varphi}{\partial x \partial y} = \int_0^T h_x(t)\, h_y(T-t)\, dt = h_{x+y}(T) = \frac{(x+y)\, e^{-(x+y)^2/2T}}{\sqrt{2\pi T^3}}$$

Hence

$$\varphi(x,y) = \int_y^\infty e^{-(x+v)^2/2T} \frac{dv}{\sqrt{2\pi T}} = \bar{\Phi}\left(\frac{x+y}{\sqrt{T}}\right)$$

Thus

$$\boxed{E\left[L(T,x) \mid X_T = a\right] = \bar{\Phi}\left(\frac{|x| + |x-a|}{\sqrt{T}}\right) \Big/ p_T(0,a)}$$

$$\equiv g(x,a), \text{ say}.$$

So we have the lower bound

$$\boxed{\text{price} \geq E\left(\left\{T^{-1}\int_{-\infty}^{\infty} (S_0 e^{\sigma x} - K)\, g(x, X_T)\, dx\right\}^+\right)}$$

(ii) Similarly, using Jensen on the integral wrt t, we obtain

$$\boxed{\text{price} \leq E\left[T^{-1}\int_{-\infty}^{\infty} (S_0 e^{\sigma x} - K)^+\, g(x, X_T)\, dx\right]}$$

(iii) Let's try to see how close these two bounds might be. Let the critical value of x where the integrand changes sign be denoted by $b \equiv \frac{1}{\sigma}\log(K/S_0)$, and define

$$Y_\pm \equiv T^{-1} \int_{-\infty}^{\infty} (S_0 e^{\sigma x} - K)^{\pm} \, g(x, X_T) \, dx ,$$

so that the bounds state

$$E \, (Y_+ - Y_-)^+ \leq \text{price} \leq E \, Y_+$$

[Can this be any good? RHS is just $E\left[T^{-1} \int_0^T (S_t - K)^+ dt \right] \ldots$]

so that the difference between the two bounds is

$$E\left[Y_+ - (Y_+ - Y_-)^+ \right] = E\left[Y_+ \wedge Y_- \right] .$$

We can also express

$$g(x, a) = \left\{ \bar{\Phi}\left(\frac{|a|}{\sqrt{T}} \right) \wedge \bar{\Phi}\left(\frac{|2x - a|}{\sqrt{T}} \right) \right\} \Big/ p_T(0, a)$$

What else do we have? Writing

$$\eta_\pm (a) \equiv T^{-1} \int (S_0 e^{\sigma x} - K)^{\pm} \, g(x, a) \, dx$$

the critical value of a where $\eta_+(a) = \eta_-(a)$ is where

$$0 = \eta_+(a) - \eta_-(a) = T^{-1} \, E\left[\int_0^T (S_0 e^{\sigma X_t} - K) \, dt \,\Big|\, X_T = a \right]$$

$$= T^{-1} \int_0^T dt \; S_0 \; e^{\sigma a t/T + \frac{1}{2}\sigma^2 t(T-t)/T} \qquad - K$$

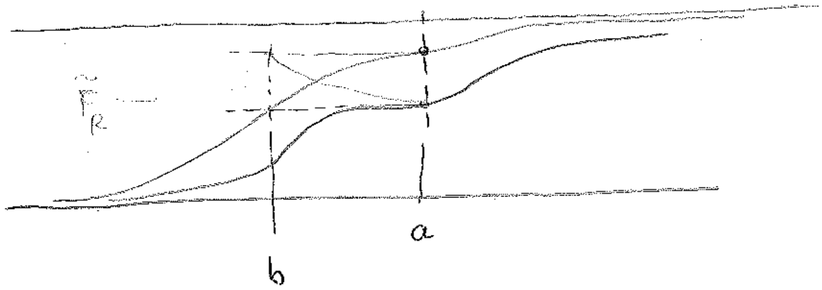
so the critical value of a is where

$$\boxed{\int_0^T dt \, \exp\left[\frac{\sigma a t}{T} + \frac{1}{2}\sigma^2 \frac{t(T-t)}{T} \right] = KT/S_0 .}$$

This cannot be solved in closed form, but by Jensen we know that

$$\frac{\sigma a}{2} \leq \log(K/S_0) - \sigma^2 T/12 .$$

Another method one could use to try to bring things closer is to look for some place where $F_R > G_R$, at a, say,



then to find

$b = \inf\{u : F_R(u) > G_R(a)\}$

and then reset

$$\tilde{F}_R(u) = F_R(a) - F_R(u) + F_R(b)$$

This way, when you use the inverse distⁿ fⁿ, you still get the law F_R, but at least you've shaken things up a bit

<u>Coupling of random walks and Lévy processes (28/6/92)</u>

(i) If one tries to couple random walks, one encounters the following question:

"<u>If</u> $F \leq_{st} G$, <u>can one find a 'good' coupling of</u> F, G <u>such that</u>

$$X \sim F, \; Y \sim G \;\; \underline{\text{and}} \;\; X \leq Y \text{ a.s.?}"$$

If F, G have densities f, g respectively, we could define the sub-probability dist^{n} $F \wedge G$ by

$$\frac{d(F \wedge G)}{dm} = f \wedge g$$

where m is the meas wrt which F and G have densities.

In this way, $\bar{F}(t) - (\overline{F \wedge G})(t) \leq \bar{G}(t) - (\overline{F \wedge G})(t)$. So we could do the following coupling; with prob^{y} $(F \wedge G)(\infty)$, we make $X = Y$, with dist^{n} $F \wedge G$ renormalise to mass 1. With the complementary prob^{y}, we take

$$X = F_R^{-1}(U), \quad Y = G_R^{-1}(U)$$

where U is $U[0, 1-(F \wedge G)(\infty)]$, $F_R(t) \equiv F(t) - (F \wedge G)(t)$. This would ensure that $X \sim F$, $Y \sim G$, and $X \leq Y$ a.s., and that X and Y were equal with maximal probability.

(ii) The specific situation of a r.w. is where F and G are shifts of one another. By the above construction, if S_n, S_n' are the two random walks with $S_0 = x$, $S_0' = x'$, then $S_n - S_n'$ is a nonneg martingale if $x > x'$, and the law has a first moment. Therefore $S_n - S_n'$ is convergent a.s.. The problem is that if at any time S_n, S_n' had reached positions from which coupling in one step were impossible, then the random walks would just move along in parallel thereafter, and never meet...

<u>Back to the Asian option</u> (2/7/92)

(i) The upper bound works out to be simply

$$e^{-rT} \; E \; T^{-1}\int_0^T (S_u - K)^+ \, du$$

$$= e^{-rT} T^{-1} \int_0^T du \left\{ S_0 e^{ru} \, \Phi\left(\frac{\log(S_0/K) + (r+\frac{1}{2}\sigma^2)u}{\sigma\sqrt{u}} \right) - K \, \Phi\left(\frac{\log(S_0/K) + (r-\frac{1}{2}\sigma^2)u}{\sigma\sqrt{u}} \right) \right\}$$

using the classical Black-Scholes thing.

The lower bound is in terms of

$$e^{-rT} \, E\left[\int_0^T (S_u - K) \frac{du}{T} \;\middle|\; X_T = a \right]$$

(*) $$= \frac{e^{-rT}}{T} \int_0^T \left(S_0 \exp\left\{ \tfrac{1}{2}\sigma^2 \frac{u(T-u)}{T} + \frac{\sigma a u}{T} \right\} - K \right) du$$

$$= e^{-rT} \frac{S_0}{T} \sqrt{\frac{2\pi T}{\sigma^2}} \, \exp\left\{ \frac{(a+\sigma T/2)^2}{2T} \right\} \left[\Phi\left(\frac{\sigma T - 2a}{2\sqrt{T}} \right) - \Phi\left(-\frac{\sigma T + 2a}{2\sqrt{T}} \right) \right] - K e^{-rT}$$

$$\equiv h(a), \text{ say.}$$

For numericals, need to know where to stop the integration (we are going to evaluate $E\left[h(X_T)^+ \right]$). Looking at the quadratic in the exponential in (*), we shall find that for $a > \sigma T/2$, the quadratic is maximised at $u = T$ to value σa. In the expectation, then, the integrand will be

$$\le e^{\sigma a} \exp\left(-(a-\mu T)^2/2T\right) (2\pi T)^{-\frac{1}{2}}$$

$$= (2\pi T)^{-\frac{1}{2}} \exp\left\{ -(a-(\mu+\sigma)T)^2/2T - \frac{\mu^2}{2T} + \frac{(\mu+\sigma)^2}{2T} \right\}$$

$$= (2\pi T)^{-\frac{1}{2}} \exp\left\{ -(a-(\mu+\sigma)T)^2/2T + rT \right\}$$

Thus we could ignore all contributions for $a \ge (\mu+\sigma)T + 6\sqrt{T}$, say.

