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Williams' characterisation of the Brownian excursion law: proof and applications

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1. Introduction.

Let $\Omega^o = \{$continuous functions from $[0,\infty)$ to $\mathbb{R}\}$, let $X_t : \Omega^o \to \mathbb{R}$ be the mapping $\omega \mapsto \omega(t)$, let $\mathcal{F}_t^o \equiv \sigma(\{X_s ; 0 \le s \le t\})$ with $\mathcal{F}^o \equiv \sigma(\{X_s ; s \ge 0\})$, and let $M_t(\omega) \equiv \max\{X_s(\omega) ; 0 \le s \le t\}$.

Let P be Wiener measure on $(\Omega^o, \mathcal{F}^o)$; then $P(X_o = 0) = 1$, and there exists $\Omega \in \mathcal{F}^o$ with $P(\Omega) = 1$ and such that for all $\omega \in \Omega$, for all $t \ge 0$, the limit

(1) $$\lim_{\varepsilon \downarrow 0} (\tfrac{1}{2}\pi\varepsilon)^{\frac{1}{2}} N(t,\varepsilon,\omega) \equiv L_t(\omega)$$

exists, defining a continuous function $L_t(\omega)$ of t. Here, $N(t,\varepsilon,\omega)$ is the number of $I_k(\omega)$ contained in $(0,t)$ and of length at least ε, where

$$K(\omega) \equiv \{t ; \omega(t) \neq 0\} = \bigcup_{k=1}^{\infty} I_k(\omega)$$

is a representation of the set $K(\omega)$ as a disjoint countable union of open intervals (for the existence of such an Ω, and other properties of L see, for example, Williams [8]). Henceforth we restrict our σ-fields $\mathcal{F}^o$, $\mathcal{F}_t^o$ to Ω, writing the restrictions as $\mathcal{F}$, $\mathcal{F}_t$.

The normalisation of local time we have made here has been chosen so that the remarkable distributional identity:

(2) $$(|X_t|, L_t) \overset{\mathcal{D}}{=} (M_t - X_t, M_t)$$

is valid.

In recent years, a number of papers have appeared dealing with the distributions of T, M_T and X_T, where T is an $(\mathcal{F}_t)$-optional time of the form

$$T \equiv \inf\{t\,;\ (B_t, M_t) \in A\}$$

for some (closed) subset A of $\mathbb{R}^2$. (See Azéma-Yor [1], Jeulin-Yor [3], Knight [4], Lehoczky [5], Taylor [7], and Williams [9]). Various approaches have been adopted by these authors; the aim of this paper is to show that Itô's excursion theory provides a natural setting for these problems, and that the explicit characterisation of the Brownian excursion law due to Williams [10] turns this natural way of considering the problems into a powerful method for solving them. No proof of this characterisation of the Brownian excursion law has yet appeared, so we devote section 3 of this paper to a proof using the path decompositions of Williams. In section 2 we see how the Azéma-Yor proof of the Skorokhod embedding theorem can be quickly established using ideas from excursion theory, and finally in section 4 we use the result of section 3 to solve the problem dealt with by Jeulin and Yor of finding a method of calculating

$$E\exp\{-a(X_S, L_S) - \int_0^{G_S} b(X_t, L_t)\,dt - \int_{G_S}^{S} c(X_t, L_t)\,dt\},$$

where a, b, c are any measurable functions from $\mathbb{R}^2$ to $\mathbb{R}^+$,

(3) $$G_t \equiv \sup\{s < t\,;\ X_s = 0\},$$

and

(4) $$S \equiv \inf\{t\,;\ h(L_t)X_t^+ + k(L_t)X_t^- = 1\};$$

here, $h, k : \mathbb{R}^+ \to \mathbb{R}^+$ are measurable, and $X_t^+ \equiv (X_t) \vee 0 \equiv X_t + X_t^-$.

We conclude this section by setting up the notation to be used for the rest of the paper.

Let $U^+ \equiv \{f \in \Omega^0\,;\ \exists\ 0 < \zeta < \infty$ with $f(t) > 0$ on $(0,\zeta)$, $f(t) = 0$ otherwise$\}$,

$U^- \equiv \{f \in \Omega^0\,;\ -f \in U^+\}$,

$U \equiv U^+ \cup U^-$.

For $f \in U$, let $\zeta(f) \equiv \sup\{t\,;\ f(t) \neq 0\}$,

$$m(f) \equiv \begin{cases} \max\{f(t)\,;\ t \geq 0\} & \text{if } f \in U^+ \\ \min\{f(t)\,;\ t \geq 0\} & \text{if } f \in U^-. \end{cases}$$

Equipping U with the topology of uniform convergence on compact sets makes U into a Polish space; let $\mathcal{U}$ denote its Borel σ-field.

Now it is a central idea of the historic paper by Itô [2] that there exists a σ-finite measure n on U, satisfying

$$(5) \qquad \int_U n(df)[1-\exp(-\zeta(f))] < \infty,$$

such that, from a Poisson process on $\mathbb{R}^+ \times U$ with measure $dt \times dn$ one can synthesize the original process X, and, conversely, by breaking the set $K(\omega)$ into its components $I_k(\omega)$ and considering the excursions of X during these intervals, one can construct a Poisson process on $\mathbb{R}^+ \times U$. In more detail, if $\omega \in \Omega$, define for each $t > 0$

$$(6) \qquad \sigma_t(\omega) \equiv \inf\{u \; ; \; L_u(\omega) > t\},$$

and use $J(\omega)$ to denote the (countable) set of discontinuities of $t \mapsto \sigma_t(\omega)$. For $t \in J(\omega)$, let f_t denote the element of U defined by

$$\begin{aligned} f_t(s) &= X(\sigma_{t-} + s) && 0 \le s \le \sigma_t - \sigma_{t-} \\ &= 0 && \text{otherwise.} \end{aligned}$$

Then $\{(t,f_t) \; ; \; t \in J(\omega)\}$ is a realisation of a Poisson point process on $\mathbb{R}^+ \times U$ with measure $dt \times dn$; in particular, defining for each measurable subset A of the Polish space $\mathbb{R}^+ \times U$, the random variable:

$$N(A) \equiv \text{number of } t \in J(\omega) \text{ for which } (t,f_t) \in A,$$

then if $A_1,\ldots,A_k$ are disjoint, $N(A_1),\ldots,N(A_k)$ are independent Poisson random variables with parameters:

$$E\, N(A_i) = \int_{A_i} dt \times dn.$$

In what follows, we will freely switch from considering the process X as a continuous function of real time to considering it as a point process in local time.

2. The Skorokhod embedding theorem.

We begin this section with a simple lemma, which can be deduced from Williams' characterisation of n, the Brownian excursion law, but which we here prove directly.

