GEOMETRIC INDICES;

A THEORY OF HEDGING AND ECONOMETRIC ANALYSIS WITH APPLICATION TO THE U.K. STOCK MARKET

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In this paper the authors describe how to hedge a contract written on a specific geometric index, and calculate the fair (no-arbitrage) price of the contract in a log-Brownian complete markets framework; results are also discussed in the more general context of semi-martingale price processes. Tests for arbitrage violations are discussed and carried out on daily U.K. data for the FT30 index from January 2, 1988 to December 31, 1991. The procedure seems to work well and be robust to jumps in a sense made precise in the paper.

Keywords : Geometric Index, Hedging, Fair Price, Semi-Martingale Process

In the UK stock market, two domestic indices that are frequently quoted are the FT100 and the FT30 indices. As a summary of the direction and the extent of average changes of stock prices, the indices provide a convenient way to indicate general market movements and a powerful source of information for investment decisions. They can be also good instruments of trading in baskets of securities for the portfolio managers and of creating derivative financial intermediaries. The Financial Times Ordinary Share Index (FT30) is the geometric average of 30 shares on an unweighted basis, the geometric average involves the product of the 30 prices of the component stocks taken to the 30th root. A curious fact about these indices is that whilst there is a myriad of contingent contracts written on the FT100 index, there is not a single one written on the FT30 index, at least not that is exchange-traded. In the United States, the Value Line Composite Index, which is a geometric mean of 1700 shares, has had both option and futures contracts written on it.

One explanation given for this phenomenon is that the FT100 index reflects the market portfolio more accurately, being value weighted and comprising more shares. A second explanation given is that the FT30 index, being a geometric average, cannot be hedged since it cannot be replicated by an appropriate portfolio of assets.[2] A third reason given for the lack of interest in the FT30 index is because of its downward bias. If we pay the geometric average out at time t, we should expect at time T, $T \geq t$, to receive a return which is lower than the geometric mean of the expected returns, reflecting the downward bias of the geometric average.

The contribution of this paper is to address the last two points; we construct a hedging portfolio for a contract whose payoff is, say, the FT30 index under the assumption that the thirty shares are joint log-normally distributed and we show that this portfolio will also hedge if the prices follow more general processes. We test the performance of our replicating portfolio using daily data from January 2, 1988 to December 31, 1991. Furthermore, we investigate the properties of the residuals from the FT30 minus its replicating portfolio, the purpose being to see if hedging is a practical reality. We discuss the problem of estimating the fair price that our contract written on the FT30 index should be sold at. This analysis answers the third anxiety mentioned earlier, our fair price and its practical analogue, the sample fair price, adjusts the price of the FT30 contract downward in order to adjust for the bias in the geometric mean.

This paper is organized as follows. In Section I, we formulate and prove the basic results on hedging the FT30 index. We first show how to hedge in the log-Brownian case, and then generalize our assumptions. In Section II, under the log-Brownian distribution, we discuss the resulting estimation problems. We then study the FT30 index from January 2, 1988 to December 31, 1991, comparing the FT30 index with its replicating portfolio. Conclusions follow in Section III.

1. FAIR PRICE OF THE FT30 INDEX

In this section we shall imagine that we are bankers offering the following contract. At time t you pay a price H to take a long position in the FT30 contract. At time $t+\tau$, $\tau \geq 0$, you receive the amount $(\prod_{j=1}^{N} S_j(t+\tau))^{\frac{1}{N}}$ where $S_j(t+\tau)$ is the price at time $t+\tau$ of the jth share and N is the number of different shares. In the empirical work on the FT30 index, the number N of shares is, of course, 30, but we keep it general for now. The issue we initially address is , what is the "fair" price, H, that the bank should charge for such contracts? Since the FT30 contract has a non-linear payoff, the contract evaluated at the expected values of each share cannot provide the fair price of the contract, since, by Jensen's inequality, $E_t[(\prod_{j=1}^{N} S_j(t+\tau))^{\frac{1}{N}}] \leq [\prod_{j=1}^{N} E_t(S_j(t+\tau))]^{\frac{1}{N}}$ where $E_t[\cdot\ ]$ is the expectation operator at time t.

Before we answer this question, we shall start with details of the distribution of the prices. We shall assume that there are N shares with time t prices $S_j(t)$. The prices follow log-normal diffusion processes, that is,

$$dS_j(t) = \alpha_j S_j(t)dt + S_j(t)\sum_{k=1}^{N}\sigma_{jk}dz_k(t) \tag{1}$$

where $\Sigma=\{\sigma_{jk}\}$ is an (N×N) non-singular matrix, α_j and σ_{jk} are constant, and $dz(t)=(dz_1(t),...,dz_k(t),...,dz_N(t))'$ is a vector of independent Brownian motions (B.M.). We shall also assume the existence of a bond, which pays a constant rate of interest r. This is recognizable as a complete markets frame-work as described in Harrison and Kreps (1979) or elsewhere. Hence, we are able to determine the no-arbitrage or "fair" price of our contract. We make the usual simplifying assumptions about zero dividends, zero transaction costs, perfect markets, etc.

Our first argument, familiar to the readers of Merton (1973), goes as follows. Let the fair price of our FT30 contract be $H=H(S(t),\tau)$, where $S(t)$ is the $(N\times 1)$ vector of prices, and the contract matures at time $t+\tau$. Since the contract has a payoff $(\prod_{j=1}^{N} S_j(t+\tau))^{\frac{1}{N}}$ which is a homogeneous function in $S_j(t+\tau)$, this contract can be hedged by holding the N shares, there is no need to hold bonds. The dynamics of H can be described via Itô's lemma as

$$
\begin{aligned}
dH &= \sum_{j=1}^{N} H_j dS_j + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} H_{ij} dS_i dS_j - H_\tau dt \\
&= (\sum_{j=1}^{N} H_j \alpha_j S_j + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} H_{ij} S_i S_j \omega_{ij} - H_\tau)dt + \sum_{j=1}^{N} H_j S_j \sum_{k=1}^{N} \sigma_{jk} dz_k(t) \\
&= H\beta\, dt + H\sum_{k=1}^{N} \gamma_k dz_k(t) \\
\text{where } \beta &= (\sum_{j=1}^{N} H_j \alpha_j S_j + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} H_{ij} S_i S_j \omega_{ij} - H_\tau)/H, \\
\omega_{ij} &= \sum_{k=1}^{N} \sigma_{ik}\sigma_{jk}, \qquad (i,j=1,\dots,N) \\
\text{and } \gamma_k &= \sum_{j=1}^{N} H_j S_j \sigma_{jk}/H .
\end{aligned}
\tag{2}
$$

Let the ECU value of your holdings in the contract be W_0, your holdings in asset j is W_j, we assume zero aggregate investment at all times t, thus $\sum_{j=0}^{N} W_j=0$, this portfolio is the self-financing portfolio of finance, it pays no interim dividends (positive or negative) until the maturity of the contract.

The instantaneous value of the portfolio, P, is

$$
dP = \frac{dH}{H} W_0 + \sum_{j=1}^{N} \frac{dS_j}{S_j} W_j . \tag{3}
$$

Substituting from (1), (2) and collecting terms we see that

$$
dP = (\beta W_0 + \sum_{j=1}^{N} \alpha_j W_j)dt + \sum_{k=1}^{N}(W_0 \gamma_k + \sum_{j=1}^{N} W_j \sigma_{jk}) dz_k(t) . \tag{4}
$$

To render dP riskless we must choose W_0 and $\{W_j\}_{j=1}^{N}$ so that

(5) $$W_0\gamma_k + \sum_{j=1}^{N} W_j\sigma_{jk} = 0 \qquad (k=1,\ldots,N)\ .$$

The implication of such a choice together with the self-financing nature of the portfolio implies that

(6) $$\beta W_0 + \sum_{j=1}^{N}\alpha_j W_j = 0\ .$$

Equation (5) can be rewritten as

(7) $$\sum_{j=1}^{N}\left(\frac{W_0 H_j S_j}{H} + W_j\right)\sigma_{jk} = 0 \qquad (k=1,\ldots,N)\ .$$

Since the matrix $\Sigma=\{\sigma_{jk}\}$ is assumed non-singular, this implies that .

(8) $$\frac{W_0 H_j S_j}{H} + W_j = 0 \qquad (j=1,\ldots,N)\ .$$

Equation (6) can also be rewritten using (2) as

(9) $$W_0\left(\sum_{j=1}^{N} H_j S_j \alpha_j + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} H_{ij}S_iS_j\omega_{ij} - H_\tau\right) + H\sum_{j=1}^{N}\alpha_j W_j = 0\ .$$

Using (8) and substituting, we see that

(10) $$\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} H_{ij}S_iS_j\omega_{ij} = H_\tau$$

is the (vector) non-linear partial differential equation for the fair price H of the contract. Our particular contract has the payoff $(\prod_{j=1}^{N} S_j(t+\tau))^{\frac{1}{N}}$, thus a trial solution might be of the form

(11) $$H(S(t),\tau) = q(\tau)\left(\prod_{j=1}^{N} S_j(t)\right)^{\frac{1}{N}} \quad where\ q(0)=1 \qquad .$$

Hence, we have

$$(12)\qquad \begin{aligned} H_j &= \frac{H}{NS_j} \\ H_{ij} &= \frac{H}{N^2 S_i S_j} \qquad i \neq j \\ &= \frac{(1-N)H}{N^2 S_i^2} \qquad i = j \\ H_\tau &= \frac{q'(\tau)H}{q(\tau)} \end{aligned}$$

thus equation (10) becomes

$$(13)\qquad \frac{1}{2}\sum_{i}^{N}\sum_{j\neq i}^{N}\frac{\omega_{ij}}{N^2} + \frac{1}{2}\sum_{i=1}^{N}\frac{\omega_{ii}(1-N)}{N^2} = \frac{q'(\tau)}{q(\tau)} .$$

The last equation is easy to solve, in fact

$$(14)\qquad q(\tau) = \exp\left(-\frac{1}{2}\left(\frac{tr(\Omega)}{N} - \frac{e'\Omega e}{N^2}\right)\tau\right)$$

so that the fair price of the contract is

$$(15)\qquad H(S(t),\tau) = \exp\left(-\frac{1}{2}\left(\frac{tr(\Omega)}{N} - \frac{e'\Omega e}{N^2}\right)\tau\right)\left(\prod_{j=1}^{N} S_j(t)\right)^{\frac{1}{N}} .$$

Note that Ω is the matrix $\{\omega_{ij}\}$, this is the unit time variance-covariance matrix of $\ln(S_j(t+1)/S_j(t))$, $j=1,\ldots,N$, $tr(\Omega)=\Sigma\omega_{ii}$ and e is an $(N\times 1)$ vector of ones.

