Large investors, takeovers, and the rule of law

J.P. Heritage* & L.C.G. Rogers†

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Abstract. There have been various studies of the effect of a large investor's position on the price of an asset, typically assuming that agents' demands are exogenously-given functions of the current values of explanatory variables, but it often turns out that the resulting actions of the agents are not optimal for the (non-linear) market they then find themselves in. In this study, we suppose that a single large investor declares at time 0 what (deterministic) proportion of output he will consume at all future times, and the remaining agents respond optimally to the residual output process. The large agent must propose a plan which he can afford, and which would induce the other agents to agree to his plan (for otherwise they would simply form a market on their own). We compare this large agent's optimal choice with the equilibrium which would obtain if he did not attempt to exploit his large size but simply entered the market on an equal footing. We find that sometimes this can be advantageous. We also investigate circumstances under which the large agent might be better off at a later stage to walk out on his original deal.

1 Introduction

It is often observed that the assumptions of the Black-Scholes paradigm are all violated in practice, among them the assumption that agents act as price-takers. Price can be influenced by positions taken, and it is a natural question to ask how this effect operates, say in the simplest situation of two groups of agents, perhaps a large homogeneous pool of price-takers, and a small group of 'large' investors who behave differently. Effects of this kind have been studied in examples where there is a group of agents who are following some trading program, as in Frey & Stremme [4], Gennotte & Leland [5], Platen & Schweizer [6], and Brennan & Schwartz [3]. The paper of Frey & Stremme is typical, in that there are 'reference' traders and 'program' traders, each with their own demand functions, which depend on time, current price, and some 'economic fundamental' process.

This is certainly one approach to the rationalisation of the demand as a function of environment, where the demand functions are in effect given exogenously and prices are derived from that. The approach of the present study is somewhat different, in that we aim to determine the demand by an *endogenous* derivation of optimal portfolio and consumption paths. We treat the single risky asset as a

*Department of Mathematical Sciences, University of Bath, Bath, BA2 7AY, UK.

†Corresponding author. Statistical Laboratory, University of Cambridge, Wilberforce Road, Cambridge, CB3 0WB, UK.

share in some productive process, generating some dividend stream δ_t. A pool of $J-1$ agents initially achieve an equilibrium using the shares which they own, while a single (large?) agent J stands aside. Suppose now that agent J decides to get involved in the market, buying and selling shares in some way that suits his purposes. It is clear that the valuation at time 0 of a share *must* depend on the whole of J's planned future holdings of shares and not just on the present holding; indeed, the share price is the net present value of all future dividends from the share, and if it were known that J was planning to squeeze the market at some later stage, the value of the share would be greatly increased. So in this situation, we see that we cannot price the share without determining what J is going to do in the future. Now in the earlier studies mentioned above, the actions of J (the program trader) are specified exogenously, but in our setting we determine J's actions through some optimality criterion. Rather than specify the number of shares which agent J wishes to hold at different times in the future, we shall suppose that J announces at the beginning that he intends to consume at rate $(1-\varphi_t)\delta_t$ at future time t, leaving remaining consumption rate $\varphi_t\delta_t$ for the pool. He chooses φ to maximise his payoff, subject to the constraint that the members of the pool would be prepared to agree to the proposed deal. To achieve this, he has to offer the members of the pool individually sufficient reward to have an incentive to agree to his proposal. We suppose that the policy φ is stated, and there is some initial redistribution of shares, after which the agents in the pool establish an equilibrium based on the declared dividend stream $\varphi\delta \equiv \tilde{\delta}$. We call the resulting solution the *J-solution* for short.

The situation just described could equally well be interpreted as the takeover of a company (ABC plc), whose shares are held by $J-1$ large shareholders, by another company (XYZ plc) operating an identical technology. XYZ (thought of as J) is assumed to have homogeneous ownership, and makes a proposal to the $J-1$ shareholders of ABC which guarantees them collectively a stated time-dependent deterministic share of the output of the combined firm. The $J-1$ shareholders of ABC then decide between them how this should be divided, according to the equilibrium that they would achieve when faced with the dividend stream $\varphi_t\delta_t$. Alternatively and equivalently, the proposal from XYZ states explicitly what the dividend streams should be for each of the $J-1$ shareholders, in accordance with this equilibrium. The shareholders of ABC now decide whether to accept the offer of XYZ on a take-it-or-leave-it basis.