Lemma 2.1.

$$n(\{f \in U \; ; \; |m(f)| > x\}) = x^{-1} \quad \text{for each} \quad x > 0.$$

Proof.

Bearing in mind that $(|X_t|, L_t) \overset{\mathcal{D}}{=} (M_t - X_t, M_t)$, and fixing $x > 0$, we see that if

$$\rho \equiv \inf\{s \; ; \; M_s - X_s > x\},$$

then M_ρ is exponentially distributed with rate $n(\{f \; ; \; |m(f)| > x\})$. An application of Itô's formula tells us that for each $\theta > 0$,

$$Z_t^\theta \equiv e^{-\theta M_t}(M_t - B_t + \theta^{-1}) \quad \text{is a local martingale.}$$

But Z^θ is bounded on $[0,\rho]$, and using the optional sampling theorem at ρ proves that M_ρ is exponential, rate x^{-1}.

Now let μ be a probability measure on $\mathbb{R}$ satisfying

$$\int_{\mathbb{R}} |t| \; \mu(dt) < \infty, \quad \int_{\mathbb{R}} t \; \mu(dt) = 0.$$

Azéma and Yor define a left continuous non-negative increasing function $\Psi : \mathbb{R} \to \mathbb{R}^+$ by

$$\text{(7)} \qquad \Psi(x) = \bar{\mu}(x)^{-1} \int_{[x,\infty)} t \, \mu(dt) \qquad \text{if} \quad \bar{\mu}(x) > 0$$

$$= x \qquad \text{if} \quad \bar{\mu}(x) = 0,$$

where $\bar{\mu}(x) \equiv \mu([x,\infty))$; they remark that $\Psi(x) \geq x \; \forall x$, $\Psi(x) = x \Rightarrow \Psi(y) = y \quad \forall y \geq x$, and $\lim_{x \to -\infty} \Psi(x) = 0$.

Now define

$$\text{(8)} \qquad T \equiv \inf\{t \; ; \; M_t \geq \Psi(X_t)\}.$$

Theorem (Skorokhod; Azéma-Yor).

The optional time T is finite a.s., and the law of X_T is μ. Moreover, if μ possesses a finite second moment, then $E\,T = \int t^2\,\mu(dt)$.

Proof.

We leave the proof of the last assertion aside until Section 4.

Define the right continuous inverse Φ to Ψ by

$$\Phi(x) \equiv \inf\{y\,;\ \Psi(y) > x\},$$

and notice that, with this definition,

$$T = \inf\{t\,;\ \Phi(M_t) \geq X_t\}.$$

Since $X_T = \Phi(M_T)$ when $T < \infty$, it is enough to find the law of M_T. We make the convention that $M_T = \infty$ if $T = \infty$.

Now look at Fig. 1 and think in terms of excursions. A sample path of (X_t, M_t) in $\mathbb{R}^2$ consists of a (countable) family of horizontal "spikes" (corresponding to excursions of X below its maximum) with their right-hand end-points on the line $x = y$. The time T occurs when one of these spikes goes far enough to the left to enter the shaded set, $\{(x,y)\,;\ y \geq \Phi(x)\}$. If

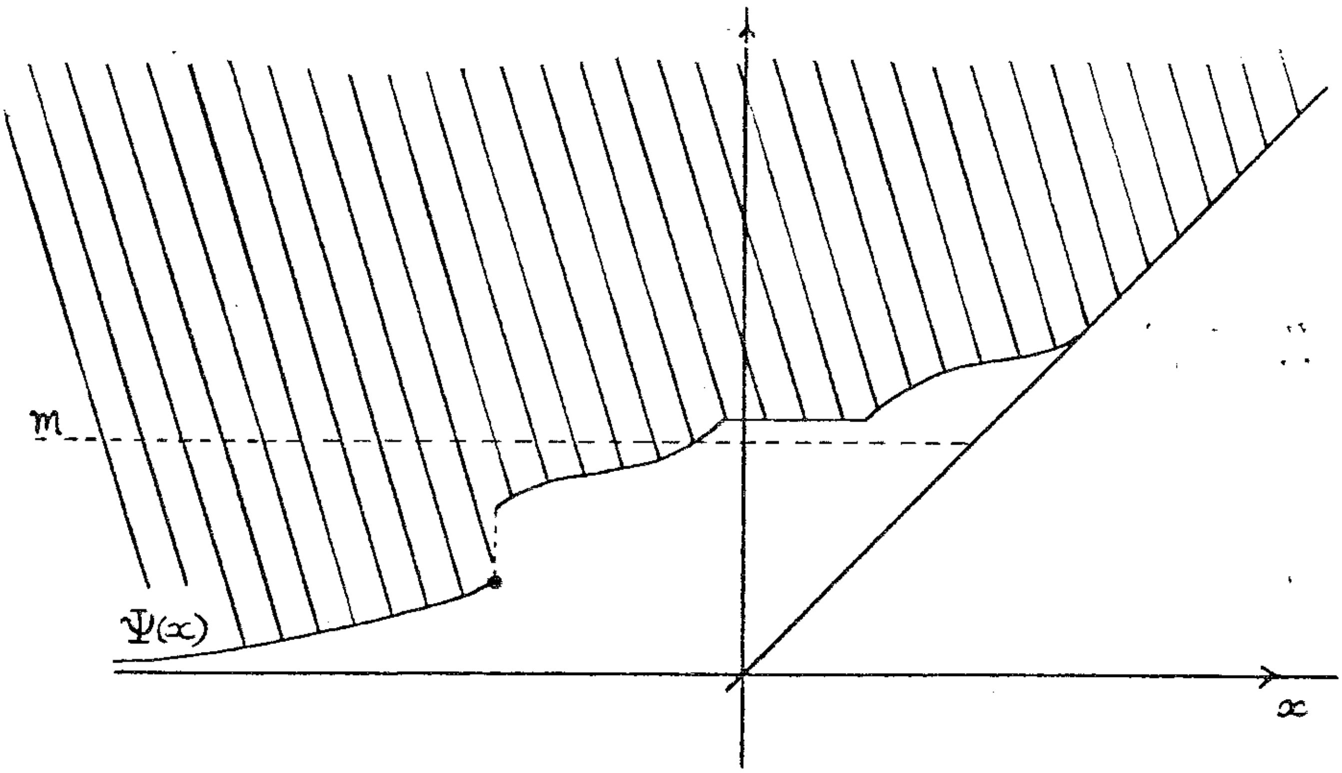

Fig. 1.

