Asymptotic Behavior of Brownian Tourists

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Introduction. Let B_t be a Brownian motion and $f : \mathbb{R}^d \to \mathbb{R}^d$ be Lipschitz continuous. In this paper we will be concerned with processes of the form

$$X_t = B_t + \int_0^t ds \int_0^s du\, f(X_s - X_u) \tag{1.1}$$

where $f(x) = g(x)x/\|x\|$ and g is real valued. When g is a nonegative, X_t is a continuous analogue of a process invented by Diaconis and studied by Pemantle (1988a,b) so stealing a metaphor from that paper (and changing the sign) we can think of X_t as the trajectory of a tourist who wants to stay away from places she has visited before. For a more serious physical motivation one can think of X_t as describing a growing polymer in which newly added units are repelled by existing ones. As a polymer model, (1.1) has two serious weakness: (i) the repulsion does not prevent self–intersections, and (ii) while real polymers can rearrange themselves to minimize energy, ours cannot. However, in contrast to Edward's model (see Westwater (1980a)) the existence of the process presents no problem. It is easy to verify that (1.1) has a pathwise unique strong solution. Furthermore, one can hope that results about the end to end displacement of our polymer will give some insight into the behavior of more realisitic models.

In this paper we will be concerned with the asymptotic behavior of X_t as $t \to \infty$. The reader will see this problem presents some interesting mathematical challenges. Most of our results are confined to one dimension but the first one is general.

Theorem 1. *Suppose $\|f(x)\| \le M$ and f has compact support. There is a constant $\Gamma < \infty$ so that*

$$\limsup_{t\to\infty} \|X_t\|/t \le \Gamma \quad \text{a.s.}$$

To see that there is something to prove, notice that the cumulative drift at time t might be as large as $Mt^2/2$. Indeed in Theorem 4 we will see that for any $\alpha < 2$, there are examples with bounded f in which X_t is of order t^α. Although Theorem 1 is not obvious, it is not difficult to prove. The key observation is that if $f(x) = 0$ for $\|x\| \ge K$ and $\|X_t\|$ grows too quickly then many annuli $\{x : (n-1)K < \|x\| < nK\}$ must be crossed quickly. However after a fast crossing the drift is small and the chances of another fast crossing are not very good.

To get lower bounds on X_t/t we have to impose some strong assumptions.

Theorem 2. *Suppose $d = 1$, $f \geq 0$ and $f(0) > 0$. Then there is constant $\gamma > 0$ so that*

$$\liminf_{t\to\infty} X_t/t \geq \gamma.$$

The condition $f(0) > 0$ is too strong; $f \not\equiv 0$ should be sufficient. A more interesting problem is to strengthen the conclusion.

Conjecture 1. *Suppose f has compact support, $f \geq 0$, and $f \not\equiv 0$. Then there is a constant $\mu > 0$ so that*

$$X_t/t \to \mu \quad \text{a.s.}$$

If one is not careful this is easy to prove. "Clearly", if $X_0 = 0$ the increment $X_t - X_s$ is larger in distribution than X_{t-s} (due to the repulsive effect of $X_r, 0 \leq r \leq s$), so the result follows from the subadditive ergodic theorem. We have put clearly in quotation marks because the conclusion, if true, is far from clear and may be false. If we let $Y_h = X_{s+h} - X_s$ and construct a copy of X_h on the same space by using the Brownian motion $B_{s+\cdot} - B_s$ to drive the two SDE's then Y_h will be larger than X_h for small h, but if τ is the first time $X_t = Y_t$ the drift in the Y process at time τ may be smaller since $Y_\tau - Y_s \leq X_\tau - X_s$ does not imply $f(Y_\tau - Y_s) \geq f(X_\tau - X_s)$. ($Y_\tau - Y_s$ may be negative!)

The assumption $f \geq 0$ is undesirable since it says that our tourist avoids familiar territory by moving North. The situation becomes very complicated when f has values of both signs.

Conjecture 2. *Suppose f has compact support, $xf(x) \geq 0$, and $f(-x) = -f(x)$ then $X_t/t \to 0$ a.s.*

Before the reader declares that this is an obvious consequence of symmetry we would like to observe that there is no zero–one law, so one might have $X_t/t \to c > 0$ on a set of probability 1/2. Indeed the last behavior occurs in Westwater's polymer model (see Kusuoka (1985)), but computer simulations of related discrete systems suggest that for our process the following scenario is more likely. X_t grows (or decreases) linearly for a while until a fluctuation overcomes the drift, which is O(1), and brings the process well below its maximum. At this point the push from above is larger than that from below and the process tends to decrease for a while. Once the process gets well inside the initial increasing segment things get complicated but can be visualized if one thinks of the graph of the local time at time t as a mountain range and X_t as an Brownian ant that drifts downhill and drops sand at rate 1. We have no idea whether X_t satisfies the central limit theorem or displays more interesting behavior but suspect that this will be very difficult to resolve.

