17

BROWNIAN LOCAL TIMES

AND

BRANCHING PROCESSES

by

L.C.G. Rogers*

1. Introduction

a) For each $k \in \mathbb{N}$, let $(Z_n^k)_{n\geq 0}$ be a critical discrete-time branching process, with $Z_o^k = k$. The offspring distribution is the same for each k, and has finite variance σ^2, with mean 1.

Define the random elements $z^k(\cdot)$ of $D[0,\infty)$ by

$$z^k(t) \equiv k^{-1} Z^k([kt]),$$

where $[x]$ denotes the integer part of x. Let z_t denote the solution of the stochastic differential equation

$$(1) \qquad z_t = 1 + \sigma \int_0^t (z_s^+)^{\frac{1}{2}} \, d B_s.$$

The solution exists, is pathwise unique (Yamada-Watanabe [16]) and is the square of a zero-dimensional Bessel process (see Pitman-Yor [11], [12] for more information on these diffusions. All the facts we shall need about Bessel processes can be found in these two papers.)

In [9], Lamperti established the convergence of the finite dimensional distributions of the z^k, and in [10], Lindvall also proved the tightness

---

* Department of Statistics, University of Warwick, Coventry CV4 7AL, G.B.

Now moved to: Department of Mathematics, University College of Swansea, Swansea SA2 8PP, G.B.

of the laws of the z^k, from which one obtains the following result.

Theorem A (Lamperti, Lindvall).

As random elements of $D[0,\infty)$,

$$z^k \Rightarrow z.$$

Remark. The intuitive interpretation of Theorem A is very appealing; if one defines $\mathfrak{F}_t^k \equiv \sigma(\{Z_n^k;\ n\le kt\})$, then, since the branching processes are critical,

$$z_t^k \text{ and } (z_t^k)^2 - \sigma^2 \int_0^{k^{-1}[kt]} z_s^k \, ds \text{ are } \mathcal{F}^k\text{-martingales.}$$

Thus one expects that if a limit process z exists, it should have the property

$$z_t \text{ and } z_t^2 - \sigma^2 \int_0^t z_s \, ds \text{ are martingales.}$$

Together with continuity of paths, these requirements uniquely characterise the law of the solution of (1). Much recent work has gone into making this intuitive notion precise (see, for example, Durrett-Resnick [4], Jacod-Memin-Metivier [7]).

Now equation (1) will not have escaped the notice of Brownian motion enthusiasts; it has the following striking interpretation.

Theorem B (Ray[13], Knight[8])

Let B_t *be Brownian motion on* $\mathbb{R}$, $B_0=0$, *and let* $\ell(x,t)$ *be (a jointly continuous version of) its local time. If* $T \equiv \inf\{t;\ell(0,t)>1\}$, *then*

$$\big(\ell(x,T)\big)_{x\ge 0} = (z_t)_{t\ge 0},$$

where z *is defined by (1) with* $\sigma=2$.

The fact that the same process is appearing as a limit of branching processes and as the local time process of Brownian motion is largely explained by the next result, which says that there is a branching process hidden in random walk!

Let $(S_n)_{n\geq 0}$ be symmetric simple random walk on $\mathbb{Z}$, with $S_0=0$. Define the "local time" of $|S|$ as follows:

$$L(j,n) \equiv \sum_{r=1}^{n} I_{\{|S_r|=j\}} \qquad (j,\ n\in\mathbb{Z}^+)$$

<u>Theorem 1</u>

<u>If</u>

$$\tau \equiv \inf\{n;L(0,n)\geq 1\},$$

<u>then</u>

(2) $$\{L(j,\tau);j\geq 1\} = \{Z_j+Z_{j-1};j\geq 1\},$$

<u>where</u> $(Z_j)_{j\geq 0}$ <u>is a critical branching process whose offspring distribution has the probability generating function</u>

$$\phi(t) \equiv (2-t)^{-1},$$

<u>and</u> $Z_0=1$.

The proof of Theorem 1 is given in the next sectiin. The idea of the proof is essentially that of Dwass [5]; the method of proof is only a little different.

