

# A BAYESIAN SOLUTION TO THE EQUITY PREMIUM PUZZLE

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leads to equilibrium prices

$$S_t = E_t \left[ \sum_{j \geq 0} \beta^j \frac{U'(y_{t+j})}{U'(y_t)} y_{t+j} \right], \quad B_t = \beta E_t \left[ \frac{U'(y_{t+1})}{U'(y_t)} \right].$$

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- stock price is only finite if  $\beta E[\exp\{(1-R)\xi_1\}] < 1$ .

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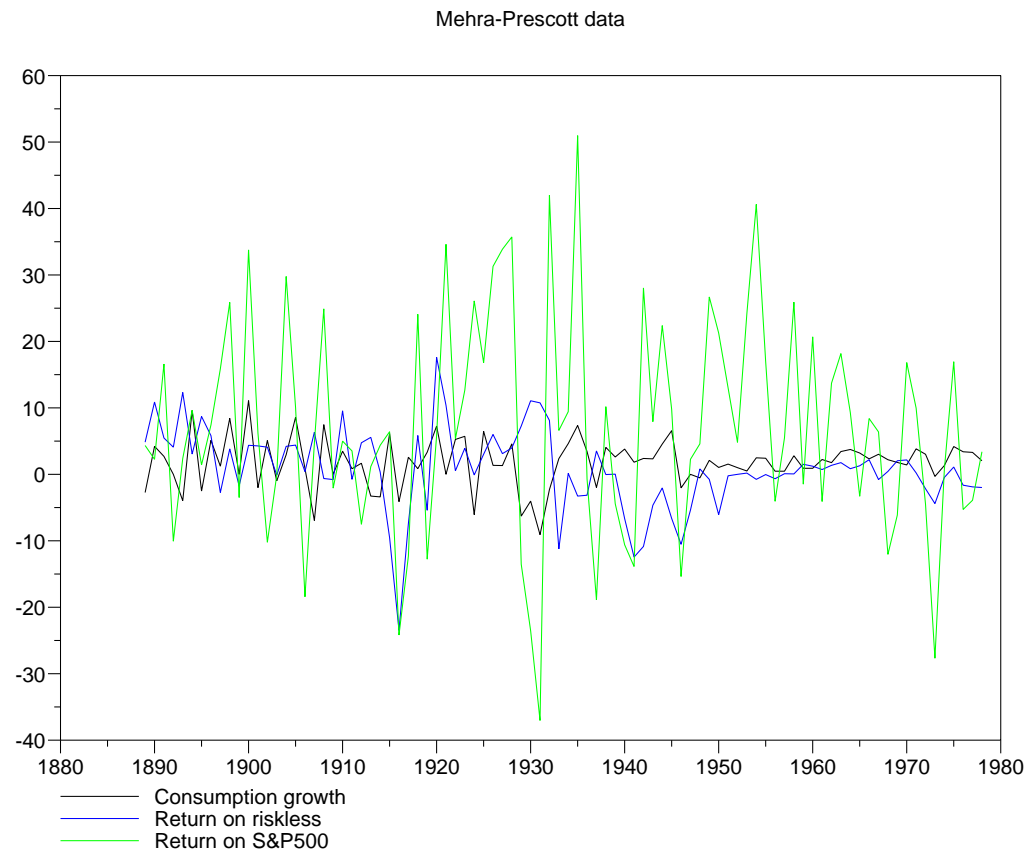
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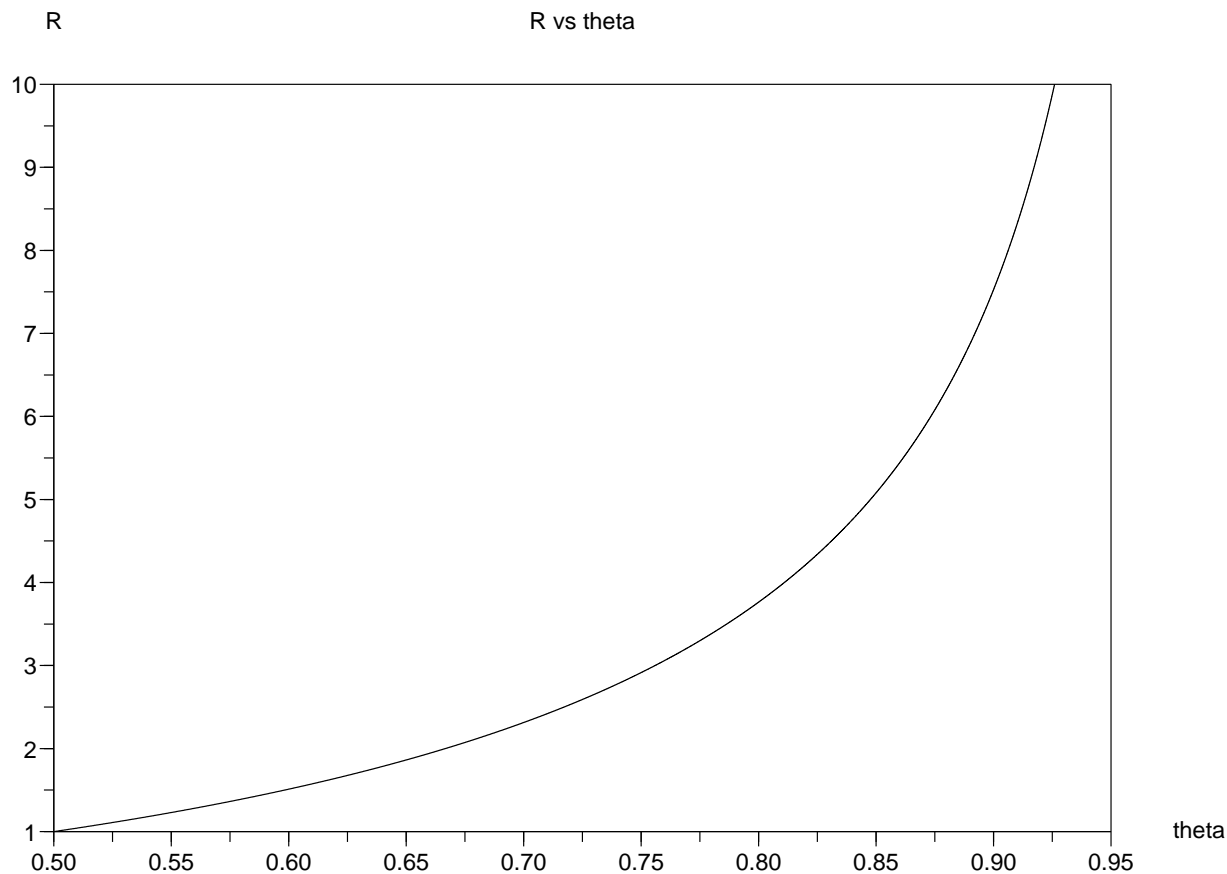
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Kocherlakota (JEL 96) and Mehra & Prescott (Handbook of the Economics of Finance 2003) can't endorse any of these alternatives ...

