When is there an eigenfunction expansion for a 1-dim diffusion? (18/5/93)

1) Let's consider a one-dimensional diffusion in natural scale with speed measure m, and some killing $V \geq 0$, and generator $\mathcal{G} = \frac{1}{2}\, d^2/dm dx \quad - V$. The resolvent has a density w.r.t. m

$$r_\lambda(x,y) = c_\lambda \; \psi_\lambda^+(x \wedge y) \;\; \psi_\lambda^-(x \vee y)$$

where $\psi_\lambda^\pm$ are increasing and decreasing sol^{ns} of $\mathcal{G}\psi = \lambda\psi$. It is not hard to see that $R_\lambda : L^2(m) \to L^2(m)$, and R_λ is self-adjoint. If we could prove that R_λ is a compact operator, then there would be an eigenfunction expansion.

We shall suppose that the interval I of definition of the diffusion is <u>either</u> $\mathbb{R}$ <u>or else</u> $[0,\infty)$ with reflection at 0 (the case of a compact statespace is largely dealt with in the classical Fredholm theory – see R.F. Curtain & A.J. Pritchard, "Functional Analysis in Modern Applied Mathematics", for example).

2) Since for $f \in L^2(m)$

$$R_\lambda f(x) = c_\lambda \psi_\lambda^-(x) \int^x \psi_\lambda^+(y) f(y)\, m(dy) + c_\lambda \psi_\lambda^+(x) \int_x^\infty \psi_\lambda^-(y) f(y)\, m(dy),$$

it follows quite easily that on any compact set K, for some constant $C(K)$

$$|R_\lambda f(x)| \leq C(K) \, \|f\|_2 \qquad (x \in K),$$

$$|R_\lambda f(x) - R_\lambda f(y)| \leq C(K)\, \|f\|_2 \;\; |x-y|^{1/2} \qquad (x, y \in K).$$

Thus if $(f_n)_{n\geq 1}$ is a bounded sequence in L^2, the sequence $(R_\lambda f_n)$ is equicontinuous and bounded on each compact subset of I. Passing to a subsequence if need be, we can assume (Arzela-Ascoli) that

$$(R_\lambda f_n)_{n \geq 1} \;\; \underline{\text{is uniformly convergent on each compact}} \;\; K \subseteq I.$$

The limit is of course continuous and in $L^2(m)$, so to prove that the operator is compact, we have to prove that not only do we get uniform convergence on compacts, <u>but also</u> L^2 <u>convergence</u>! This is not automatic, as the following example shows.

3) Example: Brownian motion. The resolvent density is $\theta^{-1} e^{-\theta|x-y|}$ w.r.t. Lebesgue measure, where $\theta = \sqrt{2\lambda}$. If we take $f_n(x) = I_{[n, n+1]}(x)$, these are bounded in L^2, and $R_\lambda f_n \to 0$ uniformly on compacts, yet $R_\lambda f_n \not\to 0$ in L^2!

4) A sufficient condition. Suppose that

$$(*) \qquad \boxed{\int m(dx) \int m(dy)\, r_\lambda(x,y)^2 < \infty;}$$

then R_λ is a compact operator. Indeed,

$$|R_\lambda f(x)| = \left| \int r_\lambda(x,y) f(y)\, m(dy) \right| \leq g(x)^{\frac{1}{2}} \|f\|_2$$

where

$$g(x) \equiv \int r_\lambda(x,y)^2\, m(dy).$$

The boxed condition says that $g \in L^1(m)$, and so if $R_\lambda f_n \to h$ uniformly on compacts, $\|f_n\| \leq 1$, the convergence is dominated by $g^{\frac{1}{2}}$, and hence the convergence is in L^2 also.

5) Is it the case that if m is a probability measure, then condition $(*)$ holds? Not always. Suppose we take $V \equiv 0$, $I = \mathbb{R}^+$, $m(dx) = (1+x)^{-2}\,dx$, then

$$\psi_\lambda^-(x) = (1+x)^{-\gamma}, \qquad \psi_\lambda^+(x) = (1+x)^{\theta} + \frac{\theta}{\gamma}(1+x)^{-\gamma}$$

where $\gamma \equiv (\sqrt{1+8\lambda}-1)/2$, $\theta \equiv (\sqrt{1+8\lambda}+1)/2 = 1+\gamma$. (We assume instantaneous reflection at 0 as the boundary condition, and so $D\psi_\lambda^+(0) = 0$).

In this case,

$$\int_0^\infty m(dx)\, \psi_\lambda^+(x)^2 \int_x^\infty m(dy)\, \psi_\lambda^-(y)^2 \geq \int_0^\infty \frac{dx}{(1+x)^2} (1+x)^{2\theta} \int_x^\infty \frac{dy}{(1+y)^2} (1+y)^{-2\gamma}$$

$$= c \int_0^\infty dx\, (1+x)^{2\theta-2-1-2\gamma} = +\infty.$$

So here the condition $(*)$ fails, even though m is a probability measure.

6) Interpreting condition (*) Observe that

$$\iint m(dx)\, m(dy)\, r_\lambda(x,y)^2 = \int m(dx) (R_\lambda^2)(x,x)$$
$$= -\frac{\partial}{\partial\lambda} \int m(dx)\, r_\lambda(x,x)$$

by the resolvent equation.

Now

$$r_\lambda(x,x) = E^x L^x(T_\lambda) = \{\text{rate in local time at } x \text{ of } \lambda\text{-marked excursions from } x\}^{-1}$$

– see, for example, § VI-54 of Rogers + Williams. Assuming for simplicity $dm/dx = \rho$, we get marks happening in the Brownian motion at rate $(\lambda+V)\rho$, so

$$r_\lambda(x,x)^{-1} = \text{rate of } (\lambda+V)\rho\text{- marked excursions of BM away from } x.$$

As a crude condition, if we set

$$\alpha_+(x) = \inf\{ (\lambda+V(y))\rho(y) : y \geq x \}$$
$$\alpha_-(x) = \inf\{ (\lambda+V(y))\rho(y) : y \leq x \},$$

we shall have

$$r_\lambda(x,x)^{-1} \geq \left(\sqrt{2\alpha_+(x)} + \sqrt{2\alpha_-(x)} \right)/2.$$

Thus

$$\boxed{\int \frac{m(dx)}{\sqrt{\alpha_+(x)} + \sqrt{\alpha_-(x)}} < \infty \Rightarrow \text{condition } (*) \text{ holds}}$$

(This is a bit too rapid. If we have this integral condition, then replacing λ by larger μ would not change the integral condition ($\alpha_\pm^\mu(x) \geq \alpha_\pm^\lambda(x)$). So if $\lambda < \mu < \theta$,

$$\iint m(dx)\, m(dy)\, r_\theta(x,y)^2 \leq \iint m(dx)\, m(dy)\, r_\theta(x,y)\, r_\mu(x,y)$$
$$= \int m(dx) \{ r_\mu(x,x) - r_\theta(x,x) \} / (\theta - \mu)$$
$$< \infty. \quad)$$

Conditioning BM not to spend long in $\mathbb{R}^-$ (20/5/93)

Let $A_t \equiv \int_0^t I_{\{B_u<0\}}\,du$. We calculate

$$\varphi(t,x,\xi) \equiv P^x[A_t \le \xi]$$

$$= \begin{cases} \int_0^\xi h(|x|,s)\,ds.\ \frac{2}{\pi}\sin^{-1}\left(\sqrt{\frac{\xi-s}{t-s}}\right) & \text{if } x<0 \\ P^x[H_0 > t-\xi] + \int_0^{t-\xi} h(x,s)\,ds\ \frac{2}{\pi}\sin^{-1}\sqrt{\frac{\xi}{t-s}} & \text{if } x>0 \end{cases}$$

where $h(x,s) \equiv x e^{-x^2/2s}(2\pi s^3)^{-1/2}$. If we condition on $\{A_T \le c\}$, with T very large, then the density of the conditioned law (on $\mathcal{F}_t$) will be

$$\frac{\varphi(T-t, B_t, c-A_t)}{\varphi(T, B_0, c)}.$$

Now observe that for $x<0$

$$\sqrt{T}\ \varphi(T,x,\xi) \longrightarrow \int_0^\xi h(|x|,s)\,ds\ \sqrt{\xi-s}\ \frac{2}{\pi} \qquad (T\to\infty)$$

and for $x>0$

$$\sqrt{T}\ \varphi(T,x,\xi) \longrightarrow x\sqrt{\frac{2}{\pi}} + \int_0^\infty h(x,s)\,ds\ \frac{2}{\pi}\sqrt{\xi} \qquad (T\to\infty)$$

$$= x\sqrt{\frac{2}{\pi}} + \frac{2}{\pi}\sqrt{\xi}$$

So if we define

$$\boxed{u(x,a) \equiv \begin{cases} x\sqrt{\frac{2}{\pi}} + \frac{2}{\pi}\sqrt{c-a} & (x\ge 0) \\ \frac{2}{\pi}\int_0^{c-a} h(|x|,s)\sqrt{c-a-s}\,ds & (x<0) \end{cases}}$$

then conditioning on $\{A_T \le c\}$ is equivalent to reweighting the law on path space by the positive martingale $u(B_t, A_t)$.

We have for $0 \le b \le c$

$$\boxed{P^0(A_T \le b \mid A_T \le c) = \frac{\sin^{-1}\sqrt{b/T}}{\sin^{-1}\sqrt{c/T}} \longrightarrow \sqrt{\frac{b}{c}} \quad \text{as } T\to\infty}$$

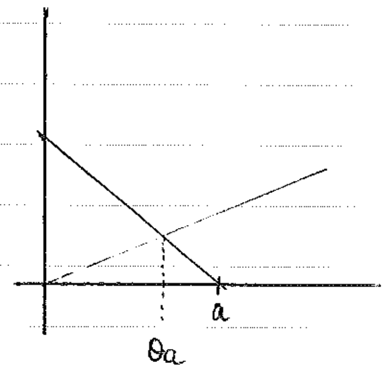
Is there some limit behaviour for $X_1/(X_1+X_2)$ given X_1+X_2 is large? (21/5/93)

1) Suppose that X_1, X_2 are i.i.d. positive random variables, and that

$$\lim_{a\to\infty} P[X_1 \le t X_2 \mid X_1+X_2 > a] \equiv \Lambda(t) \quad \text{exists}$$

for every $t>0$. What limits are possible? Assuming that the distribution has density f, we obtain

$$P\left[\frac{X_1}{X_1+X_2} < \theta ; X_1+X_2 > a\right] = P\left[X_1 < \frac{\theta}{1-\theta} X_2 ; X_1+X_2 > a\right]$$



$$= \int_0^{\theta a} f(x)\,dx\, \bar{F}(a-x)$$

$$+ \int_{\theta a}^{\infty} f(x)\,dx\, \bar{F}\left(\frac{1-\theta}{\theta} x\right)$$

so that

$$\boxed{P\left[\frac{X_1}{X_1+X_2} \in d\theta ; X_1+X_2 > a\right] / d\theta = \int_a^{\infty} y\, f(\theta y)\, f((1-\theta)y)\, dy}$$

2) Some examples.

(i) Suppose $f(x) = x^{-\beta} I_{(x>c)}$ where $\beta>1$, c chosen appropriately. Then we get

$$\int_a^{\infty} y\, (\theta y)^{-\beta} ((1-\theta)y)^{-\beta}\, dy = (\theta-\theta^2)^{-\beta} . c.\ a^{2-2\beta},$$

So in this case, a limit exists and is a Beta dist^n. But observe that $\beta>1$, so in fact we have that the limit distn degenerates to $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$.

(ii) Suppose that $f(x) = x^{-1}(\log x)^{-\beta} I_{(x>c)}$ with $\beta>1$. In this case, the limit law is $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$:

$$\int_a^{\infty} y\, f(\theta y) f(y-\theta y)\, dy = \int_a^{\infty} \frac{dy}{y\,\theta(1-\theta)\,(\log \theta y\ \log(y-\theta y))^{\beta}}$$

$$\sim \int_a^{\infty} \frac{dy}{y\,\theta(1-\theta)(\log y)^{2\beta}}$$

$$= \frac{c(\log a)^{1-2\beta}}{\theta(1-\theta)}.$$

When do we ever use continuity of the price processes?

Why do we consume everything as it comes in?

<u>An even more general model!</u> An agent holds cash, and the M commodities, holding $(\xi_t^1, \dots, \xi_t^M) = \xi_t$ at time t. Commodities are potentially perishable, or could even grow. Asset i produces commodity j with rate dy_t^{ij}. Agent receives income e_t in cash, $\mathcal{E}_t = (\mathcal{E}_t^1, \dots, \mathcal{E}_t^M)$ in commodities. There is spot prices $(\psi_t^1, \dots, \psi_t^M) \equiv \psi_t^T$. If X_t is agent's wealth at time t, then

$$dX_t = \theta_t \cdot dS_t + d\eta_t \cdot \psi_t + de_t$$

$$d\xi_t = \theta_t \, dy_t - d\eta_t - \xi_t \, d\rho_t + d\mathcal{E}_t - c_t \, dt$$

where η tracks how much has been sold of the commodities, and ρ_t is a diagonal matrix for the decay in commodities, and c is consumption. It seems reasonable to require $\xi \geq 0$.

$$F_t \equiv (f_0(t), f_1(t), \dots, f_L(t))^T, \qquad f_t \equiv f(t) \equiv (f_1(t), \dots, f_L(t))^T$$

A general model for investment / consumption (1/6/93)

(i) This is really just the model of Karatzas, Lehoczky + Shreve (Math OR $\underset{\sim}{15}$, 80–128, 1990) with a few additions. In this model, there are J agents with C^2 utility functions $U_j(t, c)$, where $U_j(t, \cdot) : (\mathbb{R}^+)^M \to \mathbb{R}$ is strictly concave and strictly increasing in each of its M arguments. Agent j receives an earnings stream $e_j(t)$, which is an M-vector, whose ith component is the earnings of commodity i. There are assets whose price processes $f_\lambda(t)$ are semimartingales,

$$df_0(t) = r_t \, f_0(t)\, dt,$$

and there is a measure $\tilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ under which adjusted* price processes are (local) martingales. Asset λ produces a dividend stream $\delta_\lambda(\cdot)$, which is an M-vector process. There is an M-vector semimartingale $\psi(\cdot)$ which is the spot price process for the commodities. Write

$$\beta_t \equiv \exp(-R_t) \equiv \exp\left(-\int_0^t r_u \, du\right), \quad Z_t \equiv \left.\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}_t}, \quad J_t \equiv \beta_t Z_t .$$

If agent j starts with wealth x, and holds portfolio $\Phi = (\varphi_0(t), \varphi_1(t), \dots, \varphi_L(t))$ of the assets (assuming there are finitely many of them), where Φ is a self-financing previsible portfolio, then his wealth process X satisfies

$$(1) \quad \boxed{\begin{aligned} dX_t &= \Phi_t \{ dF_t + \delta(t) \cdot \psi(t)\, dt \} + (e_j(t) - c(t)) \cdot \psi(t)\, dt \\ X_t &= \Phi_t \cdot F_t \quad , \quad X_0 = x \end{aligned}}$$

where c is the consumption process drawn off. The agent seeks to obtain

$$(2) \quad \sup E\left[\int_0^T U_j(t, c_t)\, dt \right]$$

over admissible consumption/wealth pairs (c, X) (in particular, $X_T \geq 0$ for admissibility).