The term $q(\tau)$ is a number between 0 and 1 since

$$(16)\qquad \begin{aligned} \frac{1}{N}\zeta &= \frac{1}{N}\left(tr(\Omega) - \frac{e'\Omega e}{N}\right) \\ &= \frac{1}{N}\left(tr\left(\Omega - \frac{\Omega ee'}{N}\right)\right) \\ &= \frac{1}{N}\left(tr\left(\Omega\left(I - \frac{ee'}{N}\right)\right)\right) \end{aligned}$$

and $I-ee'/N$ is idempotent, it can be written as CC' so that $\frac{1}{N}\zeta = \frac{1}{N}tr(C'\Omega C)$, $\zeta>0$ follows from the positivity of the trace of a non-null positive semi-definite matrix, thus it proves $0<q(\tau)<1$. Although one may believe that the market knows the value of $\exp(-\frac{\zeta}{2N})$, the economist does not and this will involve estimation, we shall deal with this in the next section.

We now calculate W_0 and W_j. From the solution (8), it is immediate that

$$W_j = -\frac{W_0}{N} \tag{17}$$

so that to hedge the FT30 contract you hold a portfolio $(W_0/N)e'$, that is , invest the value of your holdings in the FT30 contract equally in the N shares. An alternative view might be more intuitive, if you are holding the N shares long in equal investments in value terms, then a perfect hedge can be attained by taking an appropriate short position on the FT30 contract.

Our result is summarized as Theorem 1.

THEOREM 1: *Under the assumptions given by equation (1), in order to hedge the FT30 contract you invest the value of your holdings in the FT30 contract equally in the 30 shares. And the fair price of the contract is,* $H(S(t),\tau) = \exp(-\frac{\tau}{2N}(tr(\Omega) - \frac{e'\Omega e}{N}))(\prod_{j=1}^{N} S_j(t))^{\frac{1}{N}}$, *for* $N=30$.

We now move to the more general case where the log-price processes of the N shares can be any semi-martingales. As before, let $S_j(t)$ be the price at time t of the *j*th share, and write

$$Y(t) = (\prod_{j=1}^{N} S_j(t))^{\frac{1}{N}}$$

for the index at time t. We assume that for each j,

$$d(\log S_j(t)) = dX_j(t) + dA(t) \tag{18}$$

where X_j is a continuous semi-martingale, and A is a common finite-variation process, for details and definitions, see Rogers and Williams (1987). Then

$$d(\log Y(t)) = dX(t) + dA(t) \tag{19}$$

where $X(t) = (\sum_{j=1}^{N} X_j(t))/N$. One application of the general case is that the share prices depend upon mixed jump processes of random size. A special case of equation (19) is log-normal

Brownian motions with Poisson jumps that have normal magnitudes, this model has been studied by Press (1967), Beckers (1981) and Akgiray and Booth (1987). We shall assume that equation (1) has been changed to

$$(20) \qquad dS_j(t) = S_j(t)(\alpha_j dt + \sum_{k=1}^{N}\sigma_{jk}dz_k(t) + (\exp(Q)-1)dq(t))$$

where Q is a random variable. We can think of Q as the random log-rate of profit in $S_j(t)$ due to the shock dq(t). The process q(t) is a Poisson process with Poisson parameter λ, jump size exp(Q)-1, this is a compound Poisson process. It may seem that there is some limitation in assuming only one jump for the whole market, nevertheless this corresponds to a common shock such as an unpredicted change in stamp duty or a change in dealer's charges all of which are calculated as rates common to all shares rather than absolute amounts. Also, it is necessary for the semi-martingale processes that we consider that the jump parts be the same for each share. Then, equation (19) holds.

Returning to the general case (19), we can use Itô's formula to obtain the stochastic equation for Y, see Theorem (39.1) in Rogers and Williams (1987),[3]

$$(21) \qquad \begin{aligned} dY_t &= d(e^{\log Y_t}) \\ &= Y_{t-}\{d(\log Y_t) + \frac{1}{2}d\langle X\rangle_t\} + \Delta Y_t - Y_{t-}\Delta(\log Y)_t \\ &= Y_{t-}\{dX_t + dA_t + \frac{1}{2}d\langle X\rangle_t + e^{\Delta A_t} - 1 - dA_t\} \\ &= Y_{t-}\{dX_t + \frac{1}{2}d\langle X\rangle_t + e^{\Delta A_t} - 1\} \end{aligned}$$

where $\langle X\rangle_t$ is a predictable quadratic covariation process. If we similarly develop $S_j(t)$, we obtain

$$dS_j(t) = d(e^{\log S_j(t)})$$

(22)
$$= S_j(t-)\{d(\log S_j(t)) + \frac{1}{2}<X_j>_t\} + \Delta S_j(t) - S_j(t-)\Delta(\log S_j)_t$$
$$= S_j(t-)\{dX_j(t) + \frac{1}{2}<X_j>_t + e^{\Delta A_t} - 1\} .$$

Combining (21) and (22), we learn that

(23)
$$\frac{dY_t}{Y_{t-}} = \frac{1}{N}\sum_{j=1}^{N}\frac{dS_j(t)}{S_j(t-)} + \frac{1}{2}d<X>_t - \frac{1}{2N}\sum_{j=1}^{N}d<X_j>_t .$$

If we now make the simplifying assumption that for some constant $k\geq 0$

(24)
$$\frac{1}{2}d<X>_t - \frac{1}{2N}\sum_{j=1}^{N}d<X_j>_t = -kdt ,$$

an assumption that will often be true, then (23) can be rephrased as

(25)
$$d(e^{kt}Y_t) = \frac{1}{N}\sum_{j=1}^{N}\frac{dS_j(t)}{S_j(t-)}e^{kt}Y_{t-} .$$

The interpretation of this is very striking; under the assumptions (18) and (24), the process $W_t=e^{kt}Y_t$, the wealth process, is a self-financing portfolio, which at time t invests the same sum $\frac{1}{N}e^{kt}Y_{t-}\equiv\frac{1}{N}W_{t-}$ in each of the N shares.

What would be a fair price, H of the contract at time t which pays the investor an amount $Y_{t+\tau}$ at a fixed later time $t+\tau$? By using the self-financing property of the wealth process, we find that $e^{kt}Y_t=e^{k(t+\tau)}Y_{t+\tau}$. Thus, the fair price of the contract paying $Y_{t+\tau}$ is

(26)
$$H(S(t),\tau) = e^{-k\tau}(\prod_{j=1}^{N}S_j(t))^{\frac{1}{N}} .$$

In this derivation, there's no need to postulate a risk-free bond, this can be contrasted with the Black and Scholes model where to hedge an option we need a riskless bond. We present this result as Theorem 2.

THEOREM 2: *Under the assumption that prices follow a general semi-martingale process given by equation (18) and that the predictable quadratic covariation processes of the assets* *and the FT30 index satisfy equation (24), then the FT30 can be hedged by holding equal value in each asset and the fair price of the contract is given by equation (26).*

COROLLARY : *In a market where N shares are driven by N Brownian motions and a jump process which is compound Poisson in equation (20), the fair price of the FT30 contract is given by* $H(S(t),\tau)=e^{-k\tau}(\prod_{j=1}^{N} S_j(t))^{\frac{1}{N}}$ *where* $k=\frac{1}{2N}(tr(\Omega)-\frac{e'\Omega e}{N})$.

(Proof of Corollary) It is well known that the last term in equation (20) is a finite variation process and

$$
\begin{aligned}
d\langle X_j\rangle_t &= E[(dX_j(t))^2 | I_t] \\
&= E[(\alpha_j dt + \Sigma\sigma_{jk} dz_k(t))^2 | I_t] \\
&= E[\alpha_j^2 (dt)^2 + 2\alpha_j dt(\Sigma\sigma_{jk} dz_k(t)) + (\Sigma\sigma_{jk} dz_k(t))^2 | I_t] \\
&= \Sigma\sigma_{jk}^2 dt \qquad \textit{since } \langle dz\rangle_t = E[(dz(t))^2 | I_t] = dt \textit{ and } z_k's \textit{ are independent} \\
&= \omega_{jj} dt
\end{aligned}
$$

$$
\therefore \frac{1}{2N}\sum_{j=1}^{N} d\langle X_j\rangle_t = \frac{1}{2N}\sum_{j=1}^{N}\omega_{jj} dt = \frac{tr(\Omega)}{2N} dt
$$

$$
\begin{aligned}
d\langle X\rangle_t &= E[(dX(t))^2 | I_t] \\
&= E[(\frac{1}{N}\Sigma X_j(t))^2 | I_t] \\
&= \frac{1}{N^2} E[\{\sum_j(\alpha_j dt + \sum_k \sigma_{jk} dz_k(t))\}^2 | I_t] \\
&= \frac{1}{N^2} E[(\sum_j\sum_k \sigma_{jk} dz_k(t))^2 | I_t] \\
&= \frac{1}{N^2} E[(e'\Sigma dz) dz'\Sigma' e | I_t] \qquad \textit{where } \Omega = \Sigma\Sigma' \\
&= \frac{1}{N^2} e'\Sigma E[dz dz' | I_t]\Sigma' e = \frac{1}{N^2} e'\Sigma I \Sigma' e dt = \frac{e'\Omega e}{N^2} dt
\end{aligned}
$$

$$
\therefore \frac{1}{2} d\langle X\rangle_t = \frac{e'\Omega e}{2N^2} dt
$$

$$
\therefore k = \frac{1}{2}\{\frac{tr(\Omega)}{N} - \frac{e'\Omega e}{N^2}\}. \qquad Q.E.D.
$$

It is obvious that Theorem 1 is a special case of Theorem 2.

Of course, the preferred hedge for many institutions may be partial rather than total insurance in which case one can price an option on the FT30 contract, the option for a fixed exercise price will be easily determined along familiar Black and Scholes lines, we note, however, that hedging the option would now require us to hold bonds. We proceed with the calculation of the price of an European call option on the FT30 index generated by equation (1).