we fix $m > 0$, then $M_T \geq m \iff$ no excursion of $M - X$ during the local time interval $[0,m)$ has maximum greater than or equal to $u - \Phi(u)$, where u is the local time at which the excursion occurs. But this latter event occurs iff the Poisson process of excursions puts no point into the set

$$D \equiv \{(u,f)\ ;\ 0 \leq u < m,\ |m(f)| \geq u - \Phi(u)\}.$$

Now the number of excursions in D is a Poisson random variable with mean

$$\int_D dt \times dn = \int_0^m dt\ (t - \Phi(t))^{-1},$$

by Lemma 2.1. So

$$\begin{aligned} P(M_T \geq m) &= P\ (\text{no excursions in } D) \\ &= \exp[-\int_0^m dt\ (t - \Phi(t))^{-1}]. \end{aligned} \tag{9}$$

If we make the simplifying assumption that Ψ is continuous and strictly increasing, then, as $X_T = \Phi(M_T)$ when $T < \infty$, for $x < \sup\{t\ ;\ \bar{\mu}(t) > 0\}$,

$$P(T = \infty, \text{ or } X_T > x) = P(M_T > \Phi(x))$$

$$= \exp[-\int_{-\infty}^x \frac{\Psi(ds)}{\Psi(s) - s}]. \tag{10}$$

But, by the definition (7) of Ψ, for $s < \sup\{t\ ;\ \bar{\mu}(t) > 0\}$,

$$\Psi(ds) = \frac{\mu(ds)}{\bar{\mu}(s)}\ (\Psi(s) - s), \tag{11}$$

which we put into (10) and deduce that for $x < \sup\{t\ ;\ \bar{\mu}(t) > 0\}$,

$$P(T = \infty, \text{ or } X_T > x) = \bar{\mu}(x).$$

Now let $x \uparrow \sup\{t\ ;\ \bar{\mu}(t) > 0\}$ to learn that $P(T = \infty) = 0$, and $P(X_T > x) = \bar{\mu}(x)$.

To handle general Ψ, the jumps of Ψ must be accounted for separately from the continuous part. The details are not difficult, and are left to the reader.

Pierre [6] gives a proof of this point in the spirit of the original paper by Azéma and Yor.

3. Williams' characterisation of the Brownian excursion law.

Informally, Williams' [10] characterisation of the Brownian excursion law says this; pick the maximum of the excursion according to the "density" $x^{-2}\,dx$, and then make up the excursion by running an independent $BES^0(3)$ process until it reaches the maximum, and then run a second (independent) $BES^0(3)$ process down from the maximum until it hits zero. We shall here treat the excursion measure of $(X_t)_{t\geq 0}$, which is only trivially different from the case treated by Williams, that of the excursion measure of $(|X_t|)_{t\geq 0}$.

In more detail, set up on a suitable probability triple $(\Omega', \mathcal{F}', P')$ the independent processes

(i) $(R_t)_{t\geq 0}$, a $BES^0(3)$ process,

(ii) $(\tilde{R}_t)_{t\geq 0}$, another $BES^0(3)$ process.

Define for each $x > 0$ $\tau_x(R) \equiv \inf\{s\,;\ R_s > x\}$, $\tau_x(\tilde{R}) \equiv \inf\{s\,;\ \tilde{R}_s > x\}$. Now for each $x > 0$ define the process $(Z^x_t)_{t\geq 0}$ by

$$Z^x_t \equiv \begin{cases} R_t & , \quad 0 \leq t \leq \tau_x(R), \\ x - \tilde{R}(t - \tau_x(R)), & \tau_x(R) \leq t \leq \tau_x(R) + \tau_x(\tilde{R}), \\ 0 & , \quad \tau_x(R) + \tau_x(\tilde{R}) \leq t. \end{cases}$$

For $x < 0$, set $Z^x_t \equiv -Z^{-x}_t$ $(t \geq 0)$, and define the kernel $(n|m)(\cdot,\cdot)$ from $\mathbb{R}\backslash\{0\}$ to U by

$$(n|m)(x,A) = P'(Z^x_\cdot \in A) \qquad (x \in \mathbb{R}\backslash\{0\},\ A \in \mathcal{U}).$$

(It is plain that for each x, $(n|m)(x,\cdot)$ is a probability measure on $(U,\mathcal{U})$ and to prove the measurability of $(n|m)(\cdot,A)$, notice that $(Z^x_t)_{t\geq 0} \overset{\mathcal{D}}{=} (x\, Z^1_{t x^{-2}})_{t\geq 0}$ so if $\Phi : U \to \mathbb{R}$ is bounded continuous, the map $x \mapsto \int (n|m)(x,df)\,\Phi(f)$ is continuous, and measurability of $(n|m)(\cdot,A)$ follows by a standard monotone class argument). The kernel $(n|m)$ provides a regular conditional n-distribution for the excursion given its maximum.

Theorem 3.1. (Williams)

The Brownian excursion law is the σ-finite measure n on $(U,\mathcal{U})$ defined by

$$n(A) = \tfrac{1}{2}\int_{\mathbb{R}\setminus\{0\}} x^{-2}(n|m)(x,A)\,dx$$

(12)

$$= \int_{\mathbb{R}\setminus\{0\}} n\circ m^{-1}(dx)(n|m)(x,A).$$

The rest of this section is devoted to the proof.

We begin by reviewing briefly some results on random measures which we shall use. Let $\mathcal{M}$ denote the set of σ-finite measures ν on $(\mathbb{R}^+\times U,\ \mathcal{B}(\mathbb{R}^+)\times\mathcal{U})$ with values in $\mathbb{Z}^+\cup\{\infty\}$ satisfying the condition

(13) $$\nu(\{t\}\times U) \le 1 \quad \forall\, t\ge 0.$$

We equip $\mathcal{M}$ with the smallest σ-field $\mathcal{B}(\mathcal{M})$ for which all the maps

$$\nu \mapsto \nu(E) \qquad (E\in\mathcal{B}(\mathbb{R}^+)\times\mathcal{U})$$

are measurable. There is a natural 1-1 correspondence between $\mathcal{M}$ and the space of point functions considered by Itô [2]. We shall if need be phrase statements in point function language, but generally the statements in terms of $\mathcal{M}$ are cleaner. For each $t\ge 0$ define the map $\theta_t : \mathbb{R}^+\times U \to \mathbb{R}^+\times U$ by $\theta_t(s,f) = (t+s,f)$. A random measure is a random element N of $\mathcal{M}$; we say N is renewal if for all $t\ge 0$, $N\circ\theta_t^{-1}$ is independent of the restriction of N to $[0,t)\times U$ and has the same law as N. Itô proved that every renewal random measure for which the measure $A\mapsto EN(A)$ is σ-finite is a Poisson random measure, and conversely (a random measure N is a Poisson random measure if there exists a σ-finite measure λ - the characteristic measure - on $(U,\mathcal{U})$ such that

(i) $N(A)$ is Poisson with mean $\int_A dt\times d\lambda$, $A\in\mathcal{B}(\mathbb{R}^+)\times U$;

(ii) if $A_1,\dots,A_k$ are disjoint measurable subsets of $\mathbb{R}^+\times U$, then $N(A_1),\dots,N(A_k)$ are independent.).