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You see daily prices on a stock with  $\mu = 20\%$ , and  $\sigma = 20\%$ , and you want to observe for long enough that your 95% (=19/20) confidence interval for the parameters is good to 0.01 (=  $\mu/20 = \sigma/20$ ).

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**Conclusion:** We **must** treat  $\mu$  as unknown, and infer it from the observed prices.



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where  $\alpha_t, b_t, K_t$  depend on  $\alpha_0, b_0, K_0$  and  $\xi_{[1,t]}$ .

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where  $\alpha_t, b_t, K_t$  depend on  $\alpha_0, b_0, K_0$  and  $\xi_{[1,t]}$ . Same form as before:

$$\tau \sim \Gamma(\alpha_t, b_t), \quad \mu | \tau \sim N(m_t, (K_t \tau)^{-1}).$$

## Bayes ...

So (writing  $\nu = R - 1$ , assumed positive (wmlong) )

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We therefore have

$$S_t = y_t f_t(\xi_{[1,t]}; \beta, R, \alpha_0, b_0, K_0), \quad B_t = g_t(\xi_{[1,t]}; \beta, R, \alpha_0, b_0, K_0).$$

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Econometrician sees

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approximated by selecting descendent  $x_{t+1}^i$  of  $x_t^i$  according to  $p(\cdot|x_t^i)$ , and then

