The joint law of the extrema, final value and signature of a stopped random walk.

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Abstract

A complete characterization of the possible joint distributions of the maximum and terminal value of uniformly integrable martingale has been known for some time, and the aim of this paper is to establish a similar characterization for continuous martingales of the joint law of the minimum, final value, and maximum, along with the direction of the final excursion. We solve this problem completely for the discrete analogue, that of a simple symmetric random walk stopped at some almost-surely finite stopping time. This characterization leads to robust hedging strategies for derivatives whose value depends on the maximum, minimum and final values of the underlying asset.

1 Introduction.

Suppose given $h > 0$, and suppose that $(\xi_t, \mathcal{F}_t)_{t \in h\mathbb{Z}^+}$ is a symmetric simple random walk on the grid $h\mathbb{Z}$, started at zero. Define $S_t \equiv \sup_{s \le t} \xi_s$, $I_t \equiv \inf_{s \le t} \xi_s$, $g_t^+ \equiv \inf\{u \le t : \xi_u = S_u\}$, $g_t^- \equiv \inf\{u \le t : \xi_u = I_u\}$, and let

$$
\begin{aligned}
\sigma_t &= +1 &&\text{if } g_t^+ > g_t^- \\
&= -1 &&\text{else.}
\end{aligned}
\tag{1.1}
$$

The process S records the running maximum of the martingale, and the process σ records whether the martingale is currently on an excursion down from its running maximum ($\sigma = +1$)

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or on an excursion up from its running minimum ($\sigma = -1$). We refer to the process σ as the *signature* of the random walk.

Suppose that T is an almost-surely finite $(\mathcal{F}_t)$-stopping time, and write

$$X_t \equiv \xi_{t \wedge T}$$

for the stopped process. The paper is concerned with the possible joint laws m of the quadruple $(I_T, X_T, S_T, \sigma_T)$, which we will abbreviate to (I, X, S, σ) where no confusion may arise.

Clearly the law m must be defined on the set $\mathcal{X} \equiv (-h\mathbb{Z}^+) \times h\mathbb{Z} \times h\mathbb{Z}^+ \times \{-1, +1\}$, and evidently we must have $m(I \leq X \leq S) = 1$; but beyond this, is it possible to state a set of *necessary and sufficient conditions* for a probability m on $\mathcal{X}$ to be the joint distribution of $(I_t, X_T, S_T, \sigma_T)$? The motivation for this attempt is twofold. Firstly, the joint law of (X, S) has been characterized completely (for general local martingales, not assumed to be continuous or uniformly integrable) in [7]; can the methods of that paper be extended to deal with the running minimum also? The second reason to look at this problem is the interesting recent work of Cox & Obloj [3] which finds extremal martingales for various derivatives whose payoffs depend on the maximum, minimum and terminal value of the underlying asset. This builds to some extent on the earlier work of Hobson and others ([6], [1], [2]), which addresses similar questions for derivatives whose payoffs depend only on the maximum and terminal value of the underlying asset. Many of the results of this literature can be derived alternatively using the results of [7], by converting the problem into a linear program. This approach is more general, but leads to less explicit answers in the specific instances analyzed to date.

What we shall find here is that it is possible to generalize the results of [7] to cover the joint law of (I, X, S, σ), but that the statements are more involved. For this reason, we shall restrict our analysis to a symmetric simple random walk taking values in a grid $h\mathbb{Z}$ for some $h > 0$, stopped at an almost-surely finite stopping time. The main result is presented in Section 2. The proof of necessity is in Section 2.1, and requires only the judicious use of the Optional Sampling Theorem. The proof of sufficiency, in Section 2.2, is constructive, and requires suitable modification of some of the techniques of [7]. We then show in Section 3 how this characterization can lead to robust hedging schemes and extremal prices for derivatives whose payoff depends on the maximum, minimum, terminal value and signature.

2   The main result.

We take a symmetric simple random walk $(\xi_t, \mathcal{F}_t)_{t \in h\mathbb{Z}^+}$ on $h\mathbb{Z}$ for some fixed $h > 0$; in general, the filtration $(\mathcal{F}_t)$ is larger than the filtration of the random walk, to allow for additional randomization. Stopping ξ at the almost-surely finite stopping time T creates the martingale $X_t = \xi_{t \wedge T}$. We use the notation of the Introduction, and notice that

$$g_t^+ \equiv \sup\{u \leq t : S_u > S_{u-h}\}, \qquad g_t^- \equiv \sup\{u \leq t : I_u < I_{u-h}\}, \qquad (2.1)$$

emphasizing the fact that we are dealing with *strict* ascending/descending ladder epochs, to use the language of Feller [5]. The process σ is defined as before at (1.1).

Definition 2.1. *We say that the probability measure m on $\mathcal{X} \equiv -h\mathbb{Z}^+ \times h\mathbb{Z} \times h\mathbb{Z}^+ \times \{-1, +1\}$ is* consistent *if there is some almost-surely finite $(\mathcal{F}_t)$-stopping time T such that m is the law of $(I_T, X_T, S_T, \sigma_T)$.*

2.1 Necessity.

For $x \in h\mathbb{Z}$ we define the hitting time

$$H_x = \inf\{u : \xi_u = x\}, \tag{2.2}$$

with the usual convention that the infimum of the empty set is $+\infty$. In what follows, we will let a, b stand for two generic members of $h\mathbb{Z}^+$, and will be studying the exit time $H_b \wedge H_{-a} \equiv \inf\{u : \xi_u \notin (-a, b)\}$ and related stopping times. The measure m says nothing directly about these stopping times, but by way of the Optional Sampling Theorem we are able to deduce quite a lot of information about them if the law m is consistent. Indeed, assuming that m is consistent, we are able to find the probability that $H_{-a} < H_b$ (for example) in terms of m-expectations of functions defined on $\mathcal{X}$. The expressions derived make perfectly good sense even if m is not consistent, but it may be that the expressions do not in general satisfy positivity or other properties which would hold if m were consistent. For this reason, we will denote by $\bar{m}(Y)$ the expression for the m-expectation of a random variable Y which would be correct if m were consistent; if m is not consistent, all we have is an algebraic expression without the desired probabilistic meaning, and the use of the symbol $\bar{m}$ warns us not to assume properties which need not hold.