We now give a careful construction of the map $\Phi : (\Omega,\mathcal{F}) \to (\mathcal{M},\mathcal{B}(\mathcal{M}))$

which was outlined in the Introduction. Fix $n \in \mathbb{N}$, and consider the $(\mathcal{F}_t)$-optional times

$$\rho_o \equiv 0, \ \rho_{k+1} \equiv \inf\{t > \sigma_k \ ; \ |X_t| = n^{-1}\}$$

(14) $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $(k = 0,1,2,\ldots)$

$$\sigma_o \equiv 0, \ \sigma_{k+1} \equiv \inf\{t > \rho_{k+1} \ ; \ X_t = 0\}.$$

The map $\Phi_n : (\Omega,\mathcal{F}) \to (M,\mathcal{B}(M))$ takes ω to the measure which puts mass 1 on each of the points (ℓ_k, f_k), $k = 1,2,\ldots,$ where

(15)
$$\begin{aligned} \ell_k &\equiv L_{\rho_k} \\ f_k(t) &\equiv \omega(t+\eta_k) \qquad & 0 \le t \le \sigma_k - \eta_k \\ &\equiv 0 & t \ge \sigma_k - \eta_k, \end{aligned}$$

using η_k to denote $\sup\{t < \rho_k \ ; \ X_t = 0\}$.

The measure $\Phi(\omega)$ is defined by

(16) $$\Phi(\omega)(A) = \lim \Phi_n(\omega)(A), \qquad A \in \mathcal{B}(\mathbb{R}^+) \times U.$$

Proposition 3.2.

The map $\Phi : (\Omega,\mathcal{F}) \to (M,\mathcal{B}(M))$ is measurable.

Proof.

It is plainly enough to establish measurability of each Φ_n, and to prove that each Φ_n maps into M. The latter follows from the fact that the set of points of increase of $L_\cdot(\omega)$ is the zero set of $X_\cdot(\omega)$ for all $\omega \in \Omega$, and to prove the former, it is enough to prove that the probability measure putting mass 1 at the point (ℓ_k, f_k) is measurable. The σ-field on $\mathbb{R}^+ \times U$ is the product σ-field, so it is enough to prove measurability of ℓ_k, f_k separately. Measurability of ℓ_k is immediate; as for f_k, if we fix $a > 0$ and $t > 0$ and note that

$$\{f_k(t) > a\} = \bigcup_{m=1}^{\infty} \bigcup_{r=1}^{\infty} \bigcup_{j=1}^{\infty} \bigcup_{s=1}^{\infty} \bigcap_{\rho \in \mathbb{Q}} A_{mrjs\rho},$$

where

$$A_{mrjs\rho} = \Omega \qquad \text{if } \rho \notin [j\,2^{-r}+t,\ (j+1)2^{-r}+t]$$

$$= \{\omega(\rho) > a + m^{-1},\ \eta_k \in [j\,2^{-r}, (j+1)2^{-r}),\ \sup\{\omega(x)\,;\, j\,2^{-r} \le x < (j+1)2^{-r}\} < n^{-1}$$

$$\text{and } \inf\{\omega(x)\,;\,(j+1)2^{-r} \le x \le (j+1)2^{-r}+t\} > s^{-1}\} \quad \text{otherwise,}$$

is in $\mathcal{F}$, then this proves f_k to be measurable.

Remarks.

(a) By the properties of L, it is easy to see that $\Phi(\omega) \in \mathcal{M}$ always satisfies the condition

(17) $\sigma_t \equiv \int_{[0,t]\times U} \Phi(\omega)(ds,df)\ \zeta(f)$ is a strictly increasing finite-valued function of t.

Later in this section we shall give a sort of converse to Proposition 3.2 which did not appear in Itô's paper, though it is obviously very close to what Itô did prove; we shall prove that there is a subset $\mathcal{M}_o$ of $\mathcal{M}$ and a measurable function $\Psi : \mathcal{M}_o \to \Omega$ such that $\Phi(\Omega) \subseteq \mathcal{M}_o$, and $\Psi \circ \Phi(\omega) = \omega$ for all $\omega \in \Omega$. In other words, not only is it true (as Itô proved) that the Brownian path can be decomposed into its excursions from zero, but also, given a Poisson process of excursions with measure n, one can synthesize a Brownian motion from them.

(b) By the strong Markov property of X and the fact that the points of increase of L form the zero set of X, the random measure $\Phi(X)$ is renewal, and the σ-finiteness of $A \mapsto E\,\Phi(X)(A)$ is immediate, so $\Phi(X)$ is a Poisson random measure. Thus the existence of the Brownian excursion law n is not in question - nor is its uniqueness!

As stated in the Introduction, we are going to use the path decompositions of Williams [8] to prove the characterisation of n, which will follow from Theorem 3.4 and Proposition 3.3. Firstly, we give an obvious characterisation of n which we shall prove equivalent to the result stated.

Proposition 3.3.

Fix $a > 0$, and set up on a suitable probability triple the independent processes

(i) $(B_t)_{t\geq 0}$, a Brownian motion started at 0;

(ii) $(\tilde{B}_t)_{t\geq 0}$, a Brownian motion started at a and stopped when it hits 0.

Let $\tau \equiv \inf\{u\ ;\ B_u = a\}$, $\eta \equiv \sup\{t < \tau\ ;\ B_t = 0\}$ and define the process Z by

$$
\begin{aligned}
Z_t &= B_{\eta+t} && 0 \leq t \leq \tau-\eta \\
&= \tilde{B}_{t-\tau+\eta} && \tau-\eta \leq t.
\end{aligned}
$$

The process Z is a random element of U, whose law is the restriction of n to

$$U_a \equiv \{f \in U\ ;\ m(f) \geq a\},$$

normalised to be a probability measure.

Proof.

If we restrict $\Phi(X)$ to $\mathbb{R}^+ \times U_a$, we observe a discrete Poisson process whose points come at rate $n(U_a)$ and are i.i.d. with law $n(U_a)^{-1}.n$. The law of Z is nothing other than the law of the first excursion of X with maximum greater than or equal to a.

We now turn to the path decompositions of Williams [8]. The following result is a slight extension of Theorem 2.4 in that paper.

Theorem 3.4 (Williams).