The problems we encountered in the compactly supported case become somewhat simpler when f is not integrable, for then the drift grows with time. Suppose

(A1) $|f(x)| \leq M$
(A2) $f(x)$ is decreasing for $x \in [q, \infty)$

(A3) $x^\beta f(x) \to \ell > 0$ as $x \to \infty$ with $0 < \beta < 1$

Letting $x_t = T^{-\alpha} X(tT)$ and $W_t = T^{-1/2} B(tT)$ we can rewrite (1.1) as

$$x_t = T^{1/2-\alpha} W_t + T^{2-\alpha} \int_0^t ds \int_0^s du\, f(T^\alpha (x_s - x_u)). \tag{1.2}$$

If we take $\alpha = 2/(1+\beta)$ so that $2 - \alpha = \alpha\beta$ and let $T \to \infty$ we see that the limit, if strictly increasing, should satisfy

$$x_t = \int_0^t ds \int_0^s du\, \frac{\ell}{(x_s - x_u)^\beta}\,. \tag{1.3}$$

One solution is $x_t = c_0 t^\alpha$ where c_0 satisfies

$$\alpha c_0^{\beta+1} = \int_0^1 \frac{\ell du}{(1-u^\alpha)^\beta}\,. \tag{1.4}$$

Our first result says that this argument provides an upper bound

Theorem 3. *Suppose (A1)–(A3) hold and α and c_0 are as above. Then*

$$\limsup_{t\to\infty} X_t/t^\alpha \le c_0.$$

The last result when suitably reformulated holds in $\mathbb{R}^d$. A more interesting extension would be to prove

Conjecture 3. *Suppose $d = 1$ and $f(x) = x/(1+|x|^{\beta+1})$ with $0 < \beta < 1$. Then with probability $\frac{1}{2}$*

$$X_t/t^\alpha \to c_0.$$

To see the difficulties involved the reader should try the following much simpler open question.

Problem. *Under the hypotheses of Conjecture 3,*

$$\sup_t |X_t| = \infty \quad \text{a.s.}$$

Our last result gives some support for Conjecture 3.

Theorem 4. *Suppose (A1)–(A3) hold, $f \ge 0$, and $f(0) > 0$. Then*

$$X_t/t^\alpha \to c_0 \quad \text{a.s.}$$

The rest of the paper is devoted to proofs. Theorem k is proved in Section (k+1). Sections 2–4 are independent of each other and can be read in any order but the proof in Section 5 depends on results in Sections 3 and 4. In what follows c_0 is the constant in

(1.4) and c is used for constants either slighly larger or smaller than c_0, so we use D to denote dumb constants whose values are unimportant.

2. Upper bound for compactly supported f. In this section we assume that

(i) $f : \mathbb{R}^d \to \mathbb{R}^d$ is Lipshitz continuous,
(ii) $f(x) = 0$ for $\|x\| \geq 1$,
(iii) $\|f(x)\| \leq K$ for all x.

We will prove the following result that after rescaling space and then changing time to make the Brownian motion have variance t gives the version stated in the introduction.

Theorem 1. *There is a constant $\gamma > 0$ so that*

$$\limsup_{t\to\infty} \|X_t\|/t \leq 2/\gamma \quad \text{a.s.}$$

Proof: Consider the one dimensional s.d.e.

$$dY_t = dB_t + \left\{ \frac{d-1}{2(Y_t+2)} + 7\gamma K \right\} dt \qquad Y_0 = 0 \tag{2.1}$$

and impose reflecting boundary conditions at 0. We can choose the parameter $\gamma > 0$ so small that if $\tilde{H} = \inf\{t : Y_t = 2\}$ then

$$P(\tilde{H} \geq 5\gamma) \geq 1/2 \tag{2.2}$$

Now let $H_n = \inf\{t : \|X_t\| = 2n\}$, $\tau_n = H_n - H_{n-1}$, $\mathcal{G}_n = \mathcal{F}_{H_n}$ where $\mathcal{F}_t$ is the filtration generated by the Brownian motion and define events $G_n = \{\tau_n \geq 5\gamma\}$, $F_n = \{\tau_n \leq 2\gamma\}$. When F_n happens we speak of a *fast* crossing from $2n-2$ to $2n$. When G_n happens we speak of a *slow* crossing from $2n-2$ to $2n$.