To make the connection between Theorems A and B more explicit, define $s(\cdot)$ and $\lambda(\cdot)$ to be the piecewise linear interpolations of $|S_n|$ and $L(0,n)$ respectively;

$$s(t) \equiv (t-n)\ |S_{n+1}| + (n+1-t)\ |S_n| \qquad (n\leq t\leq n+1)$$

$$\lambda(t) \equiv (t-n)\ L(0,n+1) + (n+1-t)\ L(0,n) \qquad (n\leq t\leq n+1).$$

Define for each $N\in\mathbb{N}$

$$\tau_N \equiv \inf\{n;L(0,n)\geq N\},$$

$$s_N(t) \equiv N^{-1}\ s(N^2 t\wedge\tau_N),$$

$$\lambda_N(t) \equiv N^{-1}\ \lambda(N^2 t\wedge\tau_N),$$

and finally define

$$\ell_N(x) \equiv N^{-1} L([Nx], \tau_N). \qquad (Nx\geq 1)$$

$$\equiv 2 \qquad (0\leq Nx<1).$$

We shall prove the following:

Theorem 2.

As random elements of $C(\mathbb{R}^+, \mathbb{R}^+)^2 \times D[0,\infty)$,

$$\left(s_N(\cdot), \lambda_N(\cdot), \ell_N(\cdot)\right) \Rightarrow \left(|B_{T\wedge\cdot}|, \ell(0,T\wedge\cdot), \ell^*(\cdot,T)\right),$$

where $\ell^*(x,t) \equiv \ell(x,t) + \ell(-x,t)$ for $x, t\geq 0$.

Remarks (i) While it is easy to believe that the first components of these triples (and even the first two components) should converge weakly to the stated limit, this provides little help with the weak convergence of the third; one cannot use the continuous function theorem since local time is not a continuous function of the Brownian path, and we are forced to use Theorem 1 in an essential way.

(ii) For those whose courage fails them when a lengthy technical proof of tightness drifts into view, it is worth emphasising that all the tightness we need follows from Donsker's theorem and the Lamperti-Lindvall result, so the proof of Theorem 2 is not as grim as might be feared!

b) In this second half of the Introduction, we shall begin by explaining why two halves were needed.

In the first half, we took a macroscopic view of the convergence of the local time processes $\ell_N(\cdot)$ to the $BESQ^0$ limit, $\ell(\cdot,T)$. But we can also take a microscopic view. Indeed, the local time process

$$L(\cdot,\tau_N) \equiv \sum_{r=1}^{N} \{L(\cdot,\tau_r) - L(\cdot,\tau_{r-1})\}$$

can be written as a sum of n i.i.d. processes, each with the law of $L(\cdot,\tau_1)$, by the strong Markov property of S at the times τ_r. Theorem 1 tells us that each of these processes is closely related to a branching process, and the decomposition of $L(\cdot,\tau_N)$ just described is exactly analogous to the decomposition

$$Z^k \equiv \sum_{r=1}^{k} Z^{(r)}.$$

of Z^k into k i.i.d. branching processes $Z^{(r)}$ with $Z_0^{(r)} = 1$; $Z_n^{(r)}$ has the interpretation of the number of offspring of the r^{th} individual at time 0 which survive at time n.

Now the $BESQ^0$ limit can also be decomposed; the local time $\ell(x,T)$ is the sum of the local times at level x of every excursion of B before T. There are, of course, infinitely many such excursions but they are "i.i.d." in the sense that they make up a Poisson point process in excursion space with a σ-finite law. The (excursion) law of the local time process $\ell(\cdot,\zeta)$, where ζ is the lifetime of a Brownian excursion, has been characterised by Pitman and Yor (see Theorems 4.1 and 4.2 of [12]). This suggests the conjecture that the law of $L(\cdot,\tau_1)$, when suitably scaled and normalised, converges to the excursion law of $\ell(\cdot,\zeta)$. Of course, we have to be very careful about the meaning of convergence in such a setting, since the limit measure is σ-finite, but in section 3 we shall show that, suitably interpreted, the conjecture is true, which gives a lovely way of thinking of the macroscopic results; the local time processes $\ell_N(x)$ are converging to $BESQ^0$ because each of the i.i.d. constituents $N^{-1}\{L([Nx], \tau_{r+1}) - L([Nx],\tau_r)\}$ is converging to the local time process of a Brownian excursion.