The first result we need is the following, which illustrates the use of this notational convention.

Proposition 2.2. *For any $a, b \in h\mathbb{Z}^+$ we have*

$$\bar{m}(H_b < H_{-a}) = \frac{a - m(a + X; S < b, I > -a)}{a + b} \equiv \varphi(b, -a), \tag{2.3}$$

$$\bar{m}(H_{-a} < H_b) = \frac{b - m(b - X; S < b, I > -a)}{a + b} \equiv \varphi(-a, b). \tag{2.4}$$

PROOF. We use the Optional Sampling Theorem at the time $H_b \wedge H_{-a}$ to derive the two equations

$$1 = \bar{m}(H_{-a} < H_b) + \bar{m}(H_b < H_{-a}) + m(S < b, I > -a) \tag{2.5}$$

$$0 = -a\,\bar{m}(H_{-a} < H_b) + b\,\bar{m}(H_b < H_{-a}) + m(X; S < b, I > -a). \tag{2.6}$$

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Solving this pair of linear equations leads to the conclusion that

$$\begin{aligned} \bar{m}(H_b < H_{-a}) &= \left\{\, a - m(a + X; S < b, I > -a)\,\right\}/(a + b)\,, \\ \bar{m}(H_{-a} < H_b) &= \left\{\, b - m(b - X; S < b, I > -a)\,\right\}/(a + b)\,, \end{aligned}$$

as claimed.

□

If m is consistent, then we would have for any $a, b \in h\mathbb{Z}^+$ not both zero that

$$\begin{aligned} \bar{m}(H_{-a} < H_b < H_{-a-h}) &= \bar{m}(H_{-a} \leq H_b < H_{-a-h}) \\ &= \bar{m}(H_b < H_{-a-h}) - \bar{m}(H_b < H_{-a}) \\ &= \bar{m}(H_b < \infty, I(H_b) = -a). \end{aligned}$$

This is because on the event $\{H_{-a} < H_b < H_{-a-h}\}$ the hitting time H_b is finite, and so cannot be equal to H_{-a}; the second equality follows from the inclusion $\{H_b < H_{-a}\} \subseteq \{H_b < H_{-a-h}\}$. We will therefore introduce the notation

$$\begin{aligned} \psi_+(-a, b) &= \varphi(b, -a - h) - \varphi(b, -a), & (2.7) \\ \psi_-(-a, b) &= \varphi(-a, b + h) - \varphi(-a, b). & (2.8) \end{aligned}$$

Notice that $\psi_+(-a, b)$ is *defined* as an algebraic expression in terms of m via (2.7) and (2.3); if m is *consistent*, then $\psi_+(-a, b)$ is equal to $\bar{m}(H_b < \infty, I(H_b) = -a)$, but no such interpretation holds in general.

The necessary condition we derive comes from considering what may happen if the event $B_+ = \{H_b < \infty, I(H_b) = -a\}$ occurs. When this event occurs, the martingale X does reach b before being stopped, and at that time H_b the minimum value is $-a$. Thereafter, one of three things will happen:

(i) X reaches $b + h$ before reaching $-a - h$ and before T;

(ii) T happens before X reaches either $-a - h$ or $b + h$;

(iii) X reaches $-a - h$ before reaching $b + h$ and before T.

The next result derives a necessary condition from the Optional Sampling Theorem applied at $H_{-a-h} \wedge H_{b+h} \wedge T$.

Proposition 2.3. *Define the events*

$$B_+ = \{H_b < \infty, I(H_b) = -a)\}, \qquad B_- = \{H_{-a} < \infty, S(H_{-a}) = b\}, \qquad (2.9)$$

set $p_\pm = \bar{m}(B_\pm) = \psi_\pm(-a, b)$, *and set*

$$p_{+0} = m(S = b, I = -a, \sigma = +1), \qquad p_{-0} = m(S = b, I = -a, \sigma = -1). \qquad (2.10)$$

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If we denote

$$v_\pm \equiv \frac{m(X; S = b, I = -a, \sigma = \pm 1)}{p_{\pm 0}} \equiv m(X \mid S = b, I = -a, \sigma = \pm 1), \qquad (2.11)$$

then the conditions[1]

$$\frac{p_{+0}}{p_+} \leq \frac{h}{b + h - v_+} \qquad (2.12)$$

$$\frac{p_{-0}}{p_-} \leq \frac{h}{a + h + v_-} \qquad (2.13)$$

are necessary for m to be consistent.