THEOREM 3: *Under the assumptions of Theorem 1 and the existence of a riskless bond, the price of call option for an exercise price E and maturity T equal to the maturity of the contract at time 0 is* $e^{-kT}BS(\eta_0,T,e^{kT}E,r,\sigma_H^2)$ *and* $\sigma_H^2=\Sigma\Sigma\omega_{ij}/N^2$ *where* $BS(x,T,K,r,\sigma^2) = x\Phi(d_1)-Ke^{-rT}\Phi(d_2)$ *is the Black-Scholes formula.*

(Proof) If $\eta_t=e^{kt}Y_t$, then

$$\frac{d\eta_t}{\eta_t} = \frac{1}{N}\sum_{j=1}^{N}\frac{dS_j(t)}{S_j(t)} \tag{27}$$

$$\text{and} \quad \frac{de^{-rt}\eta_t}{e^{-rt}\eta_t} = \frac{1}{N}\Big(\sum_{j=1}^{N}\frac{dS_j(t)}{S_j(t)}\Big) - rdt = \frac{1}{N}\sum_{j=1}^{N}\frac{d\tilde{S}_j(t)}{\tilde{S}_j(t)}$$

where $\tilde{S}_j(t)\equiv e^{-rt}S_j(t)$. Thus under the equivalent martingale measure the existence of which requires the existence of a riskless bond, $e^{-rt}\eta_t$ is a martingale and

$$\begin{aligned}\tilde{E}_0[e^{-rT}(Y_T-E)^+] &= \tilde{E}_0[e^{-rT}(e^{-kT}\eta_T-E)^+] \\ &= e^{-kT}\tilde{E}_0[e^{-rT}(\eta_T-e^{kT}E)^+]\,. \end{aligned} \tag{28}$$

Q.E.D.

The above formula has been independently found by Cakici, Eythan and Harpaz (1988) in their pricing of a call option on the Value Line Composite Index.

2. ESTIMATION

In this section we shall work with the simple case of equation (1). Before we start our empirical analysis, we would like to discuss the statistical problems involved. The problem of computing the fair price, given a data set of daily prices $\{S_j(t)\}$, j=1,...,N and t=1,..,T, is reduced by equation (14) to estimating Ω, the variance-covariance matrix of the daily log-returns. It is well-known that under the assumptions in equation (1) the maximum likelihood estimator of ω_{ij} is

$$\hat{s}_{ij} = \frac{1}{T-1}\sum_{t=1}^{T-1}(r_{it} - \bar{r}_i)(r_{jt} - \bar{r}_j)$$
$$\text{where } r_{it} = \log(S_i(t+1)/S_i(t))$$
$$\text{and } \bar{r}_i = \sum_{t=1}^{T-1} r_{it}/(T-1) \;.$$

The obvious unbiased estimator of ω_{ij} is

$$s_{ij} = \frac{1}{T-2}\sum_{t=1}^{T-1}(r_{it} - \bar{r}_i)(r_{jt} - \bar{r}_j) \;.$$

Then, the matrix $S=\{s_{ij}\}$ is known to follow a Wishart distribution of dimension N with parameters T-1 and $\Omega/(T-2)$, which we shall denote as $W_N(T-1,\Omega/(T-2))$. The statistic is $\exp(-\frac{1}{2N}tr(S(I_N - \frac{ee'}{N})))$, so we need to know the distribution of tr(SM) where M is an idempotent matrix. We note that tr(S) has a known distribution. In fact,

$$\text{(29)} \qquad Prob(tr(S)<x) = \det(\frac{\lambda^{-1}\Omega}{T-2})^{-\frac{T-1}{2}}\sum_{k=0}^{\infty}\frac{1}{k!}Prob(\chi^2_{N(T-1)+2k} \le \frac{x}{\lambda})\sum_{\kappa}(\frac{T-1}{2})_{\kappa}C_{\kappa}(I_N - \Omega^{-1}\lambda(T-2))$$

where $0<\lambda<\infty$ is arbitrary. The second summation is over all partitions $\kappa=(k_1,k_2,\ldots,k_N)$, where $k_1 \ge \ldots \ge k_N \ge 0$, of the integer k, $C_\kappa(\cdot)$ is the zonal polynomial corresponding to κ, and

$$(\frac{T-1}{2})_{\kappa} = \prod_{i=1}^{N}(\frac{T-1}{2} - \frac{1}{2}(i-1))_{k_i}$$
$$\text{where } (x)_{k_i} = x(x+1)\cdots(x+k_i-1)$$

This result is proved in Muirhead (1982, Theorem 8.3.4). We refer to the distribution in equation (29) as the trace Wishart distribution and write it as $trW_N(T-1,\Omega/(T-2))$. If we now consider tr(SM) where M is idempotent of rank k, we find the following result.

LEMMA 1: *If tr(S) is $trW_N(T\text{-}1,\Omega/(T\text{-}2))$, then tr(SM) is $trW_k(T\text{-}1,C'\Omega C/(T\text{-}2))$, where* ***M*** *is idempotent, tr(M)=k and C is an $(N\times k)$ matrix of rank k such that $M=CC'$.*

(Proof) If M is idempotent of rank k, we can find an orthogonal P such that

$$M = P\begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} P' = [P_1 \;\; P_2]\begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} P_1' \\ P_2' \end{bmatrix}$$

where P_1 is the first k columns of P whilst P_2 is the last (N-k) columns. Hence $M=P_1P'_1$ so that $C=P_1$ is an (N×k) matrix of rank k such that $M=CC'$. Now $C'SC \sim W_k(T-1,C'\Omega C/(T-2))$ by Theorem 3.2.5 in Muirhead (1982) and so $tr(C'SC)=tr(SCC')=tr(SM)$ which completes the proof.

Q.E.D.

For the problem we are considering $M=I_N-ee'/N$, which is idempotent of rank N-1, so that we have proved that $tr(\bar{S})=tr(S(I_N-ee'/N))$ is distributed as $trW_{N-1}(T-1,C'\Omega C/(T-2))$ where $I_N-ee'/N=CC'$ and C is N×(N-1). We now wish to consider what the mean of $\exp(-\frac{1}{2N}tr(\bar{S}))$ is. We can see immediately via Jensen's inequality that

$$E[\exp(-\frac{1}{2N}tr(\bar{S}))] > \exp(-\frac{1}{2N}tr(E[\bar{S}])) \; .$$

Now $E(\bar{S})=\Omega M$ from the properties of the Wishart, so that

$$E[\exp(-\frac{\tau}{2N}tr(\bar{S}))] > q(\tau)=\exp(-\frac{\tau}{2N}tr(\Omega M)) \; .$$

An exact calculation for $\hat{q}(1) = \exp(-\frac{1}{2N}tr(\bar{S}))$ is possible,

(30)
$$\begin{aligned} E[\hat{q}(1)] &= E[\exp(-\frac{1}{2N}tr(\bar{S}))] \\ &= E[\exp(-\frac{1}{2N}trW_{N-1}(T-1,C'\Omega C/(T-2))], \end{aligned}$$

the last equality following from Lemma 1. The right hand side of (30) can be interpreted as the expectation of the moment generating function (m.g.f.) of a trace Wishart evaluated at the matrix -1/(2N). The m.g.f., ϕ, is given by $\phi(t) = \det(I-\frac{2tC'\Omega C}{T-2})^{-\frac{T-1}{2}}$, see Muirhead (1982, p.342). Therefore,

(31)
$$E[\hat{q}(1)] = \det(I + \frac{C'\Omega C}{N(T-2)})^{-\frac{T-1}{2}} \; .$$

Let $\lambda_1,\ldots,\lambda_{N-1}$ be the eigenvalues of $C'\Omega C$, then

$$
\begin{aligned}
E[\hat{q}(1)] &= \left(\prod_{j=1}^{N-1}\left(1+\frac{\lambda_j}{N(T-2)}\right)\right)^{-\frac{T-1}{2}} \\
&= \exp\left(-\frac{T-1}{2}\log\left(\prod_{j=1}^{N-1}\left(1+\frac{\lambda_j}{N(T-2)}\right)\right)\right) \\
&= \exp\left(-\frac{T-1}{2}\sum_{j=1}^{N-1}\log\left(1+\frac{\lambda_j}{N(T-2)}\right)\right) \\
&= \exp\left(-\frac{T-1}{2N(T-2)}\sum_{j=1}^{N-1}\lambda_j+\frac{T-1}{4N^2(T-2)^2}\sum_{j=1}^{N-1}\lambda_j^2+O\left(\frac{1}{T^2}\right)\right) \\
&= \exp\left(-\frac{tr(C'\Omega C)}{2N}-\frac{tr(C'\Omega C)}{2N(T-2)}+\frac{(T-1)tr(C'\Omega^2 C)}{4N^2(T-2)^2}\right)+O\left(\frac{1}{T^2}\right) \\
&= \exp\left(-\frac{tr(\Omega M)}{2N}\right)\left(1-\frac{tr(\Omega M)}{2N(T-2)}+\frac{tr(\Omega^2 M)}{4N^2(T-2)}\right)+O\left(\frac{1}{T^2}\right).
\end{aligned}
$$

Thus we can construct a crude approximation to q(1) by considering

(32) $$\hat{q}(1)\left(1+\frac{tr(\bar{S})}{2N(T-2)}\right)+o\left(\frac{1}{NT}\right).$$

Then $\frac{1}{q(1)}$ can be approximated by $\frac{1}{\hat{q}(1)}\left(1-\frac{tr(\bar{S})}{2N(T-2)}\right)+o\left(\frac{1}{NT}\right)$ by similar arguments.[4] We can actually do better than this by searching for the estimator of q(1) which is minimum variance unbiased. Such an estimator should exist for this problem since the rates of return distribution is linear exponential, for the case when N=1 we explore this question in the Appendix. The bias adjustment suggested in equation (32) is used later in the paper, there appears to be little difference between the two bias adjustments, we report results in Table 2, discussed later in the text.

We investigate the hedging performance for the FT30 index over January 2, 1988 to December 31, 1991. From the data of the 30 constituents of the index, a new index is constructed since we found discrepancies between the reported index and one calculated from the reported individual companies, we prefer the latter so that discrepancies, if they occur, will not be due to mismeasurement. We shall consider three hedging policies. The first one, reported to one of the authors in conversation, is apparently used by some City institutions.

It holds equal numbers of shares in each company, this may be motivated by a desire to reduce transactions costs or profiting out of Jensen's inequality, it is obviously simple. The second policy is to hold equal amounts of money in each company. This is the correct hedging policy as proved in Theorems 1 and 2. Under the second hedging portfolio, the number of shares held differs across the companies and intuitively this would reflect the investment opportunities or market activities more accurately since it is formed in terms of value. However, considering Jensen's inequality the fair price requires an adjustment to correct the downward bias. So our third hedging policy is to hold the amount of the fair price equally divided into 30 companies in money terms.