$$w_{t+1}^i \propto w_t^i f(Y_{t+1}|x_{t+1}^i).$$

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*Results.*

# Results.

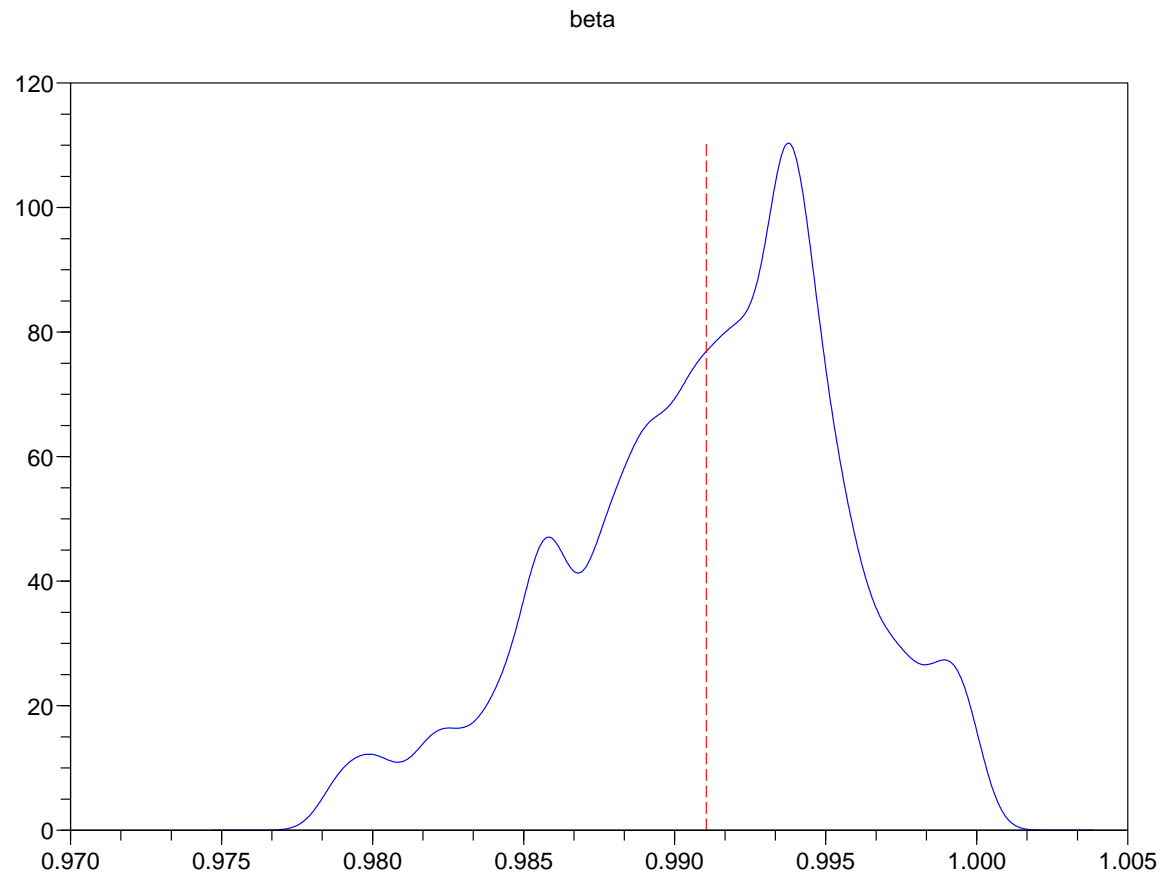


Figure 1: Posterior density for  $\beta$

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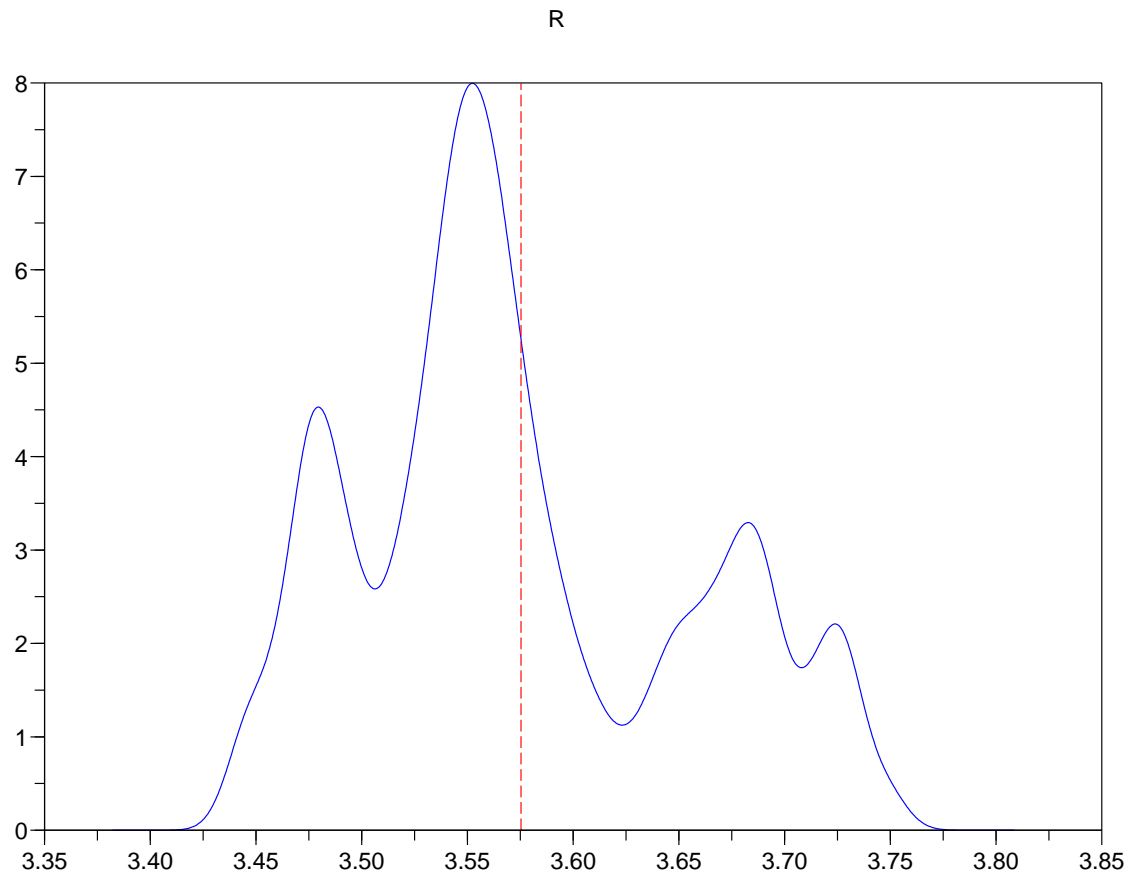


