A SIMPLE MODEL OF LIQUIDITY EFFECTS

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Summary. We consider here an agent who may invest in a riskless bank account and a share, but may only move money between the two assets at the times of a Poisson process. This models in a simplified way liquidity constraints faced in the real world. The agent is trying to maximise the expected discounted utility of consumption, where the utility is CRRA; this is the objective in the classical Merton problem. Unlike that problem, there is no closed-form solution for the situation we analyse, but certain qualitative features of the solution can be established; the agent should consume at a rate which is the product of wealth and some function of the proportion of wealth in the risky asset, and at the times of the Poisson process the agent should readjust his portfolio so as to leave a fixed proportion of wealth in the risky asset. We establish an asymptotic expansion of the solution in two slightly different formulations of the problem, which allows us to deduce that the 'cost of liquidity' is (to first order) inversely proportional to the intensity of the Poisson process.

1. Introduction. The Black-Scholes paradigm of log Brownian shares is popular, tractable, and unrealistic in almost all of its assumptions. In the real world, there are transactions costs, returns which are non-Gaussian, non-constant volatility, non-constant interest rates, jumps in the price process, large trade effects, as well as liquidity effects, which are the topic of this paper. Such effects assume importance when trying to hedge some derivative on a thinly-traded asset; the adjustments to the hedge which the trader wants to implement can only be made when there is someone prepared to buy or sell the asset, and it may be that the desired hedge can be tracked only very poorly. We shall model this situation by assuming that there is a Poisson process of times (independent of the share) at which the agent is allowed to move money freely between the bank account and the share, and at no other time can he change his portfolio. This is an oversimplified story for liquidity; in reality, the agent would only be able to buy or sell *amounts* that were offered or sought at the times the quote came onto the market. To model this is to get into the modelling of the microstructure of the market, which is altogether more involved (see Rogers & Zane (1998) for one approach). The agent is allowed to consume continuously from the bank account, and his objective is to maximise the expected discounted utility of consumption. We shall follow Merton (1969), Davis & Norman (1990) and many others in assuming that the agent has constant relative risk aversion utility (that is, $-xU''(x)/U'(x) = R$, some positive constant); this simplifies the problem to one dimensional. In some sense, in the limit as the intensity of the Poisson process increases to infinity, we are solving the original Merton problem in which there is no liquidity constraint, and this solution provides a valuable comparison. Merton found that the agent should invest a constant proportion of wealth in the risky asset, and should consume at

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a rate which was a constant multiple of current wealth. This can only be the limiting behaviour for the problem with liquidity constraints if the Merton proportion is in $[0, 1]$; for certain combinations of parameter values, the Merton solution says that the agent should short the share, or short the bank account, and this is not a feasible behaviour for the liquidity-constrained agent, who must keep non-negative quantites in both, since the value of the shares could change by an arbitrarily large amount before the agent gets the chance to react. *We make the standing assumption throughout the paper that the Merton proportion is in (0,1).* In Section 2, we review the Merton problem and its solution, introducing the notation which we shall use. Our solution to the liquidity-constrained problems is in the form of a perturbation of the Merton solution, and is most naturally expressed in terms of the solution of the basic problem.

We shall consider two slightly different formulations of the problem, which we tackle by different methods. The first form of the problem, Problem I, dealt with in Section 3, supposes that at the times of the Poisson process the agent readjusts his portfolio, and also commits himself *to consume at a fixed rate from the bank account until the next event of the Poisson process*; see Section 3 for the precise formulation. In the second form of the problem, Problem II, dealt with in Section 4, we allow the agent *to adjust the rate at which he consumes from the bank account* in between events of the Poisson process. Thus if he observed the share falling, he might decide to reduce his consumption rate, whereas in the Problem I, he would be tied in to a specified rate of consumption. One situation where Problem II might arise is where an agent were trading in a foreign asset, but with restricted access to that asset's domestic market. The agent might be able to see how the asset was performing in its own market, but might only be able to trade when orders came in on the market where he was situated. Obviously, the value of the second form of the problem is greater than the first, and is less than the value of the unrestricted (Merton) problem.

We find that under liquidity constraints the agent should consume at a rate which is the product of the wealth and some function of the fraction of wealth invested in the risky asset. At times of the Poisson process, the agent readjusts his portfolio so that the proportion of wealth in the risky asset is some constant. We obtain explicit expressions for the first few terms in the expansion of the optimal policy, under the assumption that the mean time h between events of the Poisson process is small. In particular, one conclusion from our analysis is the fact that the cost of the constraint on the agent's portfolio choice is approximately linear in h, and we give explicit expressions for the constant in each of the two problems.

Finally in Section 5 we conclude.