Formally, in policies 1 and 2 on each day t we pick a portfolio ξ_t with value $\xi_t S_t = Y_t$, and then look at its value $\xi_t S_{t+1}$ next day (policy 1 holds $\xi_t(t) = Y_t / \Sigma S_j(t)$ for j=1,..,N, policy 2 holds $\xi_j(t) = Y_t / N S_j(t)$ of share j). The values $U_t = Y_{t+1} - \xi_t S_{t+1}$ are computed, the sample mean, standard deviation and $\Sigma(U_t - \bar{U})^2$ are reported in the first three columns in Table 1. Then the t-statistic is the statistic $\sqrt{T-1}\,\bar{U}/(\Sigma(U_t - \bar{U})^2)^{1/2}$ for testing that the mean is zero, the last column is the sample correlation coefficient for the bivariate sample $(\xi_t S_{t+1}, Y_{t+1})$. In policy 3, we hold an amount $\xi_j(t) = e^{-\hat{k}} Y_t / N S_j(t)$, where the estimator $\hat{k}$ is computed from the library, and then proceed as for policies 1 and 2.

First of all, we have to estimate the variance-covariance matrix of 30 log-returns. The estimation values of equation (14) for daily data are based on 101 and 201 observations prior to our data period. As we proceed with the estimation, the fixed library size is updated. We fixed our library size because of concern about the implications of the empirical findings of heteroscedasticity and mean-reversion in stock returns discussed elsewhere in the literature. By doing this, the estimation of the variance-covariance matrix is based on pre-sample data and updated each period. Our calculations used the maximum likelihood estimators

$\hat{s}_{ij}=\Sigma(r_{it}-\bar{r}_i)(r_{jt}-\bar{r}_j)/(T-1)$ which are biased, in fact $E[\hat{S}]=\frac{(T-2)\Omega}{T-1}$. Thus Jensen's inequality applied here gives us

$$\begin{aligned} E[\exp(-\frac{1}{2N}tr(\hat{S}))] &> \exp(-\frac{T-2}{2N(T-1)}tr(\Omega M)) \\ &= (\exp(-\frac{tr(\Omega M)}{2N}))^{\frac{T-2}{T-1}} \\ &= (q(1))^{\frac{T-2}{T-1}} . \end{aligned}$$

Since $0<q(1)<1$, $(q(1))^{\frac{T-2}{T-1}} > q(1)$, so that our estimator will still overestimate. However, we found that the bias of $\hat{s}_{ij}$ from s_{ij} is negligible. We also include the bias adjustment estimation suggested by equation (32) based on the 101 library size.

We invest the same amount of money as the index value for the first and the second policies and an amount equal to the fair price for the third, which is less than the index itself. We then compare the difference between the realized payoff from our hedging portfolio and the index in the next period. Table 1 reports the replicating performance of the three policies.

Our interest is to see how well a hedging portfolio replicates the FT30 index. The correlation coefficients between the index and the hedging portfolio tell us that investing equal amounts of money in each company catches the movement of the index better than holding equal numbers of shares although both are exceptionally highly correlated. The difference is increasing with the library size. Basically, policies 2 and 3 have the same dynamics and the only difference is the magnitude, as it becomes clear when we examine the mean and variance. The sum of squared residuals is strongly suggestive that policy 3 outperforms the others in replicating the index. We note that t-statistics for testing the zero mean of the residual also supports policy 3, which is insignificant in all cases. Hence, we can consider policy 3 as the best way to hedge the FT30 index.

We now conduct a second experiment. At this stage, it is worth thinking what the properties of hedged portfolios with respect to either method would be. We concentrate on the two hedging strategies for the amount of £100,000 invested in the FT30 index. Our first strategy is going short in an equal number of shares, this will pay off $-100000\frac{\Sigma S_j(t+1)}{\Sigma S_j(t)}$. Our second strategy is to go short in equal value, then this will pay off $-\frac{100000}{N}\sum_{j=1}^{N}\frac{S_j(t+1)}{S_j(t)}$. We shall regard these as two pension funds. There are two merchant banks, one, an avaricious one, charges Y(t) for the FT30 contract, the other, a less greedy organization, charges $\theta Y(t)$, the fair price, where $Y(t)=(\prod_{j=1}^{N}S_j(t))^{\frac{1}{N}}$ and $\theta=\exp(-\frac{1}{2}(\frac{tr(\Omega)}{N}-\frac{e'\Omega e}{N^2}))$. Both pay Y(t+1) at maturity. An investor is faced with four strategies to go long in good or bad banks and short in good or bad funds. We assume that Investor 1 is long in a bad bank contract and short in a bad fund, Investor 2 is long in a bad bank and short in a good fund, Investor 3 is long in a good bank and short in a bad fund, and Investor 4 is long in a good bank and short in a good fund. The long and short positions can be switched, but it won't change any result except the sign. Each investor is long and short in the amount of £100,000 at each period, costing her zero, and she liquidates her contract in the next period, the exercise being repeated every period. We denote their returns by $\tilde{R}_i(t+1)$, i=1,...,4, where

$$
\begin{aligned}
\tilde{R}_1(t+1) &= 100000[\frac{Y(t+1)}{Y(t)} - \frac{\Sigma S_j(t+1)}{\Sigma S_j(t)}] \\
\tilde{R}_2(t+1) &= 100000[\frac{Y(t+1)}{Y(t)} - \frac{1}{N}\Sigma\frac{S_j(t+1)}{S_j(t)}] \\
\tilde{R}_3(t+1) &= 100000[\frac{Y(t+1)}{\theta Y(t)} - \frac{\Sigma S_j(t+1)}{\Sigma S_j(t)}] \\
\tilde{R}_4(t+1) &= 100000[\frac{Y(t+1)}{\theta Y(t)} - \frac{1}{N}\Sigma\frac{S_j(t+1)}{S_j(t)}] .
\end{aligned} \tag{33}
$$

We shall go more deeply into the properties of the hedging residuals of $\tilde{R}_4(t+1)$. We now consider the random variable $R_4(t+1)$ which represents "the rate of return" on the residual of the hedging portfolio, $\tilde{R}_4(t+1)$ in the following sense. We need to scale any discrepancy due to discretization, the appropriate scale being the initial investment in the contract,

$$R_4(t+1) = \frac{\tilde{R}_4(t+1)}{100000} = \frac{Y(t+1)}{\theta Y(t)} - \frac{1}{N}\Sigma\frac{S_j(t+1)}{S_j(t)}.$$

The above equation has the interpretation of a scaled geometric mean minus an arithmetic mean. It follows that $Y(t+1)/Y(t)$ is log-normal as are $S_j(t+1)/S_j(t)$, $j=1,\ldots,30$. Since $R_4(t+1)$ is a linear combination of log-normals, it has an unknown distribution and since its characteristic function is unknown it seems difficult to say much about the density of $R_4(t+1)$ numerically or analytically. We shall have to settle with an analysis of the moments of $R_4(t+1)$, before we do, we note one result.

LEMMA 2: *$R_4(t+1)$ is independently and identically distributed (i.i.d.).*

(Proof) Since $S_j(t+1)/S_j(t)$ is i.i.d. and $R_4(t+1) = \frac{1}{\theta}(\prod_{j=1}^{N}\frac{S_j(t+1)}{S_j(t)})^{\frac{1}{N}} - \frac{1}{N}\sum_{j=1}^{N}\frac{S_j(t+1)}{S_j(t)}$, it is a function of i.i.d. variables, thus it is i.i.d. Q.E.D.

We now calculate $E[R_4(t+1)]$.

LEMMA 3: *$E[R_4(t+1)]=\exp(\Sigma\alpha_j/N)-(1/N)\Sigma\exp(\alpha_j)<0$ and $\tilde{E}[R_4(t+1)]=0$ where $\tilde{E}$ is expectation taken on any measure that equates α_j leaving other parameters the same, the equivalent martingale measure being a particular example.*

(Proof)

$$R_4(t+1) = \frac{1}{\theta}\exp[\frac{\Sigma\alpha_j-\Sigma\omega_{jj}/2}{N}+\frac{1}{N}\sum_j\sum_k\sigma_{jk}(z_k(t+1)-z_k(t))]-\frac{1}{N}\Sigma\exp[\alpha_j-\frac{1}{2}\omega_{jj})+\sum_k\sigma_{jk}(z_k(t+1)-z_k(t))]$$

$$E[R_4(t+1)] = \frac{1}{\theta}\exp(\frac{\Sigma\alpha_j-tr(\Omega)/2}{N}+\frac{1}{2N^2}e'\Omega e) - \frac{1}{N}\Sigma\exp(\alpha_j)$$

$$= \exp(\frac{\Sigma\alpha_j}{N}) - \frac{1}{N}\Sigma\exp(\alpha_j) \quad \textit{since } \frac{1}{\theta} = \exp(\frac{1}{2}\frac{tr(\Omega)}{N} - \frac{1}{2N^2}e'\Omega e)$$

where we have used the properties of the moment generating function of a normal distribution. Taking expectations with respect to $\tilde{P}$ rather than P is equivalent to setting all the α_j's equal to r or indeed any other constant. Thus $\tilde{E}[R_4(t+1)]=\exp(r)-\exp(r)=0$. Finally $E[R_4(t+1)]<0$ by Jensen's inequality. Q.E.D.

The significance of Lemma 2 is theoretical rather than practical, the observed data is generated via P rather than $\tilde{P}$, we should observe a negative mean. However, lemma 3 suggests a third interpretation of $R_4(t+1)$. If all the shares are perfect complements in the sense that they all come from the same log-normal distribution, $R_4(t+1)=0$ with probability one. If we change the time interval from days to weeks or months the only change of note is that $E[R_4(t+1)]$ is decreasing as the time intervals increases, that is growing in magnitude. Also in lemmata 2 and 3, we assume that θ is known; if θ is estimated, the statistical properties of $R_4(t+1)$ will change.

