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A DIFFERENTIAL EQUATION IN WIENER-HOPF THEORY

by

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This is a heuristic introduction to some progress with certain calculations in Wiener-Hopf theory. Further details will be presented later.

PART 1. THE CASE WHEN THERE IS ONLY ONE BOUNDARY POINT

1. Let $B = \{B(t) : t \geq 0\}$ (also written $\{B_t : t \geq 0\}$) be a Brownian motion on $\mathbb{R}$. The symbol $\mathbb{P}_r$ denotes $\mathbb{P}[\cdot \mid B_0 = r]$, and $\mathbb{E}_r$ denotes $\mathbb{P}_r$ expectation. Let $V : \mathbb{R} \to \mathbb{R}$, with $V > 0$ on $(0,\infty)$ and $V < 0$ on $(-\infty,0)$. For $t \geq 0$, define:

$$\phi_t = \int_0^t V(B_s)ds , \quad \tau_t^+ = \inf\{u : \phi_u > t\} , \quad \tau_t^- = \inf\{u : -\phi_u > t\} ,$$

$$Y_t^+ = B(\tau_t^+) \in [0,\infty) , \quad Y_t^- = B(\tau_t^-) \in (-\infty,0] .$$

If $B_0 = x < 0$, then τ_0^+ is the half-winding time about the origin for the joint process $\{(\phi_t, B_t)\}$. Our primary concern is to calculate half-winding hitting probabilities:

$$\Pi^+(x,y) = \mathbb{P}_x[Y_0^+ \in dy]/dy , \quad \Pi^-(x,y) = \mathbb{P}_y[Y_0^- \in dx]/dx ,$$

where $x < 0$, $y > 0$.

Back in 1963, McKean ([3]) solved this problem for the case when $V(r) = r$, $\forall r \in \mathbb{R}$. The joint process (ϕ_t, B_t), the phase picture for McKean's resonator, is then Gaussian (as well as Markov); and McKean exploits this for Y^+ (or Y^-) then just counts the number of visits.] Let $J^+_y dy$ (resp., $J^-_x dx$) be the Lévy measure describing jumps made from 0 by $Y^+(Y^-)$. Define

(3.1) $$m^{\pm}(t) = \frac{d}{dt}\mathbb{E}_0[L^{\pm}(t)], \quad b^{\pm}_r(t) = \mathbb{P}_0[Y^{\pm}(t) \in dr]/dr .$$

Then we have the Fokker-Planck equation:

$$\frac{\partial b^{\pm}_r(t)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial r^2}[|V_r|^{-1} b^{\pm}_r(t)] + m^{\pm}(t)J^{\pm}_r ,$$

where $V_r = V(r)$. For the better formulation of the Fokker-Planck equation, introduce the Radon-Nikodym derivatives:

$$\beta^{\pm}_r(t) = b^{\pm}_r(t)/|V_r| , \quad \Lambda^{\pm}_r = J^{\pm}_r/|V_r| .$$

Then

(3.2) $$\frac{\partial \beta^{\pm}_r(t)}{\partial t} = \mathcal{G}^{\pm}_r \beta^{\pm}_r(t) + m^{\pm}(t)\Lambda^{\pm}_r , \quad \mathcal{G}^{\pm}_r = \frac{1}{2}|V_r|^{-1}\frac{\partial^2}{\partial r^2} .$$

If $H = \inf\{s : B_s = 0\}$, and $h_r(t)dt = \mathbb{P}_r[|\phi(H)| \in dt]$, then it is clear that for $x < 0$, $y > 0$,

$$\Pi^+(x,y) = \int_0^\infty h_x(t) b^+_y(t) dt ,$$

so that

(3.3) $$\Pi^+(x,y) = \rho(x,y)V(y) ,$$

where

(3.4) $$\rho(x,y) = \int_0^\infty h_x(t)\beta^+_y(t) dt .$$

The symmetry properties discovered in [2] (see Note below for a correction to [2]) make it clear that

(3.5) $$\Pi^-(x,y) = V(x)\rho(x,y) ,$$

and that we have the following dual expression for ρ :

(3.6) $$\rho(x,y) = \int_0^\infty \beta^-_x(t) h_y(t) dt .$$

where $A > 0$. Recall that

$$\phi_t = \int_0^t V(B_s)ds , \quad \tau_0^\pm = \inf\{u : \pm\phi_u > t\} , \quad Y_0^\pm = B(\tau_0^\pm) .$$