2. The classical Merton problem and its solution. To begin with, let us quickly review the Merton problem, as it is a limiting case to which we shall have repeated need to refer. In this problem, the agent may invest in a bank account (bearing a constant rate of interest r) and a share, whose log price is modelled in terms of the standard Brownian motion W as $\sigma W_t + \alpha t$, σ and α constants. If w_t denotes the agent's wealth at time t, and he consumes at rate c_t, the wealth equation is

$$dw_t = rw_t dt + \eta_t(\sigma dW_t + (\alpha - r)dt) - c_t dt, \tag{2.1}$$

where η_t is the value of the holding of shares at time t. The agent has utility function $U(x) = x^{1-R}/(1-R)$ for some $R \neq 1$ [1] and aims to achieve

$$(2.2) \qquad f(w) \equiv \max E[\int_0^\infty e^{-\rho t} U(c_t) dt | w_0 = w],$$

where ρ is some positive constant. The agent must at all times satisfy the solvency condition

$$(2.3) \qquad w_t \geq 0,$$

otherwise he could borrow without limit and the problem would be ill posed. By the martingale principle of optimal control, the process

$$\xi_t \equiv \int_0^t e^{-\rho s} U(c_s) ds + e^{-\rho t} f(w_t)$$

is a supermartingale under any control, and a martingale under optimal control. The Itô expansion of ξ gives

$$d\xi_t \doteq e^{-\rho t} \Big[U(c_t) - \rho f(w_t) + (r w_t + \eta_t(\alpha - r) - c_t) f'(w_t) + \tfrac{1}{2}\sigma^2 \eta_t^2 f''(w_t) \Big] dt,$$

where the symbol '$\doteq$' signifies that the two sides differ by the differential of a local martingale. Since ξ is a supermartingale, and a martingale under optimal control, we deduce that

$$(2.4) \qquad \sup_{c \geq 0, \eta} \Big[U(c) - \rho f(w) + (rw + \eta(\alpha - r) - c) f'(w) + \tfrac{1}{2}\sigma^2 \eta^2 f''(w) \Big] = 0,$$

and performing the maximisation over c and η leaves us with the non-linear ODE (the HJB equation for this problem)

$$(2.5) \qquad \tilde{U}(f'(w)) - \rho f(w) + rwf'(w) - \tfrac{1}{2} \frac{(\alpha - r)^2 f'(w)^2}{\sigma^2 f''(w)} = 0,$$

where $\tilde{U}$ is the concave conjugate function of the utility function U:

$$(2.6) \qquad \tilde{U}(\lambda) \equiv \sup_x \{U(x) - \lambda x\} = \frac{R}{1-R} \lambda^{1-(1/R)}$$

for $\lambda > 0$. Remarkably, the HJB equation can be solved in this case; from scaling properties of the solution, one deduces that the value function has to have the form

[1] Analogous results hold for the case $R = 1$, which is the case of logarithmic utility, but the analysis needs to be developed separately, so we leave the details to the interested reader.

$f(w) = aw^{1-R}/(1-R)$ for some constant $a > 0$, and it is easy to confirm that this does indeed solve (2.5) provided

$$a = R^R\Big(\rho + (R-1)\{r + \frac{(\alpha - r)^2}{2R\sigma^2}\}\Big)^{-R} \equiv \gamma_*^{-R}.$$

Part of this statement is the fact that we must have

$$\gamma_* \equiv (\rho + (R-1)\{r + \frac{(\alpha - r)^2}{2R\sigma^2}\})/R > 0, \tag{2.7}$$

a condition which is is always satisfied if $R > 1$, and is satisfied when $R < 1$ for $|\alpha - r|$ small enough; the condition is necessary and sufficient for the problem to be well posed.

The optimal investment and consumption decisions are determined by the optimising values of c and η in (2.4);

$$c^* = a^{-1/R}w, \qquad \eta^* = -\frac{(\alpha - r)f'(w)}{\sigma^2 f''(w)} = \frac{(\alpha - r)w}{\sigma^2 R} \equiv \pi_* w. \tag{2.8}$$

3. Problem I: Fixed consumption rate. To specify this first version of the problem, we suppose that at time t the agent is holding x_t in the bank account, and y_t in the share. At each of the times τ_n of a Poisson process of intensity λ the agent chooses (i) the proportion p of his current wealth that he wants to hold in the share and (ii) the proportional rate θ at which he plans to consume from the bank account. Once these choices have been made, the portfolio evolves without further intervention according to the dynamics

$$dx_t = rx_t dt - \theta x_t dt \tag{3.1}$$

$$dy_t = y_t(\sigma dW_t + \alpha dt) \tag{3.2}$$

until the next time τ_{n+1}, when new choices can be made. Thus, to spell it out, the consumption rate is θx_t at time t. While in principle we may allow the choices of p and θ to depend on the history of the process up to time τ_n, it is clear that for the optimal policy they must be independent of the history, and therefore they will take some constant values. So we shall restrict our consideration to such policies, characterised by the constants p and θ. Again, it is clear that the same scaling relation will hold for this problem as for the original Merton problem, so that if we define the value function V_I for the problem characterised by (p, θ)

$$V_I(x, y; p, \theta) \equiv E\left[\int_0^\infty e^{-\rho t} U(\theta x_t) dt \,\middle|\, x_0 = (1-p)(x+y), y_0 = p(x+y)\right],$$

then

$$V_I(x, y; p, \theta) = a(p, \theta) U(w) \tag{3.1}$$

for some constant $a(p, \theta)$. We have used the notation $w \equiv x + y$ for the total wealth of the investor.

