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Coupling and the tail σ-field of a one-dimensional diffusion

1. INTRODUCTION

The question of the triviality of the tail σ-field of a one-dimensional diffusion has been completely decided by work of Rösler, Fristedt and Orey, and Küchler and Lunze; the purpose of this note is essentially pedagogical, in that we shall show how the ideas of these earlier papers can be drawn together, illuminated and simplified using a little stochastic calculus.

We shall work throughout with a regular one-dimensional diffusion X with the interval $I \subset \mathbb{R}$ as statespace. Our diffusion will always be assumed to be in natural scale, with speed measure m; we refer the reader to Breiman [1], Freedman [2], Mandl [7], Rogers and Williams [8] for definitions and properties. The sample space is the canonical space $\Omega = C(\mathbb{R}^+, I)$ and we define for $t \geq 0$

$$\mathcal{G}_t \equiv \sigma(\{X_s : s \geq t\}) ,$$

where X is the canonical process. The *tail σ-field* is defined to be

$$\mathcal{G} \equiv \bigcap_{t \geq 0} \mathcal{G}_t .$$

Informally, an event is in $\mathcal{G}$ if it is determined by the *ultimate* behaviour of the path. In view of the Kolmogorov 0-1 law, it seems reasonable (and is true, as we shall see) that if X is recurrent, the tail σ-field is trivial. If X is transient, is it possible that the tail σ-field is non-trivial, and if so, how? The most illuminating explanation of the fact that $\mathcal{G}$ can be non-trivial appears in the paper of Fristedt and Orey [3]. Consider the stochastic differential equation

$$Y_t = \int_0^t \sigma(Y_s)dB_s + t \tag{1}$$

where $\sigma \in C_K^\infty$. Then the solution Y will diffuse for a while, and will ultimately leave the support of σ for ever, and follow the trajectory $Y_t = \eta + t$, *where the random variable* η *is tail measurable.* One expects that if σ were everywhere positive but tended to zero sufficiently rapidly at infinity, then the qualitative behaviour of Y would not change much, and $\mathcal{G}$ would be non-trivial. This is exactly what happens; and, as Fristedt and Orey prove, this is *all* that happens.

Let us now explain the principal results. It turns out that the only interesting case is (reducible to) the case where $\mathrm{I} = (0,1]$, and 1 is not absorbing. In this case, the diffusion tends to zero as $t \to \infty$. Defining for $x \in \mathrm{I}$

$$c(x) \equiv \mathbb{E}^1(H_x)$$

(where $H_x \equiv \inf\{t > 0 : X_t = x\}$, and $\mathbb{E}^y$ denotes expectation with respect to $\mathbb{P}^y$, the law of the diffusion started at y), the function c is finite-valued, and

$$M_t \equiv t - c(X_t)$$

is a local martingale. Here is the main result.

THEOREM 3. *The following are equivalent*:

(i) $\mathcal{G}$ *is not trivial*;

(ii) $E^x \langle M \rangle_\infty < \infty$ *for some (all)* $x \in (0,1]$;

(iii) *for some (all)* $x \in (0,1]$,

$$\lim_{y \downarrow 0} \mathrm{var}_x(H_y) < \infty;$$

(iv) $\int_0^1 y(m[y,1])^2 dy < \infty$;

(v) $(M_t)_{t \geq 0}$ *is bounded* in $L^2(\mathbb{P}^x)$ *for some (all)* $x \in (0,1]$;

(vi) $(M_t)_{t \geq 0}$ *is convergent a.s.* $\mathbb{P}^x$ *for some (all)* $x \in (0,1]$.

Remarks. Rösler [1] proved (i) $\Leftrightarrow$ (iii). Fristedt and Orey added (iv) and (vi). By var_x we mean the variance under $\mathbb{P}^x$. The third condition is illuminating; $\mathbb{E}^x(H_y) \to \infty$ as $y \downarrow 0$, yet the variances of H_y remain bounded. Compare this with (1).

The key to the proof of Theorem 3 is the following. If X and X' are independent diffusions in I, the first with law $\mathbb{P}^x$, the second with law $\mathbb{P}^{x'}$; $x \neq x'$, and if $T = \inf\{t > 0 : X_t = X'_t\}$, then

$$\mathcal{G} \text{ is trivial} \Leftrightarrow T < \infty \text{ a.s..} \tag{2}$$

This coupling characterisation of the triviality of $\mathcal{G}$ is due to Küchler and Lunze [5]; we prove it below.