* To be exact, (3) $\tilde{f}_\lambda(t) \equiv \beta_t f_\lambda(t) + \int_0^t \beta_s \, \delta_\lambda(s) \psi(s)\, ds$ must be a $\tilde{\mathbb{P}}$-martingale (or local martingale).

Note: If we fix $v \gg 0$, and set $U_j^v(t, \lambda) = \sup\{ u_j(t, c) \,;\, c \cdot v = \lambda \}$, then this is a strictly incr, strictly concave $f^{\underline{n}}$ of λ. So the assumption is very weak.

(ii) The single-agent problem First, we rework the budget (1). Since

$$d\tilde{F} = \beta\, dF - r\beta F\, dt + \beta\, \delta\cdot\psi\, dt \qquad \text{is a } \tilde{\mathbb{P}}\text{-local mg},$$

we develop the discounted wealth process $\tilde{X}_t \equiv \beta_t X_t$ as

$$\begin{aligned} d\tilde{X}_t &= \beta_t dX_t - r_t \beta_t X_t\, dt \\ &= \beta_t \{ \Phi_t (dF_t + \delta_t \psi_t\, dt) + (e_t - c_t)\cdot\psi_t\, dt \} - r_t \beta_t X_t\, dt \\ &= \Phi_t (d\tilde{F}_t + r_t \beta_t F_t\, dt) + (e_t - c_t)\cdot\psi_t\, \beta_t\, dt - r_t \beta_t X_t\, dt \\ &= \Phi_t\, d\tilde{F}_t + \beta_t (e_t - c_t)\cdot\psi_t\, dt \qquad \text{since } \Phi\cdot F = X. \end{aligned}$$

Thus

$$(4) \qquad \boxed{\beta_t X_t = \tilde{X}_t = x + \int_0^t \beta_s (e_s - c_s)\cdot\psi_s\, ds + \int_0^t \Phi_s \cdot d\tilde{F}_s}$$

If admissible portfolios are always ones for which the final term is a $\tilde{\mathbb{P}}$-supermartingale, then the budget equation implies the constraint

$$(5) \qquad \boxed{\tilde{E} \int_0^T \beta_s\, c_s\cdot\psi_s\, ds \leq x + \tilde{E} \int_0^T \beta_s\, e_s\cdot\psi_s\, ds}$$

(i.e., the discounted value of what you consume must not exceed the discounted value of what you will get.)

We now have to make the

ASSUMPTION 1 For any $v \gg 0$, $\sup_{c\geq 0} \{ U_j(t,c) - v\cdot c \} \equiv U_j^*(t,v)$ is attained.

Let the place where the sup is attained (unique, by strict concavity of U_j) be denoted $I_j(t,v)$. Notice that $U_j^*(t,\cdot)$ is convex for all $t\geq 0$, and decreasing.

The form of the constraint (5) suggests that we should consider the Lagrangian form of the problem

$$(6) \qquad \max\ E\left[\int_0^T U_j(t,c_t)\, dt + \lambda\left(x + \int_0^T \beta_s (e_s - c_s)\cdot\psi_s\, ds\right) Z_T \right]$$

now without constraint on $c \geq 0$. But the Lagrangian form can also be re-expressed as

$$\max \mathbb{E}\left[\int_0^T \{ U_j(s, c_s) - \lambda \beta_s Z_s c_s \cdot \psi_s \} ds + \lambda x + \lambda \int_0^T \zeta_s e_s \cdot \psi_s ds \right]$$

and this suggests that we should seek

(7) $\quad c^*_s = I_j(s, \lambda \zeta_s \psi_s).$

For this to work, we shall need to make

<u>ASSUMPTION 2</u> : $\psi_t \gg 0$ for all t.

For the moment, assume that λ can be chosen so that the budget constraint (5) on c is exactly satisfied for c^*;

(8) $\quad \mathbb{E} \int_0^T \zeta_s I_j(s, \lambda \zeta_s \psi_s) \cdot \psi_s ds = x + \mathbb{E} \int_0^T \zeta_s e_s \cdot \psi_s ds.$

Then for any feasible c,

$$\mathbb{E} \int_0^T U_j(s, c_s) ds \leq \mathbb{E} \int_0^T \{ c_s \cdot \lambda \zeta_s \psi_s + U_j^*(s, \lambda \zeta_s \psi_s) \} ds$$

$$\leq \mathbb{E} \int_0^T \{ \lambda e_s \zeta_s \psi_s + U_j^*(s, \lambda \zeta_s \psi_s) \} ds + \lambda x$$

since c satisfies (5)

$$= \mathbb{E} \int_0^T \{ \lambda \zeta_s I_j(s, \lambda \zeta_s \psi_s) \cdot \psi_s + U_j^*(s, \lambda \zeta_s \psi_s) \} ds$$

by (8)

$$= \mathbb{E} \int_0^T U_j(s, c^*_s) ds,$$

proving optimality of c^*. $\square$

To stick this all together, we use the following.

LEMMA. <u>Strict concavity of U implies U^* has a unique supporting hyperplane at each interior point, and is C^1, with $\nabla U^* = -I$. For any $a \gg 0$, $a \cdot I(\lambda a)$ decreases to 0, from $+\infty$ if U is unbounded on each $\frac{1}{2}$space.</u>

<u>Proof</u>. Take some $a \gg 0$, and use the form of the supporting hyperplane to U^* at a,

$$U^*(x) - U^*(a) \geq -w(x-a)$$

where $w \geq 0$. If there were some other w' which gave a supporting hyperplane, so would any convex combination z of w and w'.

In that case

$$U(z) \equiv \inf_{x \geq 0} \{ U^*(x) + z x \} = U^*(a) + z a$$

and so there's an affine piece of U, contradicting strict concavity. So there's a unique supporting hyperplane. Also, for $a \gg 0$, any $c \geq 0$

$$U(c) - c.a \leq U^*(a) = U(I(a)) - a\, I(a)$$

whence $\quad U^*(x) - U^*(a) \geq -I(a)\,(x-a)$, so that $\boxed{I(a) = -\nabla U^*(a)}$ for $a \gg 0$.

To see that I is continuous in the interior, take some $a \gg 0$ and an open bounded neighbourhood N contained in the interior. Then I must be bounded on N, otherwise U^* would be unbounded. If $a_n \to a$, and $I(a_n) \to \eta$ then for any $c \geq 0$

$$U(c) - c \cdot a_n \leq U(I(a_n)) - a_n I(a_n) \to U(\eta) - a.\eta$$

Hence $\eta = I(a)$, and I is continuous at a.

Now with $a \gg$ fixed, we consider the decreasing convex function $\lambda \mapsto U^*(\lambda a)$, which has slope $-a \cdot I(\lambda a)$. If the slope increased to $-\varepsilon < 0$, then

$$U(c) = \inf \{ U^*(x) + cx \} \leq \inf_{\lambda > 0} \{ U^*(\lambda a) + \lambda c a \} = -\infty \quad \text{if } c.a < \varepsilon,$$

which is impossible.

Finally, if $\lambda \mapsto U^*(\lambda a)$ had slope $-m > -\infty$ as $\lambda \downarrow 0$, we'd get

$$U(c) \leq \inf_{\lambda > 0} \{ U^*(\lambda a) + \lambda a.c \} = U^*(0) \quad \text{if } a.c > m. \quad ⋇ \qquad \square$$

So to get the argument on the previous page to work (i.e. we want also the budget constraint to be attainable with equality) we shall impose also

<u>ASSUMPTION 3</u>: U is not bounded in any half-space.

Elementary transformations of the CIR process (7/6/93)

Take a basic CIR process

$$dr_t = \sigma\sqrt{r_t}\, dW_t + (\alpha - \beta r_t)\, dt$$

where $\alpha, \beta, \sigma > 0$ are constant and let's now consider what happens to bond prices if we (i) multiply r by some positive C^1 function (ii) apply a deterministic C^2 time-change. So our bond prices are

$$P(t,T) = E\left[\exp - \int_{\tau_t}^{\tau_T} \theta_u r_u\, du \;\middle|\; \mathcal{F}_{\tau_t}\right]$$

$$= E\left[\exp\left(-\int_t^T \tau_s' \,\theta(\tau_s)\, r(\tau_s)\, ds\right) \;\middle|\; \mathcal{F}_{\tau_t}\right],$$

and we need to understand the process

$$\rho_t \equiv \tau_t' \,\theta(\tau_t)\, r(\tau_t).$$

If we use a $\sim$ to denote time-change by τ, and set $\varphi(t) = \theta_t\, \tau'(\tau_t^{-1})$, we get firstly $\tilde{\varphi}_t = \tau_t' \tilde{\theta}_t$, and

$$d(\varphi_t r_t) = \sigma\varphi_t\sqrt{r_t}\, dW_t + (\alpha\varphi_t - \beta\varphi_t r_t + \varphi_t' r_t)\, dt,$$

and hence

$$d\rho_t = \sqrt{\tau_t'}\,\sigma\,\tilde{\varphi}_t\sqrt{\tilde{r}_t}\, dW_t + \tau_t'\left(\alpha\tilde{\varphi}_t - \beta\tilde{\varphi}_t\tilde{r}_t + \tilde{\varphi}_t'\tilde{r}_t\right) dt$$

$$= \sigma\sqrt{\tau_t'\tilde{\varphi}_t}\,\sqrt{\rho_t}\, dW_t + \alpha\,\tau_t'\tilde{\varphi}_t\, dt - \beta\,\tau_t'\rho_t\, dt$$

$$+\left\{\left(\frac{\theta'}{\theta}\right)(\tau_t)\,\tau_t' + \frac{\tau_t''}{\tau_t'}\right\}\rho_t\, dt$$

So this is now a time-varying CIR process, with

$$\boxed{\begin{aligned} \sigma_t &\equiv \sigma\sqrt{\tau_t'\tilde{\varphi}_t} \\ \alpha_t &\equiv \frac{\alpha}{\sigma^2}\cdot\sigma_t^2 \\ \beta_t &\equiv \beta\tau_t' - \frac{\theta'}{\theta}(\tau_t)\,\tau_t' - \frac{\tau_t''}{\tau_t'} \end{aligned}}$$

Remarks (i) The ratio α_t/σ_t^2 is unchanged, so we are looking at the same subclass of time-varying CIR process that are studied by Jamshidian.
(ii) The bond price has the nice form

$$P(t,T) = \exp\{-\rho_t\, 2B(t,T)/\sigma_t^2 - A(t,T)\}$$

where

$$\begin{cases} \dot{B} - B^2 - bB + \tfrac{1}{2}\sigma^2 = 0 & , \ B(T,T) = 0 \\ \dot{A} = -aB & , \ A(T,T) = 0 \end{cases}$$

with $a_t \equiv 2\alpha_t/\sigma_t^2$, $b_t \equiv \beta_t + 2\sigma_t'/\sigma_t$. For this special class of time-dependent CIR models (with fixed dimension), we may recover the functions $\sigma_\cdot$ and $\beta_\cdot$ from knowledge of the initial yield curve and initial term-structure of volatility. Indeed, from these we can compute $B(0,\cdot)$ and hence find $A(0,\cdot)$. Now one can compute (see Jamshidian, or my own Monte Verita notes, p 76) that

$$\frac{\partial}{\partial T} A(0,T) = \frac{a}{2}\sigma_T^2 \frac{\gamma(0,T)}{\psi(0,T)}$$

$$\frac{\partial}{\partial T} \log \frac{\partial B}{\partial T}(0,T) = -\beta_T - \sigma_T^2 \frac{\gamma(0,T)}{\psi(0,T)}$$

where the exact form of the functions γ, ψ matters little, since we may now express

$$\boxed{\beta_T = -\frac{\partial}{\partial T}\log\frac{\partial B}{\partial T}(0,T) - \frac{a}{2}\frac{\partial A}{\partial T}(0,T).}$$

Also,

$$\frac{\partial^2}{\partial T^2}\log\frac{\partial B}{\partial T}(0,T) = -\beta_T' - \sigma_T^2 - \tfrac{1}{2}\beta_T^2 + \tfrac{1}{2}\left\{\frac{\partial}{\partial T}\log\frac{\partial B}{\partial T}(0,T)\right\}^2$$

so we may now work out what σ should be!
(iii) If we are told $\sigma_\cdot$, $\beta_\cdot$, can we find what τ, θ were? We have $\sigma_t = \sigma\,\tau_t'\sqrt{\tilde\theta_t}$ so that

$$D(\log\sigma_t^2) = 2\frac{\tau_t''}{\tau_t'} + \theta'(\tau_t)\,\tau_t'/\tilde\theta_t$$

so

$$\boxed{g(t) \equiv \beta_t + D(\log\sigma_t^2) = \beta\,\tau_t' + \tau_t''/\tau_t',}$$

where the function g is known, since β, σ are. Now we have the DE

$$\frac{\tau''}{\tau'} + \beta\tau' = g$$

which says

$$\frac{\partial}{\partial t} \log \frac{\partial}{\partial t} e^{\beta\tau} = g(t)$$

which can now be solved for τ, and then θ follows.

(iv) Of course, the case $\theta \equiv 1$ is the nicest by far. If we hold the parameters α, β, σ fixed and let $P_0(r, t)$ be the CIR bond-pricing formula

$$P_0(r,t) = \exp\{-r\,B(t) - A(t)\}$$

then

$$P(t,T) = P_0\left(r(\tau_t),\ \tau(T) - \tau(t)\right)$$

$$= P_0\left(\rho_t/\tau'_t,\ \tau(T) - \tau(t)\right)$$

where ρ is the spot-rate process for this situation, and all of this can be computed very easily.