Let $\{X_t\ ;\ 0 \leq t < \zeta\}$ be a regular diffusion on (A,B) with infinitesimal generator $\mathcal{G}$ satisfying the conditions

(i) $X_o = b \in (A,B)$;

(ii) the scale function s of X satisfies

$$s(A) = -\infty,\ s(B) < \infty;$$

(iii) $\zeta = \inf\{t\ ;\ X_{t-} = B\}$ a.s..

Then, defining

(18) $$\gamma \equiv \inf\{X_t\ ;\ 0 \le t < \zeta\},$$

there exists a.s. a unique ρ such that $X_\rho = \gamma$. The law of γ is

(19) $$P(\gamma < x) = \frac{s(B) - s(b)}{s(B) - s(x)} \qquad (x \le b)$$

and conditional on γ, the processes $\{X_t\ ;\ 0 \le t \le \rho\}$ and $\{X_{t+\rho}\ ;\ 0 \le t < \zeta - \rho\}$ are independent; the law of the pre-ρ process is that of a diffusion in (A,B) with generator

(20) $$[s(B) - s]^{-1}\mathcal{G}[s(B) - s]$$

started at b and stopped at γ, and the law of the post-ρ process is that of a diffusion in $[\gamma,B)$ started at γ and killed at B, with generator

(21) $$[s - s(\gamma)]^{-1}\mathcal{G}[s - s(\gamma)].$$

Finally, if $\sigma \equiv \sup\{s\ ;\ X_s = b\}$, then the process $\{X_{t+\sigma}\ ;\ 0 \le t < \zeta - \sigma\}$ has the same distribution as a diffusion in $[b,B)$ with generator

(22) $$[s - s(b)]^{-1}\mathcal{G}[s - s(b)]$$

started at b and killed at B.

Let us apply this result to the case of interest where $A = -\infty$, $B = 0$, $b = -a < 0$, and the diffusion X is Brownian motion started at b and killed on reaching 0. The scale function of X is the identity map, and the generator is $\frac{1}{2}\frac{d^2}{dx^2}$ on $C_k^2(-\infty,0)$. We can now read off the decomposition at the minimum of X from (19), (20), and (21);

(23) $$P(\inf X_t < -x) = a/x;$$

(24) $\{X_t\ ;\ 0 \le t \le \rho\}$ has generator $\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx}$, so $\{-X_t\ ;\ 0 \le t \le \rho\}$

is a $BES^a(3)$ process, stopped on reaching $-\gamma$;

(25) $\{X_{t+\rho}\,;\ 0 \le t < \zeta - \rho\}$ has generator $\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{x-\gamma}\frac{d}{dx}$, so the process

$\{X_{t+\rho} - \gamma\,;\ 0 \le t < \zeta-\rho\}$ is a $BES^0(3)$, stopped on reaching $-\gamma$.

Finally, we can read off from (22) what the law of

$$\{X_{t+\sigma}\,;\ 0 \le t < \zeta - \sigma\}$$

will be; the same argument proves that

(26)
$$\{a + X_{t+\sigma}\,;\ 0 \le t < \zeta - \sigma\}$$

is a $BES^0(3)$ process, stopped on first reaching a.

Now we can use these path decompositions and Proposition 3.3 to finish off the proof of Theorem 3.1. From Proposition 3.3, the piece of the path of Z up to the first hit on a is just a Brownian motion from its last hit on zero before its first hit on a, and this, by (26), is a $BES^0(3)$ run until it first hits a. The path of Z from its first hit on a now splits, by (23), (24) and (25), into

(27) a $BES^a(3)$ run until it first hits γ;

(28) a $BES^0(3)$ run down from γ until it hits zero,

independently of the path of Z up to the first hit on a. The law of γ is given by (23). The path of Z is now in three pieces; the last piece, (28), is what we said it would be, and the first two pieces, the path up to the first hit on a, together with (27), can now be assembled to make a $BES^0(3)$ run until it hits γ, since the two pieces are independent. This completes the proof of Theorem 3.1.

Notice that Williams deduced Theorem 3.4 from his path decomposition of $BES^b(3)$ process by change of scale and speed, which transforms the diffusion to the most general possible regular diffusion. We have taken this general result and applied it to the particular example, though it is equally possible to calculate the changes of scale and speed which transform $BES^b(3)$ into BM^{-a}. The result is the same; the diagram commutes!

As stated earlier, we return to the question of finding an inverse to the mapping Φ. The natural thing to do is to define for $\nu \in \mathcal{M}$, $t \geq 0$,

(29) $$\sigma_t \equiv \int_{[0,t]\times U} \nu(ds,df)\ \zeta(f),$$

and

(30) $$L_t \equiv \inf\{u\ ;\ \sigma_u > t\},$$

and then define the function $\Psi(\nu) : \mathbb{R}^+ \to \mathbb{R}$ by

(31) $$\begin{aligned} \Psi(\nu)(t) &= 0 && \text{if } \sigma(L_t) = \sigma(L_t-) \\ &= f(t-\sigma(L_t-)) && \text{if } \sigma(L_t) > \sigma(L_t-), \\ & && \text{where } \nu((L_t,f)) = 1. \end{aligned}$$

The problem is that the function $\Psi(\nu)$ thus defined <u>may not be continuous</u>; the solution is to restrict the set on which Ψ is defined, but we must be sure to choose the restricted domain big enough to catch all (or almost all) the $\Phi(\omega)$.

Define for $k \in \mathbb{N}$

$$U_k \equiv \{f \in U\ ;\ |m(f)| \geq \tfrac{1}{k}\}$$

$$V_k \equiv U_k \backslash U_{k-1}, \text{ with } U_0 \equiv \phi.$$

By Theorem 3.1, we know the characteristic measure n of $\Phi(X)$, and it is clear from this description of n, and the Laplace transform of the $BES^0(3)$ first passage times that for $\theta > 0$, $x \neq 0$,

(32) $$n(e^{-\frac{1}{2}\theta^2\zeta(f)}\ \big|\, m(f) = x) = (\theta x \operatorname{cosech} \theta x)^2,$$

where this equation is to be understood in the sense of regular conditional distributions. Taking $\theta = x^{-1-\varepsilon}$, trivial estimation yields for each $\varepsilon > 0$

$$n(\zeta(f).|m(f)|^{-2-2\varepsilon} < 1 \mid |m(f)| = x) \le e^{\frac{1}{2}}(x^{-\varepsilon}\operatorname{cosech} x^{-\varepsilon})^2.$$

Now consider what happens to the process $\Phi(X)|_{V_k}$; since $n(V_k) = 1$ for all k,

$$\begin{aligned} P(\exists(s,f) \in [0,t]\times V_k \text{ with } \zeta(f) < |m(f)|^{2+2\varepsilon}) &\le 1 - \exp(-t\, e^{\frac{1}{2}}(k^{\varepsilon}\operatorname{cosech} k^{\varepsilon})^2) \\ &\le 2t\, e^{\frac{1}{2}}(k^{\varepsilon}\operatorname{cosech} k^{\varepsilon})^2 \end{aligned}$$

for k large enough.