Acknowledgements. I am very grateful to K.A. Borovkov, who told me about Theorem A, and to N.H. Bingham, who told me about Dwass[5].

2. Branching processes in random walk.

The first task is to prove Theorem 1. To see why it must be true, define for each j≥0

$$Z_j \equiv \sum_{r=0}^{\tau-1} I_{\{|S_r|=j,\ |S_{r+1}|=j+1\}},$$

the number of steps up from the level j made by the random walk before the first return to zero, τ. Each step up from level j must eventually be followed by a step down from level j+1, but before this happens, the random walk will make a random number of steps up from level j+1; the distribution of this random number of upward steps is geometric with parameter $\frac{1}{2}$, since each time the random walk is at j+1, it decides with equal probability to step up or down. Thus each upward step from level j gives rise to a random number of upward steps from level j+1, the number having generating function ϕ. Hence $(Z_j)_{j\geq 0}$ is a branching process, and $L(j,\tau) = Z_j+Z_{j-1}$ is evident.

Though it is very plausible, it is not entirely obvious from this argument that the numbers of offspring of different individuals in the branching process should be independent. To deal with this point, we present an entirely computational proof of Theorem 1.

Proof of Theorem 1. Pick non-negative reals $\alpha_0,\alpha_1,\dots$ such that $\alpha_r=0$ for $r>N$, $\alpha_0=0$ and set $x_r \equiv \exp(-\alpha_r)$. Suppose $(f_k)_{k\geq 1}$ solves

$$f_0 = 1$$

(3) $$f_k = \tfrac{1}{2}\{x_{k+1}\, f_{k+1} + x_{k-1}\, f_{k-1}\} \qquad (k\geq 1),$$

$$f_k = f_{k+1} \qquad (k>N).$$

Then defining

$$M_n \equiv f(|S_{n\wedge\tau}|)\exp\{-\sum_{r\geq 1}\alpha_r\, L(r,n\wedge\tau)\},$$

$(M_n)_{n\geq 1}$ is a bounded martingale relative to the filtration $(\mathcal{F}_n)$ of $|S_n|$, and

$$M_1 = f_1\, x_1 = E[M_\infty|\mathcal{F}_1]$$

(4) $$= E\left[\prod_{r=1}^{N} x_r^{\,L(r,\tau)}\right].$$

We can solve (3) by setting $\rho_k \equiv f_{k+1}/f_k$, so that $\rho_k=1$ for $k>N$, and

$$2 = x_{k+1}\,\rho_k + x_{k-1}/\rho_{k-1} \qquad (k\geq 1)$$

or equivalently

$$\rho_{k-1} = x_{k-1}\ \phi(x_{k+1}\ \rho_k). \qquad (k\geq 1).$$

Thus if $\theta_k \equiv x_k\ x_{k+1}$, we obtain for $k\geq 0$

$$\rho_k = x_k\ \phi\big(\theta_{k+1}\ \phi(\theta_{k+2}(\ldots(\theta_{N-1}\phi(\theta_N))\ldots))\big).$$

Hence

(5) $$f_1 \equiv \rho_0 = \phi\big(\theta_1\phi(\theta_2(\ldots(\theta_{N-1}\phi(\theta_N))\ldots))\big),$$

and, by (4),

(6) $$E\left[\prod_{r=1}^{N} x_r^{L(r,\tau)}\right] = x_1\ f_1.$$

On the other hand, if $(Z_n)_{n\geq 0}$ is a branching process with $Z_0=1$, and offspring generating function ϕ, then

(7) $$E\prod_{r=0}^{N} \theta_r^{Z_r} = \theta_0\ \phi\big(\theta_1\phi(\theta_2(\ldots(\theta_{N-1}\phi(\theta_N))\ldots))\big).$$

A simple calculation based on (5), (6) and (7) yields (2), completing the proof of Theorem 1.