Futures prices for the time-dependent CIR model (7/6/93)

If we want the futures price of a bond of maturity T' to be delivered at $T < T'$, calculated at $t < T$, we must compute

$$E\left[\,P(T,T')\,|\mathcal{F}_t\right]$$

which is tantamount to computing $E[\exp -\lambda r_T \ |\mathcal{F}_t]$. Now

$$M_t \equiv E\left(e^{-\lambda r_T}\,|\mathcal{F}_t\right) = e^{-r_t C_t - \gamma_t}$$

and the Itô development of M gives us the conditions

$$\boxed{\begin{array}{ll} 0 = -\dot{C} + \beta C + \tfrac{1}{2}\sigma^2 C^2, & C_T = \lambda \\ 0 = -\dot{\gamma} - C\alpha, & \gamma_T = 0 \end{array}}$$

Writing $C_t = 2K_t/\sigma_t^2$, we get

$$\boxed{\dot{K} - bK - K^2 = 0, \quad K_T = \lambda\sigma_T^2/2}$$

where $b_t \equiv \beta_t + 2\dot{\sigma}_t/\sigma_t$.

If $L_t^{\pm}$ are the local times at $\substack{\max \\ \min}$ with inverses $\gamma_t^{\pm}$, then $Z_t^+ \equiv X(\gamma_t^+)$ and $Z_t^- \equiv -X(\gamma_t^-)$ are subordinators. The Laplace exponents can be deduced from Fristedt's identity; for the Z^+ process, it is

$$\exp \int_0^\infty \frac{dt}{t} \int_{[0,\infty)} (e^{-t} - e^{-\lambda x})\, P(X_t \in dx).$$

If ν_+ is the excursion law of the sup of upward excursions from the minimum, with ν_- analogously defined, it would be immensely helpful to have a characterisation of $\nu_\pm$. I made a guess that for suitable constants

$$\nu_+ ((x,\infty)) = \text{const} / G_-((-x,0)) \qquad (x>0)$$

$$\nu_- ((-\infty,-x)) = \text{const} / G_+((0,x)) \qquad (x>0).$$

In the case of spectrally positive Lévy processes, if s is the scale function then we know that

$$\nu_+((x,\infty)) = \text{const.}\ s'(x)/s(x)$$

$$\nu_-((-\infty,-x)) = \text{const} / s(x).$$

Assuming that $X_t \to -\infty$, we shall then have that the scale function s is the distn of $\bar{X}_\infty$, which is proportional to $G_+((0,x))$. However, $G_-((-x,0)) = cx$ evidently, and so if the guess were correct

$$\nu_+((x,\infty)) = \frac{c\, s'(x)}{s(x)} = \frac{c}{x} \quad \Rightarrow \quad s(x) = x^\alpha \quad \text{for some } \alpha > 0$$

which is manifest nonsense...

Abbreviating $f_t \equiv \exp(\int_0^t b_u\,du)$, we find

$$\boxed{\begin{aligned} K_t &= \frac{\lambda\sigma_T^2 f_t}{2f_T + \lambda\sigma_T^2\int_t^T f_s\,ds},\\ \gamma_t &= \int_t^T \frac{2\alpha_s}{\sigma_s^2} K_s\,ds. \end{aligned}}$$

Could one now use this to form an estimate of volatility of bond-prices?

Points of increase of Lévy processes (25/6/93)

(i) When does a Lévy process X have points of increase? If 0 is not regular for $(-\infty,0)$, it is trivial that there are points of increase. If 0 is not regular for $(0,\infty)$, it is trivial that (except in the compound Poisson case) there are no points of increase. So we assume that 0 is regular for $(0,\infty)$ and for $(-\infty,0)$.

(ii) Define $\bar{X}_t \equiv \sup_{u\le t} X_u$, and let L be the local time at 0 of $\bar{X}-X$, with inverse γ. Define

$$Y_t^\varepsilon \equiv I_{\{X_u \ge X(\gamma_t)-\varepsilon \text{ for all } u\in[\gamma_t,\gamma_2]\}} \qquad (0\le t\le 1)$$

$$g(x,\varepsilon) \equiv P^0[X_u \ge -\varepsilon \text{ for all } u\in[0,\gamma_x]].$$

Thus $E\,Y_t^\varepsilon = g(2-t,\varepsilon)$. One approach to the problem of finding points of increase is to try to build a local time on the set of such things. Define

$$A_\varepsilon \equiv \int_0^1 Y_t^\varepsilon\,dt \cdot c_\varepsilon \quad , \quad c_\varepsilon^{-1} \equiv E\int_0^1 Y_t^\varepsilon\,dt = \int_0^1 g(2-t,\varepsilon)\,dt.$$

Now if $\xi_t \equiv \inf\{X_u : \gamma_t \le u \le \gamma_2\}$, we have that ξ_t is right continuous, and so $C_\varepsilon \equiv \{t\in[0,1] : \xi_t \ge X(\gamma_t)-\varepsilon\}$ is closed from the right. If there is a point of increase, then $\bigcap C_\varepsilon$ is non-empty. Suppose that for each n, there is some $t_n \in C_{1/n}$. By passing to a subsequence we may suppose that (t_n) is monotone. If monotone decreasing to t, then $\xi_{t_n} - X(\gamma_{t_n}) \to \xi_t - X(\gamma_t) \ge 0$ and so γ_t is a point of increase. On the other hand, if $t_n \uparrow t$, $\gamma_{t_n}\uparrow T$, then

$$\xi_{t_n} \uparrow \inf\{X_u : T\le u\le \gamma_2\}, \quad X(\gamma_{t_n}) \uparrow X(T-) = \bar{X}(T-).$$

If $X(T) = X(T-)$, then T is a time of increase of X. On the other hand, if there is a jump of X at T, we must have $X(T) > X(T-)$. However, in reversed time, this will also be a jump of the reversed path, and immediately afterwards, the reversed path must fall below the place jumped to. This contradicts $X(T-) = \tilde{X}(T-)$, and so T is in fact a point of increase. Thus there are points of increase if and only if every C_ε is non-empty.

(ii) Now we use the same old trick. If we can prove that for some $K < \infty$

$$(*) \qquad E A_\varepsilon^2 \le K \text{ for all } \varepsilon > 0,$$

then the family $(A_\varepsilon)_{\varepsilon \ge 0}$ is UI, and $E A_\varepsilon = 1$ for all $\varepsilon \ge 0$. If there were no point of increase, then for all small enough ε, $A_\varepsilon = 0$, contradicting UI and $E A_\varepsilon = 1$. So condition (*) will guarantee that there exist points of increase.

But

$$\tfrac{1}{2} E A_\varepsilon^2 = E \int_0^1 ds \int_s^1 dt \; Y_s^\varepsilon Y_t^\varepsilon \cdot c_\varepsilon^2$$

$$= \int_0^1 ds \int_s^1 dt \; g(t-s, \varepsilon)\, g(2-t, \varepsilon) \cdot c_\varepsilon^2$$

$$= \int_0^1 dt \; g(2-t, \varepsilon) \left(\int_0^t g(s, \varepsilon)\, ds \right) c_\varepsilon^2$$

So there is a point of increase if for some $K < \infty$

$$\frac{\int_0^1 dt \; g(2-t,\varepsilon) \left(\int_0^t g(s,\varepsilon)\, ds \right)}{\left(\int_0^1 g(2-t,\varepsilon)\, dt \right)^2} \le K \quad \text{for all } \varepsilon > 0$$

But observe that for $1 \le u \le 2$

$$1 \ge \frac{g(u,\varepsilon)}{g(1,\varepsilon)} = P\left[X_s \ge -\varepsilon \text{ for } 0 \le s \le \gamma_u \,\middle|\, X_s \ge -\varepsilon \text{ for } 0 \le s \le \gamma_1 \right]$$

$$= \int P\left(X_{\gamma_1} \in dy \mid X_s \ge -\varepsilon \text{ for } 0 \le s \le \gamma_1 \right) g(u-1, \varepsilon + y)$$

$$\ge \int P\left(X_{\gamma_1} \in dy \mid X_s \ge -\varepsilon \text{ for } 0 \le s \le \gamma_1 \right) g(1, \varepsilon + y)$$

$$\ge \int P\left(X_{\gamma_1} \in dy \mid X_s \ge -\varepsilon \text{ for } 0 \le s \le \gamma_1 \right) g(1, y),$$

and it seems inconceivable that this last term should not remain bounded away from 0 as $\varepsilon \downarrow 0$. So, if true, we have to decide whether $\exists\, K < \infty$ such that

$$\boxed{\frac{\int_0^1 g(s,\varepsilon)\,ds}{g(1,\varepsilon)} \leq K \qquad \forall\, \varepsilon > 0.}$$

Remark In the case of a spectrally negative Lévy pr, $g(t,\varepsilon) = s(\varepsilon)/s(\varepsilon+t)$ where s is the scale function, so the above condition reduces to

$$\int_0^1 \frac{dt}{s(t+\varepsilon)} \leq K \qquad \forall\, \varepsilon > 0$$

that is

$$\int_0^1 \frac{dt}{s(t)} < \infty$$

which, as we know, is exactly the correct criterion in this case.

(iii) We can prove that the above condition is not only sufficient but also necessary for the existence of points of increase if we proceed as follows. Let's hold fixed some $a > 0$, and for each $\varepsilon > 0$ define

$$T_0^\varepsilon = 0, \quad T_{n+1}^\varepsilon = \inf\{ u > T_n^\varepsilon : \inf\{X_s : \gamma(T_n^\varepsilon) \leq s \leq \gamma(u)\} \leq X(\gamma(T_n^\varepsilon)) - \varepsilon \}.$$

Thus the gaps $T_{n+1}^\varepsilon - T_n^\varepsilon$ are i.i.d. random variables, with

$$P\left[T_{n+1}^\varepsilon - T_n^\varepsilon > t \right] = g(t,\varepsilon).$$

It is worth remarking also that $\{T_n^\varepsilon : n \geq 0\}$ increases as $\varepsilon \downarrow$.

Now take some r.v. Z independent of X, with an $\exp(1)$ distribution, and consider

$$p_\varepsilon = P\left[T_{n+1}^\varepsilon - T_n^\varepsilon > a \text{ for some } n \leq N_\varepsilon \right],$$

where $N_\varepsilon \equiv \inf\{k : T_k^\varepsilon > Z\}$. Then if $F(t) \equiv 1 - g(t,\varepsilon)$, we have easily

$$p_\varepsilon = \int_0^a F(dt)\, e^{-t} \cdot p_\varepsilon + e^{-a} g(a,\varepsilon),$$

from which

$$p_\varepsilon = \frac{e^{-a} g(a,\varepsilon)}{1 + \int_0^a e^{-t} g(dt,\varepsilon)}$$

$$= \frac{e^{-a} g(a,\varepsilon)}{e^{-a} g(a,\varepsilon) + \int_0^a g(t,\varepsilon) e^{-t}\, dt}.$$

Now as $\varepsilon \downarrow 0$, $p_\varepsilon \downarrow P[\text{ for some } u < L_T,\ X(\gamma_u) \le X_s \text{ for } \gamma_u \le s \le \gamma_{u+a}\ ]$, and that does it.

But can we make this more concrete?

<u>Arbitrage with a BES(3) process</u> (26/6/95)

Let's take for our sample space $\Omega = \{\omega \in C([0,1]) : \omega(0) = a\}$, where $a > 0$ is fixed, and let P be Wiener measure on Ω, so that under P the canonical process is BM started at a. Then

$$M_t \equiv P[X_1 > 0 \mid \mathcal{F}_t] = \Phi\left(\frac{X_t}{\sqrt{1-t}}\right) \qquad (\Phi \text{ is } N(0,1) \text{ dist}^{\text{n}} \text{ function})$$

and so we represent the P-martingale M as

$$M_t = M_0 + \int_0^t \varphi(s, X_s)\, dX_s \qquad \left[\varphi(s,x) = \frac{e^{-x^2/2(1-s)}}{\sqrt{2\pi(1-s)}}\right]$$

and P-a.s.

$$\boxed{M_1 \equiv I_{\{X_1 > 0\}} = M_0 + \int_0^1 \varphi(s, X_s)\, dX_s ,}$$

with $M_0 = \Phi(a)$.

Now set $T \equiv \inf\{u : X_u = 0\}$, and define a new probability measure Q by

$$\frac{dQ}{dP} = a^{-1} X_{1 \wedge T}.$$

Then under Q, X is a BES(3) process, and $Q \ll P$, though $P \not\ll Q$. Thus Q-a.s.

$$\int_0^1 \varphi(s, X_s)\, dX_s = I_{\{X_1 > 0\}} - M_0 ,$$

but Q-a.s., $I_{\{X_1 > 0\}} - M_0 = 1 - \Phi(a) > 0$, a very clear arbitrage!

—and with bounded processes!!

Conditioning B on A not too big too soon (5/7/93)

If γ is inverse to $L(t,0)$ we have

$$P^x\left[A(\gamma_t) \le y \right] = P^0\left[H_{x^- + \frac{1}{2}t} \le y \right]$$

$$= 2\,\bar{\Phi}\left(\frac{x^- + t/2}{\sqrt{y}} \right)$$

Thus

$$P^x\left[A(\gamma_t) \le ct \right] = 2\,\bar{\Phi}\left(\frac{x^- + t/2}{\sqrt{ct}} \right)$$

$$\sim \frac{2}{\sqrt{2\pi}} \exp\left(- \frac{(x^- + t/2)^2}{2ct} \right) \cdot \frac{\sqrt{ct}}{x^- + t/2}$$

and so

$$\frac{P^x\left[A(\gamma_t) \le ct \right]}{P^0\left[A(\gamma_t) \le ct \right]} \sim \exp(-x^-/2c) \qquad (t \to \infty).$$

Setting $h_T(x,a,\ell) \equiv P\left[A(\gamma_{T-\ell}) \le cT - a \mid X_0 = x \right]$

$$= 2\,\bar{\Phi}\left(\{x^- + \tfrac{1}{2}(T-\ell)\} / \sqrt{cT-a} \right),$$

get

$$\frac{h_T(x,a,\ell)}{h_T(0,0,0)} \sim \exp\left\{ -\frac{1}{2} \frac{(x^- + \frac{1}{2}(T-\ell))^2}{cT-a} + \frac{T}{8c} \right\}$$

$$= \exp\left\{ -\frac{(x^-)^2}{2(cT-a)} - \frac{x^-(T-\ell)}{2(cT-a)} + \underbrace{\frac{T}{8c} - \frac{(T-\ell)^2}{8(cT-a)}}_{\to \frac{2c\ell - a}{8c^2} \ (T \to \infty)} \right\}$$

$$\sim \exp\left\{ -\frac{x^-}{2c} + \frac{2c\ell - a}{8c^2} \right\} \qquad (T \to \infty)$$

So the effect of this is to impose a drift $1/2c$ while the process is in $(-\infty, 0)$

An example on inefficient dynamic portfolio strategies (6/7/93)

Dybvig's paper (Rev Fin Studies 1, 67-88, 1988) develops the idea that if an agent only cares about the distribution of terminal wealth, then replicating the contingent claim exactly may be inefficient.