The optimal solution to this problem will obviously be no better than the solution to the classical Merton problem of Section 2, but how much worse is it? To quantify this, it is natural to define the *efficiency* Θ by

$$\Theta \equiv \big(a(p, \theta)\gamma_*^R\big)^{1/(1-R)},$$

this being the amount of wealth that the classical Merton investor would need in order to gain the same payoff as the liquidity-constrained investor. Clearly, therefore, $\Theta \in (0, 1)$, and the closer to 1, the less liquidity affects the investor.

If the share price at time t is denoted by S_t, then $S_t = S_0 \exp(\sigma W_t + (\alpha - \frac{1}{2}\sigma^2)t)$. Following the policy determined by the constants (p, θ), by the time τ_1 of the first event in the Poisson process, the value of the holding in the bank account will be

$$x(\tau_1) = x_0 e^{(r-\theta)\tau_1} = (1-p)w_0 e^{(r-\theta)\tau_1} \tag{3.2}$$

in terms of the initial wealth w_0, and the value of the total portfolio at that time will be

$$pw_0 \exp(\sigma W(\tau_1) + (\alpha - \tfrac{1}{2}\sigma^2)\tau_1) + (1-p)w_0 e^{(r-\theta)\tau_1}. \tag{3.3}$$

At time $t \in (0, \tau_1)$, consumption has been happening at rate $\theta x_0 \exp\{(r-\theta)t\}$, so we obtain the relation

$$\begin{aligned} V_I(x, y; p, \theta) &= a(p, \theta)U(w) \\ &= E\int_0^{\tau_1} e^{-\rho s} U(\theta(1-p)we^{(r-\theta)s})ds \\ &\quad + a(p, \theta)Ee^{-\rho\tau_1}U(pwS(\tau_1) + (1-p)we^{(r-\theta)\tau_1}). \end{aligned} \tag{3.4}$$

If we let $A(p, \theta) \equiv a(p, \theta)/(1 - R)$, and we divide throughout by w^{1-R} in (3.4), we obtain the relation

$$\begin{aligned} A(p, \theta) &= E\int_0^{\tau_1} e^{-\rho s} U(\theta(1-p)e^{(r-\theta)s})ds \\ &\quad + A(p, \theta)Ee^{-\rho\tau_1}(pS(\tau_1) + (1-p)e^{(r-\theta)\tau_1})^{1-R} \\ &= \int_0^\infty e^{-(\rho+\lambda)s} \, \frac{(\theta(1-p)e^{(r-\theta)s})^{1-R}}{1-R} \, ds \\ &\quad + A(p, \theta)Ee^{-(\rho+(R-1)(r-\theta))\tau_1}(pS(\tau_1)e^{(\theta-r)\tau_1} + 1 - p)^{1-R}. \end{aligned}$$

Introducing the abbreviations

$$\beta \equiv \rho + \lambda + (R-1)(r-\theta) \tag{3.5}$$

$$k \equiv \alpha - \tfrac{1}{2}\sigma^2 + \theta - r, \tag{3.6}$$

we therefore have more compactly

$$(3.7) \qquad A(p,\theta) = \frac{(\theta(1-p))^{1-R}}{1-R}\,\beta^{-1} + A(p,\theta)\,\frac{\lambda}{\beta} E\big(pS_T e^{(\theta-r)T} + 1 - p\big)^{1-R},$$

where $T \sim \exp(\beta)$ independent of W. Defining

$$(3.8) \qquad \begin{aligned} \varphi(p,\theta) &\equiv E\big(pS_T e^{(\theta-r)T} + 1 - p\big)^{1-R} \\ &= E\big(p\exp(\sigma W_T + kT) + 1 - p)\big)^{1-R}, \end{aligned}$$

we have more simply

$$(3.9) \qquad A(p,\theta) = \frac{(\theta(1-p))^{1-R}(1-R)^{-1}}{\beta - \lambda\varphi(p,\theta)}.$$

Notice that β and λ depend on θ, so the functional dependence of A on its arguments is not quite as simple as would appear at first glance at (3.9).

Now (3.9) tells us what $A(p,\theta)$ is, but the form of φ is hard to make explicit. In fact, there does not appear to be any explicit expression for φ, but we intend to continue the analysis by supposing that the parameter λ of the Poisson process is *large*, and developing an expansion for φ as a power series in $h \equiv 1/\lambda$. In view of the application in mind, it is reasonable to suppose that h will be of the order of a few basis points at most (that is, of the order of a few trades per hour), so an asymptotic should serve well.