To prove the result it suffices to show that $\liminf_{n\to\infty} H_n/n \geq \gamma$. The first step in doing this is to observe that if $H_N \leq \gamma N$, then at least half of the $\tau_1, \ldots, \tau_N$ must be smaller than 2γ, so there are at least $N/2$ fast crossings from $2n-2$ to $2n$. The second step will be to show that after a fast crossing from $2n-2$ to $2n$ there is probability at least 1/2 that the crossing from $2n$ to $2n+2$ will be slow, so that the total time to get to $2N$ will be larger than γN with high probability.

To carry out the second step note that the process $R_t = \|X_t\| - 2n$ satisfies

$$dR_t = dB_t + \left\{ \frac{d-1}{2(R_t+2n)} + \int_0^t \frac{X_t}{\|X_t\|} \cdot f(X_t - X_u)\, du \right\} dt \tag{2.3}$$

Now suppose that F_n has happened and $n \geq 1$. Then for any $t \in [H_n, H_n + 5\gamma]$, the drift in (2.3) is bounded above by $(d-1)(2(R_t+2))^{-1} + 7\gamma K$ while the process is in $[2n, 2n+2]$. A standard comparison theorem for stochastic differential equations (see e.g. Rogers and Williams V.43) implies that we can build a process Y' identical in law to Y so that

$$Y'_t \geq \|X(H_n+t)\| - 2n \quad \text{for } t \in [0, 5\gamma]$$

so $P(G_{n+1}|\mathcal{G}_n) \geq 1/2$ on F_n. The desired conclusion now follows from a result of Dubins and Freedman (1965) (see e.g. Durrett (1990) p. 220)

(2.4) *Suppose G_n is adapted to $\mathcal{G}_n$ and let $p_n = P(G_n|\mathcal{G}_{n-1})$. Then*

$$\sum_{m=1}^{n} 1_{G_m} \Big/ \sum_{m=1}^{n} p_m \to 1 \quad \text{a.s. on} \quad \left\{ \sum_{m=1}^{\infty} p_m = \infty \right\}$$

To derive the desired conclusion now we let N_n be the number of fast crossings in the first n trials and $N_\infty = \lim N_n$. On $\{N_\infty < \infty\}$ it is clear that $\liminf_{n\to\infty} H_n/n \geq 2\gamma$. On $\{N_n = \infty\}$ we can apply (2.4) to conclude that the number of slow crossings M_n satisfies

$$\liminf_{n\to\infty} M_n/(N_n/2) \geq 1 \quad \text{a.s.}$$

The last result implies that for $n \geq n_0$,

$$M_n \geq \frac{4}{5}\,\frac{N_n}{2}\,.$$

For $n \geq n_0$ either $N_n \geq n/2$ (in which case $M_n \geq n/5$ and $H_n \geq \gamma n$) or $N_n < n/2$ (in which case there are more than $n/2$ crossings that take more than 2γ units of time and $H_n \geq \gamma n$). In all cases we have $\liminf H_n/n \geq \gamma$ and the proof is complete.

3. Lower bound when f is positive at 0. In this section we consider the one–dimensional situation with $f \geq 0$. We do not require that f have compact support but we do need f to be positive near 0. Again, scaling space and time gives the result in the introduction.

Theorem 2. *Let $A = \inf\{f(x) : |x| \leq \frac{1}{2}\}$. Then*

$$\liminf_{t\to\infty} X_t/t \geq A^{1/2}/2.$$

Proof: We will prove this result by getting a lower bound on the total drift up to time t and then observing that the contribution of the Brownian motion can be ignored. Indeed in this argument Brownian motion could be replaced by any process with $B_t/t \to 0$ as $t \to \infty$. Let $g(x) = A$ when $|x| \leq 1/2$ and $g(x) = 0$ otherwise.