Take as an example a world with riskless return r, constant, and risky asset

$$S_t = \exp\{\sigma X_t + \mu t\} \qquad (X \text{ is BM})$$

and so the EMM is

$$Z_t = \exp\{c X_t - \tfrac{1}{2}c^2 t\}, \qquad \sigma c \equiv r - \mu - \tfrac{1}{2}\sigma^2.$$

Suppose we consider a contingent claim of the form $Y = f(S_T)$, where f is increasing. Then Dybvig's result is that if we take instead $\tilde{Y} = \varphi(Z_T)$, where φ is decreasing and chosen to make the laws of $Y, \tilde{Y}$ agree, then the cost of $\tilde{Y}$ will be less than or equal the cost of Y. If $c < 0$, then obviously we just take $Y = \tilde{Y}$, but if $c > 0$, the story is different. Then we take

$$\tilde{Y} = f(\exp(-\sigma X_T + \mu T)) = f(e^{2\mu T}/S_T).$$

If we set $g(x) \equiv f(\exp(\sigma x + \mu T))$, then the fair price at time t to pay for $\tilde{Y}$ will be

$$\tilde{Y}_t \equiv e^{-r(T-t)} E^*[g(-X_T) \mid \mathcal{F}_t]$$

where E^* is EMM, and $X_T = X^*_T + ct$ under the EMM, where X^* is a BM;

$$= e^{-r(T-t)} P_{T-t}\, g(-X^*_t - cT)$$

$$\Rightarrow \quad e^{-rt}\tilde{Y}_t = e^{-rT} P_{T-t}\, g(-X^*_t - cT)$$

$$= \tilde{Y}_0 - \int_0^t \nabla P_{T-s}\, g(-X^*_s - cT)\, e^{-rT}\, dX^*_s$$

and especially,

$$\boxed{\tilde{Y}_0 = e^{-rT} P_T\, g(-cT).}$$

So let's just note one special case, the European call option, $f(x) = (x-K)^+$. Then

$$\tilde{Y}_0 = e^{-rT} E\left(e^{\sigma X_T - \frac{\sigma^2}{2}T + rT - \sigma c T} - K\right)^+$$

$$= e^{-\sigma c T} BS(\sigma, T, Ke^{\sigma c T}, r).$$

Always assuming $c>0$, $\tilde{Y}_0$ will clearly be less than the BS price of the option. Note another special case: with $K=0$,

$$\tilde{Y}_0 = e^{-\sigma c T}$$

so that <u>when</u> $c>0$, <u>it is inefficient to hold the stock!!</u>

<u>Optimal consumption with exact consumption constraint: an example (8/7/93)</u>

(1) Let's go back to the optimal consumption problem (VI, p 17). At beginning of day n $(n=0,1,\dots,N)$ we receive amount $Y_n \geq 0$, independent, $Y_n \sim \mu_n$, and choose consumption c_n for that day. We obtain utility $U_n(c_n)$ from that. Let $V_n(z)$ be maximal expected residual utility if we have wealth z at beginning of day n (after receiving Y_n). Then

$$\begin{cases} V_n(z) = \max_c \left\{ U_n(c) + \tilde{V}_{n+1}(z-c) \right\} & (n<N) \\ V_N(z) = U_N(z) \end{cases}$$

where $\tilde{V}_n(x) \equiv \int V_n(x+y)\,\mu_n(dy)$. Optimal consumption c_n satisfies

$$\boxed{U_n'(c_n(z)) = \tilde{V}_{n+1}'(z - c_n(z))}$$

and $V_n(z) = U_n(c_n(z)) + \tilde{V}_{n+1}(z - c_n(z))$. Hence

$$\boxed{V_n'(z) = \tilde{V}_{n+1}'(z - c_n(z)) = U_n'(c_n(z))}$$

Now if Z_n is the wealth at beginning of day n, then

$$\boxed{M_n \equiv U_n'(c_n(Z_n)) = \tilde{V}_{n+1}'(Z_n - c_n(Z_n)) \text{ is a martingale.}}$$

The proof is very simple:

$$E[M_n \mid \mathcal{F}_{n-1}] = E\left[U_n'(c_n(Z_n)) \mid \mathcal{F}_{n-1}\right]$$

$$= E\left[V_n'(Z_n) \mid \mathcal{F}_{n-1}\right]$$

$$= \int \mu_n(dy)\, V_n'\left(Z_{n-1} - c_{n-1}(Z_{n-1}) + y\right)$$

$$= \tilde{V}_n'\left(Z_{n-1} - c_{n-1}(Z_{n-1})\right)$$

$$\equiv M_{n-1} .$$

So if we play optimally

<u>Marginal utility of daily consumption is a martingale.</u>

(ii) This is noteworthy because of the problem of Yannis Karatzas (III, 51) of maximising $E\int_0^T U(c_s)\,ds$ subject to $\int_0^T c_s\,ds = Y$, given r.v.. It was shown there that if we could find mg M st.

$$\int_0^T (U')^{-1}(M_s)\,ds = Y$$

then $U'(c_t^*) = M_t$ defines optimal consumption plan.

(iii) If we had $U = \log$, and $Y = (1 + B_T^2)^{-1} + 1$, where B is a BM independent of the marketed world, then the problem is ill-posed; once the aggregate consumption $\int_0^t c_u\,du$ gets above 1, there is a (small) risk that the constraint Y may be smaller that $\int_0^t c_u\,du$, so would have to consume negatively.

(iv) We should instead consider $\eta_t \equiv \sup\{a : P(Y \leq a \mid \mathcal{F}_t) = 1\}$ and use the constraint $\int_0^t c_s\,ds \leq \eta_t \quad \forall t$. Phil Dybvig interprets this constraint not in terms of storing the commodity, but as a borrowing constraint.

We considered the following example. Take ξ_t to be a MC on $\{0,1\}$ with $q_{01} = \alpha$, $q_{10} = \beta$, and $U(x) = c^{1-R}/(1-R)$ and the goal is to

$$\max E \int_0^\infty e^{-\delta t} U(c_t)\,dt \quad , \quad s.t. \quad \int_0^t c_s\,ds \leq \eta_t \equiv \exp\left(k\int_0^t \xi_s\,ds\right).$$

Writing $\varphi_t \equiv \eta_t - \int_0^t c_s\,ds$, we have

$$Y_t = \int_0^t U(c_s)\,ds\; e^{-\delta s} + e^{-\delta t} V(\varphi_t, \eta_t, \xi_t)$$ is a supermg

and a martingale under optimal control. Now by scaling $V(\varphi,\eta,\xi) = \eta^{1-R} v(\varphi/\eta, \xi)$ so this will prove useful. Itô on Y gives

$$e^{\delta t} dY_t \doteq dt(U(c) - \delta V) + V_\varphi (d\eta - c\,dt) + V_\eta\, d\eta + QV\,dt$$

so that

$$0 \geq U(c) - \delta V - cV_\varphi + (V_\varphi + V_\eta)\,\eta k \xi + QV.$$

Maximising over c, we find

$$c^* = (V_\varphi)^{-1/R}$$

and then

$$0 = (V_\varphi)^{1-1/R} \frac{R}{1-R} - \delta V + (V_\varphi + V_\eta)\,\eta k\xi + QV.$$

Reworking this in terms of v, we have a factor η^{1-R} to cancel, and get

$$0 = (v')^{1-1/R} \cdot \frac{R}{1-R} - \delta v + k\xi\{(1-R)v + (1-x)v'\} + Qv$$

so in terms of $v_i(x) \equiv v(x,i)$ we get

$$\boxed{\begin{aligned} &(v_0')^{1-1/R} \cdot \frac{R}{1-R} - \delta v_0 + \alpha(v_1 - v_0) = 0 \\ &(v_1')^{1-1/R} \cdot \frac{R}{1-R} - \delta v_1 + k\{(1-R)v_1 + (1-x)v_1'\} + \beta(v_0 - v_1) = 0 \end{aligned}}$$

This appears to be hard to solve. The case $R=1$, corresponding to $U = \log$, is no easier, because when we optimise, log terms intrude.

A simple family of optimal consumption problems (22/7/93)

These are analysed in the literature; I had to work them out when on the road, so it may be worth recording the answers.

$$dS_t = S_t(\sigma\,dB_t + \mu\,dt), \quad U(x) = x^{1-R}/(1-R), \quad Z_t \equiv \exp(-\theta B_t - \tfrac{1}{2}\theta^2 t)$$

with $\theta = (\mu - r)/\sigma$.

The aim is to maximise $E\,U(X_T)$, where X is the wealth process satisfying $\quad dX_t = H_t\,(dS_t - rS_t\,dt) + rX_t\,dt, \quad X_0 = x, \quad X \geq 0.$

The first-order condition is

$$U'(X_T^*) = \lambda \rho_T = \lambda e^{-rT} Z_T,$$

and λ is chosen to make

$$x_T(\lambda) \equiv E\left[\rho_T\, I(\lambda\rho_T)\right] = x \qquad \left[I \equiv (U')^{-1}\right]$$

In this example, $I(x) = x^{-1/R}$,

$$x_T(\lambda) = \lambda^{-1/R} \exp\left[-r\,\frac{R-1}{R}\,T - \tfrac{1}{2}\theta^2 T\,\frac{R-1}{R^2}\right]$$

$$\equiv \lambda^{-1/R} \exp\left[-(R-1)T\,k_R\right], \qquad k_R \equiv \frac{r}{R} + \frac{\theta^2}{2R^2}.$$

$$= \frac{r}{R} + \frac{1}{2}\pi^2\sigma^2$$

Maximised expected utility is

$$\frac{\lambda^{1-1/R}}{1-R} \exp\left(-(R-1)\,k_R\,T\right)$$

so the value f^n is

$$\boxed{V_T(x) = U(x) \exp\left(-(R-1)\,R\,k_R\,T\right)}$$

(and for $R=1$, $V_T(x) = \log x + rT + \tfrac{1}{2}\theta^2 T$).

Optimal wealth process

$$X_t^* = x\,\rho_t^{-1/R} \exp\left[(R-1)\,k_R\,t\right]$$

and

$$dX_t^* = H_t^*\,(dS_t - rS_t\,dt) + rX_t^*\,dt$$

where optimal portfolio H^* is

$$\boxed{H_t^* = \frac{\theta}{\sigma R}\, X_t^*/S_t}$$

$$\pi = \frac{\theta}{\sigma R} = \frac{\mu - r}{\sigma^2 R}$$

An optimisation problem in an incomplete market (22/7/93)

Hyeng-Keun Koo at Washington U has been studying the problem

$$\max E\left[\int_0^\infty e^{-\delta t} U(c_t)\,dt \,\middle|\, X_0 = x,\ Y_0 = y\right] \equiv V(x,y)$$

where the wealth process X satisfies

$$dX_t = rX_t\,dt + \theta_t(dS_t - rS_t\,dt) + (Y_t - c_t)\,dt$$

and the income process Y_t satisfies

$$dY_t = Y_t(\alpha\,dB_t + \nu\,dt), \qquad \text{with } dS_t = S_t(\sigma\,dW_t + \mu\,dt)$$

where $dB\,dW = \rho\,dt$, and in general these are not perfectly correlated.

Define $V(x,y) = \varphi(x/y)\,y^{1-R}$, where we shall assume $U(x) = x^{1-R}/(1-R)$.

Then as usual we get

$$\int_0^t e^{-\delta s} U(c_s) + e^{-\delta t} V(X_t, Y_t) \quad \text{is a supermartingale,}$$

and a martingale under optimal control. So

$$0 \geq U(c) - \delta V + V_x(rX - \theta rS + y - c) + V_y\,\nu y$$
$$+ \tfrac{1}{2}\sigma^2(\theta S)^2 V_{xx} + \alpha y(\theta S)\sigma\rho V_{xy} + \tfrac{1}{2}\alpha^2 y^2 V_{yy}$$

Maximise over c to get

$$c^* = (V_x)^{-1/R}$$

and maximise over θS to get

$$\theta^* S = (rV_x - \alpha y\sigma\rho V_{xy})/\sigma^2 V_{xx},$$

to get

$$0 = (V_x)^{1-1/R}\cdot\frac{R}{1-R} - \delta V + (rx+y)V_x + \nu y V_y + \tfrac{1}{2}\alpha^2 y^2 V_{yy}$$
$$- \tfrac{1}{2}(rV_x - \alpha y\sigma\rho V_{xy})^2/\sigma^2 V_{xx}$$

But

$$V_x = y^{-R}\varphi'(z), \qquad V_y = (1-R)y^{-R}\varphi(z) - y^{-R}z\varphi'(z) \qquad (z \equiv x/y)$$

$$V_{xx} = y^{-1-R}\varphi''(z), \qquad V_{xy} = y^{-1-R}(-R\varphi'(z) - z\varphi''(z))$$

$$V_{yy} = \left[-R(1-R)\varphi(z) + 2Rz\varphi'(z) + z^2\varphi''(z)\right] y^{-1-R}$$

whence

$$\boxed{\begin{aligned} 0 = (\varphi')^{1-1/R} - \delta\varphi + (rz+1)\varphi' + \nu((1-R)\varphi - z\varphi') + \tfrac{\alpha^2}{2}\left[z^2\varphi'' + 2Rz\varphi' - R(1-R)\varphi\right] \\ - \{r\varphi' + \alpha\sigma\rho(z\varphi'' + R\varphi')\}^2/2\sigma^2\varphi'' \end{aligned}}$$

A.V. Skorohod "Studies in the theory of random processes" Addison-Wesley, Reading, Mass, 1965.

On p 103 and following, he has a result which is a multi-dimensional analogue of this; he states the conditions as being sufficient only.