Before proceeding further, we digress here to deal with $R_4(t+1)$ in the presence of the jumps generated by equation (26). Note that

$$\begin{aligned} R_4(t+1) = {} & \frac{1}{\theta}\exp\Big[\frac{1}{N}\Sigma(\alpha_j-\frac{1}{2}\omega_{jj}) + \frac{1}{N}\Sigma\Sigma\sigma_{jk}(z_k(t+1)-z_k(t)) + \sum_{k=1}^{N(t)}Q_k\Big] \\ & -\frac{1}{N}\sum_{j=1}^{N}\exp\Big[(\alpha_j-\frac{1}{2}\omega_{jj}) + \Sigma\sigma_{jk}(z_k(t+1)-z_k(t)) + \sum_{k=1}^{N(t)}Q_k\Big]\ . \end{aligned}$$

We see that Q influences the distribution of $R_4(t+1)$ in a non-trivial manner. To demonstrate the effect on $E[R_4(t+1)]$, consider

$$\begin{aligned} E[R_4(t+1)] &= \underset{z(t)}{E}[\underset{N(t)}{E}\{\underset{Q}{E}(R_4(t+1)\,|\,N(t),z(t)\,|\,z(t)\}] \\ E[R_4(t+1)\,|\,N(t),z(t)] &= \frac{1}{\theta}\exp\Big(\frac{\Sigma(\alpha_j-\omega_{jj}/2)}{N}+\frac{\Sigma\Sigma\sigma_{jk}z_k(t)}{N}\Big)\phi(1)^{N(t)} \\ &\quad -\frac{1}{N}\Sigma\exp(\alpha_j-\frac{1}{2}\omega_{jj}+\Sigma\sigma_{jk}z_k(t))\phi(1)^{N(t)} \\ E[R_4(t+1)\,|\,z(t)] &= \Big[\frac{1}{\theta}\exp\Big(\frac{\Sigma(\alpha_j-\omega_{jj}/2)}{N}+\frac{\Sigma\Sigma\sigma_{jk}z_k(t)}{N}\Big) \\ &\quad -\frac{1}{N}\Sigma\exp(\alpha_j-\frac{1}{2}\omega_{jj}+\Sigma\sigma_{jk}z_k(t))\Big]\exp(\phi(1)\lambda-\lambda) \\ E[R_4(t+1)] &= \Big[\exp\Big(\frac{\Sigma\alpha_j}{N}\Big)-\frac{1}{N}\Sigma\exp(\alpha_j)\Big]\exp(\phi(1)\lambda-\lambda) \end{aligned}$$

where $\phi(1)$ is the m.g.f. of Q evaluated at 1.

$$\underset{\Xi}{E}[E(R_4(t+1);\Xi)] = \exp(\frac{\Sigma\alpha_j}{N})\phi(1)\exp(-\lambda\frac{\Sigma k_j}{N}) - \frac{1}{N}\Sigma\exp(\alpha_j)\exp(-\lambda k_j)\phi(1) \quad .$$

For a specific example suppose that Q_l~i.i.d.$N(\mu_Q,\sigma_Q^2)$, l=1,..,N(t), where N(t) is the Poisson process,

$$\phi(1) = \underset{N(t)}{E}[\exp(\sum_{i=1}^{N(t)}(Q_i-1))]$$
$$= \underset{N(t)}{E}[\exp(\mu_Q+\frac{1}{2}\sigma_Q^2)N(t)\,|N(t)] \ .$$

Using the fact that the m.g.f. of Poisson with parameter λ is $\exp(\lambda(\exp(t)-1))$, so

$$\phi(1) = \exp(\lambda(\exp(\mu_Q+\frac{1}{2}\sigma_Q^2)-1)) \ .$$

Hence,

$$E[R_4(t+1)] = [\exp(\frac{\Sigma\alpha_j}{N})-\frac{1}{N}\Sigma\exp(\alpha_j)]\exp(\lambda(\exp(\mu_Q+\frac{1}{2}\sigma_Q^2)-1)) \ .$$

Our difficulties are complicated because estimation of Ω and $\exp(-1/(2N)\mathrm{tr}(\Omega M))$ will not be straightforward, the sample variance-covariance matrix will depend in general upon μ_Q and σ_Q^2 in the normal case; for general Q distributions, S conditional on Q will be $W_N(T-1,\exp(2\Sigma Q_k)\Omega/(T-2))$. For the case considered where Q is normal, maximum likelihood and method of moments estimators can be used to estimate the parameters, see Akgiray and Booth (1987), etc.

Because of lemma 2, we investigate the distribution of $R_4(t+1)$ over the sample period. For the i.i.d. hypothesis the BDS test is an obvious choice, this was developed in Brock, Dechert and Scheinkman (1986) as a method of testing for structure in a series. The test is based on the simple notion that for any i.i.d. time series the probability of some event at t and τ, $\mathrm{Prob}(A_t \cap A_\tau)$, is just the product, $\mathrm{Prob}(A_t)\mathrm{Prob}(A_\tau)$ of the two events. Let $x_t \in R^1$ be a time series of length N, define the following,

$$I_\varepsilon = 1 \quad if \ |x_i - x_j| \le \varepsilon \quad .$$
$$\quad = 0 \quad if \ |x_i - x_j| > \varepsilon$$
$$I_\varepsilon^m = \prod_{k=0}^{m-1} I_\varepsilon(x_{i+k}, x_{j+k}),$$
$$and \quad C(m,n,\varepsilon) = \frac{2}{n(n-1)} \sum_{1 \le i < j \le n} I_\varepsilon^m(x_i, x_j), \qquad n <+ N-m \ .$$

Now define the following expectations,

$$C = E[I_\varepsilon(x_i, x_j)],$$
$$K = E[I_\varepsilon(x_i, x_j) I_\varepsilon(x_i, x_j)] \ .$$

Brock, Dechert and Scheinkman (1986) prove that for any $m>1$, as $n \rightarrow \infty$,

$$\frac{\sqrt{n}}{\sigma}(C(m,n,\varepsilon) - C(1,n,\varepsilon)^m) \rightarrow N(0,1)$$
$$where \ \sigma^2(m,\varepsilon) = 4[K^m + 2\sum_{j=1}^{m-1} K^{m-j} C^{2j} + (m-1)^2 C^{2m} - m^2 K C^{2m-2}] \ .$$

In computing σ^2 the consistent estimators $C(1,n,\varepsilon)$ and $K(1,n,\varepsilon)$ may be used to replace C and K respectively, where $K(1,n,\varepsilon)$ can be calculated by

$$K(1,n,\varepsilon) = \frac{6}{n(n-1)(n-2)} \sum_{1 \le i < j \le n} I_\varepsilon(x_i, x_j) I_\varepsilon(x_j, x_k) \ .$$

The results of the BDS test and some statistics from the residuals are presented in Table 3 where we used the library size of 101, 201 and a bias adjustment suggested by equation (32).

The mean values for the three different time methods are very small and the t-statistics do not reject a mean of zero. The BDS test rejects the hypothesis that daily residuals are i.i.d., we did not filter the data. However, although we have 800 observations, the asymptotic critical values may not be very helpful. We could simulate the BDS test for parameter values near our maximum likelihood estimators and use the simulated critical values, but we decided to use more traditional time series methods.

Firstly we plotted the data series, see Figure 1. There is a spectacular residual on the

January 15, 1991 when the Gulf war had started. It is clear that a change in the $S_j(t)$ which is self-cancelling with respect to the multiplicative index may still generate a large number via the hedging component because of the discretization. Thus we might expect to see one or two very large negative values for $R_4(t+1)$. An alternative explanation may be that the shares are generated by a process with jump components. We shall not discuss this further and we shall stay with our maintained hypothesis and investigate the behaviour of $R_4(t+1)$ in our data period and for a subset of the first 700 observations (which excludes the outlier). The values of $R_4(t+1)$ for the first 700 values are plotted in Figure 2; they show basically the expected behaviour. Although most values are positive but very small there is an occasional large negative value. In Figure 3 we present the correlogram for $R_4(t+1)$ calculated in the 3 ways, with the different library sizes and the bias adjustment. Although we do not report any white noise tests on the correlogram, the largest t-statistic for an individual autocorrelation was about 0.3, so there appears no compelling evidence of any non-zero correlations. Our final check was the spectral density presented in Figure 4. We present spectra for the two different library sizes, both are approximately the same shape although (surprisingly) the spectrum based on a library size of 201 was uniformly above the spectrum based on a library size of 101. When we plotted the spectrum by using the first 700 observations, it is very flat and relatively small. For the test of randomness we further investigate the cumulative periodogram based on Figure 4.[5] Surprisingly the first 700 observations show a big deviation from the 5% critical range while the whole data series, including the Gulf war, is nothing but i.i.d., see Figure 5. This is an interesting finding since including the Gulf war would intuitively make the process non-i.i.d., not the other way round. Perhaps the Gulf war was not exogenous?

We finish this section by reporting the results of our second experiment describing the relative performances of our 4 strategies, see equation (33). Each investor carries out her investment strategy until the last day. Table 4 shows the performance of the investors for the period January 2, 1988 to December 31, 1991. The table is produced by using different library sizes, the purpose of the two library sizes being, as before to see how sensitive our calculations are to the way in which we calculate θ.

We summarize the results. Investors 1 and 2 always make a loss regardless of their choice of the funds since the bad bank overprices the contract. Investor 2 must make a loss since the geometric mean is less than the arithmetic. In Table 4 Investor 3 has the largest profit, which implies that the bad fund performs hedging very badly. Investor 4's results indicate that the fair price with equal amount of money in each stock is the best to replicate the FT30 index where the magnitude of her mean return is the smallest. If the bank is bad and overcharges on the contract all hedging strategies will result in a loss for all time intervals. The different library size brought quite remarkable changes to the return. In the library size of 101 Investors 3 and 4 in general make a profit, while making a loss in the library size of 201. Even though we don't report the results here, the bigger library size seems to have less fluctuation over different time intervals as we increase the holding time to weekly, fortnightly and monthly periods. The fact that the profit or loss goes to only £91 over 3 years is quite remarkable. We note that the stable average profit close to zero of Investor 4 across the different estimations means no arbitrage profit in her investment strategy. Notice that the variance of the distribution of returns is determined by the choice of pension fund, not the choice of bank, this is what statistical theory would suggest.[6] Also the returns, a maximum loss of £7567 or a gain of £679, seem little for £100,000 rolled over for approximately 3 years but the investor's net position at all times is zero. The result of including the first 700 observations is an unambiguous reduction in variance, we have eliminated the massive outlier. In conclusion, all 4 methods lead to not rejecting $E[R_4(t+1)]=0$. This is not a rejection of the model via lemma 2 as the use of an estimated θ implies an upward bias. Thus we conclude that the evidence seems to be that hedging is possible, but the lack of a proper statistical theory to base this on, must weaken this finding. This is not a fault of this paper but a fault of any method to test hedging and non-arbitrage relationships which are parameter dependent; estimation error and profit (loss) realization cannot be easily separated.