With the Brownian scaling in mind, let $c > 0$, and let

$$\tilde{B}_t = c^{-1}B(c^2t) ,$$

$$\tilde{\phi}_t = \int_0^t V(\tilde{B}_s)ds , \quad \tilde{\tau}_0^\pm = \inf\{u : \pm\tilde{\phi}_u > t\} , \quad \tilde{Y}_0^\pm = \tilde{B}(\tilde{\tau}_0^\pm) .$$

Then

$$\tilde{\phi}_t = c^{-2-\alpha}\phi(c^2t) , \quad \tilde{\tau}_t^\pm = c^{-2}\tau^\pm(c^{2+\alpha}t) , \quad \tilde{Y}_t^\pm = c^{-1}Y^\pm(c^{2+\alpha}t) .$$

Suppose for a moment that $B_0 = 0$, so that B and $\tilde{B}$ are identical in law. To avoid too heavy a notation, let us write J^+ for the Lévy measure of Y^+ at 0 (as a <u>measure</u>), as well as writing $J^+(\cdot)$ for the density of J^+ relative to Lebesgue measure. In short, $J^+(dx) = J^+(x)dx$. Let $y > 0$, and let

$$T_y = T(y) = \inf\{t : Y_{t-}^+ = 0;\ Y_t^+ > y\} .$$

Then, for $z > y$,

$$J^+(z,\infty)/J^+(y,\infty) = \mathbb{P}_0[Y^+(T_y) > z] .$$

But, with the obvious notation, $\tilde{T}(y) = c^{-2-\alpha}T(cy)$, and $\tilde{Y}^+(\tilde{T}_y) = c^{-1}Y^+(T_{cy})$.

Thus,

$$J^+(z,\infty)/J^+(y,\infty) = \mathbb{P}_0[Y^+(T_{cy}) > cz] = J^+(cz,\infty)/J^+(cy,\infty) ,$$

so that $J^+(y) \propto y^\eta$ for some η. Thus, for some constants ε and θ,

$$\Lambda^+(y) \propto |y|^\theta , \quad \Lambda^-(x) \propto |x|^\varepsilon .$$

The fundamental equation (3.9) therefore takes the form:

$$\text{(4.1)} \qquad \frac{1}{2A|x|^\alpha}\frac{\partial^2\rho}{\partial x^2} + \frac{1}{2|y|^\alpha}\frac{\partial^2\rho}{\partial y^2} = -R|x|^\varepsilon|y|^\theta .$$

The Brownian scaling gives us further information: for $x < 0$, $y > 0$,

$$\text{(4.2)} \qquad \mathbb{P}[Y_0^+ \le y | B_0 = x] = \mathbb{P}[\tilde{Y}_0^+ \le y | \tilde{B}_0 = x] = \mathbb{P}[Y_0^+ \le cy | B_0 = cx] .$$

where $\delta = \Pi/(2+\alpha)$. See §3.123 of Titchmarsh [4]. Similarly, for $y > 0$.

$$1 = \int_0^\infty K^{2+\alpha}|x|^\alpha \rho(x,y)\,dx = \delta C K^{1-\beta} \operatorname{cosec}((1-\beta)\delta) .$$

Hence β is the unique solution in $(0,1)$ of the equation.

$$\operatorname{cosec}(\beta\delta) = K \operatorname{cosec}((1-\beta)\delta) ,$$

and then

$$C = \pi^{-1}(2+\alpha)K^\beta \sin(\beta\delta) .$$

$$M^+_{ij}(t) = \frac{d}{dt}\mathbb{E}_i L^+_j(t) ,$$

and let $M^+(t)$ be the $\Gamma \times \Gamma$ matrix with (i,j)th component $M^+_{ij}(t)$.