Equivalence of Lévy processes (1/8/93)

(i) Consider two Lévy processes

$$X_j(t) = \sigma_j B_t + c_j t + \int_{|x|>1}\int_0^t x\, \Pi_j(ds,dx) + \int_{|x|\leq 1}\int_0^t x\, \tilde{\Pi}_j(ds,dx)$$

for $j=0,1$, where $\Pi_j(\cdot,\cdot)$ is a Poisson random measure with expectation measure $ds \times \mu_j(dx)$, and $\tilde{\Pi}_j$ is Π_j compensated. If P_j is the law of X_j, considered as a probability measure on path space $\mathbb{D}_0(\mathbb{R}^+,\mathbb{R})$, when can one say $P_1 \ll P_0$??

THEOREM. In order that $P_1 \ll P_0$ on each $\mathcal{F}_t$, it is necessary + sufficient that

(a) $\sigma_0 = \sigma_1$;

(b) $\displaystyle\int \Big(1 - \sqrt{\frac{d\mu_1}{d\mu_0}(x)}\Big)^2 d\mu_0(x) < \infty$;

(c) if $\sigma_0 = \sigma_1 = 0$, then

$$c_1 = c_0 - \int_{A_s} x(\rho(x)-1)\, \mu_0(dx)$$

where we write ρ for $d\mu_1/d\mu_0$, and $A_s \equiv \{x : \rho(x) > 0,\ 0 < |x| \leq 1\}$.

(ii) Let's firstly consider the necessity of (a) – (b). If we take

$$\text{p-}\lim_{n\to\infty} \sum_{j=1}^{[2^n t]} \{X_i(j2^{-n}) - X_i((j-1)2^{-n})\}^2 = \sigma_i^2 t + \sum_{s\leq t} \Delta X_i(s)^2$$

so that by taking a fast subsequence if need be, we could with probability 1 recover the value of σ_i^2 from any small piece of sample path. This ensures that (a) is necessary.

For (b), let's for the moment fix T, and write ξ_j for the number of jumps by time T whose modulus lies in $(2^{-j}, 2^{-j+1}]$ $(j=1,2,\dots)$ with ξ_0 denoting the number of jumps ≥ 1 in modulus. For either X_0 or X_1, the ξ_j form an independent sequence of random variables, and the Kakutani purity law says that the P_1-law of the ξ_j is $\ll$ P_0-law of the ξ_j if and only if

$$\lim_{n\to\infty} E\sqrt{\frac{dQ_1^n}{dQ_0^n}} > 0 ,$$

where Q_j^n is the P_j-law of $(\xi_0, \dots, \xi_n)$. Now it is elementary that

$$\frac{dQ_1^n}{dQ_0^n} = \exp\left[\int_{\{|x|>2^{-n}\}} \int_0^T \pi(ds,dx) \log \rho(x) - T\int_{\{|x|>2^{-n}\}} (\rho(x)-1)\,\mu_0(dx)\right]$$

at least if $\rho > 0$ everywhere. If this is not true, let $\Delta = \{x : \rho(x) = 0\}$, and set $A \equiv \mathbb{R} \setminus \{\Delta \cup \{0\}\}$, the set where μ_0 and μ_1 are absolutely continuous with respect to each other. Let τ be the first time some jump of the Lévy process falls in Δ; $P_1 \ll P_0$ on each $\mathcal{F}_t$ implies that $\mu_0(\Delta) \equiv \delta < \infty$. So the general statement should be

$$\frac{dQ_1^n}{dQ_0^n} = \exp\left[\int_{\{|x|>2^{-n}\}} \left(\int_0^T \pi(ds,dx) \log\rho(x) - (\rho(x)-1)\mu_0(dx)\right) I_A(x)\right] e^{\delta T} I_{\{T<\tau\}}.$$

Thus

$$E_0 \sqrt{\frac{dQ_1^n}{dQ_0^n}} = \exp\left[\int_{A\cap\{|x|>2^{-n}\}} \left(\sqrt{\rho(x)}-1\right)\mu_0(dx) - \tfrac{1}{2}\int_{A\cap\{|x|>2^{-n}\}} (\rho(x)-1)\mu_0(dx)\right]$$

$$= \exp\left[-\int_{A\cap\{|x|>2^{-n}\}} \left(\sqrt{\rho(x)}-1\right)^2 \mu_0(dx)\right],$$

and now letting $n\to\infty$ we see the necessity of (b).

To see that (c) is necessary, let's observe that if we took X_0 and formed the processes

$$Y_t^\varepsilon = \int_{\{|x|>\varepsilon\}} \int_0^t x\,\pi(ds,dx) - t\int_{\{\varepsilon<|x|\le 1\}} x\,\mu_0(dx)$$

and subtracted them from X_0, we would in the limit as $\varepsilon \downarrow 0$ obtain the process $c_0 t$. If we performed the <u>identical</u> operation on the sample paths of X_1 we would in the limit obtain the process

$$c_1 t + t\int_{\{|x|\le 1\}} x\,(\rho(x)-1)\,\mu_0(dx)$$

This explains why (c) is necessary, at least when $\rho > 0$ μ_0-a.e.. The modification needed to cope with $\rho = 0$ is not difficult in view of the modification in the method for necessity of (b).

Observe that

$$\int |x(\rho(x)-1)|\,\mu_0(dx) = \int |x(\sqrt{\rho(x)}+1)(\sqrt{\rho(x)}-1)|\,\mu_0(dx)$$

$$\leq \left(\int x^2(1+\sqrt{\rho(x)})^2\,\mu_0(dx)\right)^{\frac{1}{2}}\left(\int(1-\sqrt{\rho(x)})^2\,\mu_0(dx)\right)^{\frac{1}{2}}$$

and thus will be finite if (b) holds.

(iii) Now we prove the sufficiency of (a)–(c). I claim that the process

$$Z_t = \exp\Big[\int_0^t\int_{A_S}\log\rho(x)\,\tilde{\pi}(ds,dx) + \int_0^t\int_{A_L}\log\rho(x)\,\pi(ds,dx)$$

$$-t\int_{A_S}(\rho(x)-1-\log\rho(x))\,\mu_0(dx) - t\int_{A_L}(\rho(x)-1)\,\mu_1(dx)$$

$$+\gamma B_t - \tfrac{1}{2}\gamma^2 t\Big]\; e^{\delta t}\, I_{\{t<\tau\}}$$

will serve as the density of P_1 w.r.t. P_0 on $\mathcal{F}_t$. Here, $A_S = \{x : \rho(x)>0,\ |x|\leq 1\}$, $A_L = \{x : \rho(x)>0,\ |x|>1\}$, and γ is some constant to be determined. To make sure that the integrals over A_S exist, let's write $A_S = A_S' \cup (A_S \cap \{\rho(x)<\frac{1}{2}\})$, and note that $A_S'' \equiv A_S \cap \{\rho(x)<\frac{1}{2}\}$ must have finite μ_0-measure, from (b). So the jumps in A_S'' are finite in number, and cannot cause any trouble. Then

$$0 \leq \int_{A_S'}\{\rho(x)-1-\log\rho(x)\}\,\mu_0(dx)$$

$$= \int_{A_S'}\left[(\sqrt{\rho(x)}-1)^2 + 2\sqrt{\rho(x)} - 2 - 2\log\sqrt{\rho(x)}\right]\mu_0(dx)$$

$$\leq \text{const.}\int_{A_S'}(\sqrt{\rho(x)}-1)^2\,\mu_0(dx) < \infty.$$

As for the stochastic integral, we have

$$\frac{1}{4}\int_{A_S'}(\log\rho(x))^2\,\mu_0(dx) \leq \text{const.}\int_{A_S'}(\sqrt{\rho(x)}-1)^2\,\mu_0(dx) < \infty,$$

and so there is no problem with it.

So to finish, we compute

$$E_0\left[\exp(i\alpha X_t)\cdot Z_t\right]$$

$$= E_0\Big[\exp\Big\{i\alpha\Big(\sigma_0 B_t + c_0 t + \int_{A_S}\int_0^t x\,\tilde{\Pi}(ds,dx) + \int_{A_L}\int_0^t x\,\Pi(ds,dx)\Big)$$

$$+ \int_{A_S}\int_0^t \log\rho(x)\,\tilde{\Pi}(ds,dx) + \int_{A_L}\int_0^t \log\rho(x)\,\Pi(ds,dx)$$

$$- t\int_{A_S}(\rho(x)-1-\log\rho(x))\,\mu_0(dx) - t\int_{A_L}(\rho(x)-1)\,\mu_0(dx)$$

$$+ \gamma B_t - \tfrac{1}{2}\gamma^2 t\Big\}\; e^{\delta t}\; I_{\{t<\tau\}}\Big]$$

$$= \exp\Big[\tfrac{1}{2}(\gamma+i\alpha\sigma_0)^2 t + (i\alpha c_0 - \tfrac{1}{2}\gamma^2)t\Big].$$

$$E_0\exp\Big[\int_{A_S}\int_0^t(\log\rho(x) + i\alpha x)\,\tilde{\Pi}(ds,dx) - t\int_{A_S}(\rho(x)-1-\log\rho(x))\,\mu_0(dx)\Big].$$

$$E_0\exp\Big[\int_{A_L}\int_0^t(\log\rho(x) + i\alpha x)\,\Pi(ds,dx) - t\int_{A_L}(\rho(x)-1)\,\mu_0(dx)\Big]$$

$$= \exp\Big[-\tfrac{1}{2}\alpha^2\sigma_0^2 t + i\alpha(c_0+\gamma\sigma_0)t\Big].$$

$$\exp\Big[t\int_{A_S}(\rho(x)e^{i\alpha x} - 1 - \rho(x) + 1 - i\alpha x)\,\mu_0(dx)\Big].$$

$$\exp\Big[t\int_{A_L}(\rho(x)e^{i\alpha x} - \rho(x))\,\mu_0(dx)\Big]$$

$$= \exp\Big[-\tfrac{1}{2}\alpha^2\sigma_0^2 t + i\alpha(c_0+\gamma\sigma_0)t + t\int_{A_S}(e^{i\alpha x}-1-i\alpha x)\,\mu_1(dx)$$

$$- i\alpha t\int x(1-\rho(x))\,\mu_0(dx) + t\int_{A_L}(e^{i\alpha x}-1)\,\mu_1(dx)\Big].$$

$$= \exp\Big[-\tfrac{1}{2}\alpha^2\sigma_1^2 t + i\alpha c_1 t + t\int_{A_S}(e^{i\alpha x}-1-i\alpha x)\,\mu_1(dx) + t\int_{A_L}(e^{i\alpha x}-1)\,\mu_1(dx)\Big]$$

if conditions (a)–(c) hold. (If $\sigma_0 \neq 0$, then we pick γ so as to match the drifts; if $\sigma_0 = 0$, condition (c) guarantees that the drifts are the same!)

Asymptotic turnpike results (9/8/93)

(i) Consider a world where there is one risky asset

$$dS_t = S_t(\sigma_t\, dB_t + \mu_t\, dt)$$

and a riskless asset earning at a risk-free rate r_t. Define the deflator ζ by

$$d\zeta_t = \zeta_t\{-r_t\, dt - (\mu_t - r_t)/\sigma_t\, dB_t\}, \quad \zeta_0 = 1.$$

Assume

(A1) $\boxed{r \text{ and } \sigma^{-1}(\mu - r) \text{ are bounded processes}}$

and

(A2) $\boxed{E\,\zeta_T \to 0 \quad \text{as } T \to \infty}$

(ii) An agent with initial wealth x wants to invest so as to maximise

$$E\, U(X_T),$$

where X_t is the wealth process,

$$\tilde{X}_t \equiv \beta_t X_t = x + \int_0^t \theta_u\, d\tilde{S}_u, \qquad \beta_t \equiv \exp\{-\int_0^t r_u\, du\}$$

when the agent holds portfolio process θ in the risky asset, and $\tilde{S}_t \equiv \beta_t S_t$.

We shall suppose that U is strictly increasing + concave, $U \in C^1$, and

(A3) $\boxed{I \equiv (U')^{-1} \text{ varies regularly at } 0.}$

This means that $I(ax)/I(x) \to a^{-\alpha}$ $(x \downarrow 0)$ for each $a > 0$. Here, α is the exponent of regular variation, and must be ≥ 0 since I is decreasing. The regular variation condition is frequently satisfied, and is a common condition in limit theorems. Examples of I with this regular variation are

$$I(x) = x^{-\alpha}(\log 1/x)^{c_1}(\log\log 1/x)^{c_2}$$

Note in particular that I cannot grow faster than a power of x at 0.

It is well known that the optimal solution to the agent's problem is to take $X^*_t = \zeta_t^{-1} \, E[\zeta_T \, I(\lambda \zeta_T) \,|\, \mathcal{F}_t]$ where the multiplier λ is chosen so that

$$x = E[\zeta_T \, I(\lambda \zeta_T)].$$

Notice that because I grows at worst polynomially, and (A1) is in force, the expectation is finite for all $\lambda > 0$, and defines a continuous decreasing function of λ.

(iii) Now consider two agents with different utilities U_0, U_1 such that

(A4) $\boxed{I_0(x)/I_1(x) \to 1}$ $(x \downarrow 0)$

and all the other assumptions (A1 – A3) hold. Let $X^*_i(t;T)$ denote the optimal wealth process for agent i, and let $\lambda_i(T)$ denote the value of λ such that

$$x = E[\zeta_T \, I_i(\lambda \zeta_T)] \qquad (i = 0, 1).$$

(Frequently, the explicit dependence on T will be omitted.)

THEOREM <u>Assume A1 – A4. Then for each $t > 0$,</u>

$$\boxed{\lim_{T \to \infty} E \, \zeta_t \, | X^*_0(t;T) - X^*_1(t;T) | = 0}$$

<u>provided</u> $\alpha > 0$.