PROOF. We introduce the notation

$$p_{++} = \bar{m}(H_{-a} < H_b < H_{b+h} < H_{-a-h}), \quad p_{+-} = \bar{m}(H_{-a} < H_b < H_{-a-h} < H_{b+h}),$$
$$p_{--} = \bar{m}(H_b < H_{-a} < H_{-a-h} < H_{b+h}), \quad p_{-+} = \bar{m}(H_b < H_{-a} < H_{b+h} < H_{-a-h}).$$

Using the Optional Sampling Theorem, we have similarly to (2.5), (2.6) the equations

$$p_+ = p_{++} + p_{+0} + p_{+-} \qquad (2.14)$$
$$bp_+ = (b + h)p_{++} - (a + h)p_{+-} + m(X; S = b, I = -a, \sigma = +1). \qquad (2.15)$$

If we write $\tilde{p}_{xy} = p_{xy}/p_x$ for $x \in \{-, +\}$, $y \in \{-, 0, +\}$ the equations (2.14), (2.15) are expressed more simply in conditional form:

$$1 = \tilde{p}_{++} + \tilde{p}_{+-} + \tilde{p}_{+0} \qquad (2.16)$$
$$b = (b + h)\tilde{p}_{++} - (a + h)\tilde{p}_{+-} + \tilde{p}_{+0}v_+. \qquad (2.17)$$

The value of p_{+0} is known from m, as is the value of v_+, and since we assume that m is consistent the values of $p_\pm = \psi_\pm(-a, b)$ are also known from m. Therefore we can solve the linear system (2.16), (2.17) to discover

$$\tilde{p}_{++} = \frac{b + a + h - (a + h + v_+)\tilde{p}_{+0}}{b + a + 2h} \qquad (2.18)$$

$$\tilde{p}_{+-} = \frac{h - (b + h - v_+)\tilde{p}_{+0}}{b + a + 2h}. \qquad (2.19)$$

In order that $\tilde{p}_{+-}$ as given by (2.19) should be non-negative, we require that

$$\tilde{p}_{+0} \equiv \frac{m(S = b, I = -a, \sigma = +1)}{p_+} \leq \frac{h}{b + h - v_+}, \qquad (2.20)$$

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[1]If either of $p_\pm$ is zero, then the inequalities (2.12), (2.13) have to be understood in cross-multiplied form, when they state vacuously that $0 \leq 0$.

which is condition (2.12). Necessity of (2.13) is derived similarly.

<div style="text-align: right;">□</div>

REMARKS. (i) The necessary conditions (2.12), (2.13) come from the requirement that $\tilde{p}_{+-}$ and $\tilde{p}_{-+}$ should be non-negative. Do we know for sure that $\tilde{p}_{++}$ and $\tilde{p}_{--}$ are non-negative? The definition (2.11) of $v_{\pm}$ guarantees that $-a \le v_{\pm} \le b$, so if (2.20) holds then we know that $\tilde{p}_{+0} \le 1$. From (2.18) we see then that $\tilde{p}_{++} \ge 0$. Since all the summands on the right-hand side of (2.16) are non-negative, we learn that they are probabilities summing to 1.

(ii) Notice that we have two expressions for $\bar{m}(H_{b+h} < \infty, I(H_{b+h}) = -a)$, either as $p_{++} + p_{-+}$, or as $\psi_+(-a, b+h)$. Confirming that these are the same is an important step in the proof of sufficiency.

2.2 Sufficiency.

We have now identified necessary conditions (2.12) and (2.13) for m to be consistent. The main result of this paper is that these conditions are also sufficient.

Theorem 2.4. *The probability measure m on $\mathcal{X} \equiv -h\mathbb{Z}^+ \times h\mathbb{Z} \times h\mathbb{Z}^+ \times \{-1, +1\}$ is consistent if and only if $m(I \le X \le S) = 1$ and necessary conditions (2.12) and (2.13) hold.*

PROOF. Necessity has been proved, so what remains is to show that conditions (2.12) and (2.13) are sufficient. Not surprisingly, the proof of this is constructive.

We require a probability space $(\Omega, \mathcal{F}, P)$ rich enough to carry an IID sequence $U_0, U_1, \ldots$ of $U[0, 1]$ random variables, and an independent standard Brownian motion (B_t). Let $\mathcal{U} = \sigma(U_0, U_1, \ldots)$, and let $(\mathcal{G}_t)$ be the usual augmentation of the filtration $(\mathcal{U} \vee \sigma(B_s : s \le t))$. Define $(\mathcal{G}_t)$-stopping times

$$\alpha_0 \equiv 0, \qquad \alpha_{n+1} \equiv \inf\{t > \alpha_n : |B_t - B_{\alpha_n}| > h\},$$

the process $\xi_{nh} \equiv B(\alpha_n)$ and the filtration $\mathcal{F}_{nh} \equiv \mathcal{G}_{\alpha_n}$, so that $(\xi_t, \mathcal{F}_t)_{t \in h\mathbb{Z}^+}$ is a symmetric simple random walk. As before, define $S_t \equiv \sup_{s \le t} \xi_s$, $I_t \equiv \inf_{s \le t} \xi_s$ for $t \in h\mathbb{Z}^+$.

The construction borrows the technique of [7], where we firstly modify the given law m so that the conditional distribution of X_T given $\{S_T = b, I_T = -a, \sigma_T = s\}$ is a unit mass on the expected value $m[X_T \,|\, S_T = b, I_T = -a, \sigma_T = s]$. If we can construct a martingale with this degenerate conditional law, then we can build the required distribution of X_T given $\{S_T = b, I_T = -a, \sigma_T = s\}$ by Skorokhod embedding in a Brownian motion. So we may and shall suppose that[2]

$$m[X_T = v \,|\, S_T = b, I_T = -a, \sigma_T = s] = 1, \tag{2.21}$$

---

[2]There is no reason why v need be a multiple of h, but this does not matter; if $s = +$, say, we shall use the Brownian motion living in the original probability space, starting at b and run until it first hits either the upper barrier $b + h$ or the lower barrier, which will be *randomized*, taking value v_+ with suitably-chosen probability θ, otherwise taking value $-a - h$.

where $v = m[X_T \,|\, S_T = b, I_T = -a, \sigma_T = s\,]$.