$$\begin{aligned}\int_0^t ds \int_0^s du\; f(X_s - X_u) &\geq \int_0^t ds \int_0^s du\; g(X_s - X_u) \\ &= \tfrac{1}{2} \int_0^t ds \int_0^t du\; g(X_s - X_u) = \tfrac{1}{2}\, J(\mu)\end{aligned}$$

where μ is the occupation measure $\mu(C) = \int_0^t 1_C(X_s)ds$ and

$$J(\nu) \equiv \int \nu(dx) \int \nu(dy)\, g(x-y).$$

If ν is a probability measure supported in $[0, n]$ then

$$J(\nu) \geq A \sum_{k=0}^{2n-1} \nu((k/2, (k+1)/2])^2 \geq A/2n$$

by the Cauchy–Schwarz inequality. Let $b_t = \sup\{X_s : s \leq t\}$ and $a_t = \inf\{X_s : s \leq t\}$. Then

$$\begin{aligned} X_t - B_t &= \int_0^t ds \int_0^s du\, f(X_s - X_u) \\ &\geq \frac{At^2/2}{2(1 + b_t - a_t)} \geq \frac{t^2 A}{4(1 + b_t - \inf_{s \leq t} B_s)}\,. \end{aligned}$$

since $X_t \geq B_t$. On the other hand

$$X_t - B_t \leq 1 + b_t - \inf_{s \leq t} B_s,$$

so it follows that

$$1 + b_t - \inf_{s \leq t} B_s \geq tA^{1/2}/2. \tag{3.1}$$

The last inequality implies that

$$\liminf_{t \to \infty} b_t/t \geq A^{1/2}/2. \tag{3.2}$$

To strengthen this to the conclusion of Theorem 2 observe that if $s < t$

$$X_t - X_s = B_t - B_s + \int_s^t du \int_0^u dv\; f(X_u - X_v),$$

so taking s to be the first time $\max_{r \leq t} X_r$ is attained it follows that

$$X_t - b_t \geq \inf_{s \leq t}(B_t - B_s), \tag{3.3}$$

and

$$\liminf_{t \to \infty} X_t/t \geq \liminf_{t \to \infty} b_t/t. \tag{3.4}$$

Combining (3.4) with (3.2) completes the proof of Theorem 2.

For results in Section 4 we will need a simple extension of the results above.

$$X_t - X_s \geq B_t - B_s + \int_s^t du \int_s^u dv\; f(X_u - X_v).$$

If we let $b_t^s = \max_{s \le r \le t} X_s$ and repeat the proofs of (3.1) and (3.3) it follows that

$$1 + b_t^s - X_s - \inf_{s \le r \le t} (B_r - B_s) \ge (t-s) A^{1/2}/2 \tag{3.5}$$

$$X_t - b_t^s \ge \inf_{s \le r \le t} (B_t - B_r) \tag{3.6}$$

Adding (3.5) and (3.6) gives

$$X_t - X_s \ge (t-s) A^{1/2}/2 + \inf_{s \le r \le t} (B_t - B_r) + \inf_{s \le r \le t} (B_r - B_s) - 1. \tag{3.7}$$

4. Upper bounds for fat tailed f. The assumptions on f throughout this section are

(i) $|f(x)| \le M$
(ii) $f(x) \ge 0$ decreasing for $x \in [q, \infty)$
(iii) $x^\beta f(x) \to \ell > 0$ as $x \to \infty$ where $0 < \beta < 1$

Let $\alpha = 2/(1+\beta) \in (1,2)$, observe $2 - \alpha = \alpha\beta$, and define c_0 by

$$\alpha c_0^{1+\beta} = \int_0^1 \frac{\ell \, du}{(1-u^\alpha)^\beta} . \tag{4.1}$$

Let $\epsilon > 0$ and $c > c_0$. Our aim is to show that if T is large $P(X_T > cT^\alpha) < \epsilon$. The first step in achieving this aim is to make some choices that for the moment will seem rather mysterious. Their purposes will be revealed in the proof below. For the moment the reader should be content to check that such choices are possible. Once we have introduced the necessary definitions we will explain the idea behind the proof. Pick $\theta > 1$ so that

$$\gamma \equiv c - \theta \frac{c_0^{1+\beta}}{c^\beta} > 0. \tag{4.2}$$

Pick $N \ge q$ large enough so that $f(x) \le \theta \ell x^{-\beta}$ for $x \ge N$. Choose $\rho > 0$ and $b \in (0, 1/2)$, so that

$$\nu \equiv 2 - \alpha + 2\rho < \alpha, \tag{4.3}$$
$$\nu < \alpha(\alpha - 1/2)/(\alpha - b) \tag{4.4}$$

This is possible since $2 - \alpha < \alpha$ and the right–hand side of (4.4) is α when $b = 1/2$. Let $\epsilon_T = T^{-\rho}$, define η_T by the requirement

$$T^{2-\alpha} M(\eta_T + T^{-\alpha} N) = (c\gamma/2)\epsilon_T^2. \tag{4.5}$$

Note that the definition of η_T and (4.3) imply $\eta_T \sim (c\gamma/2M)T^{-\nu}$ and let

$$\phi_0(t) = c \int_0^t (\alpha s^{\alpha-1}) \vee \epsilon_T \, ds \qquad \phi_1(t) = \phi_0(t) + \eta_T. \tag{4.6}$$