In §2, we set out briefly some notation and basic ideas, and go on to prove (2). Then in §3 we prove Theorem 3. The final section, §4, deals with the remarkable result of

Fristedt and Orey that $\mathcal{G} = \sigma(M_\infty)$ in the case where $\mathcal{G}$ is non-trivial. We shall make use in what follows of some standard facts about one-dimensional diffusions:

(3) the semigroup $\{P_t : t \geq 0\}$ is strong Feller;

(4) there is a strictly positive continuous $p : (0,\infty) \times \text{int(I)} \times \text{int(I)} \to \mathbb{R}$ such that for f supported in int(I)

$$P_t f(x) = \int p_t(x,y) f(y) m(dy)$$

(see Itô-McKean [4], §4.11);

(5) the $\mathbb{P}^x$-distribution of H_y has a uniformly continuous density (which is even unimodal - see Rösler [10]).

2. COUPLING AND TRIVIALITY OF THE TAIL σ-field $\mathcal{G}$

Any bounded function $f : \mathbb{R}^+ \times \text{I} \to \mathbb{R}$ which is non-constant and such that for all $t,\ s \geq 0$

$$f(t,\cdot) = P_s f(t+s,\cdot) \tag{6}$$

will be called a *tail function*. Here, (P_t) is the transition semigroup of the diffusion X; since it is strong Feller, it can easily be deduced that any tail function must be jointly continuous. The terminology is explained by the following result.

Proposition. *The following are equivalent*:

(i) *There exists a tail function*;

(ii) $\mathcal{G}$ *is non-trivial under* $\mathbb{P}^x$ *for each* $x \in$ int(I);

(iii) $\mathcal{G}$ *is non-trivial under* $\mathbb{P}^x$ *for some* $x \in$ I.

There is a 1-1 correspondence between tail functions f and bounded tail-measurable random variables Y given by

$$f(t,X_t) = \mathbb{E}^y(Y \mid \mathcal{F}_t), \qquad Y = \lim_{t\to\infty} f(t,X_t) . \tag{7}$$

Proof. (i) => (ii). The process $Y_t \equiv f(t,X_t)$ is, from (6), a bounded martingale, so that $\mathbb{P}^x$-a.s. the limit Y exists. If Y were constant, so would $f(t,X_t)$ be, which contradicts the existence of a strictly positive transition density (4).

(iii) => (i). Let $Y \in L^\infty(\mathcal{G})$ be non-constant. Define f by $f(t,X_t) = \mathbb{E}^x(Y \mid \mathcal{F}_t)$; the right-hand side is a function only of X_t since $Y \in L^\infty(\mathcal{G}) \subset L^\infty(\mathcal{G}_t)$. Since Y is non-constant, and $Y = \lim_{t \to \infty} f(t,X_t)$, f cannot be constant. The fact that (7) holds is immediate from the Markov property. ◊

Now let $X_0 = x$, and let X' be an independent copy of X but with starting point x'. Define $\mathcal{G}'_t \equiv \sigma(\{X'_s : s \geq t\})$ and let $\mathcal{A}_t \equiv \mathcal{G}_t \vee \mathcal{G}'_t$. We define $\mathcal{A} \equiv \bigcap_t \mathcal{A}_t$. *Because X and X′ are independent*, we have that

$$\mathcal{A} = \mathcal{G} \vee \mathcal{G}' ;$$

with Lemma 2 of Lindvall and Rogers [6]. Without independence, this result is false in general. Let $\tilde{\mathbb{P}}^{(x,x')}$ denote the law of the bivariate process (X,X').

Here is the key result of Küchler and Lunze.

THEOREM 1. *The following are equivalent*:

(i) *for all* $x,x' \in \mathrm{I}$, $\tilde{\mathbb{P}}^{(x,x')}(T < \infty) = 1$;

(ii) *for all* $x \in \mathrm{I}$, $\mathcal{G}$ *trivial under* $\mathbb{P}^x$.