(ii) Computing shows that the bounds are OK, but not particularly close, especially for at-the-money options, not surprisingly. The upper bound is rather crude. Perhaps we can improve it by using $\|\cdot\|_1 \le \|\cdot\|_2$?

$$E\left(\int_0^T (S_u-K)\,du\right)^2 = 2\int_0^T du \int_0^u dv\; E\,(S_u-K)(S_v-K)$$

$$= 2\int_0^T du \int_0^u dv\; E\,(S_v-K)(S_v e^{(u-v)r}-K)$$

$$= 2\int_0^T du \int_0^u dv \left[S_0^2\, e^{\sigma^2 v} \cdot e^{(u+v)r} - K(1+e^{r(u-v)})S_0 e^{rv} + K^2\right]$$

$$= \frac{2S_0^2}{r+\sigma^2}\left[\frac{e^{(\sigma^2+2r)T}-1}{\sigma^2+2r} - \frac{e^{rT}-1}{r}\right]$$

$$-2KS_0\, T\,\frac{(e^{rT}-1)}{r} + K^2T^2.$$

This is an upper bound for $\left(E\,|\int_0^T(S_u-K)\,du|\right)^2$, and now we can bound

$$E\left[\left(\int_0^T(S_u-K)\,du\right)^+\right] \le \frac{1}{2}\left\{S_0\,\frac{e^{rT}-1}{r} - KT + \sqrt{E\left(\int_0^T(S_u-K)\,du\right)^2}\right\}.$$

(iii) In case it may be helpful, let's record the result

$$E\,L(T,x) = \frac{1}{\mu}\left[\bar\Phi\left(\frac{x-\mu T}{\sqrt{T}}\right) - e^{2\mu x}\,\bar\Phi\left(\frac{x+\mu T}{\sqrt{T}}\right)\right] - \frac{1-e^{-2\mu x}}{\mu}$$

which, though it looks strange, certainly checks out as $\mu \to 0$, and as $T \to \infty$.

[I got this by integrating $g(x,a)$ w.r.t. a $N(\mu T, T)$ density!]

$$\boxed{e^{-\mu x}\,E L(T,x) = \frac{1}{\mu}\left[e^{-\mu|x|} - e^{-\mu x}\,\bar\Phi\left(\frac{\mu T-x}{\sqrt{T}}\right) - e^{\mu x}\,\bar\Phi\left(\frac{\mu T+x}{\sqrt{T}}\right)\right]}$$

which is symmetric in x, as it ought to be. We may also confirm that

$$\frac{\partial}{\partial t}\left[e^{-\mu x}\,E L(t,x)\right] = \frac{e^{-\mu^2 t/2 - x^2/2t}}{\sqrt{2\pi t}},$$ as it must be,

so that really is correct.

(iv) Our goal in this is to get good bounds on $E\left(\int_0^T (S_u - K)\,du\right)^+$, or, equivalently, on $E\left|\int_0^T (S_u - K)\,du\right|$, since we know that

$$E\int_0^T (S_u - K)\,du = S_0\,(e^{rT}-1)/r \quad - KT.$$

$$= E\int_{-\infty}^{\infty} (S_0 e^{\sigma x} - K)\, L(T,x)\,dx.$$

Now suppose that this quantity is >0. Then there is some unique a such that

$$E\int_{-\infty}^{a} (S_0 e^{\sigma x} - K)\, L(T,x)\,dx = 0.$$

So if we write

$$\boxed{Y \equiv \int_{-\infty}^{\infty} (S_0 e^{\sigma x} - K)\, L(T,x)\,dx \equiv Y_1 + Y_2,}$$

where

$$Y_1 \equiv \int_{-\infty}^{a} (S_0 e^{\sigma x} - K)\, L(T,x)\,dx$$

then Y_1 is a zero-mean random variable, and we have

$$\boxed{EY - E|Y_1| \leq E|Y| \leq EY_2 + E|Y_1| = EY + E|Y_1|.}$$

Now EY we know, so it's just a matter of bounding $E|Y_1|$. One way we might do that is

$$(E|Y_1|)^2 \leq E\,Y_1^2 = E\iint f(x) f(y)\, L(T,x)\, L(T,y)\,dx\,dy$$

where $f(x) = (S_0 e^{\sigma x} - K)\, I_{\{x \leq a\}}$.

(v) So we see that the goal is to compute

$$E\big[L(T,x)\,L(T,y)\big] = \lim E\int_0^T \varphi_\varepsilon(X_u - x)\,du \int_0^T \varphi_\varepsilon(X_v - y)\,dv$$

$$= \underbrace{\int_0^T du \int_0^u dv\; p_v(y)\; p_{u-v}(x-y)}_{} + \cdots$$

$$= E\big[L(u,y) : X_T = x\big]$$

$$= e^{\mu x - \mu^2 u/2}\;\bar{\Phi}\left(\frac{|y| + |x-y|}{\sqrt{u}}\right)$$

by earlier result. Thus

$$E\,L(T,x)L(T,y) = \int_0^T dt \left\{ e^{\mu x - \mu^2 t/2}\,\bar\Phi\left(\frac{|y|+|x-y|}{\sqrt t}\right) + e^{\mu y - \mu^2 t/2}\,\bar\Phi\left(\frac{|x|+|x-y|}{\sqrt t}\right)\right\}.$$

This makes it important to be able to compute for $a>0$

$$\int_0^T e^{-\mu^2 t/2}\,\bar\Phi\left(\frac{a}{\sqrt t}\right) dt = \int_a^\infty dz \underbrace{\int_0^T e^{-\mu^2 t/2 - z^2/2t}\frac{dt}{\sqrt{2\pi t}}}_{= e^{-\mu z} E\,L(T,z)}$$

$$= \int_a^\infty \frac{dx}{\mu^2}\left[\mu e^{-\mu x} - \mu e^{-\mu x}\,\bar\Phi\left(\frac{\mu T - x}{\sqrt T}\right) - \mu e^{\mu x}\,\bar\Phi\left(\frac{\mu T + x}{\sqrt T}\right)\right] \qquad \text{by result on p.31}$$

$$= \mu^{-2}\left\{ e^{-\mu a} - \int_a^\infty \mu e^{-\mu x} dx \int_{\mu T - x}^\infty e^{-y^2/2T}\frac{dy}{\sqrt{2\pi T}} - \int_a^\infty \mu e^{\mu x} dx \int_{\mu T + x}^\infty e^{-y^2/2T}\frac{dy}{\sqrt{2\pi T}}\right\}$$

$$= \mu^{-2}\left[e^{-\mu a} - \int_{-\infty}^\infty \frac{e^{-y^2/2T}dy}{\sqrt{2\pi T}}\, e^{-\mu(\mu T - y)\vee a} - \int_{\mu T + a}^\infty \frac{e^{-y^2/2T}dy}{\sqrt{2\pi T}} \int_a^{y-\mu T} \mu e^{\mu x} dx\right]$$

$$= \frac{1}{\mu^2}\left[e^{-\mu a} - e^{-\mu a}\,\bar\Phi\left(\frac{\mu T - a}{\sqrt T}\right) - e^{-\mu^2 T/2}\,\bar\Phi\left(\frac{-a}{\sqrt T}\right) - e^{-\mu^2 T/2}\,\bar\Phi\left(\frac{a}{\sqrt T}\right) + e^{\mu a}\,\bar\Phi\left(\frac{\mu T + a}{\sqrt T}\right)\right]$$

$$= \mu^{-2}\left[e^{-\mu a}\left\{\bar\Phi\left(\frac{a-\mu T}{\sqrt T}\right) + \bar\Phi\left(\frac{a+\mu T}{\sqrt T}\right)\right\} - 2e^{-\mu^2 T/2}\,\bar\Phi\left(a/\sqrt T\right)\right]$$

$$\equiv \rho(a,T), \text{ say.}$$

Hence

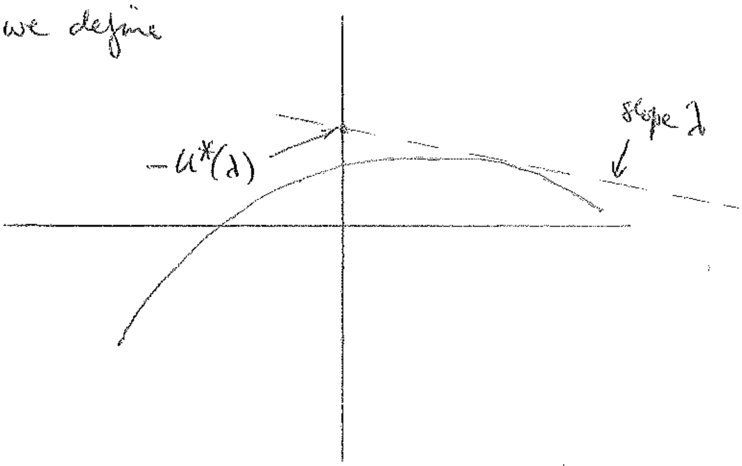
$$\boxed{E\,L(T,x)L(T,y) = e^{\mu x}\rho(|y|+|x-y|,T) + e^{\mu y}\rho(|x|+|x-y|,T)}$$

(vi) Observe that any estimate which tries to compare $E|Y|$ to $|EY|$ is certain to be lousy when the two are far apart!!

Some properties of concave conjugate functions (11/ /92)

(i) If $U: \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is concave, we define

$$U_*(\lambda) \equiv \inf_x \{\lambda x - U(x)\}$$



A nice geometric interpretation is that if the sun is shining with slope λ, then $-U_*(\lambda)$ is the height of the shadow on the y-axis.

Now some elementary properties.