The construction is sequential, and the proof that it succeeds is inductive. Let $\tau_n \equiv \inf\{t : S_t - I_t = nh\}$, and set $\sigma_n = \alpha_{\tau_n}$, the corresponding stopping time for the Brownian motion. The construction of T begins by setting $T = 0$ if $U_0 < m(S = I = 0)$, otherwise $T \geq h = \tau_1$. The sequential construction supposes[3] we have found that $T \geq \tau_n$, and $S_{\tau_n} = \xi_{\tau_n} = b$, $I_{\tau_n} = -a$. Then we place a lower barrier $\ell \in [-a - h, b + h]$ by the recipe

$$
\begin{aligned}
\ell &= v_+ &&\text{if } U_n < \theta \\
&= -a - h &&\text{else}
\end{aligned}
$$

where v_+ is defined in terms of m by (2.11), and θ is defined by

$$
\tilde{p}_{+0} \equiv \frac{m(S = b, I = -a, \sigma = +1)}{\psi_+(-a, b)} = \frac{m(S = b, I = -a, \sigma = +1)}{\bar{m}(H_b < \infty, I(S_b) = -a)} = \theta \, \frac{h}{b + h - v_+} \qquad (2.22)
$$

with the notation of Proposition 2.3; in view of the fact that we have assumed the necessary conditions (2.12) and (2.13), *we can assert[4] that θ so defined is a probability*: $0 \leq \theta \leq 1$. We now run the Brownian motion B forward from time σ_n until it first hits ℓ or $b + h$. If $\ell = v_+$ and B hits ℓ before $b + h$, then we will stop everything at that time, and declare that $X_T = v_+$; otherwise, we will reach either $-a - h$ or $b + h$ and declare that $T \geq \tau_{n+1}$. If we determine that $T \geq \tau_{n+1}$, we take a further step of the construction.

For each $n \geq 1$, let Q_n be the combined statement[5]

(i) for all $a, b \in h\mathbb{Z}^+$, $0 < a + b \leq nh$

$$
\begin{aligned}
P(H_b \leq T, \ I(H_b) = -a) &= \psi_+(-a, b) & (2.23) \\
P(H_{-a} \leq T, \ S(H_{-a}) = b) &= \psi_-(-a, b) & (2.24)
\end{aligned}
$$

(ii)
$$
P(S = x, I = -y, X = z, \sigma = s) = m(S = x, I = -y, X = z, \sigma = s) \qquad (2.25)
$$

for all $s \in \{-1, 1\}$, $x, y, z, \in h\mathbb{Z}$, $x, y \geq 0$, $x + y < nh$.

We shall prove by induction that Q_n is true for all $n > 0$, establishing the statement first for $n = 1$. We prove (2.23), leaving the analogous proof of (2.24) to the diligent reader. Taking $b = 0$, $a = h$, (2.23) says that

$$
P(H_0 \leq T, \ I(H_0) = -h) = \psi_+(-h, 0),
$$

---

[3] We provide details of what happens if $S_{\tau_n} = \xi_{\tau_n}$; the treatment of the case $I_{\tau_n} = \xi_{\tau_n}$ is analogous.

[4] We shall establish in the inductive proof that $\psi_\pm$ are non-negative.

[5] The functions $\psi_\pm$ are defined in terms of m by (2.3), (2.4), (2.7), (2.8).

and both sides are readily seen to be equal to zero; taking $b = h$, $a = 0$, (2.23) says that

$$
\begin{aligned}
P(H_h \leq T,\ I(H_h) = 0) &= \psi_+(0, h) \\
&= \varphi(h, -h) - \varphi(h, 0) \\
&= \frac{h - m(h + X; S < h, I > -h)}{2h} - 0 \\
&= \tfrac{1}{2}\big[\, 1 - m(S = X = I = 0)\,\big]
\end{aligned}
$$

which is clearly true, because if the construction does not stop immediately at time 0 (an event of probability $m(I = X = S = 0)$) then with equal probability the process steps at time 1 to $\pm h$. The second statement (2.25) holds because we have constructed the probability of $I = X = S = 0$ correctly.

Now suppose that Q_k has been proved to hold for $k \leq n$; we have to prove (2.23), (2.24) and (2.25) for $n + 1$. To prove (2.25), suppose that $x,\ y \in h\mathbb{Z}^+$ and $x + y = nh$. By construction, the random walk will be stopped before the range $S - I$ increases to $(n + 1)h$ if and only if the barrier ℓ happens to be positioned at v_+ *and* that barrier is hit before the Brownian motion rises to $b + h$. Conditional on the event $B_+ = \{T \geq \tau_n,\ S_{\tau_n} = \xi_{\tau_n} = b,\ I_{\tau_n} = -a\}$, the probability of that joint event is

$$
\theta \times \frac{h}{b + h - v_+}. \tag{2.26}
$$

By the inductive hypothesis (2.23) we have that the probability of the conditioning event B_+ is $\psi_+(-a, b)$; so from the definition (2.22) of θ we learn that

$$
P(S_T = b,\ I_T = -a,\ \sigma = +1) = m(S = b, I = -a,\ \sigma = +1).
$$

Given that this event happens, the conditional distribution of X_T is correct, by the Skorohod embedding construction of X_T with mean v_+. Therefore (2.25) has been proven for any $x,\ y \in h\mathbb{Z}$ with $x + y = nh$, and for any $z \in h\mathbb{Z}$, $s \in \{-1, 1\}$.