Proof. (i) => (ii). Recall the definition of the coupling time T given in the first section. We use the fundamental coupling inequality for the total-variation norm of $P_t(x,\cdot) - P_t(x',\cdot)$,

$$\| P_t(x,\cdot) - P_t(x',\cdot) \| \leq 2\tilde{\mathbb{P}}^{(x,x')}(T > t) ;$$

see, for example, Rogers and Williams [8], V.54.2. Suppose then that f is some tail function. Then for any $s,t \geq 0$, $x,x' \in \mathrm{I}$

$$| f(s,x) - f(s,x') | \leq \| P_t(x,\cdot) - P_t(x',\cdot) \| \,.\, \| f(s+t,\cdot) \|_\infty$$

$$\to 0 \quad \text{as} \quad t \to \infty$$

by hypothesis. Hence immediately f is constant, and so $\mathcal{G}$ is $\mathbb{P}^x$-trivial for all x.

(ii) => (i). Since $\mathcal{A} = \mathcal{G} \vee \mathcal{G}'$, it follows that $\mathcal{A}$ is $\tilde{\mathbb{P}}^{(x,x')}$-trivial for all x,x'. Now define

$$A^+ \equiv \{X_t - X'_t > 0 \quad \text{for all large enough } t\} ,$$

$$A^- \equiv \{X_t - X'_t < 0 \quad \text{for all large enough } t\} ,$$

$$\phi^{\pm}(x,x') \equiv \tilde{\mathbb{P}}^{(x,x')}(A^{\pm}) .$$

The events A^+ and A^- are tail events; indeed, they are even invariant events. By hypothesis then, $\phi^{\pm}$ take only the values 0 and 1. We shall show that $\phi^{\pm}$ are both zero throughout int(I) $\times$ int(I), leaving the extension to the boundary as an easy exercise. Suppose that $x_0, x'_0 \in$ int(I) and that $\phi^+(x_0,x'_0) = 1$. Then by (4) it must be that ϕ^+ is equal to 1, $m \times m$-a.e. in int(I) $\times$ int(I); and, since ϕ^+ is invariant, it follows immediately that ϕ^+ is 1 everywhere in int(I) $\times$ int(I). But $\phi^+(x,x') = \phi^-(x',x)$, and so ϕ^- is 1 everywhere in int(I) $\times$ int(I) - which is a contradiction because the events A^+ and A^- are disjoint. Thus $\tilde{\mathbb{P}}^{(x,x')}(A^{\pm}) = 0$ for all x,x', and the two independent diffusions X and X' keep on crossing over, so, in particular, the coupling time T must be finite. ◊

Terminology. We say that coupling is certain if for all $x,x' \in$ I, $\tilde{\mathbb{P}}^{(x,x')}(T < \infty) = 1$, and we say that $\mathcal{G}$ is trivial if $\mathcal{G}$ is $\mathbb{P}^x$-trivial for all x.

Here is a simple consequence of Theorem 1:

If X is recurrent, then $\mathcal{G}$ is trivial.

Proof. (i) If I $= \mathbb{R}$, then X and X' are independent continuous local martingales, so $\langle X - X' \rangle = \langle X \rangle + \langle X' \rangle$. Moreover, $\langle X \rangle_\infty = +\infty$ a.s. since X does not converge. Thus $\langle X - X' \rangle_\infty = +\infty$ a.s., and (since $X - X'$ is a time change of Brownian motion) $\sup_t (X_t - X'_t) = \sup_t (X'_t - X_t) = +\infty$. Thus coupling is certain.

(ii) If I $= [0,\infty)$, say, with 0 reflecting, extend the speed measure m into $(-\infty,0)$ by reflection and make up independent diffusions Y, Y' on $\mathbb{R}$ with this speed; by (i), Y and Y' are certain to couple and, since $|Y|$ has the same law as X, X and X' are also certain to couple.

(iii) The case of compact I is similar to (ii). ◊

3. CHARACTERISING THE CASES WHERE $\mathcal{G}$ IS NON-TRIVIAL

We now consider what happens when the diffusion X is transient. One of the following cases must apply (after shifting and rescaling I if necessary).

(i) I $= [0,1]$, 0 and 1 both absorbing. Then $\mathcal{G}$ is not trivial.