(a) For any constant a,

$$(U+a)_*(\lambda) = U_*(\lambda) - a.$$

(b) If we set $U_\tau(x) \equiv U(x+\tau)$, then

$$(U_\tau)_*(\lambda) = U_*(\lambda) - \tau\lambda. \qquad (\tau \in \mathbb{R})$$

(c) If $\gamma > 0$ and we define $U^\gamma(x) \equiv U(\gamma x)$, then

$$(U^\gamma)_*(\lambda) = U_*(\lambda/\gamma).$$

(d) If U is increasing, then $U_*(\lambda) = -\infty$ for $\lambda < 0$, and $U_*(0) = -U(\infty)$.

(e) If $U_1 \geq U_2$, then $(U_1)_* \leq (U_2)_*$.

(f) If $W(x) = \sup_y \{U(y) + V(x-y)\}$, then

$$W_* = U_* + V_*$$

(g) If F is a probability dist^n, and $\tilde{U}(x) \equiv \int U(x+y)\,F(dy)$, then

$$(\tilde{U}_*)(\lambda) = \sup\left\{ \int (U_*(y_t) - t y_t)\,F(dt) : \int y_t\,F(dt) = \lambda \right\}.$$

Proof If we take the second dual, using this↑ for $\tilde{U}_*$, we get

$$\inf_\lambda \{\lambda x - (\tilde{U}_*)(\lambda)\} = \inf\left\{\lambda x - \int (U_*(y_t) - t y_t)\,F(dt) : \lambda = \int y_t\,F(dt)\right\}$$

$$= \inf \left\{ - \int \{ U_*(y_t) - (t+x) y_t \} F(dt) \right\}$$

$$= \int U(t+x) F(dt)$$

$$\equiv \tilde{U}(x).$$

(h) $(U_1)_* \wedge (U_2)_*$ = greatest concave majorant of U_1, U_2.

(ii) The optimal consumption problem is

$$V_{n+1}(x) = \sup_{0 \le a \le x} \{ U(a) + \tilde{V}_n(x-a) \}, \qquad V_0 = U$$

where U is increasing concave, $U(x) = -\infty$ for $x<0$, say. If we form the concave duals, we get

$$\boxed{(V_{n+1})_* = U_* + (\tilde{V}_n)_*}$$

– does this help? Assuming $U(0) \ge 0$, we shall have

U_* is increasing, ≤ 0.

<u>Maximising expected utility: an example</u> (25/7/92)

(i) If we dropped the insistence on consumption being non-negative, and ~~took~~

$$U(x) \equiv -e^{-\alpha x}$$

then considered $V_0(x) = U(x)$,

$$V_{n+1}(x) = \max_a \{ U(a) + \tilde{V}_n(x-a) \},$$

the natural conjecture would be

$$V_n(x) = -c_n e^{-\alpha_n x},$$

so that

$$\tilde{V}_n(x) = -c_n \theta_n e^{-\alpha_n x}, \qquad \theta_n \equiv E e^{-\alpha_n Y}.$$

Certainly this is ok for $n=0$, and for the inductive step, we get that $$e^{-(\alpha+\alpha_n)a} = \frac{c_n \theta_n \alpha_n}{\alpha} e^{-\alpha_n x}$$

with maximal value

$$-e^{-\alpha\alpha_n x/(\alpha+\alpha_n)} \left[\left(\frac{\alpha_n}{\alpha}\right)^{\frac{\alpha}{\alpha+\alpha_n}} + \left(\frac{\alpha}{\alpha_n}\right)^{\alpha_n/(\alpha+\alpha_n)} \right] (c_n\theta_n)^{\alpha/(\alpha+\alpha_n)}$$

Thus we see that

$$\boxed{\begin{aligned} \alpha_{n+1} &= \frac{\alpha\alpha_n}{\alpha+\alpha_n} \\ c_{n+1} &= (c_n\theta_n)^{\alpha/(\alpha+\alpha_n)} \left\{ \left(\frac{\alpha_n}{\alpha}\right)^{\alpha/(\alpha+\alpha_n)} + \left(\frac{\alpha}{\alpha_n}\right)^{\alpha_n/(\alpha+\alpha_n)} \right\} \end{aligned}}$$

Hence easily $$\alpha_n = \frac{\alpha}{n+1}$$

$$c_{n+1} = (c_n\theta_n)^{\frac{n+1}{n+2}} \left\{ \left(\frac{1}{n+1}\right)^{\frac{n+1}{n+2}} + (n+1)^{\frac{1}{n+2}} \right\}$$

$$= (c_n\theta_n)^{\frac{n+1}{n+2}} \frac{n+2}{(n+1)^{(n+1)/(n+2)}},$$

and so

$$c_{n+1}^{n+2} = (c_n\theta_n)^{n+1} (n+2)^{n+2} / (n+1)^{n+1}$$

$$\boxed{\therefore \left(\frac{c_{n+1}}{n+2}\right)^{n+2} = \left(\frac{c_n}{n+1}\right)^{n+1} \theta_n^{n+1}, \qquad \theta_n \equiv E\, e^{-\alpha Y (n+1)^{-1}}}$$

Let's now assume that $EY^2 < \infty$, so that now

$$\left(\frac{c_n}{n+1}\right)^{n+1} = \prod_{r=1}^{n} \theta_{r-1}^{r} = E \exp\left(-\alpha \sum_{r=1}^{n} \frac{1}{r} \sum_{j=1}^{r} Y_j\right)$$

$$= e^{-n\alpha EY} E \exp\left[-\alpha \sum_{r=1}^{n} \frac{1}{r} \sum_{j=1}^{r} (Y_j - \mu)\right],$$

where $\mu = EY$.

Now if we set $E\, e^{-\alpha Y} \equiv e^{-\psi(\alpha)}$, then ψ is increasing, concave, and

$$\left(\frac{C_n}{n+1}\right)^{n+1} = \exp\left\{-\sum_{r=1}^{n} r\,\psi(\alpha/r)\right\}$$

so we're interested in

$$\log\left(\frac{C_n}{n+1}\right) = \frac{1}{n+1}\sum_{r=1}^{n} r\,\psi(\alpha/r)$$

$$= \frac{1}{n+1}\sum_{r=1}^{n}\int_0^{\alpha/r}\psi'(x)\, r\,dx$$

$$= \frac{1}{n+1}\sum_{r=1}^{n}\int_0^{\alpha}\psi'(y/r)\,dy$$

$$= \frac{1}{n+1}\sum_{r=1}^{n}\int_0^{\alpha}\left\{+\psi'(0) + \int_0^{y/r}\psi''(t)\,dt\right\}dy$$

$$= \frac{n\alpha}{n+1}\psi'(0) + \frac{1}{n+1}\sum_{r=1}^{n}\int_0^{\alpha}dy\int_0^{y/r}\psi''(t)\,dt$$

$$\to \alpha\psi'(0) = -\alpha EY,\quad n\to\infty.$$

Thus $$\boxed{\frac{C_n}{n+1}\,e^{\alpha EY} \to 1}$$ so in particular, $\frac{V_n(0)}{n} \to -e^{-\alpha EY}$

(ii) The asymptotics of this look very delicate, so let's assume further that $\boxed{Y \sim \exp(1)}$

In this case, then,

$$\theta_n = \frac{n+1}{\alpha+n+1} = 1 - \frac{\alpha}{n+1+\alpha}$$

and

$$(n+1)\log\left(\frac{C_n}{n+1}\right) = \sum_{r=1}^{n} r\log\left(1-\frac{\alpha}{r+\alpha}\right)$$

Now observe that

$$\left|\log(1+t) - t + t^2/2\right| \le \tfrac{2}{3}t^3 \qquad \text{if } |t|\le\tfrac12,$$

so we can bound the summands by considering

$$|b_r| \equiv \left|\log\left(1-\frac{\alpha}{r+\alpha}\right) + \frac{\alpha}{r} - \frac{\alpha^2}{2r^2}\right| = \left|\log\left(1-\frac{\alpha}{r+\alpha}\right) + \frac{\alpha}{r+\alpha} + \frac{\alpha^2}{2(r+\alpha)^2}\right.$$

$$\left. + \frac{\alpha}{r} - \frac{\alpha}{r+\alpha} - \frac{\alpha^2}{2r^2} - \frac{\alpha^2}{2(r+\alpha)^2}\right|$$

$$= \left| \log\left(1 - \frac{\alpha}{r+\alpha}\right) + \frac{\alpha}{r+\alpha} + \frac{\alpha^2}{2(r+\alpha)^2} - \tfrac{1}{2}\alpha^2 \left(\frac{\alpha}{r(r+\alpha)} \right)^2 \right|$$

$$\leq \left(\frac{\alpha}{r+\alpha} \right)^3 \quad \text{for large enough } r.$$

Thus $\sum r|b_r| < \infty$, and

$$(n+1)\log\left(\frac{c_n}{n+1}\right) = \sum_{r=1}^{n} \left(-\alpha + \frac{\alpha^2}{2r} \right) + \sum_1^n r b_r$$

$$= -n\alpha + \frac{\alpha^2}{2}\log n + q_n ,$$

where the q_n converge to a finite limit. Thus

$$\frac{c_n}{n+1} = \exp\left\{ -\alpha + \frac{\alpha}{n+1} + \frac{\alpha^2}{2(n+1)} \log n + \frac{q_n}{n+1} \right\}$$

and so

$$c_n - ne^{-\alpha} = (n+1)\left[\exp\left\{ \frac{\alpha + q_n}{n+1} + \frac{\alpha^2}{2(n+1)} \log n \right\} - 1 \right] e^{-\alpha} + e^{-\alpha},$$

$$\sim e^{-\alpha}\left(\alpha + q_n + \frac{\alpha^2}{2} \log n \right) + e^{-\alpha} \qquad (n \to \infty).$$

The point of this is that, contrary to an earlier conjecture,

$$\boxed{c_n - n\,U(EY) \text{ does not converge!}}$$

Of course, if we observe the constraint $a \geq 0$, it <u>might</u> be OK? No; this constraint can only <u>lower</u> the maximised expected utility, so that busts the conjecture completely.