Proceeding along this route, we write X for the random variable $p(\exp(\sigma W_T + kT) - 1)$, which should be small in some sense if λ is large. We then use the binomial expansion to re-express φ as

$$\begin{aligned} \varphi(p,\theta) &= E\Big[(1+X)^{1-R}\Big] \\ &= E\left[\sum_{j=0}^{2N+2} \frac{\Gamma(2-R)}{\Gamma(2-R-j)}\,\frac{X^j}{j!} + \frac{\Gamma(2-R)}{\Gamma(2-R-2N-3)}\,(1+\xi X)^{-R-2N-2}\,\frac{X^{2N+3}}{(2N+2)!}\right], \end{aligned}$$

where ξ is some random variable with values in (0,1). To estimate the remainder term in this expression, we firstly note the trivial bound $(1+\xi X)^{-R-2N-2} \le (1-p)^{-R-2N-2}$, and then record the following result.

PROPOSITION 1. *For each $N > 0$ there exists a constant $C_N = C_N(p,\theta)$ such that, with $m = 2N+3$, the bound*

$$(3.10) \qquad E|X|^m \le \frac{C_N}{(1+\lambda)^{N+(3/2)}}$$

holds.

Proof. See Appendix.

Thus the approach is to pick some N and then replace φ in (3.9) with the expression

$$\begin{aligned}\varphi_N(p,\theta) &\equiv E\left[\sum_{j=0}^{2N+2} \frac{\Gamma(2-R)}{\Gamma(2-R-j)}\,\frac{X^j}{j!}\right]\\ &= E\left[\sum_{j=0}^{2N+2} \frac{\Gamma(2-R)}{\Gamma(2-R-j)}\,p^j \sum_{m=0}^{j}\binom{j}{m}(-1)^{j-m}\,\frac{\beta}{\beta-km-\frac{1}{2}m^2\sigma^2}\right].\end{aligned}$$

The aim is to maximise the expression $A(p,\theta)$ over choice of p and θ, and to do this, we suppose that we can express the optimal values as $p_* = \sum_{j\geq 0} p_j h^j/j!$ and $\theta_* = \sum_{j\geq 0}\theta_j h^j/j!$. Substituting this into the expression for $A(p,\theta)$, we impose the conditions that

$$\frac{\partial A}{\partial p}(p_*,\theta_*) = 0 = \frac{\partial A}{\partial \theta}(p_*,\theta_*)$$

to determine the coefficients in the expansions of p_* and θ_*. We find in the end the following result.

THEOREM 1. *The leading terms in the expansion of p_* are given by*

$$p_0 = \pi_* \tag{3.11}$$

$$p_1 = -\pi_*(2\gamma_* + \sigma^2(1-\pi_*)(1-2\pi_*)) \tag{3.12}$$

$$p_2 = \frac{\pi_* A}{(1-\pi_*)^3\sigma^2} \tag{3.13}$$

where

$$\begin{aligned}A = &-2\,\gamma_*{}^3 + (\pi_*-1)\,\gamma_*{}^2\,(\pi_*+R\pi_*-2)\,\sigma^2\\ &+2\,\gamma_*\,(\pi_*-1)^3\left(2\,\pi_*{}^2+2\,R\pi_*{}^2-2\,R\pi_*-1\right)\sigma^4\\ &+2\,\pi_*\,(\pi_*-1)^5\,(7\,R\pi_*+7\,\pi_*-3-4\,R)\,\sigma^6.\end{aligned} \tag{3.14}$$

The leading terms in the expansion of θ_* are given by

$$\theta_0 = \frac{\gamma_*}{(1-\pi_*)} \tag{3.15}$$

$$\theta_1 = \frac{\pi_*\left[-2\,\gamma_*{}^2-2\,\gamma_*\,(2\,\pi_*-1)\,(\pi_*-1)\,\sigma^2+\pi_*\,(\pi_*-1)^3\,(-1+R)\,\sigma^4\right]}{2(1-\pi_*)^2} \tag{3.16}$$

$$\theta_2 = \frac{\pi_* B}{\sigma^2(\pi_*-1)^5}, \tag{3.17}$$

where

$$\begin{aligned}B = {}&2\,\gamma_*{}^4-\gamma_*{}^3\,(\pi_*-1)\left(4\,\pi_*{}^2+R\pi_*-7\,\pi_*+2\right)\sigma^2\\ &-\gamma_*{}^2\pi_*\,(\pi_*-1)^3\,(2\,R\pi_*+14\,\pi_*-9-R)\,\sigma^4\\ &-2\,\pi_*\,\gamma_*\,(\pi_*-1)^4\left(5\,R\pi_*{}^2+11\,\pi_*{}^2-8\,R\pi_*-14\,\pi_*+3\,R+4\right)\sigma^6\\ &+\pi_*{}^2\,(\pi_*-1)^6\,(-1+R)\,(-2\,R+2\,R\pi_*+4\,\pi_*-3)\,\sigma^8.\end{aligned} \tag{3.18}$$