(ii) I $= [0,1]$, 0 absorbing, 1 not absorbing. In this case, if f is a tail function, $f(s, 0) = f(t, 0)$ $\forall\ s,t \geq 0$ and so, since the diffusion is certain to be absorbed in 0, f is constant. Hence $\mathcal{G}$ is trivial.

(iii) $\mathrm{I} = (0,1]$, 1 is absorbing. Again, $\mathcal{G}$ is non-trivial.

(iv) $\mathrm{I} = (0,1]$, 1 is not absorbing. This is the interesting case.

(v) $\mathrm{I} = [0,\infty)$, 0 is absorbing. As in (ii), $\mathcal{G}$ is trivial.

(vi) $\mathrm{I} = (0,\infty)$. We shall see later that this can be reduced to case (iv).

Thus we shall until further notice assume that

$$\mathrm{I} = (0,1] \text{ and } 1 \textit{ is not absorbing.} \tag{8}$$

In particular, this implies that for $0 < y \leq x \leq 1$, all moments of H_y are finite under $\mathbb{P}^x$ (see, for example, Rogers and Williams [8], V.46.1) and, with $c(x) = \mathbb{E}^1(H_x)$ as before,

$$M_t \equiv t - c(X_t) \text{ is a continuous local martingale under each } \mathbb{P}^x .$$

Indeed, for $y < x$,

$$M(t \wedge H_y) = \mathbb{E}^x[H_y \mid \mathcal{F}_t] - c(y) . \tag{9}$$

It is well known that c is strictly decreasing and convex.

We are now in a position to prove the main result.

THEOREM 3. *The following are equivalent*:

(i) $\mathcal{G}$ *is not trivial*;

(ii) $\mathbb{E}^x < M >_\infty < \infty$ *for some (all)* $x \in (0,1]$;

(iii) *for some (all)* $x \in (0,1]$;

$$\lim_{y \downarrow 0} \mathrm{var}_x(H_y) < \infty;$$

(iv) $\int_0^1 y(m[y,1])^2 dy < \infty$;

(v) $(M_t)_{t\geq 0}$ *is bounded in* $L^2(\mathbb{P}^x)$ *for some (all)* $x \in (0,1]$;

(vi) $(M_t)_{t\geq 0}$ *is* $\mathbb{P}^x$-a.s. *convergent for some (all)* $x \in (0,1]$.

Proof. (ii) $\Leftrightarrow$ (iii). From (9), for $0 < y \leq x \leq 1$

$$\begin{aligned} \mathbb{E}^x < M >_{H_y} &= \mathbb{E}^x(M(H_y) - M(0))^2 \\ &= \mathbb{E}^x(H_y - \mathbb{E}^x(H_y))^2 \\ &= \mathrm{var}_x(H_y) . \end{aligned}$$

Moreover, by the strong Markov property at H_x,

$$\text{var}_1(H_y) = \text{var}_1(H_x) + \text{var}_x(H_y) ,$$

and $\text{var}_1(H_x) < \infty$ since all moments of H_x are finite under $\mathbb{P}^1$.

(ii) $\Leftrightarrow$ (v) is immediate, as is (v) => (vi).

To prove the other equivalences, we shall assume we have the diffusion X represented as a time change of a Brownian motion, as we may. Thus if B is a Brownian motion started at $x \in \mathrm{I}$, with local time process $\{l(t,x) : t \geq 0, x \in \mathbb{R}\}$,

$$A_t \equiv \int_{\mathrm{I}} m(da) l(t,a) ,$$

$$\sigma_t \equiv \inf\{u : A_u > t\} ,$$

$$X_t \equiv B(\sigma_t) .$$

Let $\tau_y \equiv \inf\{t : B_t = y\}$, $\zeta \equiv \tau_0$. Then $H_y = A(\tau_y)$ for $y < x$. Hence, defining $c(x) = 0$ for $x \geq 1$,

$$\begin{aligned} M(A_{t \wedge \tau_y}) &= A_{t \wedge \tau_y} - c(B_{t \wedge \tau_y}) \\ &= \mathbb{E}^x[A_{\tau_y} \mid \mathcal{B}_t] - c(y) \end{aligned}$$

is a $(\mathcal{B}_t)$-martingale (where $(\mathcal{B}_t)$ is the filtration of B) and hence $M \circ A$ is a $(\mathcal{B}_t)$-local martingale on $[0, \zeta)$. Since c is convex, we may apply the generalised Itô formula (see Rogers and Williams [8], IV.45.1) to deduce that for $t < \zeta$