Note also
$$\begin{pmatrix} \frac{2}{\varepsilon^2} V & -I \\ \frac{2}{\varepsilon^2} Q & 0 \end{pmatrix} \begin{pmatrix} I & -I \\ \Gamma_+ & \Gamma_- \end{pmatrix} \begin{pmatrix} \Gamma_+ & \cdot \\ \cdot & -\Gamma_- \end{pmatrix} = \frac{2}{\varepsilon^2} \begin{pmatrix} Q & \cdot \\ \cdot & Q \end{pmatrix} \begin{pmatrix} I & -I \\ \Gamma_+ & \Gamma_- \end{pmatrix}$$

If $S = \begin{pmatrix} \Gamma_+ & \Gamma_- \\ -I & I \end{pmatrix}$, then $S^{-1} = \begin{pmatrix} \Gamma_+ + \Gamma_- & \cdot \\ \cdot & \Gamma_+ + \Gamma_- \end{pmatrix}^{-1} \begin{pmatrix} I & -\Gamma_- \\ I & \Gamma_+ \end{pmatrix}$

Re expressing the first, with $2/\varepsilon^2$ set to 1 for simplicity,
$$\begin{pmatrix} 0 & Q \\ I & V \end{pmatrix} S \begin{pmatrix} \Gamma_+ & \cdot \\ \cdot & -\Gamma_- \end{pmatrix} = \begin{pmatrix} -Q & \cdot \\ \cdot & Q \end{pmatrix} S$$

and this is actually ~~an~~ equivalent way to state the WH factorisation.

Reversing noisy WH (28/7/92)

(i) The equations satisfied by $\Gamma_\pm$

$$\left.\begin{array}{l} \frac{1}{2}\varepsilon^2 \Gamma_+^2 - V\Gamma_+ + Q = 0 \\ \frac{1}{2}\varepsilon^2 \Gamma_-^2 + V\Gamma_- + Q = 0 \end{array}\right\}$$

Can be combined into a single matrix equation

$$\begin{pmatrix} \frac{2}{\varepsilon^2}V & \frac{2}{\varepsilon^2}Q \\ -I & 0 \end{pmatrix}\begin{pmatrix} \Gamma_+ & \Gamma_- \\ -I & I \end{pmatrix} = \begin{pmatrix} \Gamma_+ & \Gamma_- \\ -I & I \end{pmatrix}\begin{pmatrix} \Gamma_+ & \cdot \\ \cdot & -\Gamma_- \end{pmatrix}.$$

Let's now assume that $\boxed{m(E_+) > m(E_-)}$ so that Γ_+ is recurrent, Γ_- is transient ($\therefore$ invertible). If we multiply on the left by $(m, \frac{2}{\varepsilon^2}mV)$, we get zero on LHS, and on RHS

$$\left(m\Gamma_+ - \frac{2}{\varepsilon^2}mV,\ m\Gamma_- + \frac{2}{\varepsilon^2}mV\right)\begin{pmatrix} \Gamma_+ & \cdot \\ \cdot & -\Gamma_- \end{pmatrix} = 0$$

Hence
$$m\Gamma_- + \frac{2}{\varepsilon^2}mV = 0, \text{ and therefore } \boxed{m(\Gamma_+ + \Gamma_-)\,\Gamma_+ = 0.}$$

Since $\Gamma_+ + \Gamma_-$ is a Q-matrix which is ~~transient~~, it is certainly invertible, so we conclude

$$\boxed{\text{the invariant law of } \Gamma_+ \text{ is } \propto\ m(\Gamma_+ + \Gamma_-).}$$

(ii) Suppose now we set $\hat{Q} = M^{-1}Q^T M$, the reversal of Q, and set $\hat{V} = -V$. Then I claim that if $\hat{\Gamma}_\pm$ are the forms of $\Gamma_\pm$ for $\hat{Q}$, then

$$\boxed{\hat{\Gamma}_\pm = M^{-1}\left[(\Gamma_+ + \Gamma_-)\Gamma_\mp(\Gamma_+ + \Gamma_-)^{-1}\right]^T M.}$$

The first way I had of proving this was by excursion arguments. The essence of these can be re-expressed in the following way, which is much more direct. There will be a local time at levels for the φ process, and if we consider excursions away from level 0 which start with the X process in state i, but finish with X in a different state j, then rate of xcns up from 0, starting in i and finishing in j

$$= \lim \frac{1}{2\varepsilon} P^i(Y_\varepsilon^- = j)$$

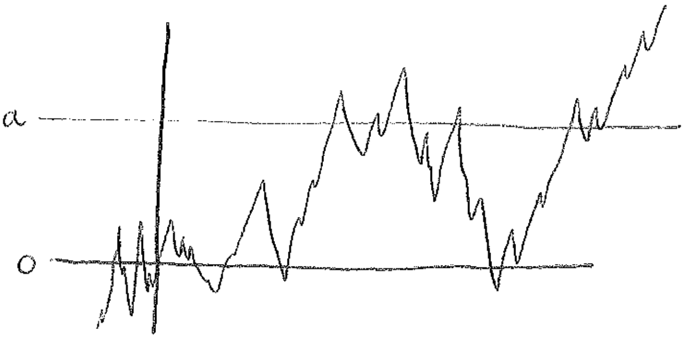
$$= \tfrac{1}{2}\Gamma_-(i,j),$$

and likewise the rate of excns down from 0, starting in i, returning in $j \neq i$, is $\frac{1}{2}\Gamma_+(i,j)$. So if we set

$$J \equiv \tfrac{1}{2}(\Gamma_+ + \Gamma_-),$$

then time-change X by $L(\cdot, 0)$, we see a ~~transient~~ Markov chain with Q-matrix J.

(iii) Now let's fix $a>0$, and time-change X by $L(t,0)+L(t,a)$. We see a MC on $E_0 \cup E_a$, where E_0 and E_a are copies of E. The chain is transient, and has Green f^n



$$\begin{pmatrix} I & e^{a\Gamma_+} \\ e^{a\Gamma_-} & I \end{pmatrix}\begin{pmatrix} -J^{-1} & \cdot \\ \cdot & -J^{-1} \end{pmatrix}.$$

Thus the Q-matrix is simply

$$\begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}\begin{pmatrix} I-e^{a\Gamma_+}e^{a\Gamma_-} & \cdot \\ \cdot & I-e^{a\Gamma_-}e^{a\Gamma_+} \end{pmatrix}^{-1}\begin{pmatrix} I & -e^{a\Gamma_+} \\ -e^{a\Gamma_-} & I \end{pmatrix}$$

$$\equiv \begin{pmatrix} Z_{00} & Z_{0a} \\ Z_{a0} & Z_{aa} \end{pmatrix},$$

say, with

$$\begin{cases} Z_{00} = J(I-e^{a\Gamma_+}e^{a\Gamma_-})^{-1}, & Z_{0a} = -J(I-e^{a\Gamma_+}e^{a\Gamma_-})^{-1}e^{a\Gamma_+} \equiv -Z_{00}e^{a\Gamma_+} \\ Z_{a0} = -J(I-e^{a\Gamma_-}e^{a\Gamma_+})^{-1}e^{a\Gamma_-} = -Z_{aa}e^{a\Gamma_-}, & Z_{aa} = J(I-e^{a\Gamma_-}e^{a\Gamma_+})^{-1} \end{cases}$$

If we now adjoin a graveyard state ∞ to the space $E_0 \cup E_a$, we see the augmented Q-matrix as

$$\begin{array}{c} \left(\begin{array}{c|c|c} Z_{00} & Z_{0a} & 0 \\ Z_{a0} & Z_{aa} & Z_{a\infty} \\ \hline 0 & 0 & 0 \end{array}\right) \end{array} \begin{array}{c} E_0 \\ E_a \\ \infty \end{array}$$

where

$$Z_{a\infty} = -\tfrac{1}{2}\Gamma_- 1 = -J1 .$$

We can check this either by a simple excursion argument of the type used in (ii) or else by confirming that this makes row sums zero