Proof of Theorem 2.

(i) We shall firstly prove that

$$\big(s_N(\cdot),\ \lambda_N(\cdot)\big) \Rightarrow \big(|B_{T\wedge\cdot}|,\ \ell(0,T\wedge\cdot)\big).$$

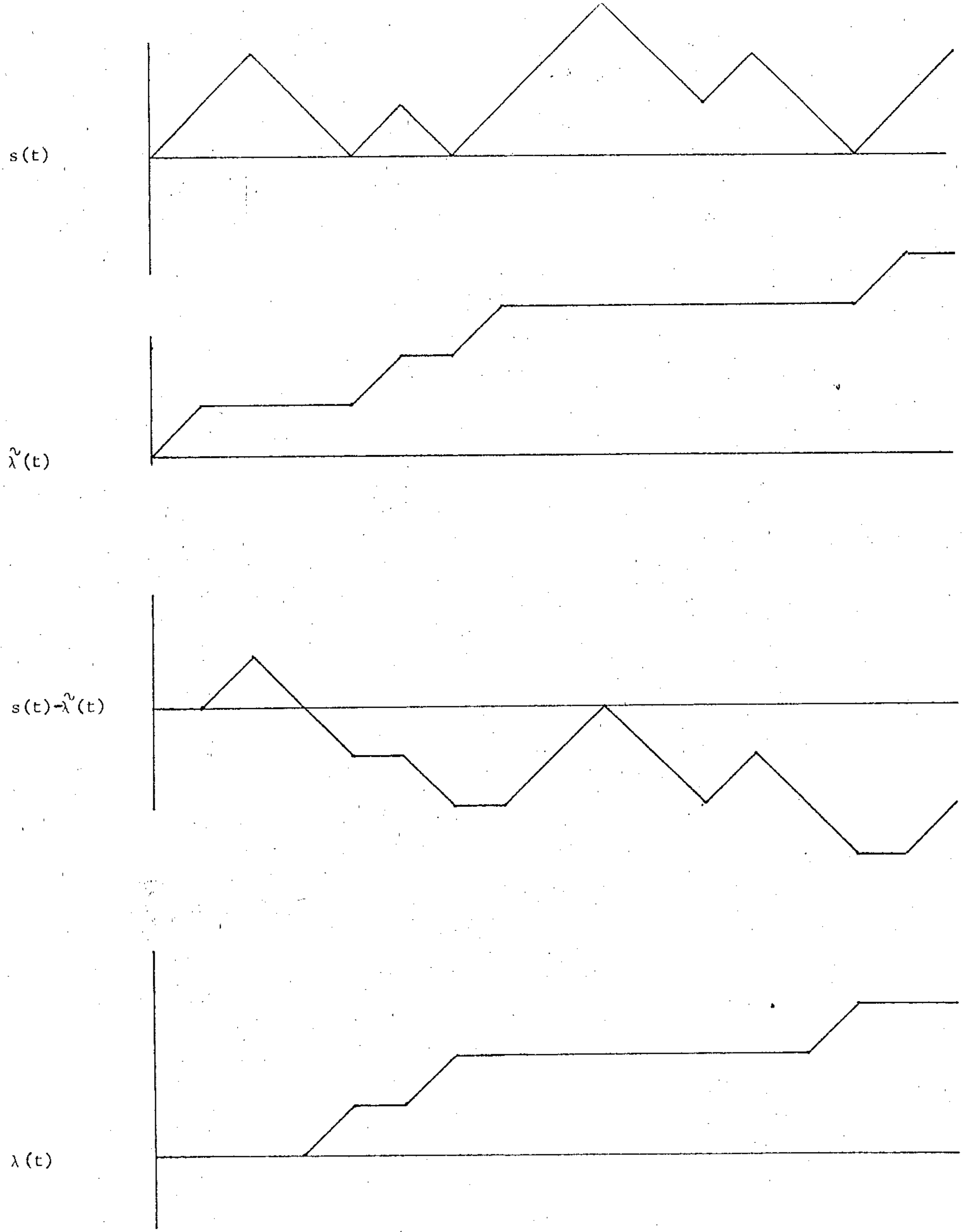
For this, it is enough to prove

(8) $$\big(s_N(\cdot) - \lambda_N(\cdot),\ \lambda_N(\cdot)\big) \Rightarrow \big(|B_{T\wedge\cdot}|-\ell(0,T\wedge\cdot),\ \ell(0,T\wedge\cdot)\big).$$

Just as we defined $\lambda(t)$ to be the piecewise linear interpolation of $L(0,n)$, we define $\tilde{\lambda}(t)$ to be the piecewise linear interpolation of the sequence $\sum_{r=0}^{n-1} I_{\{S_r=0\}}$, and we notice that $|\lambda(t)-\tilde{\lambda}(t)|\leq 1$ for all t (see the picture). Thus $\tilde{\lambda}_N(t) \equiv N^{-1}\ \tilde{\lambda}(N^2 t\wedge\tau_N)$ is uniformly within N^{-1} of $\lambda_N(t)$ so to prove (8) it is sufficient to prove

$$\big(s_N(\cdot)-\tilde{\lambda}_N(\cdot),\lambda_N(\cdot)\big) \Rightarrow \big(|B_{T\wedge\cdot}|-\ell(0,T\wedge\cdot),\ \ell(0,T\wedge\cdot)\big).$$

But notice that $\lambda(t) \equiv \min\{s(u)-\tilde{\lambda}(u);\ u\leq t\}$, so it is sufficient to prove



s(t)
$\tilde{\lambda}(t)$
$s(t)-\tilde{\lambda}(t)$
λ(t)

$$s_N(\cdot) - \tilde{\lambda}_N(\cdot) \Rightarrow |B_{T\wedge\cdot}| - \ell(0,T\wedge\cdot).$$