$$M \circ A_t = \int_0^t c'(B_s) dB_s ,$$

and that $\frac{1}{2}c'' = dm$ as measures. This tells us that for each $x \in \mathrm{I}$, $c'(x) = -2m([x, 1])$, and that

$$<M \circ A>_t = \int_0^t c'(B_s)^2 ds = <M>_{A_t} .$$

Now $<M \circ A>_t$ is an additive functional of B, and the criterion for this to converge as $t \uparrow \zeta$ is well known from the study of boundary behaviour of diffusions; we have

$$\begin{aligned} <M>_\infty = <M \circ A>_\zeta < \infty &\Leftrightarrow \mathbb{E}^x(<M \circ A>_\zeta) = \mathbb{E}^x(<M>_\infty) < \infty \qquad (10) \\ &\Leftrightarrow \int_{0+} y\, c'(y)^2 dy < \infty , \end{aligned}$$

and if this condition fails, $<M>_\infty = +\infty$, $\mathbb{P}^x$-a.s.. See, for example, Rogers and

Williams [8], V.51.2. Equivalence of (ii) and (iv) is now immediate, and, since $M_t = W(<M>_t)$ for some Brownian motion W, (vi) implies that $\mathbb{P}^x$-a.s. $<M>_\infty < \infty$ for some x so that, from (10), $\mathbb{E}^x <M>_\infty < \infty$, which is (ii).

All that now remains is to prove that (i) is equivalent to all the other statements. If (vi) holds, then M_∞ is a non-degenerate tail-measurable random variable (M cannot be constant); thus $\mathcal{G}$ is not trivial. Conversely, if (iv) is false, $<M>_\infty = +\infty$, $\mathbb{P}^x$-a.s. for each x, and so if X,X' are independent copies of the diffusion with different starting points then $<M - M'>_\infty = +\infty$ a.s., where $M'_t \equiv t - c(X'_t)$. Hence, as before, for some t

$$c(X_t) - c(X'_t) = 0 ,$$

and so X and X' couple, since c is strictly decreasing. ◊

Finally, to dispose of the case where $I = (0,\infty)$, notice that if $\mathcal{G}$ is non-trivial there is some tail function, which is a tail function for the diffusion obtained by time-changing out the time spent in $(1,\infty)$. Hence for this new diffusion we can apply Theorem 3 and conclude that

$$\int_0^1 y^2 (m[y,1])^2 dy < \infty . \tag{11}$$

Conversely, if this integral is finite, N_∞ is a non-trivial $\mathcal{G}$-measurable random variable, where N is the martingale

$$N_t = A_t - \bar{c}(X_t) ,$$

with $A_t \equiv \int_0^t I_{(0,1]}(X_s)ds$, $\bar{c}(x) \equiv \mathbb{E}^1[A(H_x)]$. In summary then, when $I = (0,\infty)$, non-triviality of $\mathcal{G}$ is equivalent to (11).

4. THE STRUCTURE OF $\mathcal{G}$ IN THE CASES OF NON-TRIVIALITY

We shall suppose still that we are dealing with the interesting case where $I = (0,1]$ and 1 is reflecting; all others can be reduced to this.

If $\mathcal{G}$ is non-trivial, and $\Lambda \in \mathcal{G}$, $\mathbb{P}^1(\Lambda) \in (0,1)$, we let f denote the corresponding tail function, and note that for each x

$$\Lambda = \{f(t,X_t) \to 1\} = \{f(H_{1/n}, n^{-1}) \to 1\} \quad \mathbb{P}^x\text{–a.s..}$$

Thus if $Y_k \equiv H_{1/k} - c(1/k)$, we have that

$$\Lambda \in \bigcap_n \mathcal{A}_n \equiv \bigcap_n \sigma(\{Y_k : k \geq n\}) .$$

Now since $Z_k \equiv Y_k - Y_{k-1}$ defines a sequence of independent random variables, and since $Y_k \to Y \equiv \lim_{t \to \infty} M_t \equiv M_\infty$, we have that

$$\Lambda \in \bigcap_n \mathcal{A}_n = \bigcap_n \sigma(\{Y, Z_k : k > n\}) \equiv \mathcal{A} .$$

In view of Kolmogorov's 0-1 law, we expect that $\mathcal{A} = \sigma(Y)$; as an example below will show, this is not correct. However, a special feature of the current situation saves us.