Finally, the efficiency of the investor has an expansion $\Theta = \sum_{j\geq 0} \Theta_j h^j / j!$ *whose leading terms are given by*

$$\Theta_0 = 1 \tag{3.19}$$

$$\Theta_1 = -\frac{R\sigma^2 \pi_*^2 (\sigma^2(1-\pi_*)^2 + \gamma_*)}{2\gamma_*} \tag{3.20}$$

$$\Theta_2 = \frac{\pi_*^2 RC}{\gamma_*^2 (1-\pi_*)^2}. \tag{3.21}$$

where

$$\begin{aligned} C = (&-4\,{\gamma_*}^4 + 4\,(-1+\pi_*)\,{\gamma_*}^3\,(-2\,\pi_* + \pi_*\,R + 1)\,\sigma^2 \\ &- {\gamma_*}^2 \pi_*\,(-1+\pi_*)^2 \left(\pi_*\,R^2 - 2\,\pi_*\,R + 8\,\pi_* - 8\right)\sigma^4 \\ &- 2\,\gamma_* \pi_*\,(-1+\pi_*)^4\,(3\,\pi_*\,R + 4\,\pi_* - 4 - 4\,R)\,\sigma^6 \\ &+ {\pi_*}^2\,(-1+\pi_*)^6\,\sigma^8). \end{aligned} \tag{3.22}$$

Notice how the efficiency falls off linearly with h; the coefficient Θ_1 is always strictly negative. It appears that the signs of p_1 and θ_1 are indeterminate.

4. Problem II: Variable consumption rate. In this Section, we study a problem *similar* to Problem I in that the investor is only allowed to move wealth between the bank account and the share at the times of an independent Poisson process, and all consumption must be taken from the bank account, but *different* in that the rate at which the agent consumes from the bank account may be varied between events of the Poisson process in the light of what is happening to the share. It is clear that the payoff of this problem must lie between the payoff of Problem I and the payoff of the classical Merton problem. Our first objective here is to obtain the HJB equation for this problem; this turns out not to have any closed-form solution, and even numerical analysis is difficult in view of the singularities at 0 and 1. However, regarding $h \equiv 1/\lambda$ as a small parameter and carrying out an expansion in h, we are able to obtain good information about the solution, and shall explicitly determine the cost of liquidity up to quadratic terms in h.[2]

As before, let x_t denote the amount in the bank account at time t, and let y_t denote the value of the holding of shares at time t. The pair (x_t, y_t) now evolves according to

$$dx_t = (rx_t - c_t)dt + \theta_t dN_t \tag{4.1}$$

$$dy_t = y_t(\sigma dW_t + \alpha dt) - \theta_t dN_t \tag{4.2}$$

where the process θ determines the amounts moved between the two assets at the jump times of N, and the other symbols have the same meanings as in (2.1). Still using the CRRA utility

$$U(x) = x^{1-R}/(1-R), \qquad (R > 0, R \neq 1)$$

[2] In fact, since the expansion is carried out using Maple, there is no difficulty in principle in obtaining higher-order terms, but the expressions quickly become too cumbersome to report in an article.

we suppose that the objective of the agent is once again to achieve

$$V(x,y) \equiv \max E[\int_0^\infty e^{-\rho t} U(c_t) dt | x_0 = x, y_0 = y],$$

where ρ is some positive constant.

Using the martingale principle of optimal control, the process

$$Y_t \equiv \int_0^t e^{-\rho s} U(c_s) ds + e^{-\rho t} V(x_t, y_t)$$

is a supermartingale under any control, and a martingale under the optimal control. The Itô expansion of Y is

$$\begin{aligned} dY_t \doteq e^{-\rho t} \Big[& U(c_t) - \rho V(x_t, y_t) + (r x_t - c_t) V_x(x_t, y_t) + \alpha y_t V_y(x_t, y_t) \\ & + \tfrac{1}{2} \sigma^2 y_t^2 V_{yy}(x_t, y_t) + \lambda \{ V(x_t + \theta_t, y_t - \theta_t) - V(x_t, y_t) \} \Big] dt, \end{aligned} \tag{4.3}$$

where perhaps the only term needing comment is the final term in the brackets, which arises because of the possibility of the process jumping from (x_t, y_t) to $(x_t + \theta_t, y_t - \theta_t)$ at the time of an event in the Poisson process N, which are occurring at rate λ. For more background on stochastic calculus for processes with jumps, see any of Brémaud (1981), Elliott (1982), Meyer (1976), Rogers & Williams (1987). From (4.3) therefore we deduce that

$$\begin{aligned} \sup_{c \ge 0, \theta} \Big[& U(c) - \rho V(x,y) + (rx - c) V_x(x,y) + \alpha y V_y(x,y) \\ & + \tfrac{1}{2} \sigma^2 y^2 V_{yy}(x,y) + \lambda \{ V(x+\theta, y-\theta) - V(x,y) \} \Big] = 0. \end{aligned} \tag{4.4}$$

We can carry out the maximisation with respect to c as before to obtain

$$\begin{aligned} \sup_{\theta} \Big[& \tilde{U}(V_x(x,y)) - \rho V(x,y) + r x V_x(x,y) + \alpha y V_y(x,y) \\ & + \tfrac{1}{2} \sigma^2 y^2 V_{yy}(x,y) + \lambda \{ V(x+\theta, y-\theta) - V(x,y) \} \Big] = 0, \end{aligned} \tag{4.5}$$

but the remainder of the analysis of the HJB equation is now not as easy as for the Merton problem, and we have to resort to special features of the problem to get further.