The theory of this optimal choice is presented in Sections 2 and 3. We then investigate a number of examples numerically, comparing the J-solution with the global equilibrium, which would be achieved if J simply entered the market and did not attempt to exploit his power to remain aloof. We find that there is always at least one member of the pool who prefers the global equilibrium to the J-solution; this is a simple consequence of the absence of a blocking coalition for the global equilibrium. Most of the examples are cases where J prefers the J-solution to the global equilibrium. However, an example is given where J prefers the global equilibrium.

The optimal choice for J will of course only work if the rule of law prevails, so that a deal agreed at the start is enforceable. There is nothing in the specification of the J-solution which guarantees that we might not at a later stage find that some subset of the pool might prefer to walk out of the agreed deal and set up a market on their own. Likewise, there is nothing which guarantees that J might not later prefer to take his existing holding of shares and walk out on the deal agreed originally. In Section 5 we show that in fact the members of the pool will *never* choose to walk out of the original deal, but that J may under certain circumstances prefer to abandon the deal and consume only the dividend from the shares that he currently holds.

2 Equilibrium for the pool

2.1 General case

Suppose that the economy consists of a single infinitely-divisible commodity and $J-1$ agents. The supply of the commodity is the dividend of a business which is modelled by the stochastic process δ_t on the time interval $[t_0, \infty)$. This dividend is continuously distributed to each agent at a rate proportional to his share holding, thus at time t agent j receives the commodity at rate $\theta_j(t)\delta_t$, where $\theta_j(t)$ is his share holding. Each agent consumes the commodity on the time interval $[t_0, \infty)$ and aims to maximise the total expected utility of his consumption stream, given by

$$E\int_{t_0}^{\infty} U_j(t, c_j(t))dt,$$

where $c_j(t)$ denotes the rate of agent j's consumption at time t. The utility function $U_j(t, \cdot)$ will be concave and increasing with

$$U_j'(t, 0) = \infty, \qquad U_j'(t, \infty) = 0.$$

The total consumption of the agents is determined by a market clearing condition. As agent J consumes at rate $c_J(t) = (1-\varphi_t)\delta_t$, the consumption rates of the remaining agents must satisfy

$$\sum_{i<J} c_i(t) = \tilde{\delta}_t \equiv \varphi_t \delta_t. \tag{1}$$

In order to achieve their desired consumption paths, the agents trade the commodity amongst themselves in return for shares or bonds. Both the share price S_t and the bond price B_t are endogenous. The time-t wealth of agent j is defined by

$$w_j(t) \quad = \quad \theta_j(t)S_t + \xi_j(t)B_t \tag{2}$$

where $\xi_j(t)$ denotes the bond holding of agent j at time t. The usual self-financing conditions give the dynamics of the wealth process to be

$$dw_j(t) \quad = \quad \theta_j(t)\left(dS_t + \delta_t dt\right) + \xi_j(t)dB_t - c_j(t)dt. \tag{3}$$

Finally, we require that the wealth process of each agent is always positive. This bounds the consumption of an agent and makes the problem well defined.

We now consider the consumption paths that the agents will follow. The market is complete and so the consumption paths of each agent will satisfy

$$\zeta(t_0, t) \quad = \quad p_j(t_0)U_j'(t, c_j(t)) \tag{4}$$

for some state-price density process $\zeta(t_0, t)$ and positive constants $p_i(t_0), i = 1, \ldots, J-1$, (see, for example, Breeden [2]). Combining this with market clearing (1) gives

$$\tilde{\delta}_t = \sum_{j<J} I_j\left(t, \frac{\zeta(t_0,t)}{p_j(t_0)}\right) \tag{5}$$

where $I_j(t, \cdot)$ is the inverse function to $U_j'(t, \cdot)$. The conditions on the utility function mean that (5) determines $\zeta(t_0, t)$ uniquely in terms of $\tilde{\delta}_t$ and the constants $p_i(t_0), i = 1, \ldots, J-1$.

The share price is the expected net present value of the dividend stream, thus

$$S_t = \frac{1}{\zeta(t_0,t)} E_t \int_t^\infty \zeta(t_0,u)\delta_u du. \tag{6}$$

Similarly the wealth of agent j is the expected net present value of his future consumption stream

$$w_j(t) = \frac{1}{\zeta(t_0,t)} E_t \int_t^\infty \zeta(t_0,u)c_j(u)du. \tag{7}$$

Given the processes S_t, $w_j(t)$ and $c_j(t)$, (3) determines the share holding process $\theta_j(t)$.