THEOREM 4. *Let $\{Z_k; k \geq 1\}$ be independent random variables whose partial sums Y_n converge a.s. to Y. Suppose that for some k, Z_k has a uniformly continuous density. Then the tail σ-field of $\{Y_n\}$ is $\sigma(Y)$.*

Proof. (i) Without loss of generality, suppose that Z_1 has a uniformly continuous density f. Notice that for any probability distribution F, the convolution of F with the law of Z_1 has the density $(F * f)(x) = \int F(dt) f(x-t)$, which is again *uniformly continuous with modulus of continuity no larger than that of f.* Thus if ϕ_k is the density of Y_k, and ϕ is the density of Y, we have for all $a < b$

$$\int_a^b \phi_k(x)dx \to \int_a^b \phi(x)dx ,$$

and so by the equi-uniform continuity of $\{\phi, \phi_k : k \geq 1\}$, it follows that $\phi_k(a) \to \phi(a)$ for every a, and hence that $\phi_k(y_k) \to \phi(y)$ if $y_k \to y$.

(ii) It suffices to prove that for bounded $X \in \mathcal{F}_n = \sigma(\{Y_k : k \leq n\})$ and bounded $\mathcal{A}$-measurable V,

$$E(XV) = E(E(X \mid Y)V) . \tag{12}$$

Now $E(X \mid \mathcal{A}_n) = E(X \mid Y_n) = g(Y_n)$ for some bounded Borel function g, so for $k \geq n$

$$\begin{aligned} E(XV) &= E(E(X \mid \mathcal{A}_k)V) \\ &= E[E(E(X \mid \mathcal{A}_n) \mid \mathcal{A}_k)V] \\ &= E[E(g(Y_n) \mid \mathcal{A}_k)V] \\ &= E[E(g(Y_n) \mid Y_k)V] . \end{aligned}$$

If $F_{n,k}$ is the distribution function of $Y_k - Y_n \equiv Z_{n+1} + \cdots + Z_k$, then

$$E[g(Y_n) \mid Y_k] = \int F_{n,k}(dt)\, g(Y_k - t)/\phi_k(Y_k) . \tag{13}$$

Now take bounded uniformly continuous $\tilde{g}$ such that $E\,|(g-\tilde{g})(Y_n)| < \varepsilon$, and notice that $E[\tilde{g}(Y_n)\,|\,Y_k] \to E[\tilde{g}(Y_n)\,|\,Y]$ a.s. as $k \to \infty$; indeed, the denominator is a.s. convergent since $Y_k \to Y$, and the convergence of the numerator is ensured by the uniform continuity of $\tilde{g}\,\phi_n$. The equality (12) follows. ◊

Because the first passage time $H_{1/k}$ has a uniformly continuous density under each $\mathbb{P}^x$, $x > 1/k$, we can apply Theorem 4 to obtain the pleasing conclusion $\Lambda \in \sigma(M_\infty)$; the *only* non-trivial information in the tail σ-field $\mathcal{G}$ comes from the limit of $t - c(X_t)$.

Finally, we provide an example which shows that Theorem 4 fails without the assumption of a uniformly continuous density for some Z_k. Let $p_1 < p_2 < \cdots$ be the primes in ascending order, and suppose that

$$P(Z_k = \frac{1}{p_k}) = P(Z_k = -\frac{1}{p_k}) = \tfrac{1}{2} .$$

Since p_k is of order $k \log k$, the martingale $Y_k = Z_1 + \cdots + Z_k$ is almost surely and L^2 convergent. Now

$$Y_n + \sum_1^n (p_k)^{-1} = 2 \sum_1^n I_{\{Z_k > 0\}} (p_k)^{-1} ,$$

and so from Y_n we can deduce all of $Z_1, \ldots, Z_n$, because if we take primes $q_1, \ldots, q_s$ and combine $\sum_1^r (q_j)^{-1}$ over a common denomnator, then that denominator is $\prod q_j$. However, knowing Y does not allow us to deduce the sign of Z_1, for example.

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