It remains to prove assertion (i) of Q_{n+1}, and for this we recall some of the notation of the proof of Proposition 2.3. For $a,\ b \in h\mathbb{Z}^+$, $a + b = nh$, we write

$$
\begin{aligned}
p_+ &= P(B_+) \equiv P(H_b \leq T,\ I(H_b) = -a), \\
p_- &= P(B_-) \equiv P(H_{-a} \leq T,\ S(H_{-a}) = b)
\end{aligned}
$$

which in view of the truth of Q_n we know are equal to $\psi_+(-a, b)$ and $\psi_-(-a, b)$ respectively.

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If we now define

$$
\begin{aligned}
p_{++} &= P(B_+, H_{b+h} \le T \wedge H_{-a-h}) \\
p_{+-} &= P(B_+, H_{-a-h} \le T \wedge H_{b+h}) \\
p_{+0} &= P(B_+, T < \tau_{n+1}) \\
p_{-+} &= P(B_-, H_{b+h} \le T \wedge H_{-a-h}) \\
p_{--} &= P(B_-, H_{-a-h} \le T \wedge H_{b+h}) \\
p_{-0} &= P(B_-, T < \tau_{n+1})
\end{aligned}
$$

then by exactly the same Optional Sampling argument which led to (2.18), (2.19), we conclude that

$$
p_{++} = \frac{(b+a+h)p_+ - (a+h+v_+)\,p_{+0}}{b+a+2h} \tag{2.27}
$$

$$
p_{+-} = \frac{hp_+ - (b+h-v_+)\,p_{+0}}{b+a+2h} \tag{2.28}
$$

$$
p_{-+} = \frac{hp_- - (a+h+v_-)p_{-0}}{a+b+2h} \tag{2.29}
$$

$$
p_{--} = \frac{(a+b+h)p_- - (b+h-v_-)p_{-0}}{a+b+2h} \tag{2.30}
$$

and now the task is to prove (after cross-multiplying by $a+b+2h$) that

$$
(a+b+2h)\{\, p_{++} + p_{-+} \,\} = (a+b+2h)\psi_+(-a, b+h), \tag{2.31}
$$

and the minus analogue, which is just the same argument *mutatis mutandis*. Firstly we develop the left-hand side using (2.27), (2.28) and their analogues for B_- to obtain

$$
\begin{aligned}
LHS ={}& (a+b+h)\psi_+(-a,b) - (a+h+v_+)p_{0+} + h\psi_-(-a,b) - (a+h+v_-)p_{-0} \\
={}& (a+b+h)\{\, \varphi(b,-a-h) - \varphi(b,-a) \,\} + h\{\, \varphi(-a,b+h) - \varphi(-a,b) \,\} \\
& -(a+h)m(S=b, I=-a) - m(X; S=b, I=-a) \\
={}& a+h - m(a+h+X; S<b, I>-a-h) - \{\, a - m(a+X; S<b, I>-a) \,\} \\
& -h(\varphi(b-a)+\varphi(-a,b)) + h\varphi(-a,b+h) - m(a+h+X; S=b, I=-a) \\
={}& h - m(a+h+X; S<b, I>-a-h) + m(a+X; S<b, I>-a) \\
& -h\{1 - m(S<b, I>-a)\} + h\varphi(-a,b+h) - m(a+h+X; S=b, I=-a) \\
={}& -m(a+h+X; S<b, I>-a-h) + m(a+h+X; S<b, I>-a) \\
& -m(a+h+X; S=b, I=-a) + h\varphi(-a,b+h) \\
={}& -m(a+h+X : (A_2 \cup A_3)\backslash A_1) + h\varphi(-a,b+h)
\end{aligned}
$$

where $A_1 = \{S<b, I>-a\}$, $A_2 = \{S<b, I>-a-h\}$ and $A_3 = \{S=b, I=-a\}$. Noticing that $A_1 \subseteq A_2$ and A_3 is disjoint from A_1, the region of integration is

$$
(A_2 \cup A_3)\backslash A_1 = \{S<b, I=-a\} \cup A_3 = \{S \le b, I=-a\} = \{S<b+h, I=-a\}.
$$

Hence the left-hand side is equal to

$$LHS = -m(a+h+X; S < b+h, I = -a) + h\varphi(-a, b+h). \qquad (2.32)$$

Turning now to the right-hand side of (2.31), we have

$$
\begin{aligned}
RHS &= (a+b+2h)\{\varphi(b+h, -a-h) - \varphi(b+h, -a)\} \\
&= a+h - m(a+h+X : S < b+h, I > -a-h) - h\varphi(b+h, -a) \\
&\quad -\{a - m(a+X : S < b+h, I > -a)\} \\
&= h - m(a+h+X : S < b+h, I > -a-h) + m(a+h+X; S < b+h, I > -a) \\
&\quad -hm(S < b+h, I > -a) - h\varphi(b+h, -a) \\
&= h\{1 - m(S < b+h, I > -a) - \varphi(b+h, -a)\} \\
&\quad -m(a+h+X; S < b+h, I = -a). \qquad (2.33)
\end{aligned}
$$

Comparing (2.32) and (2.33), we see that we have to prove

$$\varphi(b+h, -a) + \varphi(-a, b+h) = 1 - m(S < b+h, I > -a), \qquad (2.34)$$

which is evidently true from the definition (2.3), (2.4) of φ.

$\square$

3 Hedging.

Theorem 2.4 provides us with necessary and sufficient conditions for a measure m on $\mathcal{X}$ to be consistent. In principle, this allows us to construct extremal martingales, and robust hedges for derivatives.

Let us firstly see how this works in the context of the joint law of (S, X) studied in [7]. We begin by recalling some of the results of that paper. We let $X_t = B_{t \wedge T}$ be a Brownian motion stopped as an almost-surely finite stopping time T, with $S_t = \sup_{u \le t} X_u$, and with $S \equiv S_\infty$, $X \equiv X_\infty$. With this terminology, Theoren 3.1 of [7] says the following.