2.2 CRRA utility

We now specialise by making the following assumptions, in force for the rest of the paper:

(A1) the utility functions are given by

$$U_j'(t,x) = e^{-\rho_j t} x^{-R}$$

for positive constants R and ρ_i, $i = 1, \ldots, J-1$;

(A2) the dividend process is of the form

$$\delta_t = \exp(\sigma W_t + \mu t) \tag{8}$$

for some constants σ and μ;

(A3) the process φ is deterministic.

These assumptions are a significant reduction in generality; agents have a common coefficient of relative risk aversion, and differ only in their impatience parameters ρ_i. Stochastic divisions of the dividend process are disallowed. As we shall see, these two assumptions make the problem tractable; allowing different coefficients of relative risk aversion would greatly increase the computational complexity. Although such problems *can* be handled effectively numerically (see Rogers & Yousaf [7]), it is not our current purpose to get involved in such complications. The assumption (A2) is probably the least substantive of the three and could be relaxed quite easily, but this seems pointless in view of the other two assumptions being made.

The following proposition summarises the simplifications which result.

Proposition 1. *Under assumptions (A1), (A2) and (A3), all agents keep all of their wealth in shares at all times. There are positive constants $p_j(t_0)$, $j = 1, \ldots, J-1$ in terms of which the share price may be expressed as*

$$S_t = \delta_t \varphi_t^R \, \frac{\psi(t_0, t)}{\tilde{\gamma}(t_0, t)}, \tag{9}$$

where

$$\tilde{\gamma}(t_0, t) = \left(\sum_{i<J} \left(p_i(t_0) e^{-\rho_i t} \right)^{1/R} \right)^R .$$

and

$$\psi(t_0, t) = \int_t^\infty \tilde{\gamma}(t_0, u) \varphi_u^{-R} e^{\alpha(u-t)} du, \tag{10}$$

with

$$\alpha = (1-R) \left(\frac{\sigma^2}{2}(1-R) + \mu \right). \tag{11}$$

The optimal consumption streams of the agents are given by

$$c_j(t) = \frac{\left(p_j(t_0) e^{-\rho_j t} \right)^{1/R}}{\tilde{\gamma}(t_0, t)^{1/R}} \tilde{\delta}_t. \tag{12}$$

and the holdings of shares of agent j at time t is

$$\theta_j(t) = \frac{1}{\psi(t_0, t)} \int_t^\infty \left(p_j(t_0) e^{-\rho_j u} \right)^{1/R} \varphi_u^{1-R} \tilde{\gamma}(t_0, u)^{1-1/R} e^{\alpha(u-t)} du. \tag{13}$$

Proof. See Appendix.

The vector $p(t_0)$ determines the state-price density and the consumption paths, and hence the holdings of shares. In what follows, we have taken as given the initial share holdings $\theta(t_0)$ of the agents, and then computed the values of $p(t_0)$ which match (13) to the given $\theta(t_0)$. We shall write

$$\begin{aligned}
\pi_j^{t_0} &= E \int_{t_0}^\infty U_j(t, c_j(t)) dt \\
&= \frac{1}{1-R} E \int_{t_0}^\infty e^{-\rho_j t} \left(\left(\frac{p_j(t_0) e^{-\rho_j t}}{\tilde{\gamma}(t_0, t)} \right)^{1/R} \delta_t \left(1 - \theta_J(t_0)\right) \right)^{1-R} dt \\
&= \frac{\left(\left(1 - \theta_J(t_0)\right) \delta_{t_0} \right)^{1-R}}{1-R} \int_{t_0}^\infty e^{(\alpha - \rho_j)t} \left(\frac{p_j(t_0) e^{-\rho_j t}}{\tilde{\gamma}(t_0, t)} \right)^{(1/R)-1} dt
\end{aligned}$$

for the payoffs of the different agents in the original equilibrium.

3 J's optimisation problem

We now consider a Jth agent who will follow the consumption path

$$c_J(t) = (1 - \varphi_t) \delta_t.$$

The problem for agent J is to choose the function φ_t which maximises his total expected utility of consumption. However, he is constrained in the choice of φ_t. One possible consumption stream is given by taking $\varphi_t = 1-\theta_J(t_0)$. This choice of φ_t does not require any trading with the pool as J is consuming his share of the dividend as he receives it. But for all other choices of φ_t agent J requires the cooperation of the pool in attaining the desired consumption stream. We will suppose that the pool will accept a particular φ_t if each member of the pool prefers, or is indifferent to, that choice of φ_t over taking $\varphi_t = 1 - \theta_J(t)$. The preferences of an agent between various proposed functions φ_t are deduced from the relative total expected utility of the corresponding consumption streams. Agent J, therefore, has the following problem

$$\sup_{\varphi,\{p_i(t_0)\}} E\int_{t_0}^{\infty} U_J(t,(1-\varphi_t)\delta_t)dt \tag{14}$$

subject to the constraints

$$E\int_{t_0}^{\infty} U_j(t,c_j(t))dt \geq \pi_j^{t_0} \qquad j=1,\dots,J-1 \tag{15}$$

where $\pi_j^{t_0}$ is the total expected utility agent j obtains when $\varphi_t = 1-\theta_J(t_0)$.