By Borel-Cantelli, we deduce that, for each $0 < t < \infty$ P-almost surely there exists a constant $K(t)$ such that

$$\Phi(X)(\{(s,f)\, ;\ 0 \le s \le t,\ \zeta(f)\ K(t) < |m(f)|^{2+2\varepsilon}\}) = 0.$$

So we define M_o to be the set of $\nu \in M$ for which the following two conditions hold:

(i) $t \mapsto \sigma_t$ is finite and strictly increasing (σ_t defined at (29));

(ii) for each $t > 0$, $\exists\, K(t) \in (0,\infty)$ with

$$\nu(\{(s,f)\, ;\ 0 \le s \le t,\ \zeta(f)\ K(t) < |m(f)|^{2+2\varepsilon}\}) = 0, \quad \text{each } \varepsilon > 0.$$

With ν restricted to lie in M_o, definition (31) makes sense; the function $\Psi(\nu)$ is continuous. (Indeed, continuity in the open intervals where L is constant is immediate, and, at the end points, the fact that $|m(f)|^{2+2\varepsilon}$ is dominated by a multiple of $\zeta(f)$ for all f implies continuity). An argument similar to that used in Proposition 3.2 proves that $\Psi : M_o \to \Omega^o$ is measurable; the details are left to the reader. The final observation is that for $\omega \in \Omega$, $\Psi \circ \Phi(\omega) = \omega$, since, by the construction of Φ, the function σ defined at (29) is the right continuous inverse to Brownian local time. We conclude that any Poisson process with characteristic measure n maps under Ψ to Brownian motion, which is the converse to Theorem 3.1.

4. Functionals of the Brownian path

Recall the notation of the Introduction; $a,b,c : \mathbb{R}^2 \to \mathbb{R}^+$ and $h,k : \mathbb{R}^+ \to \mathbb{R}^+$ are fixed measurable functions, and for each $t \geq 0$, $G_t \equiv \sup\{s < t;\, X_s = 0\}$. Define the optional time

$$T \equiv \inf\{t;\, h(L_t)X_t^+ + k(L_t)X_t^- = 1\}\,, \tag{33}$$

the random variables

$$\begin{aligned} Y_1 &\equiv \exp\{-a(X_T,L_T) - \int_{G_T}^{T} c(X_s,L_s)ds\}\,, \\ Y_2 &\equiv \exp\{-\int_0^{G_T} b(X_s,L_s)ds\}\,, \end{aligned} \tag{34}$$

and consider the expected value of

$$Y \equiv Y_1 Y_2\,.$$

We impose the condition

$$\lim_{t\to\infty}\int_0^t \{h(x)+k(x)\}dx = \infty\,, \tag{35}$$

whose interpretation will become obvious shortly.

Let $(R_t)_{t\geq 0}$ be a $\mathrm{BES}^{\circ}(3)$ process with first hitting times $\{\tau_x(R);\, x \geq 0\}$, and define the measurable functions $\beta,\gamma : \mathbb{R}\times\mathbb{R}^+ \to [0,1]$ by

$$\begin{aligned} \beta(x,\ell) &= E\exp[-\int_0^{\tau_x(R)} b(R_s,\ell)ds] \quad \text{if } x \geq 0 \\ &= E\exp[-\int_0^{\tau_{-x}(R)} b(-R_s,\ell)ds] \quad \text{if } x < 0\,, \end{aligned} \tag{36}$$

with γ defined similarly, replacing b with c.

In this section, we shall incline to the point function description of Poisson random measures (equivalently, Poisson point processes), since this accords more directly with intuition, though, as remarked before, the two are equivalent.

Now let $N \equiv \{N(F);\, F \in \mathcal{B}(\mathbb{R}^+\times U)\}$ be the Poisson process of excursions of X. For $C \in \mathcal{B}(\mathbb{R}^+\times U)$ we write:

$$N|_C \equiv \{N(F);\, F \subseteq C\}\,.$$

For each $x > 0$ define the Borel subsets of $\mathbb{R}^+ \times U$:

$$A_x \equiv \{(t,f);\ 0 \le t \le x,\ -k(t)^{-1} < m(f) < h(t)^{-1}\},$$

$$\bar{A}_x \equiv ([0,x] \times U) \setminus A_x ,$$

with $A \equiv \bigcup_{x>0} A_x$, $\bar{A} \equiv \bigcup_{x>0} \bar{A}_x$. Now look at Fig.2 and think in terms of excursions. If we map N to a Poisson process $m \circ N$ on $\mathbb{R}^+ \times \mathbb{R}$ by sending (t,f) to $(t,m(f))$, then the excursions lying in A go to the (open) unshaded region of Fig.2, and the others go to the (closed) shaded region. It is clear from the Poisson process description of N that

(37)

(i) $N|_A$ and $N|_{\bar{A}}$ are independent;

(ii) $L_T = \inf\{x;\ N(\bar{A}_x) > 0\}$; in particular, L_T is independent of $N|_A$;

(iii) $P(L_T \in d\ell,\ X_T > 0)/d\ell = \frac{1}{2}h(\ell)\exp[-\frac{1}{2}\int_0^\ell \{h(x) + k(x)\}dx]$,

$P(L_T \in d\ell,\ X_T < 0)/d\ell = \frac{1}{2}k(\ell)\exp[-\frac{1}{2}\int_0^\ell \{h(x) + k(x)\}dx]$;

(iv) $X_T = h(L_T)^{-1}$ if $X_T > 0$,

$\quad = -k(L_T)^{-1}$ if $X_T < 0$.

Let us now note some consequences of properties (37). From (iii) we see that

$$P(L_T > \ell) = \exp[-\tfrac{1}{2}\int_0^\ell \{h(x) + k(x)\}dx],$$

explaining condition (35) - it is to ensure that $T < \infty$ a.s.. The random variable Y_1 is measurable on the σ-field generated by $N|_{\bar{A}}$ so, conditional on L_T and the sign of X_T, Y_1 and Y_2 are independent, since Y_2 is measurable on the σ-field generated by $N|_{A_{L_T}}$.