Theorem 3.1. *The probability measure μ on $\mathbb{R}^+ \times \mathbb{R}^+$ is the joint law of $(S, S-X)$ for some almost-surely finite stopping time T if and only if*

$$\left(\iint_{(t,\infty) \times \mathbb{R}^+} \mu(ds, dy) \right) dt \ge \int_{(0,\infty)} y \, \mu(dt, dy). \qquad (3.1)$$

If $(X_t)_{t \ge 0}$ is also uniformly integrable, then inequality (3.1) holds with equality:

$$\left(\iint_{(t,\infty) \times \mathbb{R}^+} \mu(ds, dy) \right) dt = \int_{(0,\infty)} y \, \mu(dt, dy). \qquad (3.2)$$

10

Finally, if (3.2) holds, and if $X \in L^1$,

$$\iint |t - y| \, \mu(dt, dy) < \infty, \tag{3.3}$$

then μ is the joint law of $(S, S - X)$ for a uniformly integrable martingale $(X_t)_{t \geq 0}$.

PROOF. See [7]. The final assertion is not in [7], but can easily be deduced. In view of the first assertion, there is some stopping time $T < \infty$ such that μ is the joint law of $(S, S - X)$. By multiplying (3.2) by some non-negative test function φ and integrating with respect to t we discover that

$$\mu(\Phi) = \mu(\,(S - X)\varphi(S)\,) \tag{3.4}$$

where $\Phi(t) = \int_0^t \varphi(y) \, dy$. Taking $\varphi(x) = I_{\{x > b\}}$ for some $b \geq 0$ we find that

$$b\mu(S > b) = \mu(X : S > b). \tag{3.5}$$

Using the fact that $X \in L^1$, we can let $b \uparrow \infty$ in (3.5) to prove that $\lim_{b \uparrow \infty} b\mu(S > b) = 0$. Lemma 2.3 of [7] gives the result.

$\square$

REMARK. Standard monotone class arguments show that (3.1) is equivalent to the statement that

$$\mu(\Phi) \geq \mu(\,(S - X)\varphi(S)\,) \tag{3.6}$$

for all non-negative test functions, which again is equivalent to the statement that

$$b\mu(S > b) \geq \mu(X : S > b) \tag{3.7}$$

for all $b \geq 0$. Likewise, (3.2) is equivalent to (3.4) for all non-negative test functions φ, which again is equivalent to the statement (3.5):

$$\mu(\,X - b : S > b\,) = 0 \qquad \forall b \geq 0. \tag{3.8}$$

An important and typical[6] use of this would be to try to find an *extremal* martingale, which would in turn lead to a maximum possible derivative price and a robust hedging strategy. So, for example, suppose that we observe call option prices $C(K)$ for every strike K at a common fixed expiry time[7] for some (discounted) asset, and suppose that the asset has continuous paths $(X_t)_{0 \leq t \leq 1}$, and is a uniformly-integrable martingale in the pricing measure.

Suppose now that we are given some derivative whose payoff at time 1 is $G(S_1, X_1)$, where $S_1 = \sup_{0 \leq t \leq 1} X_t$; *what is the most expensive the time-0 price of this derivative can be?*

---

[6]The papers Hobson [6], ... give examples of this kind.

[7]Let us suppose that the expiry is 1.

The time-0 price of the derivative is given by

$$\iint G(s,x)\, q(ds,dx) \tag{3.9}$$

where q is the joint law[8] of (S,X). Now provided the law q satisfies the conditions

$$\iint (x-K)^+\, q(ds,dx) = C(K) \qquad \forall K \tag{3.10}$$

and (see (3.8))

$$\iint_{s>b} (x-b)\, q(ds,dx) = 0 \qquad \forall b > 0 \tag{3.11}$$

then q is the joint distribution of (S,X) for *some* continuous martingale whose law at time 1 agrees with the data contained in the call prices. The problem of finding the most expensive time-0 price is therefore the problem of maximizing the *linear* objective (3.9) over non-negative probability measures q subject to the *linear* constraints (3.10) and (3.11). Writing the problem in Lagrangian form[9], we seek

$$L(\alpha,\eta,\lambda) = \sup_{q\geq 0}\left[\iint \Big\{ G(s,x) - \alpha - \int (x-K)^+\, \eta(dK) + \int_0^\infty (x-b)I_{\{s>b\}}\, \lambda(db) \Big\}\, q(ds,dx) \right.$$
$$\left. + \alpha + \int C(K)\, \eta(dK) \right]. \tag{3.12}$$

From standard linear programming results, we would expect that for dual feasibility we must have

$$G(s,x) \leq \alpha + \int (x-K)^+\, \eta(dK) - \int_0^\infty (x-b)I_{\{s>b\}}\, \lambda(db) \tag{3.13}$$

everywhere, with equality everywhere that the optimal q places mass; and that the dual problem will be

$$\inf\left[\alpha + \int C(K)\, \eta(dK) \right] \tag{3.14}$$

over (α,η,λ) satisfying (3.13). These equations have a simple and beautiful interpretation. The dual-feasibility relation (3.13) expresses a *robust hedge*; if we hold α in cash, $\eta(dK)$ calls of strike K, and *sell forward $\lambda(db)$ units of the underlying when S reaches the level b*, then we generate a contingent claim at the terminal time which will always dominate the claim G which we have to pay out. The dual form of the linear program (3.14) says that the cost of constructing such a hedge, which is of course $\alpha + \int C(K)\, \eta(dK)$, must be minimized.

The primal problem seeks to find the most expensive that the derivative $G(S,X)$ can be, given the market prices $C(K)$; and the dual problem seeks the cheapest super-replicating

---

[8]As before, when the time subscript of a process is omitted, we understand it to be 1.