Thus by Lévy's identification of the laws of $|B_t|-\ell(0,t)$ and B, we must equivalently prove

(9) $$s_N(\cdot)-\tilde{\lambda}_N(\cdot) \Rightarrow B(T'\wedge\cdot),$$

where $T' \equiv \inf\{t;\ B_t = -1\}$.

But $s-\tilde{\lambda}$ is the piecewise linear interpolation of a symmetric simple ranodm walk which is held still for one unit of time immediately after each strict descending ladder epoch (look at the picture!). More explicitly,

$$s(t) - \tilde{\lambda}(t) \equiv \xi\left(t-\tilde{\lambda}(t)\right)$$

defines the piecewise linear interpolation ξ of a symmetric simple random walk. By Donsker's theorem, $\xi_N(\cdot) \equiv N^{-1}\ \xi(N^2) \Rightarrow B$, and

$$s_N(t) - \tilde{\lambda}_N(t) \equiv N^{-1}\ \xi\left(N^2\Phi_N(t)\right) \equiv \xi_N\left(\Phi_N(t)\right),$$

where $\Phi_N(t) \equiv t_{\wedge}(N^{-2}\tau_N) - N^{-2}\ \tilde{\lambda}(N^2 t\wedge\tau_N)$. Now clearly

$$0\le t\wedge(N^{-2}\tau_N) - \Phi_N(t)\le N^{-1}$$

since $\tilde{\lambda}(\cdot\wedge\tau_N)\le N$. Hence one shows easily

$$(\xi_N,\Phi_N) \Rightarrow (B,T'\wedge\cdot)$$

and from this one deduces, following Billingsley [2], p. 145, that

$$\xi_N\circ\Phi_N \equiv s_N-\tilde{\lambda}_N \Rightarrow B(t'\wedge\cdot),$$

which is (9) as required.