Figure 2: Posterior density for  $R$

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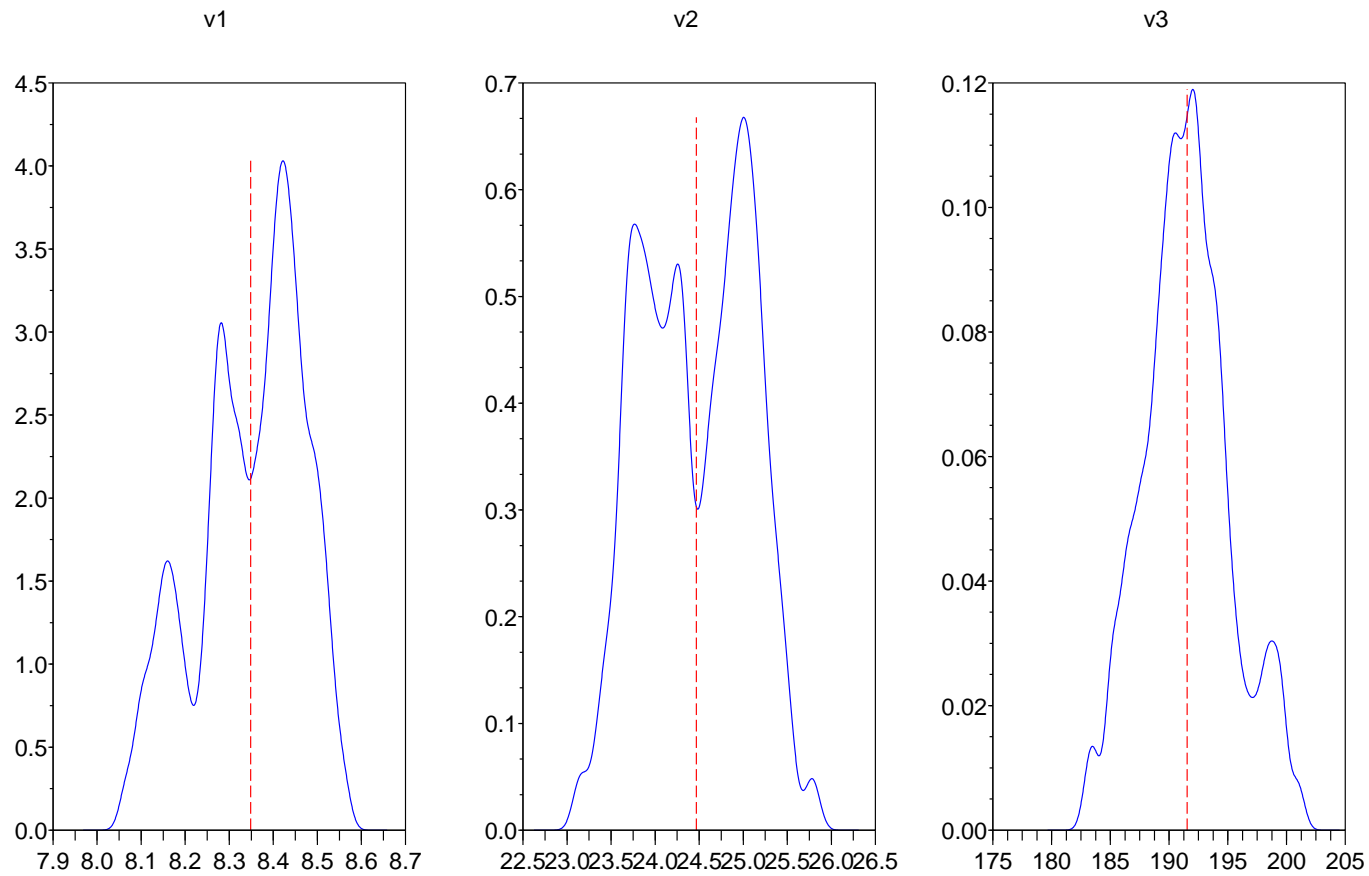


Figure 3: Posterior densities for  $v_1$ ,  $v_2$  and  $v_3$ , measured in bp. Original SSQs for first variable is 12.86 bp, for second is 35.04 bp, and for third is 273.6 bp

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.. but the apparently large value of the equity premium is **not** one of them, once you acknowledge and properly handle the enormous uncertainty in the rate of return of the stock index.