[9]This linear programming approach to the problem is also used in [4].

hedge. The characterization (3.8) of the possible joint laws of (S, X) *tells us what the form of the hedge* (3.13) *must be.*

Our goal now is to try to use Theorem 2.4 to similarly bound the price of, and to super-replicate, contingent claims which depend on the maximum, terminal value, *minimum, and direction of the final excursion* for a stopped symmetric simple random walk. To understand how this is to be done, we focus on the 'plus' versions of the necessary and sufficient conditions (2.12). We shall also suppose that the martingale X is *uniformly integrable*, to avoid having to bother about side issues.

The condition (2.12) can be restated in terms of the measure m as

$$
\begin{aligned}
m(b + h - X : S = b, I = -a, \sigma = +1) &\leq h\psi_+(-a, b) \\
&= h\{\varphi(b, -a - h) - \varphi(b, -a)\}
\end{aligned}
\tag{3.15}
$$

in the notation of Section 2. From the definition (2.3) of $\varphi(b, -a)$, from the fact that $m(X) = 0$, and the Optional Sampling Theorem result that $m(a + X : I \leq -a) = 0$, we have

$$
\begin{aligned}
(a + b)\varphi(b, -a) &= a - m(a + X : S < b, I > -a) \\
&= m(a + X : S \geq b \text{ or } I \leq -a) \\
&= m(a + X : S \geq b, I > -a) \\
&= (a + b)m(S \geq b, I > -a) - m(b - X : S \geq b, I > -a).
\end{aligned}
$$

Thus the inequality (3.15) may be re-expressed after some simple rearrangement as

$$
\begin{aligned}
0 \leq\ & hm(S \geq b, I = -a) - \frac{h}{a + b + h} m(b - X : S \geq b, I > -a - h) + \\
& + \frac{h}{a + b} m(b - X : S \geq b, I > -a) - m(b + h - X : S = b, I = -a, \sigma = +1).
\end{aligned}
$$

This inequality for all $a, b \in h\mathbb{Z}^+$ not both zero, together with the 'minus' analogues, is necessary and sufficient for a probability measure m to be the joint law of (I, X, S, σ). Just as we did at (3.12) for derivatives depending only on (X, S), we can construct the Lagrangian for this problem, which would give us terms of the form

$$
\begin{aligned}
\lambda_{ab}^+ (Z - w) \equiv\ & \lambda_{ab}^+ \Bigg[hI_{\{S \geq b, I = -a\}} - \frac{h}{a + b + h} (b - X)I_{\{S \geq b, I > -a - h\}} + \\
& + \frac{h}{a + b} (b - X)I_{\{S \geq b, I > -a\}} - (b + h - X)I_{\{S = b, I = -a, \sigma = +1\}} - w \Bigg],
\end{aligned}
\tag{3.16}
$$

where $w \geq 0$ is a non-negative slack variable to handle the inequality constraint. Dual feasibility will therefore require that $\lambda_{ab}^+ \geq 0$, and at optimality we will have the complementary slackness condition $\lambda_{ab}^+ w = 0$.

13

In the situation of derivatives depending only on (X, S), we had terms of the form $\lambda_a(X - a)I_{\{S>a\}}$, which were interpreted as forward purchase of the underlying asset when the supremum process reaches a new level. This forward purchase interpretation determines a hedging strategy which *can be implemented in an adapted fashion.* However, it is very far from clear that the random variable Z defined at (3.16) can be realized by some adapted trading strategy. For example, the term involving $(b - X)I_{\{S\geq b, I>-a\}}$ could be interpreted as a forward sale of the underlying when the price first gets to b; but this trade should only be put on if $I > -a$, and it is not known at time H_b whether or not the ultimate infimum I will be greater than $-a$ or not.

Nevertheless, we can specify an adapted trading strategy which will subreplicate the random variable Z, as follows. We construct a random variable Y which is the final value of the adapted hedging strategy made up of three component positions:

1. At H_b, buy forward $h/(a + b + h)$ units of the underlying if $I(H_b) > -a - h$, and come out of the position at time H_{-a-h};

2. At H_b, buy forward $-h/(a + b)$ units of the underlying if $I(H_b) > -a$, and come out of the position at time H_{-a};

3. At H_b, buy forward 1 unit of the underlying if $I(H_b) = -a$, and come out of the position at time $H_{b+h} \wedge H_{-a-h}$.

Now clearly the random variable

$$
\begin{aligned}
Z \quad &\equiv \quad hI_{\{S\geq b, I=-a\}} - \frac{h}{a + b + h}(b - X)I_{\{S\geq b, I>-a-h\}} + \\
&\quad + \frac{h}{a + b}(b - X)I_{\{S\geq b, I>-a\}} - (b + h - X)I_{\{S=b, I=-a, \sigma=+1\}}
\end{aligned}
\tag{3.17}
$$

will be zero if $S < b$ or if $I \leq -a - h$, so to understand Z we may suppose that $H_b < \infty = H_{-a-h}$.

But before we narrow our attention down to the event $\{H_b < \infty = H_{-a-h}\}$, we should consider what happens off that event to Y. If $H_b = \infty$, then none of the component positions of Y is ever entered, so $Y = 0$ in that case. If $H_b < \infty$ and $H_{-a-h} < \infty$, then we have three cases to consider:

(i) When $I(H_b) > -a$, the strategy enters positions 1 and 2 at time H_b, and closes out both when the infimum falls to $-a$ and then to $-a - h$; position 1 loses h, position 2 gains h, so altogether $Y = 0$;

(ii) When $I(H_b) = -a$, the strategy enters positions 1 and 3. If $H_{b+h} < H_{-a-h}$, then position 3 makes a gain of h when it is closed out, but position 1 makes a loss of h when it is closed out, so overall zero gain. On the other hand, if $H_{-a-h} < H_{b+h}$, then position 1 makes a loss of h when it is closed out, and position 3 makes a loss of $(a + b + h)$ when

it is closed out, so overall $Y = -(a + b + h) - h < 0$, and as we shall subsequently see, *this is the only situation in which Y is strictly less than Z*;

(iii) When $I(H_b) \leq -a - h$, none of the positions is entered, and $Y = 0$.