(ii) Now we consider the full statement of Theorem 2. If Z_n^N denotes the number of steps up from level j made by the random walk before τ_N,

$$Z_j^N \equiv \sum_{r=0}^{\tau_N-1} I_{\{|S_r|=j,\ |S_{r+1}|=j+1\}} \qquad (j\ge 0)$$

then it follows from Theorem 1 that Z^N is a branching process with $Z_0^N = N$, and offspring generating function ϕ.

Define

$$z_N(x) \equiv N^{-1} Z^N([Nx]) \qquad (x\geq 0).$$

If we set $Z_{-1}^N \equiv N$, then $\ell_N(x) \equiv z_N(x) + z_N(x-N^{-1})$, so Theorem 2 will follow if we can prove

$$\left(s_N(\cdot),\lambda_N(\cdot),2z_N(\cdot)\right) \Rightarrow \left(|B_{T\wedge\cdot}|,\ell(0,T\wedge\cdot),\ell^*(\cdot,T)\right).$$

Now, as we have seen, the laws of (s_N,λ_N) are tight, and so are the laws of z_N, since these are transformed branching processes converging weakly, by Theorem A. Hence the laws of (s_N,λ_N,z_N) are tight, and it is enough to prove that only one limit law is possible.

Now take any smooth $h\colon \mathbb{R}^+ \to [0,1]$ of compact support in $(0,\infty)$ and notice that if we define for $s\in C(\mathbb{R}^+, \mathbb{R}^+)$, $z\in D[0,\infty)$

$$\psi_1(s) \equiv \int_0^\infty h\big(s(t)\big)dt$$

$$\psi_2(z) \equiv \int_0^\infty h(x)z(x)dx,$$

then

$$\psi_1(s_N) = \psi_2(2z_N).$$

I claim this also holds in the limit. In more detail, if μ_N is the law on $C(\mathbb{R}^+, \mathbb{R}^+)^2 \times D[0,\infty)$ of (s_N,λ_N,z_N), and μ is the weak limit of (some subsequence of) the μ_N, then

$$\mu(F) = \mu_N(F) = 1,$$

where $F \equiv \{(s,\lambda,z);\ \psi_1(s) = \psi_2(2z)\}$. Indeed, given $\varepsilon>0$ there exists M so large that for every N

$$\mu_N(A_M)\leq\varepsilon,$$

where $A_M \equiv \{(s,\lambda,z);\ z(t)>M$ for some t, or $s(t)>0$ for some $t>M\}$. Now A_M is open, and on the closed set A_M^c, ψ_1 and ψ_2 are continuous.

Thus $F_M \equiv F \cap \Lambda_M^c$ is a closed set, and for each N

$$\mu_N(F_M) = \mu_N(\Lambda_M^c) \geq 1-\varepsilon$$

Thus $\mu(F_M) \geq \limsup \mu_N(F_M) \geq 1-\varepsilon$, and letting $\varepsilon \downarrow 0$ we deduce $\mu(F)=1$, as claimed.

Thus if μ is a possible limit of the μ_N, by passing to a subsequence if necessary, we may take on some $(\Omega,\mathfrak{F},P)$ random elements (s_N,λ_N,z_N) with laws μ_N converging a.s. to (s,λ,z) with law μ. By part (i) of the proof,

$$(s,\lambda) \overset{\mathcal{D}}{=} \left(|B_{T\wedge\cdot}|, \ell(0,T\wedge\cdot)\right).$$

By Theorem A, 2z has the law of a $BESQ^0$ process started at 2 and by what we have just proved, for any smooth h with compact support in $(0,\infty)$

$$\int_0^\infty h\big(s(t)\big)dt = \int_0^\infty 2h(x)z(x)dx.$$

That $2z(x) = \ell^*(x,\tau)$ follows from the definition of local time as an occupation density, completing the proof of Theorem 2.