We can solve this problem with a Lagrangian. The $y_i, i=1,\dots,J-1$ are non-negative Lagrange multipliers.

$$\begin{aligned} L &= E\int_{t_0}^{\infty}\left\{U_J(t,(1-\varphi_t)\delta_t) + \sum_{i<J} y_i U_i(t,c_i(t))\right\}dt - \sum_{i<J} y_i \pi_i^{t_0} \\ \frac{\partial L}{\partial \varphi_t} &= E\int_{t_0}^{\infty}\left\{-\delta_t U_J'(t,(1-\varphi_t)\delta_t) + \sum_{i<J} y_i U_i'(t,c_i(t))\frac{\partial c_i(t)}{\partial \varphi_t}\right\}dt. \end{aligned}$$

Setting this derivative to zero gives the optimal φ_t. In the case of constant relative risk aversion a solution is given by the roots of

$$e^{-\rho_J t}(1-\varphi_t)^{-R} = \sum_{i<J} y_i e^{-\rho_i t}\frac{\tilde{\gamma}(t_0,t)^{1-1/R}}{(p_i(t_0)e^{-\rho_i t})^{1-1/R}}\varphi_t^{-R}. \tag{16}$$

φ_t is determined up to the choice of constants y_i and $p_i(t_0)$. These are chosen to give equality in each constraint (15) and to maximise (14). Typically $\theta_j(t)$ will not be continuous at t_0 and so there will be a reallocation of shares at t_0.

4 Numerical results

In this section we present some examples. The Lagrange multipliers can be calculated numerically when the initial conditions of the problem are specified. We will take $t_0 = 0$. In the case of logarithmic utility, the form of δ_t only influences the payoff by an additive constant, and so this is omitted. For non-logarithmic utility, δ_t has to be specified. We choose δ_t to be of the form given by (8) and report the value of the constant α, defined in (11). The bold typeface indicates the largest payoff for an agent. The final row shows the proportional change to the equilibrium consumption path that would be required to match the J-solution payoff.

Logarithmic utility ($R = 1$)

Agent	1	2	J
ρ	1.2	0.9	1.8
$\theta(0)$	0.35	0.1	0.55
$p(0)$	1	0.1992	
y	0.4086	0.08139	
Equilibrium payoff	**−0.8623**	**−2.439**	−0.3182
J-solution payoff	−0.8728	−2.5312	**−0.3059**
Change	0.98754	0.92061	1.0223

Logarithmic utility ($R = 1$)

Agent	1	2	J
ρ	1.2	0.6	0.38
$\theta(0)$	0.15	0.15	0.7
$p(0)$	1	0.6579	
y	0.4919	0.3236	
Equilibrium payoff	**−1.323**	−3.104	−0.8893
J-solution payoff	−1.529	**−3.086**	**−0.7762**
Change	0.78143	1.0105	1.0439

Logarithmic utility ($R = 1$)

Agent	1	2	3	4	J
ρ	1.0	1.4	0.7	1.9	1.1
$\theta(0)$	0.2	0.1	0.3	0.25	0.15
$p(0)$	1	0.7006	1.045	2.375	
y	1.212	0.8503	1.267	2.879	
Equilibrium payoff	−1.603	−1.618	**−1.606**	−0.6615	−1.719
J-solution payoff	**−1.601**	**−1.617**	−1.610	**−0.6610**	**−1.718**
Change	1.0015	1.0022	0.99697	1.0010	1.0009

$R = 3 \quad \alpha = -0.12$

Agent	1	2	J
ρ	1.5	1.9	2.2
$\theta(0)$	0.35	0.1	0.55
$p(0)$	1	0.02972	
y	0.1755	0.005216	
Equilibrium payoff	**−2.479**	−24.75	−0.7081
J-solution payoff	−2.517	**−24.51**	**−0.7025**
Change	0.99232	1.0048	1.0040