3. CONCLUSION

It might be argued that it is inappropriate to assume that the 30 shares in the FT30 index are multivariate log-normal. In a study of the distributional properties of the FT30, Yoon (1991) found substantial evidence of kurtosis in daily log-returns for the period of January 2, 1988 to December 31, 1990. However, our results will still hold if we assume that the instantaneous means in equation (1) are no longer assumed constant but allowed to depend on S_j and t. In particular, our hedging portfolio and the fair price of the contract will go through in exactly the same manner. The change that will occur will be for the estimation of Ω; in this case the distribution of S will no longer be Wishart.

Given that we can relax our assumptions about the means, is it possible that the distribution of daily rates of returns will exhibit the kurtosis and conditional heteroscedasticity that one observes empirically. Both these factors can be explained by changing the running time of the B.M., see Stock (1988) and Yoon (1991). Another possible solution is to add jumps of random size to the process as in Section I, see Perraudin et. al.(1991), yet a third is to consider the presence of ARCH effects as being due to the discretization of the process, see Nelson (1991) and Melino (1990). We shall defend our position by invoking the last of these three explanations, thus casting doubts upon our estimation of Ω but preserving the validity of our hedging portfolio. It is well known, see Akgiray and Booth (1987), that the presence of jumps will increase kurtosis so Theorem 2 allows us to justify our hedging procedure, again statistical estimation of the fair price will be flawed.

On balance the overall conclusion of this paper is that it is possible to hedge the FT30 index. One can calculate a fair price, dependent on parameter estimation, in much the same way as the Black and Scholes option price. We did find that there is a fundamental difficulty in testing the validity of a non-arbitrage relationship based on unknown parameters; various attempts to evaluate the efficiency of hedging foundered on the extra variability introduced due to statistical estimation. Notwithstanding the caveats, our fair price performs well and has certain robustness properties that make it potentially very attractive.

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APPENDIX

Consider the situation where the N-vector is just a 1-vector, so we have that $X_1, X_2, \ldots, X_T$ are i.i.d. $N(\mu,\sigma^2)$, and the aim is to get an unbiased estimator of $\exp(-\lambda\sigma^2)$. Then the joint density of $\exp(X_1), \exp(X_2), \ldots, \exp(X_T)$ is

$$\exp\{-\frac{1}{2\sigma^2}[\sum_{j=1}^{T}(X_j-\mu)^2]-\frac{T}{2}\log(2\pi\sigma^2)\}$$
$$=\exp\{-\frac{1}{2\sigma^2}[\sum_{j=1}^{T}(X_j-\bar{X})^2+T(\bar{X}-\mu)^2]-\frac{T}{2}\log(2\pi\sigma^2)\}$$
$$=\exp\{-\frac{1}{2\sigma^2}\sum X_j^2+\frac{\mu}{\sigma^2}\sum X_j-\frac{T\mu^2}{2\sigma^2}-\frac{T}{2}\log(2\pi\sigma^2)\}$$

so we have the exponential family with natural parameters $(1/\sigma^2, \mu/\sigma^2)\in \mathbf{R}^+\times\mathbf{R}$. The idea is that $(t_1(x), t_2(x))\equiv(\sum X_j^2, \sum X_j)$ are sufficient statistics for the parameters (μ,σ^2), so if we have one unbiased estimator of $\exp(-\lambda\sigma^2)$, thereby taking the conditional expectation of that estimator, given (t_1,t_2), we get another unbiased estimator and this has a variance which is no larger. In the case of an exponential family whose parameter space contains an open set, there is only one estimator which is unbiased and a function of the sufficient statistics, and this estimator is minimum variance. One unbiased estimator of $\exp(-\lambda\sigma^2)$ is $\cos(\sqrt{\lambda}(X_1-X_2))$. In general, if we Rao-Blackwell this, and take

$$E[\cos(\sqrt{\lambda}(X_1-X_2))|\sum_{j=1}^{T}X_j^2=R^2,\sum_{j=1}^{T}X_j=a]$$
$$=E[\cos(\sqrt{\lambda}(X_1-X_2))|\Sigma(X_j-\bar{X})^2=R^2-\frac{a^2}{T},\Sigma X_j=a]$$
$$=E[\cos(\sqrt{\lambda}(\xi_1-\xi_2))|\Sigma\xi_j^2=R^2-\frac{a^2}{T},\Sigma\xi_j=0]$$

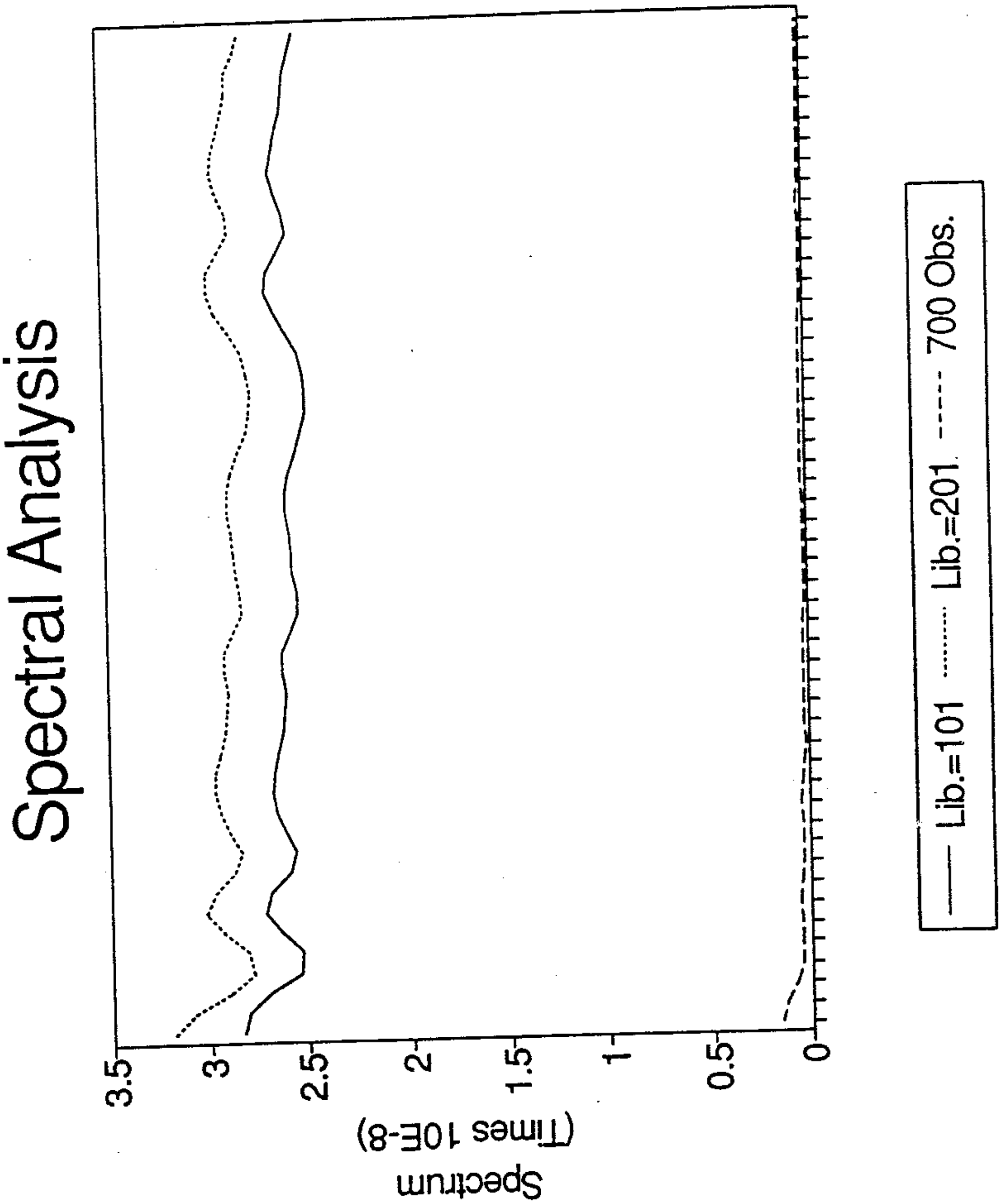
where ξ is uniformly distributed on the space of radius $r=\sqrt{(R^2-a^2/T)}$ intersected with the hyperplane $\xi\cdot 1=0$; the above becomes

$$=E[\cos(\sqrt{2\lambda}\,e\xi)|\Sigma\xi_j=0,\Sigma\xi_j^2=r^2]$$

where e is the unit vector $(1/\sqrt{2},-1/\sqrt{2},0,...,0)$. Now rotate so that the vector 1 lies along the direction $(0,0,...,1)$ and then the rotated e will be orthogonal to this, and we get

$$=E[\cos(\sqrt{2\lambda}\,\xi_1|\sum_{j=1}^{T-1}\xi_j^2=r^2]$$
$$=J_\nu(r\sqrt{2\lambda})\Gamma(\nu+1)(r\sqrt{2\lambda})^{-\nu}$$

where $\nu\equiv(T/2)-1$ and J_ν is the Bessel function of index ν, see Revuz and Yor (1991). Thus, one can obtain the minimum variance unbiased estimator but it is not non-negative since J_ν oscillates in sign. So it doesn't seem like too good an estimator of a non-negative quality.



Spectral Analysis
Spectrum
(Times 10E-8)
3.5
3
2.5
2
1.5
1
0.5
0
Lib.=101
Lib.=201
700 Obs.