<u>Remarks</u> (i) The sense of convergence is in fact much stronger. In the proof, we show

$$E \, | \zeta_T \, I_0(\lambda_0(T) \zeta_T) - \zeta_T \, I_1(\lambda_1(T) \zeta_T) | \to 0,$$

so by Doob's submartingale maximal inequality, for any $K < \infty$, $\delta > 0$

$$P\Big[\sup_{0 \le t \le K} \zeta_t \, | X^*_0(t;T) - X^*_1(t;T) | > \delta \Big] \to 0 \qquad (T \to \infty)$$

(ii) It would be nice to get rid of the assumption $\alpha > 0$; if we assume also that for some $0 < a < b$ we have for <u>all</u> $x > 0$

(A5) $\quad I_0(ax) \ge I_1(x) \ge I_0(bx),$

then the result holds – but this is rather a heavy restriction! See later

<u>Proof</u> We firstly prove that

$$\frac{\lambda_0(T)}{\lambda_1(T)} \to 1 \qquad (T\to\infty).$$

If this were false, then there would be some $\rho<1$, say, and $T_n \to \infty$ such that

$$\lambda_0(T_n)/\lambda_1(T_n) \leq \rho \qquad \forall\ n.$$

In that case, for $T=T_n$,

$$x = E[\zeta_T I_0(\lambda_0\zeta_T)] \geq E[\zeta_T I_0(\rho\lambda_1\zeta_T)]$$

$$= E[\zeta_T\ I_0(\rho\lambda_1\zeta_T) : \lambda_1\zeta_T > \delta]$$

$$+ E[\zeta_T\ I_0(\rho\lambda_1\zeta_T) : \lambda_1\zeta_T \leq \delta]$$

where δ is so chosen that for $x \leq \delta$, $\dfrac{I_0(\rho x)}{I_1(x)} \geq \rho^{-\alpha/2}$;

$$\geq E[\zeta_T I_0(\rho\lambda_1\zeta_T) : \lambda_1\zeta_T > \delta]$$

$$+ \rho^{-\alpha/2}\ E[\zeta_T\ I_1(\lambda_1\zeta_T)\ ;\ \lambda_1\zeta_T \leq \delta].$$

Now the first term goes to 0 (A2), and the second term likewise differs by something $o(1)$ from

$$\rho^{-\alpha/2}\ E[\zeta_T I_1(\lambda_1\zeta_T)] = \rho^{-\alpha/2}\ x > x$$

which is a contradiction. Hence $\lambda_0(T)/\lambda_1(T) \to 1$.

To finish the proof, take some $\rho<1$, very close to 1, and some T_0 so large that for $T \geq T_0$,

$$\rho \leq \frac{\lambda_1(T)}{\lambda_0(T)} \leq \rho^{-1}.$$

Now find δ so small that for $x \in (0,\delta]$,

$$1 \leq \frac{I_0(\rho x)}{I_1(x)} \leq \rho^{-2\alpha}, \qquad \rho^{2\alpha} \leq \frac{I_0(\rho^{-1}x)}{I_1(x)} \leq 1.$$

Silly example: if the price process is already a martingale, constant spot-rate r, we have simply $\zeta_t = e^{-rt}$ and so

$$x = e^{-rT} I_j(\lambda_j e^{-rT}) \quad \therefore \quad \lambda_j = e^{rT} U_j'(x e^{rT}).$$

If we try to weaken (A4) to "I_0/I_1 slowly varying" and take $U_0' = x^{-R}$, then we'd get

$$\frac{\lambda_1(T)}{\lambda_0(T)} = \ell(x e^{rT})$$

and this could oscillate between 0 and ∞ as $T \to \infty$.

Then

$$E\, \zeta_T \,| I_0(\lambda_0 \zeta_T) - I_1(\lambda_1 \zeta_T)|$$

$$= E\left[\zeta_T I_0(\lambda_0 \zeta_T) \left| 1 - \frac{I_1(\lambda_1 \zeta_T)}{I_0(\lambda_0 \zeta_T)} \right| ; \ \lambda_1 \zeta_T \le \delta\rho \right]$$

$$+ \ E\left[\zeta_T \,| I_0(\lambda_0 \zeta_T) - I_1(\lambda_1 \zeta_T)| ; \ \lambda_1 \zeta_T > \delta\rho \right]$$

Using (A2) as before, the second expectation goes to zero. As for the first, on the event $Y \equiv \lambda_1 \zeta_T \le \delta\rho$, we have

$$\frac{I_1(Y)}{I_0(\lambda_0 \zeta_T)} \le \frac{I_1(Y)}{I_0(Y/\rho)} \le \rho^{-2\alpha}$$

and

$$\frac{I_1(Y)}{I_0(\lambda_0 \zeta_T)} \ge \frac{I_1(Y)}{I_0(\rho Y)} \ge \rho^{2\alpha}.$$

Since $E\, \zeta_T I_0(\lambda_0 \zeta_T) = x$ for all T, the conclusion follows.

Remarks (i) For the slowly varying case ($\alpha = 0$), assuming (A5), we would then have easily

$$b\, \lambda_1(T) \ge \lambda_0(T) \ge a\, \lambda_1(T),$$

and the second part of the proof above can be used again.

(ii). It is possible for ℓ to vary slowly at infinity, and to have $\overline{\lim}\ \ell(x) = +\infty$, $\underline{\lim}\ \ell(x) = 0$ $(x \to \infty)$. To see this, take a slowly varying f^{n}

$$\ell(x) = \exp\left(\int_1^x \varepsilon(u)\, du/u \right)$$

where $\varepsilon(u) = (-1)^n/n$ in $(a_n, a_{n+1}]$, and the a_n are chosen well. In fact, if $a_{n+1}/a_n = e^{n^2}$, we get $\int_{a_n}^{a_{n+1}} \varepsilon(u) \frac{du}{u} = (-1)^n . n$, and can show quite easily that

$$\sum_{j=1}^{n} (-1)^{j-1} . j = (-1)^{n-1} \frac{n+1}{2} + \frac{1-(-1)^{n+1}}{4}.$$

Blacks consol rate conjecture (14/8/93).

(i) This is suggested by a preprint of Darrell Duffie, Jin Ma, Jiongmin Yong with the above title. The idea is to try to model the spot rate r in such a way that (r_t, Y_t) is a bivariate diffusion, where Y is the price of the consol,

$$Y_t \equiv E_t \left[\int_t^\infty \exp\left(-\int_t^s r_u du\right) ds \right].$$

Of course, one way this can be done is to assume r is itself a diffusion process, whereupon one has $Y_t = f(r_t)$ for some f, and DMY prove that under some restrictive assumptions, this is all that can happen. However, these assumptions rule out many interesting examples; generally, if (r_t, ξ_t) is a diffusion $(r_t \geq 0 \;\forall t)$ then $Y_t = \psi(r_t, \xi_t)$ for some function ψ, and if we can invert ψ and write $\xi_t = \varphi(r_t, Y_t)$, then (r_t, Y_t) will also be a diffusion.

(ii) Here is one interesting class of such processes. Suppose

$$dr_t = \sigma \sqrt{r_t}\, dW_t + (\alpha_t - \beta r_t)\, dt$$

where σ, β are constants, and α is a non-negative diffusion independent of W. It is not hard to prove (see my talk for Minneapolis "Which model for the term-structure of interest rates should one use?") that if α were a deterministic non-negative function, then

$$E_t \exp\left(-\int_t^{t+\tau} r_u du\right) = \exp\left[-r_t B(\tau) - \int_t^{t+\tau} \alpha_u\, B(t+\tau-u)\, du\right]$$

where as in CIR

$$B(\tau) \equiv \sinh \gamma\tau \left\{ \gamma \cosh \gamma\tau + \tfrac{1}{2}\beta \sinh \gamma\tau \right\}, \qquad 2\gamma \equiv (\beta^2 + 2\sigma^2)^{1/2}$$

We could, for example, take α itself to be a CIR process:

$$d\alpha_t = \tilde{\sigma} \sqrt{\alpha_t}\, dB_t + (\tilde{\alpha} - \tilde{\beta}\alpha_t)\, dt$$

and then for deterministic non-negative f

$$E_t \exp\left(-\int_t^T \alpha_u f_u du\right) = \exp\left[-\alpha_t \tilde{B}(t,T) - \tilde{A}(t,T)\right]$$

where
$$\begin{cases} \dfrac{\partial \tilde{B}}{\partial t} - \tfrac{1}{2}\tilde{\sigma}^2 \tilde{B}^2 - \tilde{\beta}\tilde{B} + f = 0, & \tilde{B}(T,T) = 0 \\[2mm] \dfrac{\partial \tilde{A}}{\partial t} = -\tilde{\alpha}\,\tilde{B}(t,T), & \tilde{A}(T,T) = 0. \end{cases}$$

These give us that the log of the consol price is an affine function of (r_t, d_t) so that we can write d_t as a function of (r_t, Y_t). It seems unlikely that any simple expressions will result.

Asymptotic turnpike results: postscript (7/9/93).

If $X_i^*(\cdot\,; T)$ was the optimal wealth process for agent i with horizon T, and if we define

$$\Delta(t;T) \equiv \beta_t \{X_0^*(t;T) - X_1^*(t;T)\},$$

then $\Delta(\cdot\,;T)$ is a martingale with respect to $(P_T^*, (\mathcal{F}_t)_{0\le t\le T})$, where

$$\frac{dP_T^*}{dP} = Z_T \quad \text{on } \mathcal{F}_T,$$

and even has a representation

$$\Delta(t;T) = \int_0^t \eta(u;T)\, d\tilde{S}_u \qquad , \quad \eta(u,T) \equiv \theta_0^*(u;T) - \theta_1^*(u;T).$$

What we proved before is

$$E_T^* \, |\Delta(T;T)| \longrightarrow 0 \qquad (T \to \infty).$$

Now fix $K < \infty$, and (for $T > K$) use Doob's submartingale maximal inequality

$$P_T^*\Big(\sup_{t\le K} |\Delta(t;T)| > \lambda\Big) \le \lambda^{-1} E_T^* \, |\Delta(K;T)|$$

$$\le \lambda^{-1} E_T^* \, |\Delta(T;T)|.$$

Hence for $0 < p < 1$

$$E_T^*\Big[\sup_{t\le K} |\Delta(t;T)|^p\Big] = \int_0^\infty p\lambda^{p-1}\, d\lambda\; P_T^*\Big[\sup_{t\le K} |\Delta(t;T)| > \lambda\Big]$$

$$\le \int_0^\infty p\lambda^{p-1}\, d\lambda \left(\lambda^{-1} E_T^* |\Delta(T;T)| \wedge 1\right)$$

$$\to 0 \qquad (T\to\infty).$$

But $\Delta(\cdot\,;T)$ is a continuous local martingale, so by the Burkholder-Davis-Gundy inequalities (see, for example, Rogers + Williams IV.42) we have for some absolute constant c_p.

$$E_T^*\left([\Delta(\cdot;T)]_K^{p/2} \right) \le c_p\, E_T^*\left[\sup_{t\le K} |\Delta(t;T)|^p \right] \to 0 \qquad (T\to\infty).$$

But the lefthand side of this inequality is

$$E_T^*\left\{ \left(\int_0^K \eta(u;T)^2\, d[\tilde S]_u \right)^{p/2} \right\}$$

$$= E\left\{ Z_K \left(\int_0^K \eta(u;T)^2\, d[\tilde S]_u \right)^{p/2} \right\},$$

so we have in particular that <u>for each</u> $K<\infty$,

$$\boxed{\int_0^K \eta(u;T)^2\, d[\tilde S]_u \xrightarrow{P} 0}$$

– so not only are the optimal wealth processes converging, but the optimal portfolio processes are also converging (in the above sense).

<u>Remarks</u> (i) Cox + Huang get a stronger result than the above, in that they claim to prove

$$E_T^*\,[\Delta(\cdot;T)]_K \to 0 \qquad (T\to\infty)$$

(i.e. our statement, but with $p=2$ instead). However, they are assuming constant coefficients, and $U_0(x) = \frac{1}{1-b}x^{1-b}$, which is much more restrictive. <u>However, their "proof" contains a huge hole</u>.

(ii) In any case, since we only aim to establish the <u>principle</u> that for large T we should do the same thing for both utilities, does it really matter what convergence we get...?!

(iii) Restricting the time-set to $[0,K]$ has not necessary; we could as well have used $[0,T]$!

Hence if $\rho = \binom{\rho_+}{\rho_-}$, we can conclude

$$\left.\begin{aligned} Az + \rho_+ &= -\Pi_- w \\ Cz + \rho_- &= w \end{aligned}\right\} \Rightarrow \quad (A + \Pi_- C)\, z = -\Pi_- \rho_- - \rho_+$$

The generator $\hat{G}_+$ of $\hat{y}^+$, the plus process of the reversal, is $(A+\Pi_- C)^*$, so we shall have

$$\begin{aligned} f(0,i) &= \quad 0 \qquad\qquad (i \in E^-) \\ &= -(A+\Pi_- C)^{-1}(\rho_+ + \Pi_- \rho_-)_i \qquad (i \in E^+), \end{aligned}$$

and a little arithmetic reduces the expression on E^+ to

$$-\rho^T M \binom{I}{\hat{\Pi}_+} \hat{G}_+^{-1} M^{-1} = \left(-(A+\Pi_- C)^{-1}(I \;\; \Pi_-)\rho\right)^T.$$

In the special case $\rho \equiv 1$, we even have $\Pi_- 1 = 1$, so we have the simple expression

$$-2(A+\Pi_- C)^{-1} 1 .$$

[Remark. If we kill at rate λ, and do the WH factorisation for this, we obtain $\Pi_- \equiv \Pi_-(\lambda)$ satisfying

$$\Pi_- D_\lambda + A_\lambda \Pi_- + B + \Pi_- C \Pi_- = 0$$

$D_\lambda \equiv D - \lambda$, $A_\lambda \equiv A - \lambda$. Differentiating w.r.t. λ, letting $\lambda \downarrow 0$, we find that, with
$Z \equiv -\frac{\partial}{\partial\lambda}\Pi(\lambda)\big|_{\lambda=0} = E^{\cdot}[\tau_0^- ; X(\tau_0^-) = \cdot]$,

$$Z G_- + (A + \Pi_- C) Z = -2\Pi_-$$

which has the unique solution

$$Z = 2\int_0^\infty e^{t(A+\Pi_- C)}\, \Pi_-\, e^{tG_-}\, dt$$

so

$$Z 1 = -2(A+\Pi_- C)^{-1} 1,$$

confirming the above calculation.]

Where is BM when its range first exceeds 1? (10/10/93)

Let's write $\bar{B}_t \equiv \sup_{u \le t} B_u$, $\underline{B}_t \equiv \inf_{u \le t} B_u$, and set $\tau \equiv \inf\{t : \bar{B}_t - \underline{B}_t > 1\}$. What's the law of B_τ?