Let $\tau = \inf\{t : x_t = \phi_1(t)\}$, and let $\sigma = \sup\{u < \tau : x_u = \phi_0(u)\}$. We will show that $P(\tau \leq 1)$ is small by getting an upper bound on the drift at times $s \in [\sigma, \tau]$ which shows that the crossing from ϕ_0 at time σ to ϕ_1 at time τ must be due to an abnormally large fluctuation in the Brownian motion. To bound the drift we let

$$v = \sup\{u : \phi_1(u) + T^{-\alpha} N < \phi_0(s)\}$$

with $\sup \emptyset = 0$. (Note that v depends on s.) Since $x_t < \phi_1(t)$ for $t < \tau$ and $x_s > \phi_0(s)$ using (i) and (ii) (and recalling $N \geq q$) gives

$$\int_0^s f(T^\alpha(x_s - x_u))\, du \leq M(s - v) + \int_0^v f(T^\alpha(\phi_0(s) - \phi_1(u)))\, du. \tag{4.7}$$

To estimate the integral on the right we observe that the choice of N and the definition of v imply

$$f(T^\alpha(\phi_0(s) - \phi_1(u))) \leq \theta\ell\{T^\alpha(\phi_0(s) - \phi_1(u))\}^{-\beta} \leq \theta\ell\{T^\alpha(\phi_1(v) - \phi_1(u))\}^{-\beta}.$$

Now

$$\phi_1(v) - \phi_1(u) = \phi_0(v) - \phi_0(u) \geq c(v^\alpha - u^\alpha),$$

so

$$\int_0^v f(T^\alpha(\phi_0(s) - \phi_1(u)))\, du \leq \frac{\theta}{c^\beta T^{\alpha\beta}} \int_0^v \frac{\ell\, du}{(v^\alpha - u^\alpha)^\beta} = \frac{\theta}{c^\beta T^{\alpha\beta}}\, v^{\alpha-1} \alpha c_0^{\beta+1}, \tag{4.8}$$

by the definition of c_0 given in (4.1). Now the convexity of ϕ_0 implies

$$(s - v)\phi_0'(v) \leq \phi_0(s) - \phi_0(v) = \eta_T + T^{-\alpha} N,$$

and recalling the definition of ϕ_0 gives

$$(s - v) \leq (\eta_T + T^{-\alpha} N)/(c\epsilon_T).$$

Using the last inequality and $v \leq s$ with (4.7) and (4.8) gives

$$\int_0^s f(T^\alpha(x_s - x_u))\, du \leq \left(M(\eta_T + T^{-\alpha} N)/(c\epsilon_T) + \frac{\theta}{c^\beta T^{\alpha\beta}}\, c_0^{\beta+1} \alpha s^{\alpha-1} \right).$$

Recalling $2 - \alpha = \alpha\beta$ and using the definition of η_T in (4.5), and γ in (4.2), we have

$$T^{2-\alpha} \int_0^s f(T^\alpha(x_s - x_u))\, du \leq (\gamma/2)\epsilon_T + (c - \gamma)\alpha s^{\alpha-1} \tag{4.9}$$

The last inequality gives an upper bound on the drift that is smaller than $\phi_0'(s)$. To complete the proof we will now bound the contribution of the Brownian motion. Since we have chosen $b < 1/2$, it follows from Lévy's modulus of continuity for the Brownian path that we can pick K so that

$$P(|B_t - B_s| > K|t-s|^b \text{ for some } s, t \le 1) \le \epsilon.$$

Hence with probability at least $1-\epsilon$,

$$\begin{aligned} K(\tau-\sigma)^b T^{1/2-\alpha} &\ge T^{1/2-\alpha}(B_\tau - B_\sigma) \\ &\ge x_\tau - x_\sigma - (\gamma/2)\epsilon_T(\tau-\sigma) - (c-\gamma)(\tau^\alpha - \sigma^\alpha). \end{aligned} \tag{4.10}$$

by (1.2) and (4.9). Now

$$\phi_1(\tau) - \phi_0(\sigma) = \eta_T + c\int_\sigma^\tau (\alpha s^{\alpha-1}) \vee \epsilon_T \, ds,$$