FIGURE 4

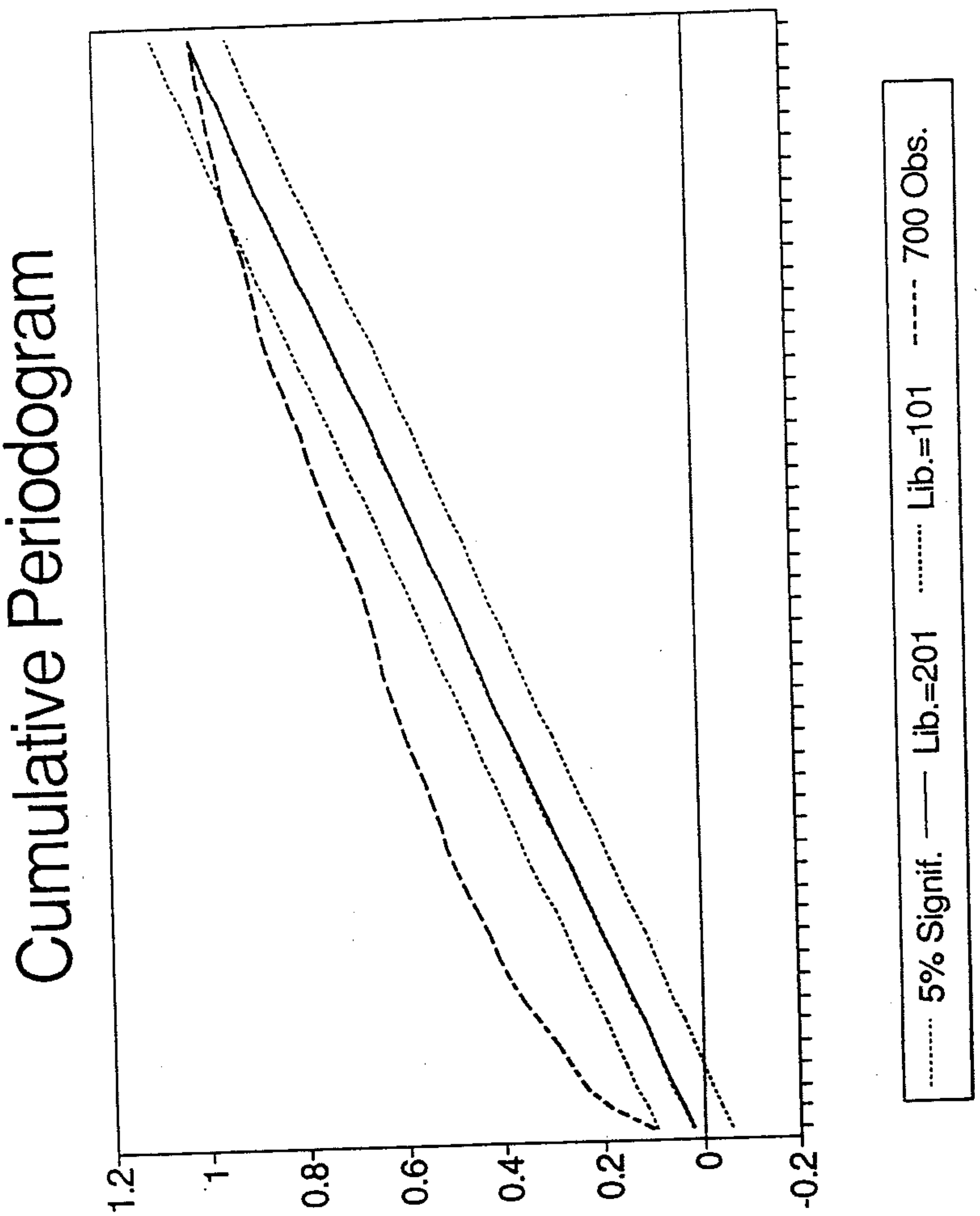

Cumulative Periodogram
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
5% Signif.
Lib.=201
Lib.=101
700 Obs.

FIGURE 5

TABLE I

Test of FT30(t)-H.P.(t) from 1/1/88 to 31/12/91

	Mean	Std. Dev.	S.S.R.	t-stat.	Corr.Coef.
Lib.=101					
Policy 1	-0.0301	0.5201	247.0	-1.7475	0.999867
Policy 2	-0.0295	0.1349	17.37	-6.6048	0.999991
Policy 3	0.0007	0.1360	16.83	0.1646	0.999991
Lib.=201					
Policy 1	-0.0349	0.5414	238.4	-1.8396	0.999807
Policy 2	-0.0311	0.1429	17.34	-6.2066	0.999986
Policy 3	-0.0001	0.1434	16.67	-0.0226	0.999986
Adjustment[a]					
Policy 1	-0.0301	0.5201	247.0	-1.7475	0.999867
Policy 2	-0.0295	0.1349	17.37	-6.6048	0.999991
Policy 3	0.0004	0.1359	16.82	0.0967	0.999990

[a]Adjustment was carried out with Lib.=101

TABLE II

Different Adjustments

	Type	Profit	Mean	Std. Dev.
Library=101				
Adjustment 1	Investor 3	599.94	0.6590	148.27
	Investor 4	91.24	0.1001	40.32
Adjustment 2	Investor 3	599.94	0.6590	148.27
	Investor 4	91.24	0.1001	40.32
Library=201				
Adjustment 1	Investor 3	-512.97	-0.6322	153.03
	Investor 4	-161.63	-0.1993	42.56
Adjustment 2	Investor 3	-512.97	-0.6322	153.03
	Investor 4	-161.63	-0.1993	42.56

TABLE III

Test of the Residuals

		Lib.=101	Lib.=201	Adjustment
Mean		0.1880E-5	-0.1555E-5	0.1000E-5
Std. Dev.		0.4032E-3	0.4256E-3	0.4032E-3
t-Stat		0.1407	-0.1040	0.0749
BDS Test[a] Embedding Dimension	2	-28.39	-2.04	-3.91
	3	4.58	-3.32	97.35
	4	20.59	-6.26	29.23
	5	-12.65	-10.06	4.43

[a]The BDS statistics are for ε equal 1/2 standard deviation. These are asymptotically distributed N(0,1). Hsieh and LeBaron (1988) have found that the small sample 5% critical values are 2.5, 3.3, 3.8, 4.6 for samples of 500, and 2.1, 2.2, 2.5, 2.9 for samples of 1000, under simulated normals and N=2,3,4,5. We note that the assumption of normality is a very strong restriction.

TABLE IV

Investment Strategies

	Type	Profit[a]	Mean	Std. Dev.
Library = 101	Investor 1	-7271.53	-7.9820	148.43
	Investor 2	-7780.29	-8.5409	40.05
	Investor 3	679.95	0.7468	148.27
	Investor 4	171.27	0.1880	40.32
Library = 201	Investor 1	-7567.85	-9.3317	153.33
	Investor 2	-7216.46	-8.8988	42.42
	Investor 3	-477.47	-0.5884	153.03
	Investor 4	-126.13	-0.1555	42.56
Adjustment with Lib.=101	Investor 1	-7271.53	-7.9820	148.43
	Investor 2	-7780.29	-8.5409	40.05
	Investor 3	599.94	0.6590	148.27
	Investor 4	91.24	0.1001	40.32
First 700 Observations with Lib.=101	Investor 1	490.19	0.8183	106.98
	Investor 2	-3672.21	-6.1305	4.27
	Investor 3	4103.81	6.8512	107.02
	Investor 4	-58.50	-0.0976	4.18

[a]Column 3 is total profit $\sum_i \bar{X}_i(t+1)$ from carrying out the strategy taken over different investors and time periods. Column 4 is the sample mean $\bar{X}_i = \sum_{t=1}^{T-1} \bar{X}_i(t+1)/(T-1)$. Column 5 is the sample variance $s^2 = \Sigma(\bar{X}_i - \bar{\bar{X}}_i)^2/(T-2)$.