(i) Here's a simple excursion proof that

$$\boxed{P[B_\tau \in dx] = |x|\, dx} \qquad (-1 \le x \le 1).$$

Clearly enough to do for $x > 0$. We have (thinking of excursions up from $\underline{B}$)

$$P[x \le B_\tau \le x+h] = \int_{1-x-h}^{1-x} dv \, \exp\left(-\int_0^v \frac{dy}{x+y}\right) \cdot \frac{1}{x+v} + o(h)$$

$$= h \cdot \exp\left(-\int_0^{1-x} \frac{dy}{x+y}\right) + o(h)$$

$$= h \cdot x + o(h).$$

(ii) Simply using the scale function of B,

$$P[\bar{B}_\tau > x] = P[B \text{ reaches } x \text{ before } -(1-x)]$$

$$= 1 - x$$

so that $\boxed{\bar{B}_\tau \text{ is } U[0,1].}$ (David Hobson first made me aware of this.)

(iii) Here is an alternative proof of (i) above, using various sample path transformations which could be useful elsewhere.

Let τ_t be inverse to $\bar{B}_t - \underline{B}_t$, so that $B(\tau_t)$ is a martingale, which, when positive, rises at unit speed, and jumps across from $x \in (0,t)$ to $x-t$ at rate $1/t$. If we now define for $t \in \mathbb{R}$

$$X_t \equiv I_{\{B(\tau(e^t)) > 0\}}$$

then X is a Markov chain on $\{0,1\}$ with Q-matrix $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$, and

$$\bar{B}(\tau_t) = \int_0^t I_{\{B(\tau_u)>0\}}\, du = \int_{-\infty}^{\log t} I_{\{B(\tau(e^s))>0\}}\, e^s\, ds$$

$$\equiv \int_{-\infty}^{\log t} e^s X_s\, ds.$$

So what is the law of $\bar{B}(\tau_1)$ given that $B(\tau_1)>0$? This is like finding the law of $\int_{-\infty}^0 e^s X_s\, ds$ given that $X_0 = 1$. If we condition X on $X_0=1$, and run time backwards from 0, we see the chain X started at 1. Thus

$$\mathcal{L}(\bar{B}(\tau_1) \mid B(\tau_1)>0) = \mathbb{P}^1\text{-law of } \int_0^\infty e^{-s} X_s\, ds.$$

What is the law of this r.v.??

(iv) To take this in more generality, let's try to characterise the law of

$$Z \equiv \int_0^\infty \lambda e^{-\lambda t} I_{\{a\}}(X_t)\, dt$$

where now X is a general finite Markov chain with statespace $I \ni a$. Abbreviate $I_{\{a\}}$ to e_a. Then

$$\frac{1}{n!} E^a Z^n = E^a \int_0^\infty \lambda e^{-\lambda t_1} e_a(X_{t_1}) \int_{t_1}^\infty \lambda e^{-\lambda t_2} e_a(X_{t_2}) \cdots \int_{t_{n-1}}^\infty \lambda e^{-\lambda t_n} e_a(X_{t_n})\, dt$$

$$= \lambda^n \left(R_{n\lambda}\, e_a\, R_{(n-1)\lambda}\, e_a \cdots e_a\, R_\lambda\, e_a\right)(a),$$

Now if ν is the Lévy measure of excursion lifetimes from a, we get

$$\lambda R_\lambda e_a(a) = \frac{\lambda}{\lambda + \int \nu(dt)(1-e^{-\lambda t})} \equiv \frac{\lambda}{\psi(\lambda)},$$

and hence

$$\boxed{E^a Z^n = \prod_{j=1}^n j\lambda / \psi(j\lambda).}$$

It's hard to make a lot of this in general, but if we have two state chain with

$$Q = \begin{array}{c} \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix} \end{array}\begin{array}{c} a \\ b \end{array}$$

then $\psi(\lambda) = \lambda + \alpha\left(1 - \frac{\beta}{\lambda+\beta}\right) = \frac{\lambda}{\lambda+\beta}(\lambda+\alpha+\beta)$

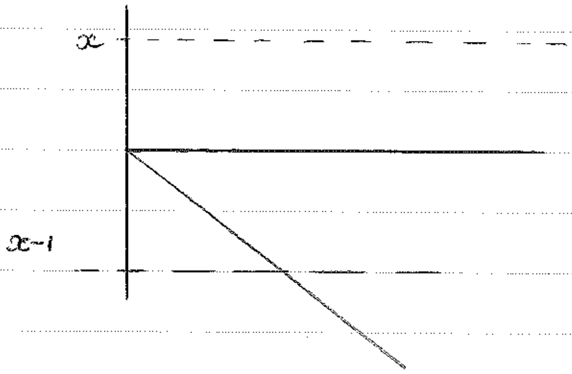
and so

$$E^a Z^n = \prod_{j=1}^n \frac{j\lambda+\beta}{j\lambda+\alpha+\beta} = \prod_1^n \frac{j+\beta/\lambda}{j+(\alpha+\beta)/\lambda} = \frac{\Gamma(n+1+\beta/\lambda)}{\Gamma(1+\beta/\lambda)} \cdot \frac{\Gamma(1+(\alpha+\beta)/\lambda)}{\Gamma(n+1+(\alpha+\beta)/\lambda)}$$

This implies that $Z \sim B\left(1+\frac{\beta}{\lambda}, \frac{\alpha}{\lambda}\right)$, so in the special case of interest where $\alpha = \beta = \lambda = 1$, we have the desired result that $Z \sim B(2,1)$.

(25/10/93) We can also obtain the joint law of (τ, B_τ)! Suppose we kill at rate $\lambda \equiv \frac{1}{2}\theta^2$.

For $0 < x < 1$, by excursions



$$E\left[e^{-\lambda\tau} : B_\tau \in dx\right]$$

$$= \exp\left[-\int_x^1 dv \left(\underbrace{\theta\coth\theta v}_{\substack{\text{rate of xcns which are} \\ \text{either killed or rise above } v}}\right)\right] \cdot \frac{dx}{1} \cdot \frac{\theta}{\sinh\theta}$$

$$= \frac{\theta\sinh\theta x}{\sinh^2\theta}\, dx.$$

This checks out when we let $\theta \downarrow 0$. Also, if we integrate in x, we obtain

$$E\left[e^{-\lambda\tau}\right] = \frac{2(\cosh\theta - 1)}{\sinh^2\theta} = \operatorname{sech}^2 \theta/2$$

– so τ has the same law as $T + T'$, where T and T' are i.i.d., with T having the distribution of the exit time from $[-\frac{1}{2}, \frac{1}{2}]$ of BM... Simple explanation of this fact??

Asymptotic turnpike examples (25/10/93)

Huberman + Ross treat a discrete-time situation (Econometrica 51, 1345–1361) with independent returns in different periods, and a utility whose derivative varies regularly: $U'(x) = x^{-R} L(x)$. They manage to prove that if the horizon $T \to \infty$, then the optimal investment for the first day converges to what it would be if the utility were exactly x^{1-R}; in other words, the slowly-varying part of U' has no effect in the limit. Phil asks to what extent this may be a general phenomenon.

Example 1. By breaking the independence assumption badly, we can ruin the result. Suppose that on day 1 a coin is tossed, prob of $H = p \equiv 1-q$. The investor wins his stake if coin falls H, else loses stake. Let Y be the outcome of the toss; $P(Y=1) = p$, $P(Y=-1) = q$. On all subsequent days, the stock and the bond grow by e^{1+Y} (so no more randomness ever). Thus the horizon-T problem is

$$\max_{-1 \le \theta \le 1} \; p\, U\big((1+\theta)e^{2T}\big) + q\, U(1-\theta),$$

and if $U(x) = x^{1-R}/(1-R)$, optimality criterion is

$$\left(\frac{1+\theta}{1-\theta}\right)^R = p\, e^{2(1-R)T}/q .$$

So if $0<R<1$, we get that as $T \to \infty$, the optimal behaviour takes θ closer and closer to 1!

Example 2. Let's go back to the continuous-time setting, where we consider a class of examples, those where σ, μ and r are deterministic functions of time. Thus the deflator ζ will be a log-Gaussian process, and if we define

$$\psi(a,b) \equiv E\left[e^Y I(e^Y)\right], \qquad Y \sim N(a,b),$$

then the optimal wealth process is

$$X_t^* = \zeta_t^{-1} E\left[\zeta_T I(\lambda \zeta_T) \mid \mathcal{F}_t\right] = (\lambda \zeta_t)^{-1} \psi(a_{t,T}, b_{t,T})$$

Special case:

$$\log \lambda_T = -R\log x_0 + (1-R)\, R_T + \tfrac{1}{2} b_T \left(\tfrac{1}{R} - 1\right)$$

$$\lambda_T \, \zeta_T = x_0^{-R} \exp\left[-R\, R_T - \int_0^T \theta_u \, dW_u - b_T + \tfrac{1}{2} b_T \cdot \tfrac{1}{R} \right]$$

$$dX_t^* = \theta_t X_t^* \tfrac{1}{R}(dW_t + \theta_t dt) + r_t X_t^* dt$$

$$\text{so } X_t^* = x_0 \exp\left[\tfrac{1}{R}\int_0^t \theta_u \, dW_u + \int_0^t \left(r_u + \tfrac{\theta_u^2}{R}\right) du - \tfrac{1}{2}\int_0^t \theta_u^2 \, du / R^2 \right]$$

$$= x_0 \exp\left[\tfrac{1}{R}\int_0^t \theta_u \, dW_u + R_t + \frac{b_t}{R} - \frac{b_t}{2R^2} \right]$$

(What happens if R is very small?)

Optimally, hold $\dfrac{\theta_t X_t^*}{R\sigma_t S_t}$ shares, so hold $\boxed{\dfrac{\theta_t}{R\sigma_t} \text{ of wealth in shares}}$

where $\begin{cases} a_{t,T} \equiv \log \lambda_T - \int_0^T (r_u + \frac{1}{2}\theta_u^2)\, du - \int_0^t \theta_u \, dW_u \\ b_{t,T} \equiv \int_t^T \theta_u^2 \, du \end{cases}$ $\quad [\theta \equiv \sigma^{-1}(\mu - r)]$

and where λ_T is chosen to satisfy the budget constraint

initial wealth $x_0 = \lambda_T^{-1} \psi(a_{0,T}\ b_{0,T})$.

Notice that $\psi(W_t, T-t)$ would be a martingale, so ψ satisfies

$$\boxed{\frac{1}{2} \frac{\partial^2 \psi}{\partial a^2} = \frac{\partial \psi}{\partial b}}$$

A little Itô calculus on $X_t^* = (\lambda_T \zeta_t)^{-1} \psi(a_{t,T}\ b_{t,T})$ reduces to

$$dX^* = \frac{\theta(\psi - \nabla\psi)}{\lambda\zeta}(dW + \theta dt) + rX^* dt$$

so that optimal portfolio is

$$\boxed{\pi_t^* = \frac{\theta_t}{\sigma_t S_t} X_t^* \left(1 - \nabla \log \psi\right)(a_{t,T}, b_{t,T})} \qquad \left(\nabla = \frac{\partial}{\partial a}\right)$$

In the special case $U(x) = x^{1-R}/(1-R)$, $\nabla \log \psi = 1 - 1/R$, agreeing with known result.

[If we take

$$\boxed{I(x) = x^{-1/R} \left(1 + (\log x)^2\right)} \qquad (0<R<1)$$

then I is a decreasing function, and

$$\psi(a,b) = E\, e^{(1-1/R)Y} (1 + Y^2)$$

$$= \exp\left[\gamma a + \tfrac{1}{2}\gamma^2 b\right] \left\{1 + b + (a+\gamma b)^2\right\}. \qquad \left(1 - \tfrac{1}{R} \equiv \gamma\right)$$

So

$$\boxed{\nabla \log \psi = \gamma - \frac{2(a+\gamma b)}{1 + b + (a+\gamma b)^2}}$$

This is probably the simplest case of regularly varying I, apart from powers.]

Let's abbreviate $R_T \equiv \int_0^T r_u\, du$, $b_T \equiv b_{0,T} \equiv \int_0^T \theta_u^2\, du$, and make the assumptions

$$\boxed{\begin{array}{l} \text{(B1)} \quad R_T \to \infty \quad (T \to \infty); \\[2ex] \text{(B2)} \quad \text{There exists } K \text{ such that for all } T \geq K \\ \qquad\qquad b_T \leq K R_T ; \\[2ex] \text{(B3)} \quad \text{The function} \\ \qquad x \longmapsto \frac{1}{R} + \frac{x I'(x)}{I(x)} \text{ is bounded and tends to } 0 \text{ as } x \to 0. \end{array}}$$

(i) The first aim is to understand the behaviour of λ_T for large T. To do this, fix $\delta > 0$ so small that

$$\boxed{\frac{1}{R} - \delta > 1, \qquad 1 + R\delta < R/K\delta}$$

Now for each $\varepsilon > 0$, take

$$c_\varepsilon^+ = \sup_{x \leq \varepsilon} I(x) / x^{-1/R - \delta}$$

$$c_\varepsilon^- = \inf_{x \leq \varepsilon} I(x) / x^{-1/R + \delta}.$$

We have

$$\begin{aligned} 1 &= E\, \zeta_T I(\lambda_T \zeta_T) \\ &= E\left[\zeta_T I(\lambda_T \zeta_T) : \lambda_T \zeta_T \leq \varepsilon\right] + E\left[\zeta_T I(\lambda_T \zeta_T) : \lambda_T \zeta_T > \varepsilon\right] \\ &\geq c_\varepsilon^- \lambda_T^{-1/R + \delta} E\left[\zeta_T^{1 - 1/R + \delta} ; \lambda_T \zeta_T \leq \varepsilon\right] \\ &= c_\varepsilon^- \lambda_T^{-1/R+\delta} \left\{ E\left[\zeta_T^{1-1/R+\delta}\right] - E\left[\zeta_T^{1-1/R+\delta} ; \lambda_T \zeta_T > \varepsilon\right] \right\} \\ &\geq c_\varepsilon^- \Big[\lambda_T^{-1/R+\delta} \exp\{-(1 - \tfrac{1}{R} + \delta)(R_T + \tfrac{1}{2} b_T) + \tfrac{1}{2}(1 - \tfrac{1}{R} + \delta)^2 b_T\} \\ &\qquad\qquad - E\,\zeta_T \cdot \varepsilon^{-1/R+\delta} \Big] \end{aligned}$$

Rearranging gives

$$\boxed{\log \lambda_T \geq \frac{1-R-R\delta}{1-R\delta} \cdot R_T + \tfrac{1}{2} b_T \left(\tfrac{1}{R} - 1 - \delta\right) - \frac{R}{1-R\delta} \gamma_\varepsilon}$$

where $\gamma_\varepsilon \equiv \log \left[\frac{1}{c_\varepsilon^-} + \varepsilon^{\delta - 1/R} \right]$.