so the right-hand side of (4.10) is at least

$$\begin{aligned} &\eta_T + c\{(\tau^\alpha - \sigma^\alpha) \vee \epsilon_T(\tau-\sigma)\} - (\gamma/2)\epsilon_T(\tau-\sigma) - (c-\gamma)(\tau^\alpha - \sigma^\alpha) \\ &\qquad \ge \eta_T + \frac{\gamma}{2}\{(\tau^\alpha - \sigma^\alpha) \vee (\epsilon_T(\tau-\sigma))\} \end{aligned} \tag{4.11}$$

since for $a, b, c, x, y > 0$, $c(x \vee y) - ax - by \ge (c - (a+b)) \cdot (x \vee y)$ (consider two cases: $x \ge y$, $x < y$.) Combining (4.10) and (4.11), recalling the definition of η_T, and using

$$\tau^\alpha - \sigma^\alpha = \int_\sigma^\tau \alpha s^{\alpha-1} \, ds \ge \int_0^{\tau-\sigma} \alpha s^{\alpha-1} \, ds = (\tau-\sigma)^\alpha$$

gives

$$K(\tau-\sigma)^b T^{1/2-\alpha} \ge DT^{-\nu} + (\gamma/2)(\tau-\sigma)^\alpha. \tag{4.12}$$

We will now show that our choice of ν makes this inequality impossible for large T. To do this we observe that

$$\begin{aligned} Kh^b T^{1/2-\alpha} &\le DT^{-\nu} \text{ when } h \le (D/K)^{1/b} T^{-(\nu-\alpha+1/2)/b} \\ Kh^b T^{1/2-\alpha} &\le (\gamma/2)h^\alpha \text{ when } h \ge (2K/\gamma)^{1/(\alpha-b)} T^{-(\alpha-1/2)/(\alpha-b)} \end{aligned}$$

Our choice of ν and b in (4.4) implies that

$$\frac{\nu}{b} < \frac{\alpha(\alpha-1/2)}{b(\alpha-b)} = \frac{\alpha-1/2}{\alpha-b} + \frac{\alpha-1/2}{b} \tag{4.13}$$

so

$$\frac{\nu-\alpha+1/2}{b} < \frac{\alpha-1/2}{\alpha-b}$$

and the inequality in (4.12) is impossible for large T. This shows that when $|B_t - B_s| \le K|t-s|^b$ for all $0 \le s \le t \le 1$ it is impossible to have $x_t = \phi_1(t)$ for $t \le 1$ and the proof is complete.

5. Lower bound for fat tailed f. Throughout this section we will suppose

(i) $|f(x)| \le M$
(ii) $f(x)$ is decreasing for $x \in [q, \infty)$
(iii) $x^\beta f(x) \to \ell > 0$ as $x \to \infty$ where $0 < \beta < 1$
(iv) $f(x) \ge 0$ and $f(0) > 0$

The proof of Theorem 4 is similar to that of Theorem 3 but requires more computation. As in the last section we begin by making a number of choices whose purposes will become clear later. After we have enough defintions we will explain the idea behind the proof. Let $\epsilon > 0$ and $c \in (0, c_0)$. We choose $\delta > 0$ to satisfy

$$\gamma \equiv (1-\delta)^2 c_0^{1+\beta} c^{-\beta} - c > 0, \tag{5.1}$$

and then pick N large enough so that

$$f(x) \ge \ell(1-\delta)x^{-\beta} \quad \text{for } x \ge N \tag{5.2}$$

Next choose $b \in (0, 1/2)$, $0 < \lambda < \nu < \alpha$ to satisfy

$$\frac{\alpha}{2(1-b)} > \lambda \tag{5.3}$$

$$\alpha - \frac{1-2b}{2(1-b)} > \lambda \tag{5.4}$$

$$\frac{1}{2(1-b)} - \nu\beta > \lambda(\alpha-1)/\alpha \tag{5.5}$$

$$\nu < \{\alpha - 1/2 - \lambda b(\alpha-1)/\alpha\}/(1-b) \tag{5.6}$$

To see that such choices can be made, note that $\alpha > 1$ implies $2 - \alpha < 1$, so when $\lambda = 0$ and $b = 1/2$ we can pick ν small enough so that inequalities (5.3)–(5.6) hold strictly.