Let

$$T^+_\Gamma = \inf\{t \geq 0 : Y^+_t \in \Gamma\} .$$

For $y \in \mathrm{Int}(E^+)$, and $i \in \Gamma$, let

$$h^+_{yi}(t) = \frac{d}{dt}\mathbb{P}_y[T^+_\Gamma \leq t;\ Y^+(T^+_\Gamma -) = i] .$$

Let $h^+_{y\cdot}(t)$ be the row vector $\{h^+_{yi}(t) : i \in \Gamma\}$. Let $J^+_{iy}dy$ be the Lévy measure describing jumps made by Y^+ from i, and let $J^+_{\cdot y}$ be the column vector $\{J^+_{iy} : i \in \Gamma\}$. Introduce the Radon-Nikodym derivative

$$\Lambda^+_{iy} = |V(y)|^{-1} J^+_{iy} .$$

Define b^+ and β^+ via

$$b^+_{iy}(t)dy = \beta^+_{iy}(t)|V(y)|dy = \mathbb{P}_i[Y^+(t) \in dy] .$$

Introduce the column vectors $b^+_{\cdot y}(t)$, $\beta^+_{\cdot y}(y)$, $\Lambda^+_{\cdot y}$ in the obvious way.

The Fokker-Planck equation

$$\frac{\partial}{\partial t} b^+_{\cdot y}(t) = \frac{1}{2}\frac{\partial^2}{\partial y^2}[|V(y)|^{-1} b^+_{\cdot y}(t)] + M^+(t)J^+_{\cdot y}(t)$$

holds, and, as at (3.2) transforms to

$$(5.1) \qquad \frac{\partial}{\partial t}\beta^+_{\cdot y}(t) = \mathcal{G}^+_y \beta^+_{\cdot y}(t) + M^+(t)\Lambda^+_{\cdot y}(t) ,$$

where

$$\mathcal{G}^+_y = \frac{1}{2}|V(y)|^{-1}\frac{\partial^2}{\partial y^2} , \quad \text{for } y \in \mathrm{Int}(E^+) .$$

Let ${}_\Gamma p^+$ be the taboo transition function on $E^+ \times E^+$:

$${}_\Gamma p^+(t,y_1,y_2)dy_2 = \mathbb{P}_{y_1}[Y^+(t) \in dy_2;\ T^+_\Gamma > t] .$$

Then, the symmetry property:

$$|V(y_1)|\,{}_\Gamma p^+(t,y_1,y_2) = |V(y_2)|\,{}_\Gamma p^+(t,y_2,y_1)$$

Exactly as in the argument following (3.7), we can deduce from (5.3) and (5.1) that

$$\left(\mathcal{G}_x^- + \mathcal{G}_y^+\right)\rho = \left(\int h_{x\cdot}^-(t)M^+(t)dt\right)\Lambda_{\cdot y}^+ .$$

And we can again appeal to the symmetry result in [2] to obtain

$$(5.6) \qquad \left(\int h_{x\cdot}^-(t)M^+(t)dt\right)\Lambda_{\cdot y}^+ = \left(\int h_{y\cdot}^+(t)M^-(t)dt\right)\Lambda_{\cdot x}^- .$$

We claim that for some constants $a_i (i \in \Gamma)$,

$$(5.7) \qquad \int h_{x\cdot}^-(t)M_{\cdot i}^+(t)dt = a_i\Lambda_{ix}^- ,$$

$$(5.8) \qquad \int h_{y\cdot}^+(t)M_{\cdot i}^-(t)dt = a_i\Lambda_{iy}^+ .$$

This is one of several claims in this paper for which full justification will have to wait to a later paper. The reader should believe our results because the analogues for symmetric Markov chains are true, and we have tested out that one can force through weak-convergence results.

Let us explain briefly a direct method of deducing (5.7) and (5.8) from (5.6) in the case when i is a regular boundary point both for Y^+ and Y^- (so that each of these processes has a true continuous local time at i). As mentioned in a Note at the start of §3, this will be the situation in all but extremely pathological cases. The point is that as $y \to i$,

$$\Lambda_{jy}^+ = o(\Lambda_{iy}^+) , \quad j \neq i ,$$

and

$$\lim \int h_{yj_1}^+(t)M_{j_1j_2}^-(t)dt \quad \begin{array}{ll} = & \infty \text{ if } j_1 = j_2 = i , \\ < & \infty \text{ otherwise .} \end{array}$$

These results allow us to infer (5.7) from (5.6).

It will simplify the algebra to assume, as we may plainly do, that the normalizations of L_i^+ and L_i^- are made compatible for each i , so that

$$a_i = 1 , \ \forall i .$$

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