The main feature of the problem is the *scaling* property of the solution: for any $\beta > 0$, we shall have $V(\beta x, \beta y) = \beta^{1-R} V(x,y)$, because any solution to (4.1)-(4.2) from initial conditions (x,y) can be converted into a solution to (4.1)-(4.2) from initial conditions $(\beta x, \beta y)$ just by multiplying c and θ by β, and the effect of this is to multiply the objective by β^{1-R}. The argument is spelled out in more detail in Davis & Norman (1990) if you want to see it; we shall use the scaling property to rewrite

$$V(x,y) \equiv (x+y)^{1-R} \, g(p), \tag{4.6}$$

where $p \equiv p/(x+y)$ is the proportion of wealth in the share. Using this, we can convert the HJB equation (4.5) into the form

$$\begin{aligned}
&\tilde{U}((1-R)g(p) - pg'(p)) - \rho g(p) + (1-R)\{r(1-p) + \alpha p - \tfrac{1}{2}\sigma^2 R p^2\}g(p) \\
(4.7) \qquad &+ p(1-p)(\alpha - r - \sigma^2 R p)g'(p) + \tfrac{1}{2}\sigma^2 p^2 (1-p)^2 g''(p) + \lambda\{\kappa - g(p)\} = 0, \\
(4.8) \qquad &\kappa = \sup_{0 \le t \le 1} g(t).
\end{aligned}$$

The final term in (4.7) comes from the jumps, and corresponds to resetting the portfolio to have the optimal proportion in the share; if we have $\sup_{0 \le t \le 1} g(t) = g(\bar{\pi})$, then the best thing is to reset at each opportunity to have proportion $\bar{\pi}$ in the share. Similarly, if we were to decide to reset at each opportunity to proportion π, the value function V_π for this problem would be of the form $V_\pi(x, y) = (x+y)^{1-R} h_\pi(p)$, where h_π solves

$$\begin{aligned}
&\tilde{U}((1-R)h(p) - ph'(p)) - \rho h(p) + (1-R)\{r(1-p) + \alpha p - \tfrac{1}{2}\sigma^2 R p^2\}h(p) \\
(4.9) \qquad &+ p(1-p)(\alpha - r - \sigma^2 R p)h'(p) + \tfrac{1}{2}\sigma^2 p^2 (1-p)^2 h''(p) + \lambda\{\kappa - h(p)\} = 0, \\
(4.10) \qquad &\kappa = h(\pi).
\end{aligned}$$

Thus (4.7)-(4.8) is the special case of (4.9)-(4.10) when the proportion $\pi = \bar{\pi}$. The value function is always increasing in both arguments, and is concave, so we deduce from this that h must be concave, and the functions $t \mapsto t^{R-1} h(t)$ and $t \mapsto t^{R-1} h(1-t)$ are both decreasing. Thus we have for some range of the parameter κ a family of non-linear ODEs with solutions which are concave on $[0, 1]$ and satisfy these monotonicity properties.

Closed-form solution of these equations appears to be hopeless, so we now embark on an asymptotic expansion of the solution which proves to be extremely informative. We are assuming that the parameter $h \equiv 1/\lambda$ is small, so that the problem under study is a perturbation of the classical Merton problem, so it is natural to attempt to express the solution of the HJB equation (4.7) as

$$(4.11) \qquad g(p) = \frac{\gamma_*^{-R}}{1-R} + \sum_{j \ge 1} \psi_j (p - \pi_*) h^j / j!\ ,$$

the maximised value $\kappa \equiv \sup_{0 \le t \le 1} g(t)$ as

$$(4.12) \qquad \kappa = \frac{\gamma_*^{-R}}{1-R} + \sum_{j \ge 1} \alpha_j h^j / j!\ ,$$

and the proportion $\bar{\pi}$ at which the maximum is attained as

$$(4.13) \qquad \bar{\pi} = \pi_* + \sum_{j \ge 1} \beta_j h^j / j!$$

in terms of unknown functions ψ_j and constants α_j, β_j which are to be determined.

Using the facts that g solves (4.7), and that $g'(\bar{\pi}) = 0$ and $g(\bar{\pi}) = \kappa$, we are able to deduce the forms of the unknowns appearing in (4.11)-(4.13). We summarise our finding in the following result.