From the characterisation (Theorem 3.1) of the Brownian excursion law, we have that, conditional on L_T, and $X_T > 0$,

$$\{X_s;\ G_T \le s \le L_T\} \text{ is distributed as } \{R_s;\ 0 \le s \le \tau_{h(L_T)^{-1}}\},$$

where R is a $\mathrm{BES}^{\circ}(3)$ process. Thus

(38) $\quad E(Y_1 | L_T,\ X_T > 0) = \gamma(h(L_T)^{-1}, L_T)\exp[-a(h(L_T)^{-1}, L_T)]$ a.s.,

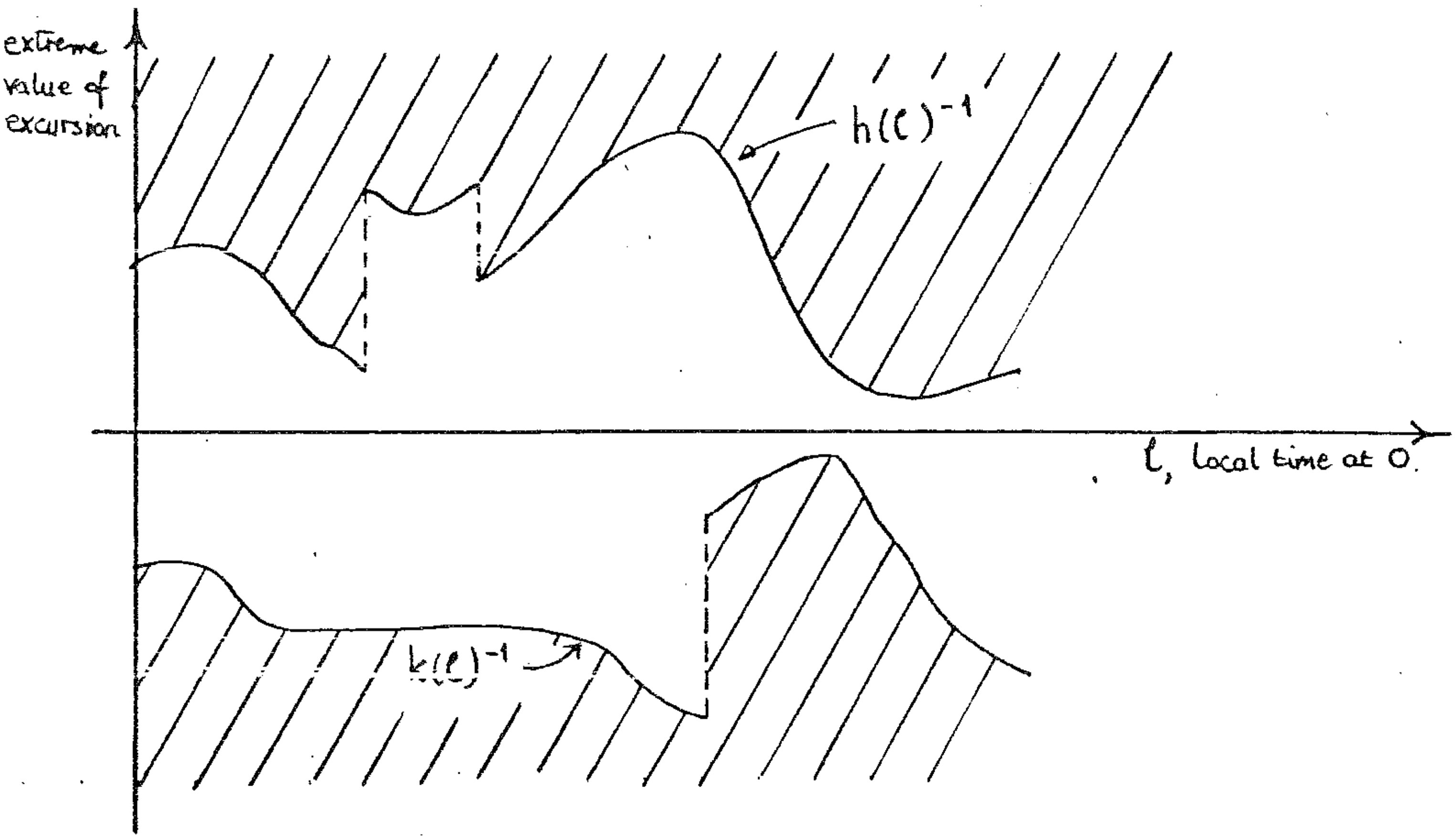

Fig. 2.

with a corresponding expression for $E(Y_1|L_T, X_T < 0)$.

Turning to $E(Y_2|L_T, X_T > 0)$, we have to think of the process in another way. If the function b was equal everywhere to $\xi > 0$, we could take a Poisson process in $\mathbb{R}^+$ of rate ξ independent of the process X and superimpose it on X to give a Brownian motion marked at the points of an independent Poisson process, and then

(39) $$Y_2 = P(\text{no mark in } [0,G_T] | X_s;\ 0 \le s) \quad \text{a.s.}.$$

The case of general b is only a little more complicated; the rate of the Poisson process of marks is no longer constant, but is equal to $b(X_t, L_t)$. Y_2 still has the interpretation (39). We now think of building up the marked Brownian motion from marked excursions. Informally, the Poisson process N^* of marked excursions is obtained from the Poisson process N of unmarked excursions by taking each unmarked excursion and independently inserting marks at rate $b(X.,L.)$. In more detail, if, in the unmarked excursion process, an excursion $f \in U$ appears at local time ℓ, then the number of marks which go into it is a Poisson random variable with mean

$$\int_0^{\zeta(f)} b(X_s, \ell)\,ds$$

independently of all the other excursions.

In particular, the probability that the excursion receives no mark is

$$\exp\left[-\int_0^{\zeta(f)} b(X_s, \ell)\,ds\right],$$

and so, by the characterisation of the Brownian excursion law (Theorem 3.1), the probability that the excursion receives no mark <u>conditional on</u> <u>$m(f)$</u>, its extreme value, is

$$\beta(m(f), \ell)^2.$$

If we now project the marked excursion process $N|_A^*$ into the marked Poisson process $m \circ N|_A^*$ on $\mathbb{R}^+ \times \mathbb{R}$ as before (by identifying excursions with the same extreme value), we observe a Poisson process with measure $dt \times \frac{1}{2}x^{-2}dx$, whose points (t,x) are independently marked with probability $1 - \beta(x,t)^2$, and unmarked with probability $\beta(x,t)^2$. Thus the number of marked excursions before time ℓ is a Poisson random variable with mean

$$\theta(\ell) \equiv \int_0^\ell dt \int_{-k(t)^{-1}}^{h(t)^{-1}} \frac{dx}{2x^2}\,[1-\beta(x,t)^2]\,. \tag{40}$$