Let $x_t = T^{-\alpha} X(tT)$, $a_T = T^{-\lambda}$, $\eta_T = T^{-\nu}$ and

$$\phi_0(t) = ct^\alpha - a_T \qquad \phi_1(t) = \phi_0(t) - \eta_T$$
$$\tau = \inf\{t : x_t = \phi_1(t)\}$$
$$\sigma = \sup\{u < \tau : x_u = \phi_0(u)\}$$
$$\rho = \inf\{t : x_t = \phi_0(t)\}$$
$$t_T = \inf\{t : \phi_0(t) = 0\} = c^{1/\alpha} T^{-\lambda/\alpha}$$

Our aim will be to show that if

$$W_t = T^{-1/2} B_{tT} \text{ satisfies } |W_t - W_s| \le K|t-s|^b, \tag{$\star$}$$

then for large T, $\tau \le 1$ is impossible. We will do this in two steps. First we will show $\rho \ge t_T$ and then we will show that $t_T \le \sigma < \tau \le 1$ is impossible. In each part of the proof we will use the assumption $f(0) > 0$ to give a lower bound on the rate of growth of X_t.

Lemma 5.1. *If ($\star$) and and $T \ge T_0(K,b)$, we must have $\rho \ge t_T$.*

Proof: When $t < t_T$, $\phi_0(t) \le 0$. To get a lower bound on x_t we observe that (3.7) with $s = 0$ and $t = u$ implies

$$X_u \ge uA^{1/2} + \inf_{r\le u}(B_u - B_r) + \inf_{r\le u} B_r - 1.$$

Changing the time scale $u = tT$, using ($\star$), and dividing by T^α gives

$$x_t = T^{-\alpha}X(tT) \ge \psi(t) \equiv T^{-\alpha}(tTA^{1/2}/2 - 2T^{1/2}Kt^b - 1).$$

Let $\kappa = (4K/A^{1/2})^{1/(1-b)}$. When $t = \kappa T^{-1/2(1-b)}$,

$$tTA^{1/2}/2 = 2T^{1/2}Kt^b.$$

So if $u_T = 2\kappa T^{-1/2(1-b)}$, $t \ge u_T$, and T is large then $\psi(t) > 0$. To take care of $[0, u_T]$ we notice that over this interval

$$\psi(t) \ge -T^{-\alpha}(2T^{1/2}Ku_T^b + 1) = -T^{-\alpha}(DT^{(1-2b)/2(1-b)} + 1)$$
$$\phi_0(t) \le D'T^{-\alpha/2(1-b)} - T^{-\lambda}$$

(5.3) guarantees that for large T, $\phi_0(t) \le -T^{-\lambda}/2$, so using (5.4) we see that for large T $\psi(t) > \phi_0(t)$ for $t \in [0, u_T]$ and the proof of Lemma 5.1 is complete.

To finish the proof of Theorem 4 now it suffices to show:

Lemma 5.2. *If ($\star$) and $T \ge T_1(K,b)$ then it is impossible to have $t_T \le \sigma < \tau \le 1$.*

Proof: Suppose that $t_T \le \sigma < \tau < 1$. and let $\sigma < t < \tau$. We want to get a lower bound on the drift at time t. To do this using (ii) we have to know $X_t - X_s \ge q$ so our first step is to observe that (3.7) says

$$X_t - X_s \ge (t-s)A^{1/2}/2 + \inf_{s\le r\le t}(B_t - B_r) + \inf_{s\le r\le t}(B_r - B_s) - 1,$$

so it follows from the proof of Lemma 5.1 that if $t - s \ge u_T$ and W_t satisfies ($\star$) then

$$X_{tT} - X_{sT} \ge (DT^{(1-2b)/2(1-b)} - 1) \ge q \tag{5.7}$$

for large T. Let $v = t - u_T$. To estimate the drift of $x_t = T^{-\alpha}X(tT)$ we observe that $x_t \le \phi_0(t)$ and if $s \le v$,

$$q \le T^\alpha(x_t - x_s) \le T^\alpha(\phi_0(t) - \phi_1(s)),$$

so (ii) and (iv) imply

$$\int_0^t f(T^\alpha(x_t - x_s))\,ds \geq \int_0^v f(T^\alpha(\phi_0(t) - \phi_1(s))\,ds. \tag{5.8}$$

Using the defintion of the ϕ_i and then (5.2) we see that for large T

$$\begin{aligned}\int_0^v f(T^\alpha(ct^\alpha - cs^\alpha + \eta_T))\,ds &\geq \frac{1-\delta}{c^\beta T^{\alpha\beta}} \int_0^v \frac{\ell\,ds}{(t^\alpha - s^\alpha + \eta_T/c)^\beta} \\ &= \frac{1-\delta}{c^\beta T^{\alpha\beta}} \left[\int_0^t \frac{\ell\,ds}{(t^\alpha - s^\alpha + \eta_T/c)^\beta} - \int_v^t \frac{\ell\,ds}{(t^\alpha - s^\alpha + \eta_T/c)^\beta}\right].\end{aligned} \tag{5.9}$$