3. Convergence of the local time process for individual excursions.

For the present purposes, an <u>excursion</u> is a map $\rho: \mathbb{R}^+ \to \mathbb{R}^+$ which is right continuous with left limits and such that for some $\zeta \in (0,\infty]$, called the <u>lifetime</u> of the excursion ρ, $\rho^{-1}\big((0,\infty)\big) = (0,\zeta)$. Let U denote the space of all excursions; under the Skorokhod topology, U is a Polish space. Let U^c be the subspace of U consisting of continuous excursions.

The Brownian excursion law n is a σ-finite measure on U^c which can be characterised in various ways (see, for example, Williams [15] II.66-67, Ikeda-Watanabe [6] III.4.3, Rogers [14]). An important property is that n-a.e. excursion ρ has a local time process, a continuous map $\ell: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that for bounded Borel f,

$$\int_0^t f(\rho_s)ds = \int_0^\infty \ell(a,t)f(a)da.$$

Abbreviating $\ell(a,\zeta)$ to ℓ_a, the process $(\ell_a)_{a\geq 0}$ takes values in U^c, and its distribution under n is known (see, for example Pitman-Yor [12] Theorems 4.1 and 4.2); explicitly, for $0<t_1<t_2<\ldots<t_n$, with $s_k \equiv t_{k+1}-t_k$,

$$\text{(10)} \qquad n\left(\{\ell_{t_i}\in dx_i; i=1,\ldots,n\}\right) = q_{t_1}(x_1)dx_1 \prod_{i=1}^{n-1} p(s_i;x_i,x_{i+1})dx_{i+1},$$

where $q_t(x)$ is the density of the entrance law,

$$q_t(x) \equiv (2t)^{-2} \exp(-x/2t),$$

and $p(\cdot;\cdot,\cdot)$ is the transition density of a $BESQ^0$ process, characterised by

$$\int_0^\infty p(t;x,y)\, e^{-\alpha y}\, dy = \exp\{-x\alpha/(1+2\alpha t)\}.$$

Let μ denote the law of $(\ell_a)_{a\geq 0}$ under n; that is,

$$\mu\left(\{\rho_{t_i}\in dx_i; i=1,\ldots,n\}\right) = n\left(\{\ell_{t_i}\in dx_i; i=1,\ldots,n\}\right)$$

Now suppose that $(Z_n)_{n\geq 0}$ is a branching process with offspring generating function $\phi(t) \equiv (2-t)^{-1}$ and $Z_0=1$. Define the random elements ℓ^k of U by

$$\ell_t^k \equiv 2k^{-1}\, Z_{[kt]},$$

and let P_k be the law of ℓ^k, a probability measure on U. Defining

$$\mu_k \equiv \tfrac{1}{2}\, k\, P_k,$$

we have the following result.

Theorem 3.

As measures on U, $\mu_k \Rightarrow \mu$.

Remarks (i) This statement must be understood in the following sense. If $U_n \equiv \{\rho\in U;\ \zeta>{}^1/n\}$, then $\mu|_{U_n}$ is a finite measure; by $\mu_k \Rightarrow \mu$ we mean $\mu_k|_{U_n} \Rightarrow \mu|_{U_n}$ as $k\to\infty$ for each n.

(ii) This is a crude definition of weak convergence to a σ-finite limit which it would obviously be difficult to generalise to an arbitrary Polish space U. One can very quickly write down at least five different possible definitions of $\mu_k \Rightarrow \mu$ which agree with the usual definition if μ is finite, and one can almost as quickly find examples to show that the concepts are all different if μ is allowed to be σ-finite. Finding the correct definition (if there is one) is a problem well worth study; in some sense, the law of symmetric simple random walk started at 1 and killed on first reaching 0 must, when suitably transformed, converge to the Brownian excursion law, and one even expects the analogue of Theorem 2 to hold. However, we restrict ourselves for the time being to more modest objectives.