(iv) Now let's find

$$P[\,X = j \text{ when } \varphi \text{ last leaves } 0,\ X = k \text{ when } \varphi \text{ last leaves } a \,/\, X_0 = i,\ \varphi_0 = 0\,]$$

$$= \sum_{n\geq 0} \left(-Z_{00}^{-1} Z_{0a} (-Z_{aa})^{-1} Z_{a0}\right)^n (-Z_{00})^{-1}(i,j)\ Z_{0a}(-Z_{aa})^{-1}(j,k)\ Z_{a\infty}(k)$$

$$= -\left(I - Z_{00}^{-1} Z_{0a} Z_{aa}^{-1} Z_{a0}\right)^{-1} Z_{00}^{-1}(i,j)\ \ Z_{0a}(-Z_{aa})^{-1}(j,k)\, Z_{a\infty}(k)$$

$$= -(I - e^{a\Gamma_+} e^{a\Gamma_-})^{-1}\ Z_{00}^{-1}(i,j)\left(Je^{a\Gamma_+}J^{-1}\right)(j,k)\ (-J1)_k$$

$$= -J^{-1}(i,j)\ \left(J e^{a\Gamma_+} J^{-1}\right)(j,k)\ (-J1)_k$$

and if we mix this over the law of X_0, which should be taken as the invariant law of Γ_+, which is expressed as

$$\nu = c\, m J$$

for some constant c, $c \equiv (mJ1)^{-1}$, we obtain

$$\boxed{\begin{aligned} &P[\,X = j \text{ when } \varphi \text{ last leaves } 0,\ X = k \text{ when } \varphi \text{ last leaves } a\,] \\ &= -c\, m_j\ \left(Je^{a\Gamma_+}J^{-1}\right)(j,k)\ (-J1)_k . \end{aligned}}$$

Summing over j yields the helpful information that

$$\boxed{P[X = k \text{ when } \varphi \text{ last leaves } a] = P[\hat{Y}_- = k] = c\, m_k (J1)_k}$$

and from this

$$\boxed{\begin{aligned} &P(X = j \text{ when } \varphi \text{ last leaves } 0 \mid X = k \text{ when } \varphi \text{ last leaves } a) \\ &\quad = m_j \, (J e^{a\Gamma_+} J^{-1})(j,k) / m_k \end{aligned}}$$

From this, the asserted form of $\hat{\Gamma}_-$ follows immediately.

(v) And now to find $\hat{\Gamma}_+$. The argument is structurally similar, except that now if we set $\sigma_0 = \sup\{t : \varphi_t = 0\}$, $\gamma_a = \sup\{t < \sigma_0 : \varphi_t = a\}$, the goal is to calculate

$$P\big(X(\sigma_0) = k,\ X(\gamma_a) = j \,\big|\, X_0 = i,\ \varphi_0 = 0 \big)$$

$$= \sum_{n \geq 0} \left(Z_{00}^{-1} Z_{0a} Z_{aa}^{-1} Z_{a0} \right)^n (-Z_{00}^{-1}) Z_{0a} (-Z_{aa}^{-1})(i,j) \; Z_{a0}(-Z_{00}^{-1})(j,k) \,.$$

$$Z_{0a}(-Z_{aa}^{-1}) Z_{a\infty} \; (k)$$

$$= \left(I - Z_{00}^{-1} Z_{0a} Z_{aa}^{-1} Z_{a0} \right)^{-1} Z_{00}^{-1} Z_{0a} Z_{aa}^{-1} \, (i,j) \, \left(Z_{a0} Z_{00}^{-1} \right)(j,k) \; Z_{0a} Z_{aa}^{-1} Z_{a\infty} \, (k)$$

$$= \left(-e^{a\Gamma_+} J^{-1} \right)(i,j) \, \left(-J e^{a\Gamma_-} J^{-1} \right)(j,k) \, (J1)(k) \,.$$

Mixing now over i with law ν yields

$$\boxed{P\big(X(\sigma_0) = k,\ X(\gamma_a) = j \big) = c\, m_j \left(J e^{a\Gamma_-} J^{-1} \right)(j,k) \; (J1)_k}$$

Hence

$$\boxed{P\big(X(\gamma_a) = j \mid X(\sigma_0) = k \big) = m_j \left(J e^{a\Gamma_-} J^{-1} \right)_{jk} / m_k \,.}$$

The confirms the other claim – BUT IS THERE A PURELY ALGEBRAIC PROOF?

<u>Some thoughts on Hans Bühlmann's discrete-time asset-pricing model</u> (3/8/92).

Take a discrete-time filtered prob^y space $(\Omega, (\mathcal{F}_n), P)$ on which is defined an adapted strictly positive process φ. The definition of the price at time m of a (bounded) $\mathcal{F}_n$-meas r.v. X received at time $n \geq m$ is

$$\frac{1}{\varphi_m} E(\varphi_n X \mid \mathcal{F}_m)$$

(i) How should we interpret this? The no-arbitrage pricing paradigm says that there is some stochastic discounting process β_n and an equivalent martingale measure $\tilde{P}$ such that the fair price to pay at time $m < n$ for amount X at time n will be

$$\beta_m^{-1} \tilde{E}(\beta_n X \mid \mathcal{F}_m).$$

In this case, if $d\tilde{P}/dP = Z$, and $Z_n = E(Z/\mathcal{F}_n)$, we have

$$\beta_m^{-1} \tilde{E}(\beta_n X \mid \mathcal{F}_m) = (\beta_m Z_m)^{-1} E(\beta_n Z_n X \mid \mathcal{F}_m)$$

so we have the identification

$$\boxed{\varphi_n = \beta_n Z_n.}$$

(ii) Notice particularly that the stochastic discounting process β must be <u>previsible</u> (a fact which is rather obscured in the continuous semimartingale setting). We can understand this through a simple hedging argument using one-period bonds. Let's suppose that the price on day t of a bond yielding 1 on day $t+1$ is $P(t, t+1) \in \mathcal{F}_t$. Suppose now that a marketed asset has price Y_n on day n. Then I claim that

$$P(Y_{t+1} P(t,t+1) \geq Y_t) = 1$$

gives arbitrage, unless $Y_{t+1} P(t,t+1) = Y_t$. Why? At time t, buy one unit of the asset (cost Y_t) by selling $Y_t / P(t,t+1)$ of the bonds. At time $t+1$, your portfolio is worth

$$Y_{t+1} - Y_t / P(t, t+1).$$

So to preclude arbitrage, it is necessary that

$$P(\, Y_{t+1}\; P(t, t+1) \geq Y_t\,) < 1$$

and this, by a familiar argument, is enough to ensure that there is an equivalent (martingale) measure such that

$$\tilde{E}\left(\, Y_{t+1}\; P(t, t+1) \;\middle|\; \mathcal{F}_t \,\right) = Y_t \,.$$

Thus the stochastic discount process is

$$\beta_n = \prod_{k=1}^{n} P(k-1, k) \quad , \; \beta_0 = 1,$$

which is $\mathcal{F}_{n-1}$-measurable, and $\beta_n Y_n$ is a $\tilde{P}$-martingale.

(iii) We can just as well express β, Z in terms of φ, rather than the other way around:

$$E(\varphi_n \,|\, \mathcal{F}_{n-1}) = \beta_n Z_{n-1} \qquad (\beta_n \in \mathcal{F}_{n-1},\; Z \text{ is a } P\text{-mg})$$

so that

$$\boxed{\begin{aligned} \beta_n / \beta_{n-1} &= \frac{E(\varphi_n \,|\, \mathcal{F}_{n-1})}{\varphi_{n-1}} \\ \text{and} \quad Z_n / Z_{n-1} &= \varphi_n \,/\, E(\varphi_n \,|\, \mathcal{F}_{n-1}) \end{aligned}}$$

More transparently,

$$\mathrm{Cov}\left(\begin{pmatrix} B_s \\ B_t \end{pmatrix} \middle| A\right) = \begin{pmatrix} s - \frac{3s^2}{4T^3}(2T-s)^2 & s - \frac{3st}{4T^3}(2T-s)(2T-t) \\ s - \frac{3st}{4T^3}(2T-s)(2T-t) & t - \frac{3t^2}{4T^3}(2T-t)^2 \end{pmatrix}$$

For large t, small s, B_s and B_t are negatively correlated.

More tricks for bounding the price of the Asian option? (14/8/92)

(i) Let's try conditioning on the value of $\int_0^T X_u\,du$, where $X_t = B_t + \mu t$ as before. If $A \equiv \int_0^T B_u\,du$ then for $0 < s < t < T$, we get

$$E\left(\begin{pmatrix} B_s \\ B_t \end{pmatrix} \Big| A = a \right) = \frac{3a}{2T^3} \begin{pmatrix} s\,(2T-s) \\ t\,(2T-t) \end{pmatrix}$$

$$\mathrm{cov}\left(\begin{pmatrix} B_s \\ B_t \end{pmatrix} \Big| A = a \right) = \begin{pmatrix} s\left(\frac{T-s}{T}\right)^3 + \frac{s^4}{4T^3} & s - \frac{3st}{4T^3}(2T-s)(2T-t) \\ s - \frac{3st}{4T^3}(2T-s)(2T-t) & t\left(\frac{T-t}{T}\right)^3 + \frac{t^4}{4T^3} \end{pmatrix}$$

This gives us for the lower bound

$$E\left[\int_0^T (S_0\, e^{\sigma X_u} - K)\,du \;\Big|\; \int_0^T X_u\,du = x \right]$$

$$\left[a \equiv x - \mu T^2/2 \right]$$

$$= -KT + S_0 \int_0^T \exp\left\{ \sigma\mu u + \frac{3a}{2T^3}\,\sigma\, u(2T-u) + \tfrac{1}{2}\sigma^2\left(u\left(\frac{T-u}{T}\right)^3 + \frac{u^4}{4T^3} \right) \right\} du$$

$$= -KT + S_0 T \int_0^1 \exp\left[\sigma\mu T s + \frac{3a\sigma}{2T}\, s(2-s) + \tfrac{1}{2}\sigma^2 T\left\{ s(1-s)^3 + s^4 \right\} \right] ds.$$

This will be hard to evaluate, except numerically. For the second moment, we need to calculate

$$E\left[\int_0^T du \int_0^T dv\; e^{\sigma(X_u + X_v)} \;\Big|\; \int_0^T X_u\,du = x \right]$$

$$= 2\int_0^T du \int_u^T dv\; \exp\Big[\sigma\,\mu(u+v) + \frac{3a\sigma}{2T^3}\left\{ u(2T-u) + v(2T-v) \right\}$$

$$+ \tfrac{1}{2}\sigma^2 \left\{ 3u + v - \frac{3}{4T^3}\left(u(2T-u) + v(2T-v) \right)^2 \right\} \Big].$$

Again, only numerics are feasible.