THEOREM 2. *Assuming as always that the Merton proportion $\pi_* \equiv (\alpha - r)/\sigma^2 R$ is in (0,1), we have*

$$\psi_1(t) = -\tfrac{1}{2}\sigma^2 R\gamma_*^{-1-R}\big(\sigma^2\pi_*^2(1-\pi_*)^2 + \gamma_* t^2\big) \tag{4.14}$$

$$\begin{aligned}\psi_2(t) = -\tfrac{1}{4}\sigma^2 R\gamma_*^{2-R}\Big(& 2\sigma^2\gamma_*^2(R+1)(R+2)t^4 + 8\sigma^2\gamma_*^2(R+1)(2\pi_*-1)t^3 \\ & + (4\gamma_*^2(\sigma^2+\gamma_*) - 8\sigma^2\gamma_*^2(R+3)\pi_* + 2\gamma_*\sigma^2(\sigma^2 R(R-1) \\ & + 4\gamma_*(R+3))\pi_*^2 - 4R\gamma_*\sigma^2(R-1)\pi_*^3 + 2R\gamma_*\sigma^4(R-1)\pi_*^4)t^2 \\ & \quad + 8\gamma_*^2\pi_*(\gamma_* + \sigma^2(2\pi_*-1)(\pi_*-1))t \\ & \quad + \pi_*^2\big[4\gamma_*^2(\sigma^2+\gamma_*) - 8\gamma_*\sigma^2(\gamma_* + \sigma^2(R+1))\pi_* \\ & \quad + \sigma^2(24\gamma_*\sigma^2(R+1) + 4\gamma_*^2 + \sigma^4(R^2-1))\pi_*^2 \\ & \quad - 4\sigma^4(R+1)(\sigma^2(R-1) + 6\gamma_*)\pi_*^3 \\ & \quad + 2\sigma^4(R+1)(3\sigma^2(R-1) + 4\gamma_*)\pi_*^4 \\ & \quad - 4\sigma^6(R^2-1)\pi_*^5 + \sigma^6(R^2-1)\pi_*^6\big]\Big)\end{aligned} \tag{4.15}$$

for the functions ψ_1, ψ_2,

$$\alpha_1 = -\tfrac{1}{2}\sigma^4\gamma_*^{-1-R}R\pi_*^2(1-\pi_*)^2 \tag{4.16}$$

$$\begin{aligned}\alpha_2 = -\tfrac{1}{4}\sigma^2\pi_*^2 R\gamma_*^{-2-R}\Big(& 4\,{\gamma_*}^2\left(\sigma^2+\gamma_*\right) - 8\,\gamma_*\sigma^2\left(\gamma_* + \sigma^2 R + \sigma^2\right)\pi_* \\ & + \sigma^2\left(24\,\gamma_*\sigma^2 + 4\,{\gamma_*}^2 - \sigma^4 + \sigma^4 R^2 + 24\,\gamma_*\sigma^2 R\right){\pi_*}^2 \\ & - 4\,\sigma^4\left(R+1\right)\left(\sigma^2 R - \sigma^2 + 6\,\gamma_*\right){\pi_*}^3 \\ & + 2\,\sigma^4\left(R+1\right)\left(3\,\sigma^2 R - 3\,\sigma^2 + 4\,\gamma_*\right){\pi_*}^4 \\ & - 4\,\sigma^6\left(R-1\right)\left(R+1\right){\pi_*}^5 + \sigma^6\left(R-1\right)\left(R+1\right){\pi_*}^6\Big)\end{aligned} \tag{4.17}$$

for the constants α_1 and α_2, and

$$\beta_1 = -\pi_*\big(\gamma_* + \sigma^2(\pi_*-1)(2\pi_*-1)\big) \tag{4.18}$$

$$\begin{aligned}\beta_2 = -\sigma^2\pi_*^2\Big[& -2\,R\gamma_* - 6\,\sigma^2 - 8\,\sigma^2 R + \left(31\,\sigma^2 R + 25\,\sigma^2 + 2\,R\gamma_*\right)\pi_* \\ & - 2\,\sigma^2\left(19\,R + 16\right){\pi_*}^2 + \sigma^2\left(15\,R + 13\right){\pi_*}^3\Big]\end{aligned} \tag{4.19}$$

for the constants β_1 and β_2.

The efficiency $\Theta = \sum_{j\geq 0} \theta_j h^j/j!$ has an expansion whose leading terms are

$$\Theta_0 = 1 \tag{4.20}$$

$$\Theta_1 = -\frac{R\sigma^4\pi_*^2(1-\pi_*)^2}{2\gamma_*} \tag{4.21}$$

$$\begin{aligned}\Theta_2 = -\frac{\sigma^2\pi_*^2 R}{4\gamma_*^2}\Big[&4\,{\gamma_*}^2\left(\sigma^2+\gamma_*\right) - 8\,\gamma_*\sigma^2\left(\gamma_*+\sigma^2 R+\sigma^2\right)\pi_* \\ &+\sigma^2\left(-\sigma^4+24\,\gamma_*\sigma^2+24\,\gamma_*\sigma^2 R+4\,{\gamma_*}^2\right){\pi_*}^2 \\ &-4\,\sigma^4\left(-\sigma^2+6\,\gamma_*+6\,R\gamma_*\right){\pi_*}^3+2\,\sigma^4\left(-3\,\sigma^2+4\,\gamma_*+4\,R\gamma_*\right){\pi_*}^4 \\ &\quad+4\,\sigma^6{\pi_*}^5-\sigma^6{\pi_*}^6\Big]\end{aligned} \tag{4.22}$$

Remarks. From (4.21) we see that the loss of efficiency is to first order $R\sigma^4\pi_*^2(1-\pi_*)^2/(2\gamma_*)$, which is less than the first-order loss of efficiency in Problem I, (3.20). This had to be the case, but it is reassuring to see it happening.