We now seek a bound on the other side by again splitting the expectation.

$$1 = E\, \zeta_T I(\lambda_T \zeta_T)$$

$$\leq c_\varepsilon^+ \, E\left[\zeta_T^{1-1/R-\delta} \lambda_T^{-1/R-\delta} ; \lambda_T \zeta_T \leq \varepsilon \right] + E\left[\zeta_T I(\lambda_T \zeta_T) : \lambda_T \zeta_T > \varepsilon \right]$$

$$\leq c_\varepsilon^+ \lambda_T^{-1/R-\delta} E\left[\zeta_T^{1-1/R-\delta} \right] + I(\varepsilon)\, E[\zeta_T]$$

$$= c_\varepsilon^+ \lambda_T^{-1/R-\delta} \exp\left\{ -(1-\tfrac{1}{R}-\delta)(R_T + \tfrac{1}{2} b_T) + \tfrac{1}{2}(1-\tfrac{1}{R}-\delta)^2 b_T \right\} + I(\varepsilon)\, e^{-R_T}$$

Hence for large T, using (B1), we obtain the bound

$$0 < (1 - I(\varepsilon) e^{-R_T}) / c_\varepsilon^+ \leq \lambda_T^{-1/R-\delta} \exp\left[R_T (\tfrac{1}{R} + \delta - 1) + \tfrac{1}{2} b_T (\tfrac{1}{R} + \delta - 1)(\tfrac{1}{R} + \delta) \right]$$

so that

$$\boxed{\log \lambda_T \leq \frac{1-R+R\delta}{1+R\delta} R_T + \tfrac{1}{2} b_T \left(\tfrac{1}{R} + \delta - 1 \right) + \frac{R}{1+R\delta} \gamma'_\varepsilon}$$

with $\gamma'_\varepsilon \equiv \log 2 c_\varepsilon^+$.

This bounds λ_T very effectively.

(ii) Now I want to prove that as $T \to \infty$, for each $t > 0$

$$\boxed{1 - \nabla \log \psi (a_{t,T}, b_{t,T}) \to \frac{1}{R}} \quad \text{a.s.}$$

Recalling the form of the optimal portfolio π^*, this explains why the asymptotic optimal portfolio is as it would be if $U(x) = x^{1-R}$.

To begin with, let's observe that

$$\frac{1}{R} - 1 + \nabla \log \psi(a,b) = \frac{1}{R} - 1 + \frac{1}{\psi(a,b)} \frac{\partial}{\partial a} E\left[e^{a+\sqrt{b}Z} \, I(e^{a+\sqrt{b}Z}) \right]$$

where $Z \sim N(0,1)$

$$= \frac{1}{R} + E\left[e^{a+\sqrt{b}Z} \, e^{\sqrt{b}Z} \, I'(e^{a+\sqrt{b}Z}) \right] / E\left[e^{\sqrt{b}Z} \, I(e^{a+\sqrt{b}Z}) \right]$$

$$= E\left[e^{\sqrt{b}Z} I(e^{a+\sqrt{b}Z}) \left\{ \frac{1}{R} + \frac{e^{a+\sqrt{b}Z} I'(e^{a+\sqrt{b}Z})}{I(e^{a+\sqrt{b}Z})} \right\} \right] / E\left[e^{\sqrt{b}Z} I(e^{a+\sqrt{b}Z}) \right]$$

Thus in view of (B3), it is enough to prove for each $\varepsilon > 0$ that

(*) $$E\left[e^{\sqrt{b}Z} \, I(e^{a+\sqrt{b}Z}) : a+\sqrt{b}Z > \varepsilon \right] / E\left[e^{\sqrt{b}Z} \, I(e^{a+\sqrt{b}Z}) \right] \to 0$$

as $T \to \infty$, where we abbreviate $a_{\varepsilon,T}$ to a, $b_{\varepsilon,T}$ to $\mathbf{b}$. The numerator in this expression is at most

$$I(\varepsilon) \, E e^{\sqrt{b}Z} = I(\varepsilon) \, e^{\frac{1}{2}b},$$

so the goal is to show that the denominator is a lot bigger. Indeed,

$$E\left[e^{\sqrt{b}Z} I(e^{a+\sqrt{b}Z}) \right] \geq E\left[e^{\sqrt{b}Z} I(e^{a+\sqrt{b}Z}) ; \; a+\sqrt{b}Z \leq \varepsilon \right]$$

$$\geq c_\varepsilon^- \, E\left[e^{\sqrt{b}Z} \, e^{-(\frac{1}{R}-\delta)(a+\sqrt{b}Z)} ; \, a+\sqrt{b}Z \leq \varepsilon \right]$$

$$\geq c_\varepsilon^- \, E\left[e^{\sqrt{b}Z} \, e^{-(\frac{1}{R}-\delta)(a+\sqrt{b}Z)} \right]$$

$$- c_\varepsilon^- \, E\left[e^{\sqrt{b}Z} \right] e^{-\varepsilon(\frac{1}{R}-\delta)}$$

$$= c_\varepsilon^- \exp\left\{ -(\tfrac{1}{R}-\delta)a + (1-\tfrac{1}{R}+\delta)^2 \cdot \tfrac{1}{2}b \right\} - c_\varepsilon^- \, e^{-\varepsilon(\frac{1}{R}-\delta)} \, e^{\frac{1}{2}b}.$$

Now $a = \log \lambda_T - R_T - \frac{1}{2} b_T + O(1)$, $b = b_T + O(1)$, so the first term is

$$c_\varepsilon^- \exp\left[(\tfrac{1}{R}-\delta)R_T - (\tfrac{1}{R}-\delta)\log \lambda_T + (\tfrac{1}{R}-\delta) \cdot \tfrac{1}{2} b_T + (\tfrac{1}{R}-1-\delta)^2 \cdot \tfrac{1}{2} b_T + O(1) \right]$$

$$\geq c_\varepsilon^- \exp\left[\frac{1-R\delta}{1+R\delta} R_T + \tfrac{1}{2} b_T \left\{ 1 - \frac{2\delta}{R} + 2\delta^2 \right\} + O(1) \right],$$

substituting in the upper bound for $\log \lambda_T$ and doing some rearranging. This is

$$c_\varepsilon^- e^{\frac{1}{2}b_T} \exp\left[\frac{1-R\delta}{1+R\delta} R_T - 2\delta\left(\tfrac{1}{R}-\delta\right) \cdot \tfrac{1}{2} b_T + O(1) \right]$$

$$\geq c_\varepsilon^- e^{\frac{1}{2}b_T} \exp\left[\frac{1-R\delta}{1+R\delta} R_T - \delta K \left(\tfrac{1}{R}-\delta\right) R_T + O(1) \right]$$

using (B2);

$$= c_\varepsilon^- e^{\frac{1}{2}b_T} \exp\left[(1-R\delta) R_T \underbrace{\left\{ \frac{1}{1+R\delta} - \frac{\delta K}{R} \right\}}_{\text{positive by choice of small } \delta} + O(1) \right]$$

Thus, since $R_T \to \infty$, this thing is obviously of larger order than $e^{\frac{1}{2}b_T}$. Assembling all this, the denominator in (*) is a lot bigger than $e^{\frac{1}{2}b_T}$, whereas an upper bound for the numerator. We conclude that the ratio (*) does indeed ~~tend~~ to 0 as $T \to \infty$.

Example 3. This is more exploratory, and really doesn't have a lot to do with the asymptotic turnpike stuff. Here we shall suppose that the spot rate and the stock are quite general, assuming only that r and $\theta \equiv (\mu - r)/\sigma$ are bounded processes, but now the utility is $U(x) = x^{1-R}/(1-R)$ for some $R \in (0,1)$.

Mimicking the HJB equation in this non-Markovian situation, we can consider a value function V_t defined informally by

$$V_t(x) = \operatorname{ess\,sup}\left\{ E\left(U(X_T) \mid \mathcal{F}_t\right) ;\ X_t = x \right\}.$$

It is clear by scaling that we should have

$$V_t(x) = x^{1-R} \cdot Y_t \,/(1-R),$$

and also since we could pursue a policy of "hold all in the bond between s and t, then switch to optimal policy from t onward" it must be that

$$\eta_t \equiv Y_t \exp\left((1-R)\int_0^t r_u\,du\right) \text{ is a supermartingale.}$$

Assuming η is continuous and we live in a world with just one BM, we have

$$d\eta = \eta\,(\psi\, dW - dA)$$

where A is continuous increasing (this needs a little good behaviour from η, like, class(D)).

If X is the wealth process we get by using portfolio φ, we have

$$dX = rX\,dt + \varphi\,\sigma S(dW + \theta dt) \equiv rX\,dt + \rho(dW + \theta dt)$$

so that

$$d\,V_t(X_t) = d\,U(X_t)Y_t = Y_t\left[(\rho U' + \psi U)\,dW + \left\{\tfrac{1}{2}\rho^2 U'' + \rho(\theta+\psi)U'\right\}dt - U\,dA\right]$$

by Itô. So the optimal is to take $\boxed{\rho = \rho^* \equiv -(\theta+\psi)U'/U''}$ giving

$$d\,V_t(X_t^*) = Y_t\left[(\rho U' + \psi U)\,dW + \left\{-\tfrac{1}{2}(\theta+\psi)^2\,U'^2/U''\right\}dt - U\,dA\right],$$

under optimal control. But under optimal control, LHS is a mg, so $dA = \alpha\,dt$, where the relation

$$\boxed{\tfrac{1}{2}(\theta+\psi)^2/R = \alpha/(1-R)}$$

must hold. Could we choose ψ in such a way as to make this happen?

8) If Z is an RBM in a wedge (or, more generally, some corner) with constant directions of reflection on edges, $Z_0 = 0$, what can one say of $tZ(1/t)$?

9) What about recurrence/transience, or coupling, of time-inhomogeneous 1-dim! diffusions?

10) Going back to fractional BM questions, the only "semimartingale, yes or no?" question unresolved is the case of $X_t = \int_0^t \sqrt{t-s}\, dB_s$. We know that the quadratic variation is zero, and

$$E \sum_1^{2^n} |X(j2^{-n}) - X((j-1)2^{-n})| = 2^{-n} \sum_1^{2^n} E\, |X(j) - X(j-1)| \sim 2^{-n} \sum_1^{2^n} \left(\tfrac{1}{2} + \tfrac{1}{4}\log j\right)^{\frac{1}{2}} \ldots$$

but is the sum going to diverge a.s. ...?

11) I wonder whether $\exists$ Lévy process (X_t), $\mu_t \equiv \mathcal{L}(X_t)$, s.t. $\mu_t \perp$ Leb for $t<1$, and $\mu_t \sim$ Leb for $t>1$? Try $X_t = \sum 2^{-n} Y_n(t)$ where the Y_n are indept. Poisson processes rates λ_n. One idea one might attempt to exploit is to have $\lambda_n = 0$ except along an increasingly sparse sequence. Now $Y_n(t) \approx N(\lambda_n t, \lambda_n t)$, so if we had λ only non-zero on the s/seq n_j, then if $\lambda_{n_j} t < 2^{n_j - n_{j-1}}$, the binary expansions Y_{n_j} will not bridge the gap from n_{j-1} to n_j, and if $\lambda_{n_j} t$ is like $4^{n_j - n_{j-1}}$, then the fluctuations of $Y_{n_j}(t)$ are $O(2^{n_j - n_{j-1}})$ .. can one put the parameters together correctly...?

[Answer is Yes! Stratton TAMS $\underset{\sim}{132}$ 1-29 (1968)]

12) Worth recording? The function $f_n(k) = \left(\sum_{r=0}^{n-1} e^{ir\,2\pi k/n}\right) n^{-1}\, e^{i\pi k/n}$ will be nonzero only if $n \mid k$, and will be 1 if $k = 2jn$, -1 if $k = (2j+1)n$. Thus the function $\sum_{\ell=0}^{n-1} f_n(k-\ell)$ will give 1 for $k = 0, \ldots, n-1$, -1 for $k = n, \ldots, 2n-1, \ldots$

Good questions/outstanding problems.

1) Suppose $u: \{x \in \mathbb{R}^n : |x| \le 1\} \to \mathbb{R}$ solves $\Delta u + f(u) = 0$, where f is very general. Prove that u is in fact radial (DW says this is a big result of Nirenberg, proved by an intricate argument (22/6/98)).

2) It's not enough to find an equivalent measure making all price processes local martingales. For example, if $S_t = B\left(\frac{t}{1-t} \wedge H_0\right)$ where B is BM started at 1, then S is already a loc. mg, but just by going short in S one makes arbitrage.

Question: if $S_t \ge 0$, S is cts loc mg but not mg, can we make an arbitrage opportunity?

3) If we tried to discretise by taking $\left(S(j2^{-n}), \mathcal{F}(j2^{-n})\right)_{j=0}^{2^n}$, then we could make an EMM P_n for each discretisation, $Z_n \equiv dP_n/dP$. How would this help?

Note (i) If an EMM exists, it will be in L^1 but no more, so the very best one could hope for is $Z_n \xrightarrow{L^1} Z_\infty$ (down a s/seq perhaps). More likely is to get (Z_{n_j}) is UI, and $Z_n \to Z_\infty$ $(\sigma(L^1, L^\infty))$. Even if we got this, how would we prove that we'd got an EMM, i.e. $E\left[Z_\infty (S_u - S_t) : A_t\right] = 0$ $\forall\, 0 \le t \le u \le 1$, $\forall\, A \in \mathcal{F}_t$? Is the thing in the expectation integrable?

(ii) If we take $B_t + \sqrt{1-t} - 1$, then there's no EMM for this, and yet for each discretisation there is ...

4) Rodrigo Banuelos + Elizabeth Housworth are interested in the following question. If $D \subseteq \mathbb{C}$ is convex, and has in-radius 1, is it true that $E^0(\tau_D)$ is maximal when D is a strip? Here, $0 \in D$ is the centre of a circle radius 1 contained entirely in D.

5) Take RBM in a wedge with (constant) directions of reflection to make the process transient. Can one obtain upper/lower function test for escape to infinity?

6) Arising from conversations with Phil Dybvig: what are the possible laws of $B(\tau)$ where τ is a stopping time ≤ 1? [Answer: any μ s.t. $\int \mu(dy)\, |y-x| \le E\,|B_1 - x|$ $\forall x$]

7) If X is a subordinator, $f: \mathbb{R}^+ \to \mathbb{R}^+$, when do we have $\int_{0+} f(X_t)\, dt < \infty$ a.s.??