(ii) Suppose we condition on $\bar{X}_T = b$, $\bar{X}_T - X_T = a$; what can we get? If we were to make T an indep't $\exp(\lambda)$ r.v., then the piece of Brownian path until it hits b is a BM with drift $\sqrt{\mu^2+2\lambda}$, and run back from T until it hits b, we again see a BM with this same drift. If we run a BM with drift $-\gamma$ started at b until it first hits 0, the local time process solves

$$dZ_x = 2\sqrt{Z_x}\,dW_x + 2\{I_{[0,b]}(x) - \gamma Z_x\}\,dx, \quad Z_0 = 0.$$

Thus if

$$\rho_x \equiv E Z_x$$

we have

$$\rho'_x = 2\left(I_{[0,b]}(x) - \gamma \rho_x\right), \quad \rho_0 = 0$$

so that

$$\boxed{\rho(x) = \frac{1}{\gamma} e^{-2\gamma x}\left\{ e^{2\gamma(x\wedge b)} - 1\right\}}$$

Hence

$$E\left(\int_0^T f(X_u)\,du \;\middle|\; \bar{X}_T = b,\ \bar{X}_T - X_T = a\right)$$

$$= E\left[\int_0^T f(X_u)\,du : \bar{X}_T = b,\ \bar{X}_T - X_T = a\right] \Big/ P(\bar{X}_T = b,\ \bar{X}_T - X_T = a)$$

$$= \int_0^\infty \frac{e^{-2\gamma y}}{\gamma}\left[e^{2\gamma(y\wedge b)} + e^{2\gamma(y\wedge a)} - 2\right] f(b-y)\,dy,$$

so that

$$E\left[\int_0^T f(X_u)\,du : \bar{X}_T = b,\ \bar{X}_T - X_T = a\right]$$

$$= 2\lambda e^{-b(\sqrt{\mu^2+2\lambda}-\mu) - a(\sqrt{\mu^2+2\lambda}+\mu)} \cdot \int_0^\infty \frac{e^{-2\gamma y}}{\gamma}\left[e^{2\gamma(y\wedge b)} + e^{2\gamma(y\wedge a)} - 2\right] f(b-y)\,dy$$

where $\gamma \equiv \sqrt{\mu^2+2\lambda}$, of course. Now this Laplace transform can be inverted; for $\xi > 0$,

$$e^{-\gamma\xi} = \int_0^\infty e^{-(\lambda+\frac{1}{2}\mu^2)t}\,\frac{\xi e^{-\xi^2/2t}}{\sqrt{2\pi t^3}}\,dt,$$

so

$$\frac{e^{-\gamma\xi}}{\gamma} = \int_0^\infty e^{-\lambda t - \mu^2 t/2}\, e^{-\xi^2/2t}\,\frac{dt}{\sqrt{2\pi t}},$$

so that

$$E\left[L(t,b-y) : \bar{X}_t = b,\ \bar{X}_t - X_t = a\right] = \frac{2e^{\mu(b-a)-\mu^2 t/2}}{\sqrt{2\pi t}} \cdot \left\{ e^{-(b+a+2(y-b)^+)^2/2t} + e^{-(b+a+2(y-a)^+)^2/2t} - 2e^{-(b+a+2y)^2/2t} \right\}.$$

The density of $(\bar{X}_t,\ \bar{X}_t - X_t)$ is

$$e^{\mu(b-a)-\mu^2 t/2} \cdot \frac{2(a+b)\, e^{-(a+b)^2/2t}}{\sqrt{2\pi t^3}},$$

so we get

$$\boxed{\begin{aligned} &E\left[L(t,b-y) \mid \bar{X}_t = b,\ \bar{X}_t - X_t = a\right] \\ &= \frac{t}{a+b}\, e^{(a+b)^2/2t} \left\{ e^{-(b+a+2(y-b)^+)^2/2t} + e^{-(b+a+2(y-a)^+)^2/2t} - 2e^{-(b+a+2y)^2/2t} \right\} \end{aligned}}$$

(iii) Can we now compute

$$E\left[\left(\int_0^T f(X_u)\,du\right)^2 \,\Big|\, \bar{X}_T = b,\ \bar{X}_T - X_T = a \right] ?$$

Let's write the local time process, conditioned by $\bar{X}_T = b,\ \bar{X}_T - X_T = a$, as

$$L(T, b-y) = Z_y + \tilde{Z}_y$$

where $Z, \tilde{Z}$ are independent, with the dist^n described previously. What we want is

$$2\,E\left[\int_0^\infty f(b-y)\,dy \int_y^\infty f(b-v)\,dv\ (Z_y + \tilde{Z}_y)(Z_v + \tilde{Z}_v) \,\Big|\, \bar{X}_T = b,\ \bar{X}_T - X_T = a \right]$$

for which we need to compute for $0 \le y \le v$

$$E[Z_y Z_v] = E\left[Z_y\, E(Z_v \mid \mathcal{F}_y) \right]$$

$$= E\left[Z_y \left\{ e^{-2\gamma(v-y)} Z_y + \frac{e^{-2\gamma v}}{\gamma} \left(e^{2\gamma(b\wedge v)} - e^{2\gamma(b\wedge y)} \right) \right\} \right]$$

A few calculations yield

$$\boxed{E\, Z_y^2 = \frac{2e^{-4\gamma y}}{\gamma^2}\left(e^{2\gamma y}-1\right)\left(e^{2\gamma(y\wedge b)}-1\right)}$$

Hence for $0 \le y \le v$, conditional on $\bar{X}_T = b$, $\bar{X}_T - X_T = a$, we have

$$E\left[L(T,y)L(T,v)\right] = E\left[Z_y Z_v + Z_y \tilde{Z}_v + \tilde{Z}_y Z_v + \tilde{Z}_y \tilde{Z}_v\right]$$

$$= \frac{e^{-2\gamma(v+y)}}{\gamma^2}\Big[\left(e^{2\gamma(y\wedge b)}-1\right)\left\{2e^{2\gamma y}-2+e^{2\gamma(v\wedge b)}-e^{2\gamma(y\wedge b)}\right\}$$

$$+ \left(e^{2\gamma(y\wedge b)}-1\right)\left(e^{2\gamma(v\wedge a)}-1\right) + \left(e^{2\gamma(y\wedge a)}-1\right)\left(e^{2\gamma(v\wedge b)}-1\right)$$

$$+ \left(e^{2\gamma(y\wedge a)}-1\right)\left\{2e^{2\gamma y}-2+e^{2\gamma(v\wedge a)}-e^{2\gamma(y\wedge a)}\right\}\Big].$$

This can in principle be inverted using

$$\frac{e^{-\gamma\xi}}{\gamma^2} = \int_0^\infty e^{-\lambda t - \mu^2 t/2}\, \bar{\Phi}\left(\xi/\sqrt{t}\right) dt.$$

(iv) Suppose that we try to condition on the Gaussian variable

$$Z = \int_0^T \varphi_u \, dB_u$$

where φ is deterministic, in $L^2[0,T]$, and then

$$E(Z B_t) = \int_0^t \varphi_u \, du.$$

Thus

$$E\left[B_t \mid Z\right] = \tilde{\sigma}^{-2} Z \int_0^t \varphi_u \, du \qquad \left[\tilde{\sigma}^2 = \int_0^T \varphi_u^2 \, du\right]$$

and

$$E\left(B_s B_t \mid Z\right) - E(B_s \mid Z)E(B_t \mid Z)$$

$$= s \wedge t - \tilde{\sigma}^{-2}\left(\int_0^t \varphi_u \, du\right)\left(\int_0^s \varphi_v \, dv\right)$$

Thus $E\left[S_0 e^{\sigma X_t} \,\middle|\, Z=z\right] = S_0\, e^{rt + \sigma z g_t - \frac{1}{2}\sigma^2 g_t^2}$.

(v) It may also be worth recording that

$$P\left[X_t \in dx \,\middle|\, \bar{X}_T = b,\ \bar{X}_T - X_T = a\right] / dx$$

$$= \left\{ h(t, 2b-x)\ {}_b p(T-t;\, x,\, b-a) + {}_b p(t;\, 0, x)\ h(T-t,\, b-x+a) \right\} / h(T, b+a)$$

where $h(t,a)$ is Brownian first passage density.