5. Conclusions. We have managed to establish the expansion for the optimal solution to two liquidity-constrained investment problems, where the investor is only able to adjust his portfolio at the times of an independent Poisson process. Throughout, we assume that the Merton proportion is in (0,1), since the problems are otherwise ill posed, and this situation is the only one of any practical interest. In particular, we have shown that for Problem I (where the investor may adjust his portfolio at the times of the Poisson process, and must consume at constant proportional rate from the bank account in between) to achieve the same expected utility as the constrained investor with initial wealth 1, the investor *not* constrained will need (up to first order) only initial wealth

$$1 \;-\; \frac{R\sigma^2\pi_*^2(\sigma^2(1-\pi_*)^2+\gamma_*)}{2\gamma_*}\;h.$$

In Problem II, where the rate of consumption may be varied in the light of what is happening to the share, we find that the unconstrained investor will need (to first order in h) initial wealth

$$1 \;-\; \frac{R\sigma^4\pi_*^2(1-\pi_*)^2}{2\gamma_*}\;h$$

to match the expected utility of the constrained investor.

In both problems, at each event of the Poisson process the investor always returns his portfolio mix to the same optimal value, which is not the Merton proportion π_*, but rather (to first order in h)

$$\pi_* - h\pi_*\big(2\gamma_* + \sigma^2(\pi_*-1)(2\pi_*-1)\big),$$

for Problem I, and

$$\pi_* - h\pi_*\big(\gamma_* + \sigma^2(\pi_*-1)(2\pi_*-1)\big)$$

for Problem II.

Appendix: proof of Proposition 1. Recall that we require to prove that for large enough λ, writing $m = 2N + 3$,

$$E|X|^m \le \frac{C_N}{(1+\lambda)^{N+(3/2)}}$$

for some constant $C_N = C_N(p, \theta)$, where $X = p(\exp(\sigma W_T - kT) - 1)$. We have

$$\begin{aligned}
E|X|^m &= p^m E|\exp(\sigma W_T - kT) - 1|^m \\
&= p^m E|e^{\sigma W_T - kT} - e^{kT + \frac{1}{2}\sigma^2 T} + e^{kT+\frac{1}{2}\sigma^2 T} - 1|^m \\
&\le 2^m p^m \big[E e^{(k+\frac{1}{2}\sigma^2)mT} |e^{\sigma W_T - \frac{1}{2}\sigma^2 T} - 1|^m + E|e^{kT+\frac{1}{2}\sigma^2 T} - 1|^m \big]. \qquad (A1)
\end{aligned}$$

We estimate the two parts of this separately. For the first, we note that (holding T fixed to begin with)

$$\begin{aligned}
E|e^{\sigma W_T - \frac{1}{2}\sigma^2 T} - 1|^m &= E\Big|\int_0^T \sigma e^{\sigma W_s - \frac{1}{2}\sigma^2 s} dW_s\Big|^m \\
&\le c_m E\Big(\int_0^T \sigma^2 e^{2\sigma W_s - \sigma^2 s} ds\Big)^{m/2} \\
&\le c_m E\Big[T^{(m/2)-1} \int_0^T \sigma^m e^{m\sigma W_s - \frac{1}{2}m\sigma^2 s} ds\Big] \\
&\le c_m E\Big[T^{(m/2)} \sigma^m \exp(m(m-1)\sigma^2 T/2)\Big]
\end{aligned}$$

Now letting T be independent of W with an $\exp(\beta)$ distribution, we get an upper bound (for a constant c_m varying from place to place)

$$\begin{aligned}
E e^{(k+\frac{1}{2}\sigma^2)mT} |e^{\sigma W_T - \frac{1}{2}\sigma^2 T} - 1|^m &\le \frac{c_m \beta}{(\beta - m(k + m\sigma^2/2))^{(m/2)+1}} \\
&\le \frac{c_m}{(1+\lambda)^{m/2}}
\end{aligned}$$

We estimate the second term in (A1) by noting that for non-negative x, $0 \le e^x - 1 \le xe^x$, so that

$$\begin{aligned}
E|e^{kT+\frac{1}{2}\sigma^2 T} - 1|^m &\le E(k + \tfrac{1}{2}\sigma^2)^m T^m e^{m(k+\frac{1}{2}\sigma^2)T} \\
&\le \frac{c_m \beta}{(\beta - m(k + \sigma^2/2))^{m+1}} \\
&\le \frac{c_m}{(1+\lambda)^m},
\end{aligned}$$

which is of smaller order than the first term of (A1). Thus completes the proof of Proposition 1.

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