Proof. If Z_n is a critical branching process with the variance of the offspring distribution equal to σ^2, and such that $Z_0=1$, then it is well known (see, for example, Athreya-Ney [1] p.19) that as $n\to\infty$.

$$P(Z_n>0) \sim 2/n\sigma^2.$$

Thus if we fix n, set $\varepsilon \equiv {}^1/n$ and consider some bounded continuous $f\colon U_n \to \mathbb{R}$, then as $k\to\infty$,

$$\int_{U_n} f d\mu_k \sim \frac{1}{2\varepsilon} E\left[f\left(2k^{-1} Z_{[kt]}\right) \middle| Z_{[k\varepsilon]}>0 \right], \tag{11}$$

bearing in mind that $\sigma^2=2$ in this example. Now Durrett [3], p. 813-815, has obtained the limit law of $k^{-1} Z_{[k\cdot]}$ given $Z_k>0$, at least on the interval [0,1]. Modifying his results to the present context, we find, combining with the Lamperti-Lindvall result, that as $k\to\infty$, the law of $2k^{-1} Z_{[k\cdot]}$ given $Z_{[k\varepsilon]}>0$ converges weakly to the law of a continuous inhomogeneous Markov process X_t, governed by the entrance law

$$P(X_t \in dx) = \frac{\varepsilon}{2t^2} e^{-x/2t} h(x,t)\, dx \qquad (0<t\leq\varepsilon), \tag{12}$$

where

$$h(x,t) \equiv 1-\exp\{-x/2(\varepsilon-t)\},$$

and by the transition densities

$$P(X_t \in dy | X_s=x)/dy = p\,(t-s;x,y)\, h(y,t)/h(x,s) \qquad (0<s<t\leq\varepsilon); \tag{13}$$

$$P(X_t \in dy | X_s=x)/dy = p(t-s;x,y) \qquad (\varepsilon\leq s<t). \tag{14}$$

We leave it to the reader to check these calculations. The theorem follows immediately on inspection of (10), (11), (12), (13) and (14).

References

[1] ATHREYA, K.B. and NEY, P.E. Branching Processes. Springer, Berlin, 1972.

[2] BILLINGSLEY, P. Convergence of Probability Measures. Wiley, New York, 1968.

[3] DURRETT, R. Conditioned limit theorems for some null recurrent Markov processes. Ann. Probability 6, 798-828, 1978.

[4] DURRETT, R. and RESNICK, S.I. Functional limit theorems for dependent variables. Ann. Probability 6, 829-846, 1978.

[5] DWASS, M. Branching processes in simple random walk. Proc. Amer. Math. Soc. 51, 270-274, 1975.

[6] IKEDA, N. and WATANABE, S. Stochastic differential equations and diffusion processes. North Holland-Kodansha, Amsterdam and Tokyo, 1981.

[7] JACOD, J. MEMIN, J., and METIVIER, M. Stopping times and tightness. Stoch. Procs. and App. 14, 109-146, 1982.

[8] KNIGHT, F.B. Random walks and a sojourn density of Brownian motion. Trans. Amer. Math. Soc. 109, 56-86, 1963.

[9] LAMPERTI, J. The limit of a sequence of branching processes. Z.f. Wahrscheinlichkeitsth. 7, 271-288, 1967.

[10] LINDVALL, T. Convergence of critical Galton-Watson processes. J. Appl. Probability 9, 445-450, 1972.

[11] PITMAN, J.W. and YOR, M. Bessel processes and infinitely divisible laws. Stochastic Integrals SLN 851, 285-370, Ed. D. Williams. Springer, Berlin, 1981.

[12] PITMAN, J.W. and YOR, M. A decomposition of Bessel bridges. Z.f. Wahrscheinlichkeitsth. 59, 425-457, 1982.

[13] RAY, D.B. Sojourn times of diffusion processes. Ill. J. Math. 7, 615-630, 1963.

[14] ROGERS, L.C.G. Williams' characterisation of the Brownian excursion law; proof and applications. Sem. Prob. XV 227-250, Springer, Berlin, 1981.

[15] WILLIAMS, D. Diffusions, Markov Processes, and Martingales. Vol. I. Wiley, Chichester 1979.

[16] YAMADA, T. and WATANABE, S. On the uniqueness of solutions of stochastic differential equations. J. Math. Kyoto Univ. 11, 155-167, 1971.