(vi) Notice also that

$$E\,|Y| - |EY| \leq \sqrt{\operatorname{var}(Y)},$$

if it helps.

<u>Matrix proof for the noisy WH reversal?</u> (14/10/92)

Let's here write

$$J \equiv \Gamma_+ + \Gamma_-$$

(which is double the previous definition, but doesn't alter the suspected form of $\hat{\Gamma}_\pm$).
We'll also suppose when necessary that Q is non-singular. Write

$$S \equiv \begin{pmatrix} \Gamma_+ & \Gamma_- \\ -1 & 1 \end{pmatrix}, \quad \text{so} \quad S^{-1} = \begin{pmatrix} J & \cdot \\ \cdot & J \end{pmatrix}^{-1} \begin{pmatrix} 1 & -\Gamma_- \\ 1 & \Gamma_+ \end{pmatrix}.$$

Assume for notational ease that $\varepsilon = \sqrt{2}$. The WH factorisation says

$$\boxed{\begin{pmatrix} V & Q \\ -1 & 0 \end{pmatrix} S = S \begin{pmatrix} \Gamma_+ & \cdot \\ \cdot & -\Gamma_- \end{pmatrix}},$$

which we now have to rework to the analogous thing for $\hat{V} = -V$, $\hat{Q} \equiv M^{-1} Q^T M$.
Re-express the WH factorisation as

$$\begin{pmatrix} -\hat{V} & M^{-1}\hat{Q}^T M \\ -1 & 0 \end{pmatrix} S = S \begin{pmatrix} \Gamma_+ & \cdot \\ \cdot & -\Gamma_- \end{pmatrix} \equiv S\,\mathcal{U}, \text{ say},$$

$$U = \begin{pmatrix} \Gamma_+ & \cdot \\ \cdot & -\Gamma_- \end{pmatrix} \qquad \tilde{S} = S^T \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}$$

$$\hat{S}^{-1} = \begin{pmatrix} \hat{J} & \cdot \\ \cdot & \hat{J} \end{pmatrix}^{-1} \begin{pmatrix} 1 & -\hat{\Gamma}_- \\ 1 & \hat{\Gamma}_+ \end{pmatrix} \qquad \hat{J} \equiv \hat{\Gamma}_+ + \hat{\Gamma}_- \equiv M^{-1} J^T M .$$

Essential problem is to prove that

$$\boxed{J \Gamma_-^2 J^{-1} + J \Gamma_- J^{-1} V + Q = 0 .}$$

so that $$S^T \begin{pmatrix} -\hat{V} & -1 \\ M\hat{Q}M^{-1} & 0 \end{pmatrix} = U^T S^T$$

$$\therefore \quad S^T \begin{pmatrix} M & \cdot \\ \cdot & M \end{pmatrix} \begin{pmatrix} -\hat{V} & -1 \\ \hat{Q} & 0 \end{pmatrix} = U^T S^T \begin{pmatrix} M & \cdot \\ \cdot & M \end{pmatrix}$$

$$\therefore \quad \begin{pmatrix} -\hat{V} & -1 \\ \hat{Q} & 0 \end{pmatrix} = \left(S^T \begin{pmatrix} M & \cdot \\ \cdot & M \end{pmatrix}\right)^{-1} U^T \left(S^T \begin{pmatrix} M & \cdot \\ \cdot & M \end{pmatrix}\right) \equiv \tilde{S}^{-1} U^T \tilde{S}$$

Now invert both sides:

$$\begin{pmatrix} 0 & 1 \\ -1 & -\hat{V} \end{pmatrix} \begin{pmatrix} 1 & \cdot \\ \cdot & \hat{Q}^{-1} \end{pmatrix} = \tilde{S}^{-1} (U^T)^{-1} \tilde{S}$$

so that

$$\begin{pmatrix} 0 & 1 \\ -1 & -\hat{V} \end{pmatrix} = \tilde{S}^{-1} (U^T)^{-1} \tilde{S} \begin{pmatrix} 1 & \cdot \\ \cdot & \hat{Q} \end{pmatrix}$$

and $$\begin{pmatrix} -\hat{V} & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tilde{S}^{-1} (U^T)^{-1} \tilde{S} \begin{pmatrix} 1 & \cdot \\ \cdot & \hat{Q} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

whence $$\begin{pmatrix} \hat{V} & \hat{Q} \\ -I & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tilde{S}^{-1} (U^T)^{-1} \tilde{S} \begin{pmatrix} 0 & 1 \\ \hat{Q} & 0 \end{pmatrix} \begin{pmatrix} 1 & \cdot \\ \cdot & \hat{Q} \end{pmatrix}$$

$$= -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tilde{S}^{-1} (U^T)^{-1} \tilde{S} \begin{pmatrix} 0 & \hat{Q} \\ \hat{Q} & 0 \end{pmatrix}. \qquad (*)$$

Now we <u>think</u> that

$$\hat{S} \equiv \begin{pmatrix} \hat{\Gamma}_+ & \hat{\Gamma}_- \\ -I & I \end{pmatrix} = \begin{pmatrix} M^{-1}(J\Gamma_- J^{-1})^T M & M^{-1}(J\Gamma_+ J^{-1})^T M \\ -I & I \end{pmatrix}$$

$$(R \equiv J^T M) \qquad = \begin{pmatrix} R & \cdot \\ \cdot & R \end{pmatrix}^{-1} \begin{pmatrix} \Gamma_-^T & \Gamma_+^T \\ -1 & 1 \end{pmatrix} \begin{pmatrix} R & \cdot \\ \cdot & R \end{pmatrix},$$

so let's right-multiply (*) by this matrix and simplify.

A problem considered by Jean Bertoin (30/10/92).

1) Take a spectrally negative Lévy process X with scale function s:

$$P^x [\, X \text{ hits } a \text{ before } 0 \,] = s(x)/s(a) \qquad 0 \le x \le a.$$

When does this process have points of increase? Jean says that it should have points of increase iff

$$(*) \qquad \boxed{\int_{0+} \frac{dx}{s(x)} < \infty}.$$

2) If we consider the Lévy measure of the depths of excursions down from the maximum, then

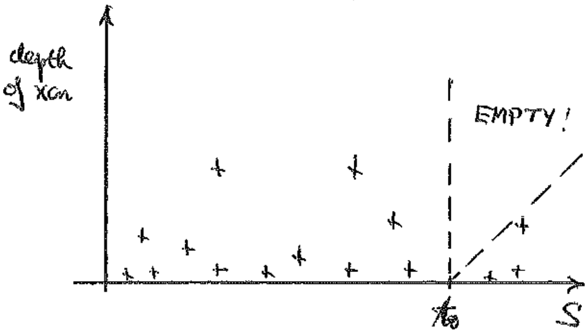
$\bar{\mu}(x) \equiv$ mass of excursions getting down to $-x < 0$

$$= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \frac{s(x) - s(x-\varepsilon)}{s(x)}$$

$$= s'(x)/s(x).$$

3) We prove:

$$\text{if } \int_{0+} \frac{dx}{s(x)} = +\infty \quad \underline{\text{then}} \quad X \ \underline{\text{has no points of increase}}.$$



Let

$$S_t = \sup\{X_s : s \le t\}$$

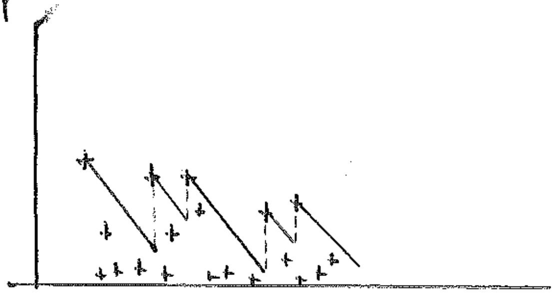
which is a local time at zero for the process $S - X$.

To say that X has a point of increase is equivalent to the statement that there exists some t_0 such that the region $\{0 \le x - t_0 \le y \le 1\}$ contains no points.

Each point (x,y) of the point process therefore blocks out an interval $(x-y, x)$ of the S-axis; there can be no point of increase in such an interval. To prove that there is no point of increase, then, we must prove that all of the S-axis gets blocked out.

An equivalent and more helpful way to view it is that each point (x,y) of the point process blocks out $(x, x+y)$. Now we can make a Markov jump process ξ_t, where ξ_t = time we would have to wait until S-axis is unblocked if no further jumps (= points) arrive.

A picture explains:



The process jumps from x to $y > x$ at rate $\mu(dy)$.

If

$$\varphi(x) \equiv P^x(\text{reach } 0 \text{ before } [1,\infty))$$

then

$$\varphi(\xi_{t\wedge T}) \text{ \underline{is a martingale}} \qquad T \equiv \inf\{t : \xi_t \geq 1\}$$

Thus

$$\mathcal{G}\varphi(x) \equiv -\varphi'(x) + \int_x^\infty \mu(dy)(\varphi(y) - \varphi(x)) = 0$$

whence

$$-\varphi''(x) - \varphi'(x)\,\bar{\mu}(x) = 0$$

$$\therefore \quad \varphi'(x) = A \exp\left(\int_x^1 \bar{\mu}(t)\,dt\right)$$

$$= A \exp\left([\log S(t)]_x^1\right)$$

$$= A / S(x).$$

Since $\int_{0+} S(x)^{-1}\,dx = +\infty$, the only possibility is $\varphi \equiv 0$.