We now have to compare the values of Z and Y on the event $\{H_b < \infty = H_{-a-h}\}$, breaking the comparison down into seven cases as presented in the following table. In the first two rows, we see what happens if $I > -a$, and in the remaining rows, we are considering situations where $I = -a$. The reader is invited to check through each of the entries of the table, and confirm the findings reported there. The only entry that requires comment is the penultimate row, in the column for Z. In this row, we are in the situation where $S = b$ and $I = -a$, so we get a contribution to Z from the first term in (3.17), and from the second term, none from the third term, and *none from the fourth term*, because if $H_b < H_{-a} < H_{b+h} = \infty$ it must be that *the signature σ is -1* ! What we see from the table is that in every case the value of Z is equal to the value of Y.

$H_{-a-h} = \infty$	Z	Y
$H_b < H_{b+h} < \infty = H_{-a}$	$\frac{h(b-X)}{a+b} - \frac{h(b-X)}{a+b+h}$	$\frac{h(X-b)}{a+b+h} - \frac{h(X-b)}{a+b}$
$H_b < H_{b+h} = \infty = H_{-a}$	$\frac{h(b-X)}{a+b} - \frac{h(b-X)}{a+b+h}$	$\frac{h(X-b)}{a+b+h} - \frac{h(X-b)}{a+b}$
$H_{-a} < H_b < H_{b+h} < \infty$	$h - \frac{h(b-X)}{a+b+h}$	$\frac{h(X-b)}{a+b+h} + h$
$H_{-a} < H_b < H_{b+h} = \infty$	$h - \frac{h(b-X)}{a+b+h} + (X - b - h)$	$\frac{h(X-b)}{a+b+h} + X - b$
$H_b < H_{-a} < H_{b+h} < \infty$	$h - \frac{h(b-X)}{a+b+h}$	$\frac{h(X-b)}{a+b+h} + h$
$H_b < H_{-a} < H_{b+h} = \infty$	$h - \frac{h(b-X)}{a+b+h}$	$\frac{h(X-b)}{a+b+h} + h$
$H_b < H_{b+h} < H_{-a} < \infty$	$h - \frac{h(b-X)}{a+b+h}$	$\frac{h(X-b)}{a+b+h} + h$

Thus we may conclude that $Y \leq Z$ in all instances, and the only situation in which the inequality is strict is when $H_{-a} < H_b < H_{-a-h} < H_{b+h}$.

Now we explain how these observations lead to a super-replicating hedging strategy. For this, let us denote by Z_{ab}^+ then random variable we have been calling Z up til now; this is because in the Lagrangian we have to consider such random variables (and their 'minus' analogues) for all $a, b \in h\mathbb{Z}^+$ not both zero. Suppose that we have some derivative $G(I, X, S, \sigma)$ whose price we wish to maximize subject to the distribution of X matching call price data, just as we did for derivatives depending only on (X, S) in the first part of our discussion in

this Section. We would find ourselves with a Lagrangian form similar to (3.12):

$$
\begin{aligned}
L(\alpha, \lambda, \eta) \;=\; \sup_{m \geq 0} \Bigg[\int \Big\{ \; & G(I, X, S, \sigma) - \alpha - \int (X - K)^{+} \eta(dK) + \\
& + \sum_{a,b,\pm} \lambda_{ab}^{\pm} (Z_{ab}^{\pm} - w_{ab}^{\pm}) \; \Big\} \, dm(I, X, S, \sigma) + \alpha + \int C(K)\, \eta(dK) \; \Bigg]
\end{aligned}
\tag{3.18}
$$

with obvious notation. Now dual feasibility imposes the condition

$$
G(I, X, S, \sigma) \;\leq\; \alpha + \int (X - K)^{+} \eta(dK) - \sum_{a,b,\pm} \lambda_{ab}^{\pm} Z_{ab}^{\pm}
\tag{3.19}
$$

$$
\leq\; \alpha + \int (X - K)^{+} \eta(dK) - \sum_{a,b,\pm} \lambda_{ab}^{\pm} Y_{ab}^{\pm}
\tag{3.20}
$$

in another obvious notation. The interpretation of (3.20) is that *the derivative G is super-replicated by the adaptively-realizable hedge given by a position in calls and a position in the Y-hedges.*

At optimality, complementary slackness tells us that if $\lambda_{ab}^{+} > 0$ then $w_{ab}^{+} = 0$, and therefore the inequality (3.15) must hold with equality. Tracing this back to the condition (2.12), and its derivation from (2.19), we find that equality in (3.15) is equivalent to the statement that $\tilde{p}_{+-} = 0$. What this means is that on the event $\{H_{-a} < H_b < H_{-a-h}\}$ we *cannot have* $H_{-a-h} < H_{b+h}$, and as we saw, this was *the only situation where $Y < Z$*. We may therefore conclude that for the optimal m^{*}, not only will (3.19) hold with equality m^{*}-a.e., but also (3.20) will hold with equality m^{*}-a.e.. In other words, if the joint law m is the optimal joint law, the hedging strategy expressed by (3.20) is a perfect replication of the contingent claim - there is no slack.

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