Thus

$$Y_2 = P(\text{no mark in } [0,G_T] \,|\, X_s;\ 0 \le s) = e^{-\theta(L_T)}\,,$$

and finally we can, by the independence of Y_1 and Y_2 conditional on L_T and $\operatorname{sgn}(X_T)$, and the explicit expression (37)(iii) for the density of L_T put everything together and get

$$\begin{aligned} EY = \int_0^\infty \tfrac{1}{2}d\ell\, e^{-\rho(\ell)-\theta(\ell)}\{&h(\ell)\gamma(h(\ell)^{-1},\ell)\, e^{-a(h(\ell)^{-1},\ell)} \\ &+ k(\ell)\gamma(-k(\ell)^{-1},\ell)\, e^{-a(-k(\ell)^{-1},\ell)}\}\,, \end{aligned} \tag{41}$$

where

$$\rho(\ell) \equiv \tfrac{1}{2}\int_0^\ell \{h(x)+k(x)\}\,dx\,.$$

This is really the whole story, though the functions γ and θ which appear in (41) are as yet in no very explicit form. Jeulin and Yor give a characterisation of γ and θ through solutions of certain differential equations. Our approach also leads naturally to a differential equations characterisation of β and γ; indeed, referring back to the definition (36) of β, we see that for each $\ell \ge 0$, $\beta(.,\ell)$ is the reciprocal of the solution to

$$\begin{gathered} \tfrac{1}{2}\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - b(x,\ell)y = 0 \qquad (x>0) \\ y(0)=1\,,\quad y \text{ increasing}\,, \end{gathered} \tag{42}$$

with the analogous differential equation in $(-\infty,0)$. It can be shown that the differential equations obtained by Jeulin and Yor are equivalent to (42); as in their work, we understand (42) in the distributional sense if $b(.,\ell)$ is not continuous. The easy way to see that (42) is true, at least in the case where $b(.,\ell)$ is continuous, is to note from (36) that for each ℓ

(43) $$\beta(R_t,\ell)^{-1}\exp[-\int_0^t b(R_s,\ell)ds] \quad \text{is a local martingale.}$$

Itô's formula now gives (42) as a necessary and sufficient condition for (43).

Let us now apply this to the final assertion of the statement of Skorokhod's embedding result, as promised. In fact, we shall do more; we shall obtain the Laplace transform of (M_T,T), as do Azéma and Yor.

Let us fix ξ, $\eta > 0$ and take the measurable functions a, b, and c of (34) to be defined by

$$b(x,\ell) = c(x,\ell) = \tfrac{1}{2}\xi^2, \quad a(x,\ell) = \eta\ell \qquad (x \in \mathbb{R},\ \ell \geq 0)\,.$$

The measurable functions h and k of (33) are defined by

$$h(\ell) = k(\ell) = \phi(\ell)^{-1} \qquad (\ell \geq 0)\,,$$

where $\phi(\ell) \equiv \ell - \Phi(\ell)$. It is possible that ϕ may vanish; in this case, we replace ϕ by $\phi \vee \varepsilon$, solve, and let $\varepsilon \downarrow 0$. Plainly, the optional times T^ε defined by (33) with h and k replaced by $h \wedge \varepsilon^{-1}$, $k \wedge \varepsilon^{-1}$ will converge almost surely to T, so we lose no generality by assuming that ϕ is bounded away from zero.

These definitions of a, b, c, h and k cast the problem of this Section into the problem of Section 2; all that remains is a few trivial calculations. From (36) or (42), we obtain

$$\beta(x,\ell) = \gamma(x,\ell) = \xi x \operatorname{cosech} \xi x$$

so that, from (40),

$$\theta(\ell) = \int_0^\ell dt\,\{\xi \coth \xi\phi(t) - \phi(t)^{-1}\}\,,$$

and from (41)

$$\rho(\ell) = \int_0^\ell dt\,\phi(t)^{-1}\,.$$

Putting this all into (41) gives

(44) $$E\exp(-\eta M_T - \tfrac{1}{2}\xi^2 T) = \xi\int_0^\infty dx \operatorname{cosech} \xi\phi(x)\exp(-\int_0^x (\xi\coth\xi\phi(t) + \eta)dt)\,,$$

which agrees with the result of Azéma and Yor in the case where $\bar{\mu}(x) > 0$ for all x ; the remaining case is handled by the approximation argument outlined above.

If we are interested in the expected value of T , we can differentiate (44) with respect to ξ , divide by $-\xi$ and let ξ and η drop to zero, giving

(45) $$ET = \tfrac{1}{3}\int_0^\infty dx\ [\phi(x) + \phi(x)^{-1}\int_0^x 2\phi(t)dt]\exp(-\int_0^x \phi(t)^{-1}dt)\ .$$

If we now suppose that Ψ is continuous and strictly increasing, we can change variables in (45) and we obtain after a few calculations that

(46) $$ET = \int_{-\infty}^\infty \mu(dt)(\Psi(t) - t)^2\ .$$

By Schwarz' inequality, and the assumption that μ has a second moment,

(47) $$\bar{\mu}(t)\ \Psi(t)^2 \quad \leq \quad \int_t^\infty x^2\mu(dx) \quad \to 0 \quad \text{as} \quad t \to \infty\ ,$$

so we can integrate by parts to give for each $N \in \mathbb{N}$ that

(48) $$\int_{-N}^N \mu(dt)\Psi(t)^2 = \bar{\mu}(-N)\Psi(-N)^2 - \bar{\mu}(N)\Psi(N)^2 + \int_{-N}^N 2\Psi(t)(\Psi(t) - t)\mu(dt)\ ,$$

using (11). Rearranging (48) gives

(49) $$2\int_{-N}^N \mu(dt)t\Psi(t) = \int_{-N}^N \mu(dt)\Psi(t)^2 \quad + \quad o(1)\ ;$$

applying Schwarz' inequality to the left-hand side of (49), we see from the fact that μ has a second moment that the right-hand side of (49) remains bounded as $N \to \infty$, and, taking the limit, we deduce from (46) and (49) that

$$ET = \int_{-\infty}^\infty \mu(dt)\ t^2\ ,$$

as required. The case where Ψ is not continuous and strictly increasing can be handled directly, or by appeal to the results of Pierre [6] , as in Section 2.

Using the results of this Section, we can provide alternative proofs of the results of Knight [4] ; these are concerned with the case where

$$h(\ell) = k(\ell) = 0 \qquad (0 \le \ell \le \alpha)$$
$$= +\infty \qquad (\alpha < \ell) ,$$

and

$$b(x,\ell) = c(x,\ell) = \lambda I_{[0,g_1(\ell))}(x) + \mu I_{[g_1(\ell),g_2(\ell))}(x) + \nu I_{[g_2(\ell),\infty)}(x) \quad (x \in \mathbb{R}, \ell \ge 0)$$

where g_1, g_2 are given measurable functions, and λ, μ, ν are positive.

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