4) We prove:

$$\text{if } \int_{0+} \frac{dx}{S(x)} < \infty \text{ \underline{then points of increase happen.}}$$

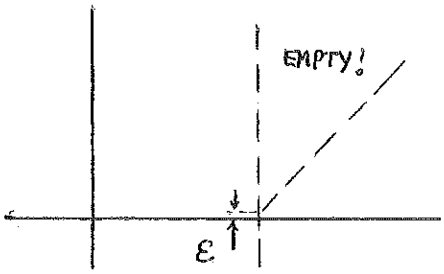
We aim to construct a random measure on the points of increase, which will be non-trivial, and therefore the set of points of increase is not a.s. empty.

Let N be the Poisson process in $\mathbb{R}^+ \times \mathbb{R}^+$ with intensity $dx \times \mu(dy)$.

For $\varepsilon > 0$, define

$$A_\varepsilon \equiv \int_0^1 ds \; I_{\{ N(\{(s+x, y) : 0 \le x \le 1, \; y \ge x+\varepsilon\}) = 0 \}}$$

which counts the time that there is an "almost-point-of-increase"



Easily,

$$E A_\varepsilon = P\left[N\{(x,y) : 0 \le x \le 1, \; y \ge x+\varepsilon\} = 0\right]$$

$$= \exp(-\alpha_\varepsilon),$$

where $\alpha_\varepsilon = \int_0^1 \bar{\mu}(x+\varepsilon)\,dx$.

The idea now is similar to that used in "Multiple points of Markov processes in a complete metric space" – we shall prove that

$$\{ A_\varepsilon / EA_\varepsilon \; : \; 0 < \varepsilon < 1 \} \quad \underline{\text{is bounded in}} \; L^2$$

and therefore is UI. If there were no points of increase, then $A_\varepsilon \to 0$ a.s. and so

$$A_\varepsilon / EA_\varepsilon \xrightarrow[L^1]{a.s.} 0 \qquad \text{by uniform integrability.}$$

But $E(A_\varepsilon / EA_\varepsilon) = 1$, a contradiction.

So now we must estimate $E[A_\varepsilon^2.]$ Define

$$Y_\ell \equiv Y_\ell^\varepsilon = I_{\{ N(\{(\ell+x, y) : 0 \le x \le 1, \; y \ge x+\varepsilon\}) = 0 \}}$$

so that $A_\varepsilon = \int_0^1 Y_s^\varepsilon \, ds$.

Then

$$E(A_\varepsilon^2) = 2 \int_0^1 ds \int_s^1 dt \; E(Y_s^\varepsilon Y_t^\varepsilon)$$

$$= 2 \int_0^1 ds \int_s^1 dt \; E(Y_t^\varepsilon) \exp\left\{ - \int_0^{t-s} \bar{\mu}(x+\varepsilon)\,dx \right\}$$

so that

$$E(A_\varepsilon^2)/(EA_\varepsilon)^2 = 2\int_0^1 ds \int_s^1 dt \; \exp\left(\alpha_\varepsilon - \int_0^{t-s} \bar\mu(x+\varepsilon)\,dx\right)$$

$$= 2\int_0^1 ds \int_s^1 dt \; \exp\left(\int_{t-s}^1 \bar\mu(x+\varepsilon)\,dx\right)$$

$$\uparrow 2\int_0^1 ds \int_s^1 dt \; \exp\left(\int_{t-s}^1 \bar\mu(x)\,dx\right) < \infty \qquad \text{as } \varepsilon \downarrow 0.$$

In fact, the proof is complete after §3 ; we don't need the last argument. Jean has a proof of this in Stochastics 37 247–251, relying on earlier work of Slepp, but essentially using the same ideas.

Some ideas on monotone couplings of random walks. (3/11/92)

The aim is to take two random walks with identical step distributions but different starting points $x \le x'$, and make them couple if possible, or come close if not.

1) Suppose we fix some large N, and make the two random walks S_n and S_n' jump the same amount if $|X_n| \equiv |S_n - S_{n-1}| \ge N$. Only if there are smaller jumps will $S_n' - S_n$ then change.

We thus reduce to the situation where the jump distribution is compactly supported, in $[-N, N]$.

2) We now describe a method which makes the two rws come close together, while maintaining the inequality $S_n' \ge S_n$. Actual coupling will require more work.

PROPOSITION. Consider the problem

$$v(a) \equiv \max\left\{ \mathrm{var}(X-Y) \,;\quad Y - X \ge -a, \quad X, Y \sim F \right\} \qquad (a \ge 0)$$

where F is the given (compactly-supported) jump distribution. Then v is right continuous and increasing, and the maximum is attained.

Proof Since supp(F) is compact, variance of $X-Y$ is a bounded continuous functional on the set of probability measures on $[-N,N]^2$. The extremum is therefore attained, and monotonicity is obvious. ☐

Let μ_a be a maximising measure for the problem:

$$\mu_a((X,Y) : Y - X \ge -a) = 1,$$

$X, Y \sim F$ under μ_a, $\quad var(X-Y) \equiv \iint (x-y)^2 \mu_a(dx, dy) = v(a)$.

Now the idea is that if $S_n' - S_n = a$, we choose $(X_{n+1}, X_{n+1}') \sim \mu_a$, and then

$$S_{n+1}' - S_{n+1} = S_n' - S_n + X_{n+1}' - X_{n+1} = X_{n+1}' - X_{n+1} + a \geq 0.$$

This way, S_n and S_n' are random walks with the desired step dist^n and $S_n' \geq S_n \ \forall n$.

<u>What about convergence</u>? Let $\eta \equiv \inf\{a > 0 : v(a) > 0\}$, and suppose that $\delta > \eta$, $2\varepsilon \equiv v(\delta) > 0$. I claim that $M_n \equiv S_n' - S_n$ is a non-negative martingale, whose a.s. limit M_∞ will be $\leq \delta$.

The martingale property of M is obvious. Also, the jumps of M are bounded in modulus by $2N$.

Now if $|Z| \leq 2N$, $var(Z) = 2\varepsilon$, then

$$4N^2 \, P(|Z| > \sqrt{\varepsilon}) + \varepsilon \, P(|Z| \leq \varepsilon) \geq E|Z|^2 = 2\varepsilon$$

$$\therefore \quad (4N^2 - \varepsilon) \, P(|Z| > \sqrt{\varepsilon}) + \varepsilon \geq 2\varepsilon$$

$$\therefore \quad P(|Z| > \sqrt{\varepsilon}) \geq \gamma \equiv \varepsilon / (4N^2 - \varepsilon) .$$

Let $T \equiv \inf\{n : M_n \leq \delta\}$. By Lévy's 0-1 law,

$$\{ |\Delta M_n| > \sqrt{\varepsilon} \text{ i.o.} \} \overset{a.s.}{=} \{ \sum_n P(|\Delta M_n| > \sqrt{\varepsilon} \mid \mathcal{F}_{n-1}) = +\infty \},$$

so on the event $\{T = \infty\}$, $P(|\Delta M_n| > \sqrt{\varepsilon} \mid \mathcal{F}_n) \geq \gamma$ for all n, and so $|\Delta M_n| > \sqrt{\varepsilon}$ i.o. But M converges, therefore

$$P(T = \infty) = 0.$$

3) There remain some good (but deeper) questions!

1) Gérard Ben Arous is interested in

$$i\dot{\psi}_t = Q\psi_t + V\psi_t \quad ; \qquad \psi_0 = \delta_0$$

where Q is generator of rw on $\mathbb{Z}^d$ ($q_{ii} = 1$, $q_{ij} = 1/2d$, $|i-j| = 1$) and V is diagonal with i.i.d. $N(0,\sigma^2)$ entries. The goal is to find the large t asymptotics of

$$\sum_k |k|^2 \, |\psi_t(k)|^2$$

There's the Molchanov formula

$$\psi_t(k) = E^k\left[e^{(1-i)t} \, i^{N_t} \exp\left(i\int_0^t V(X_s)\,ds\right) \delta_0(X_t)\right]$$

which might help.

2) Here's a good question.

(a) Given a kernel $K(x,dy)$ on a suitable space, when is it the Green kernel of some Markov process?

(b) When it is, how could one reconstruct the process from the kernel?

3) Ron Doney asks: are there any Lévy processes apart from the symmetric ones and the stable ones for which

$$P(X_t > 0) = c \in (0,1) \quad \text{for all } t > 0 ?$$

Riemann ξ-fn satisfies

$$\xi(z) = E\left[\left(\sqrt{\tfrac{2}{\pi}}\, H\right)^z\right] = E\left[\left(\tfrac{\pi}{2} T\right)^{z/2}\right] = z(z-1)\,\pi^{-z/2}\,\Gamma(z/2)\,\zeta(z)$$

$$= \xi(1-z),$$

where H is height of scaled Brownian xcn, T is duration of Br xcn of height 1.

$$H^2 \overset{\mathcal{D}}{=} \frac{\pi^2}{4} T \qquad (\text{Chung})$$