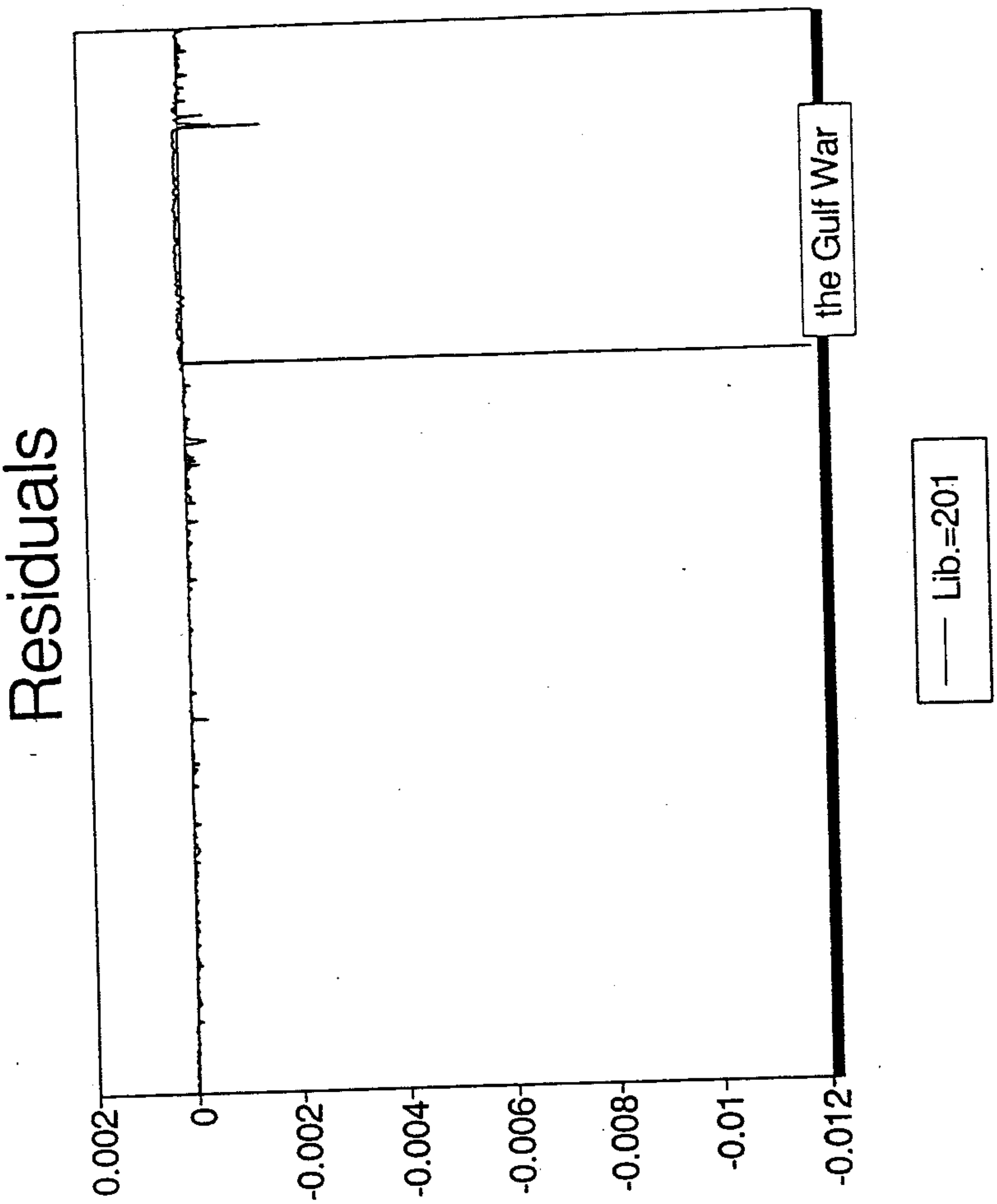
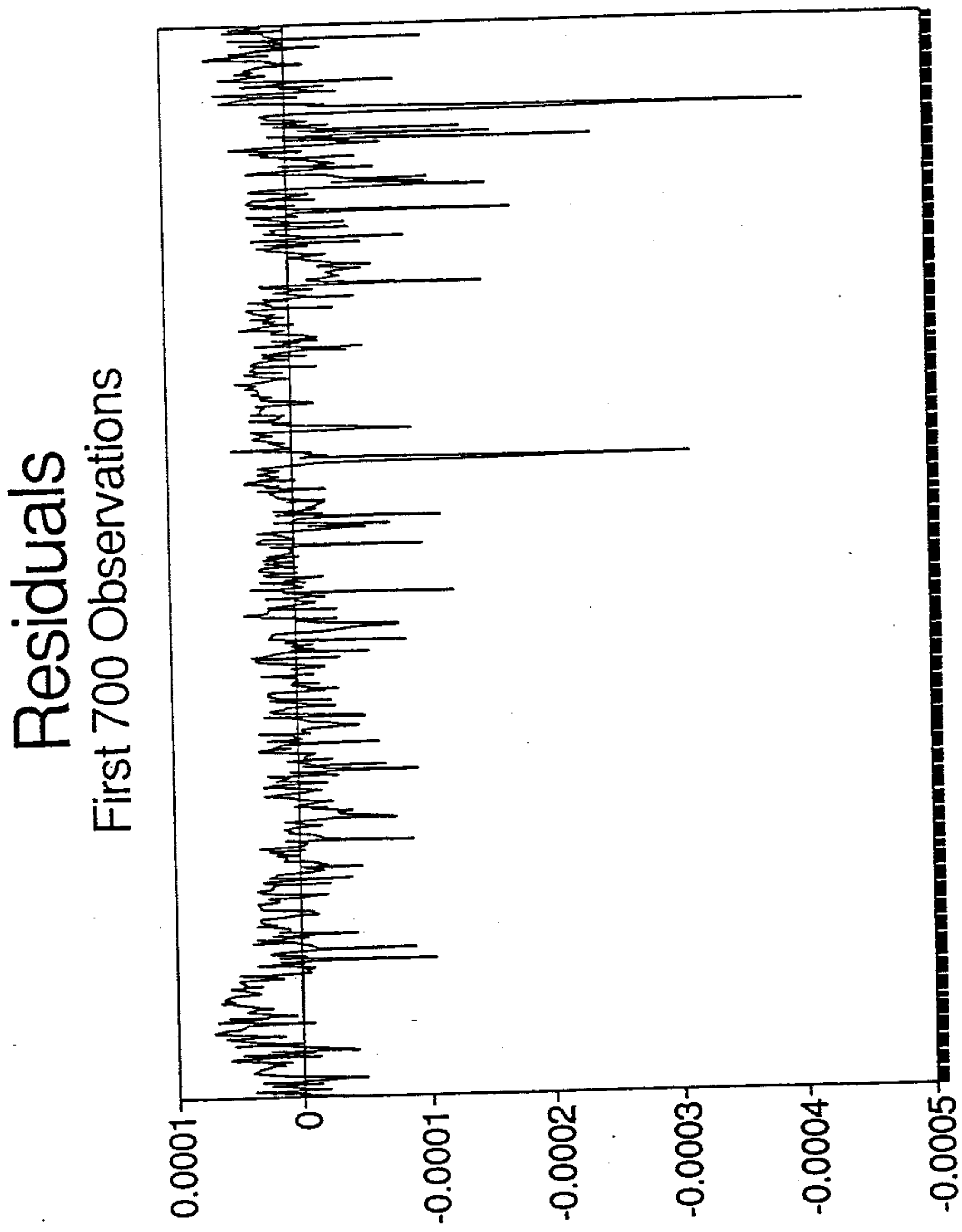

Residuals
0.002
0
-0.002
-0.004
-0.006
-0.008
-0.01
-0.012
the Gulf War
Lib.=201

FIGURE 1



Residuals
First 700 Observations
0.0001
0
-0.0001
-0.0002
-0.0003
-0.0004
-0.0005

FIGURE 2

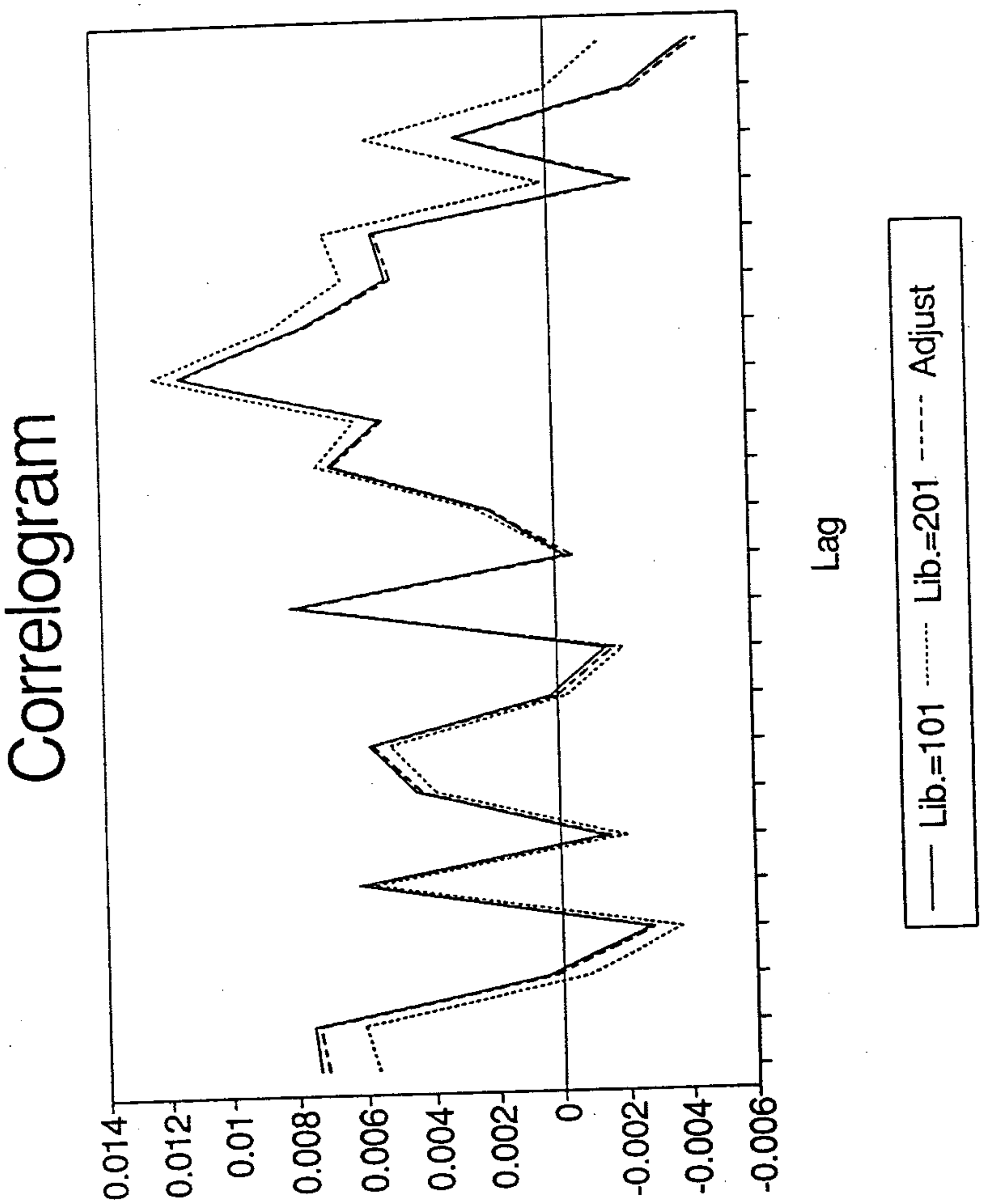

Correlogram
0.014
0.012
0.01
0.008
0.006
0.004
0.002
0
-0.002
-0.004
-0.006
Lag
Lib.=101
Lib.=201
Adjust

FIGURE 3

1. We would like to thank G. Hillier and W. Perraudin for helpful comments. Financial support from the Newton Trust and INQUIRE (UK) is gratefully acknowledged.

2. This argument is reproduced in many parts of the literature, see Gibson (1991, p. 223) that "we will never be able to perfectly hedge a long (or short) position in a Value Line Composite Index."

3. The Theorem says that

$$f(X_t)-f(X_0)=\int_{(0,t]} D_i f(X_{s-})dX_s+\frac{1}{2}\int_{(0,t]} D_{ij} f(X_{s-})d<X^i,X^j>_s+\sum_{0<s\leq t}\{f(X_s)-f(X_{s-})-D_i f(X_{s-})\Delta X_s^i\}$$

where $X_{\cdot}=\{X_{s-}:s>0\}$ is left-continuous process with limits from right, $\Delta X_s=X_s-X_{s-}$, and

$$<X^i,X^j>_t = E[X^i(0)X^j(0)]-\sum_{s=1}^{t}\{[X^i(s)-X^i(s-)][X^j(s)-X^j(s-)]\,|I_{s-}\} .$$

The adapted process $X_{\cdot}$ is previsible and also locally bounded, so the integral exists.

4. We found that the inclusion of the second-order term didn't improve matters, so our bias adjustment is only based upon the first-order approximation.

5. The cumulative periodogram is a frequency domain concept based on the periodogram ordinates, $p_j=a_j^2+b_j^2$ *where* $a_j=\sqrt{\frac{2}{T}}\sum_{t=1}^{T}x_t\cos(\frac{2\pi jt}{T})$ *and* $b_j=\sqrt{\frac{2}{T}}\sum_{t=1}^{T}x_t\sin(\frac{2\pi jt}{T})$. Then the test procedure is constructed based on a series of statistics, $s_i=\sum_{j=1}^{i}p_j/\sum_{j=1}^{n}p_j$, $(i=1,\ldots,n)$, see Durbin (1969) for more details.

6. Different hedging policies produce different distributions while the fair price is a simple multiplication by a fixed scalar.

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