Changing variables $s = tu$ and using $\alpha\beta = 2 - \alpha$

$$\begin{aligned}\int_0^t \frac{\ell\,ds}{(t^\alpha - s^\alpha + \eta_T/c)^\beta} &= t^{\alpha-1}\int_0^1 \frac{\ell\,du}{(1 - u^\alpha + \eta_T/ct^\alpha)^\beta} \\ &\geq (1-\delta)t^{\alpha-1}\alpha c_0^{1+\beta}\end{aligned} \tag{5.10}$$

for large T, since $t > t_T$ and $\nu > \lambda$ imply $\eta_T/ct^\alpha \leq \eta_T/ct_T^\alpha \to 0$. To estimate the second integral on the right hand side of (5.9) we observe that $t^\alpha - s^\alpha \geq 0$ implies

$$\int_v^t \frac{\ell\,ds}{(t^\alpha - s^\alpha + \eta_T/c)^\beta} \leq c^\beta \ell(t-v)/\eta_T^\beta = 2\ell c^\beta u_T/\eta_T^\beta. \tag{5.11}$$

Combining (5.9)–(5.11) and using the definition of γ in (5.1) gives

$$\int_0^v f(T^\alpha(ct^\alpha - cs^\alpha + \eta_T))\,ds \geq \frac{(\gamma+c)t^{\alpha-1}\alpha}{T^{\alpha\beta}} - \frac{(1-\delta)2\ell u_T}{T^{\alpha\beta}\eta_T^\beta}. \tag{5.12}$$

Now we use (1.2) and the relationship $2 - \alpha = \alpha\beta$ to write

$$\begin{aligned}-T^{1/2-\alpha}(W_\tau - W_\sigma) &= -x_\tau + x_\sigma + T^{2-\alpha}\int_\sigma^\tau dt \int_0^t f(T^\alpha(x_t - x_s))\,ds \\ &\geq -c(\tau^\alpha - \sigma^\alpha) + \eta_T + (\gamma + c)(\tau^\alpha - \sigma^\alpha) \\ &\qquad - (1-\delta)2\ell u_T\,\eta_T^{-\beta}(\tau - \sigma)\end{aligned} \tag{5.13}$$

Letting $\xi_T = 2(1-\delta)\ell u_T \eta_T^{-\beta}$, then using the convexity of x^α and the fact that $\sigma \geq t_T$ we can write (5.13) as

$$\begin{aligned}-T^{1/2-\alpha}(W_\tau - W_\sigma) &\geq \gamma(\tau^\alpha - \sigma^\alpha) + \eta_T - \xi_T(\tau - \sigma) \\ &\geq \eta_T + (\gamma\alpha\sigma^{\alpha-1} - \xi_T)(\tau - \sigma) \\ &\geq \eta_T + (\gamma\alpha t_T^{\alpha-1} - \xi_T)(\tau - \sigma).\end{aligned} \tag{5.14}$$

Now (ignoring constants)

$$t_T^{\alpha-1} = T^{-\lambda(\alpha-1)/\alpha} \qquad \eta_T^{-\beta} = T^{\nu\beta} \qquad u_T = T^{-1/2(1-b)}$$

and we have supposed (see (5.5))

$$-\frac{1}{2(1-b)} + \beta\nu < -\lambda(\alpha-1)/\alpha$$

so $\xi_T = o(t_T^{\alpha-1})$ and the right hand–side of (5.14) is positive for large T. Using ($\star$) now, it follows that for large T

$$KT^{1/2-\alpha}|\tau-\sigma|^b \geq \eta_T + \frac{\gamma\alpha}{2} t_T^{\alpha-1}(\tau-\sigma). \tag{5.15}$$

To see that this is impossible for large T we notice that

$$\begin{aligned} KT^{1/2-\alpha}h^b \leq \eta_T \quad &\text{when} \quad h \leq DT^{(-\nu-1/2+\alpha)/b} \\ KT^{1/2-\alpha}h^b \leq \frac{\gamma\alpha}{2} t_T^{\alpha-1}h \quad &\text{when} \quad h \geq DT^{(1/2-\alpha+\lambda(\alpha-1)/\alpha)/(1-b)} \end{aligned}$$

and (5.6) implies that

$$\frac{-\nu-1/2+\alpha}{b} > \frac{1/2-\alpha+\lambda(\alpha-1)/\alpha}{1-b}.$$

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