$R = 0.5 \quad \alpha = 0.06125$

Agent	1	2	3	J
ρ	1.1	1.4	1.7	2.0
$\theta(0)$	0.2	0.3	0.3	0.2
$p(0)$	1	1.671	2.047	
y	0.5196	0.8683	1.064	
Equilibrium payoff	**0.8991**	**0.8205**	0.6764	0.4791
J-solution payoff	0.8853	0.8195	**0.6816**	**0.4819**
Change	0.96945	0.99742	1.0152	1.0115

$R = 2.3 \quad \alpha = -0.0962$

Agent	1	2	3	4	J
ρ	0.9	1.1	1.5	1.8	1.4
$\theta(0)$	0.2	0.1	0.15	0.3	0.25
$p(0)$	1	0.2372	0.7980	4.750	
y	0.3277	0.1878	0.1328	2.229	
Equilibrium payoff	**−6.0065**	**−12.682**	−5.6642	−1.9051	−3.1163
J-solution payoff	−6.0154	−12.691	**−5.6618**	**−1.9033**	**−3.1160**
Change	0.99886	0.99949	1.0003	1.0007	1.0001

$R = 0.8 \quad \alpha = 0.07$

Agent	1	2	J
ρ	0.2	1.4	1.7
$\theta(0)$	0.35	0.49	0.16
$p(0)$	1	22.99	
y	0.9216	2.160	
Equilibrium payoff	**34.77**	3.446	**2.273**
J-solution payoff	33.93	**3.497**	2.263
Change	0.88522	1.0768	0.97826

In each example, the constraints in (15) are met with equality. This means that the payoff for an agent in the pool under the J-solution is equal to the payoff which that agent would obtain if J chose φ_t to be given by $\varphi_t = 1 - \theta_J(0)$. If each agent in the pool preferred the J-solution to the global equilibrium, the pool would be a blocking coalition. The absence of blocking coalitions therefore implies that at least one agent in the pool will prefer the global equilibrium to the J-solution.

If each agent in the pool prefers global equilibrium to the payoff obtained when $\varphi_t = 1 - \theta_J(0)$, as in the first example, then the choice of φ_t leading to the global equilibrium satisfies the constraints. Therefore agent J's payoff under the J-solution will be greater than under global equilibrium, as the J-solution gives J his maximum payoff over functions φ_t which satisfy the constraints.

The final example shows that J does not always prefer the J-solution to global equilibrium.

5 Breakdown of the rule of law

In section 3 agent J made the choice of φ_t at time t_0 and it was assumed that each agent would follow the consumption path implied by φ_t. In this section we consider whether the deal reached at time t_0 will ever break down at a future time. This would occur if one of two conditions holds, either

1. a subset of the pool prefers to stop trading outside the subset and forms its own equilibrium, or
2. agent J prefers to stop trading and consumes the dividend as he receives it.

The following lemma, which is proved in the appendix, shows that condition 1 is never satisfied in the case of constant relative risk aversion.

Lemma 1. *Without loss of generality assume that $\rho_1 < \rho_2 < \ldots < \rho_{J-1}$. Let each agent have constant relative risk aversion, suppose that the dividend process is of the form given by (8), and assume that $\rho_J \neq \rho_1$. Then every subset of the pool contains at least one agent who prefers the J-solution to the subset equilibrium.*

Now we look at the second condition. The lemma below gives conditions under which J will break away. It is necessary to show that it is possible for the conditions of this lemma to be satisfied. We do this by presenting an example.

Lemma 2. *Let each agent have log utility. If $\rho_J < \rho_1 < \ldots < \rho_{J-1}$, $\rho_1 - \rho_J > \rho_2 - \rho_1$ and $p_2(t_0)/p_1(t_0) - y_2/y_1 < 0$ then agent J will eventually prefer to break away from the J-solution and hold onto his shares consuming the dividend as he receives it.*

Proof. At time τ agent J's share holding is given by

$$\theta_J(\tau) \quad = \quad 1 - \frac{\int_\tau^\infty \tilde{\gamma}(t_0, u)du}{\psi(t_0, \tau)}$$

and so the condition for J to break away at time τ is

$$\log\left(1 - \frac{\int_\tau^\infty \tilde{\gamma}(t_0, u)du}{\psi(t_0, \tau)}\right) - \int_\tau^\infty \rho_J e^{-\rho_J(u-\tau)} \log\left(1 - \varphi_u\right) du > 0. \tag{17}$$

We begin by applying Jensen's inequality to the left-hand-side of (17).

$$\begin{aligned} & \log\left(\frac{\int_\tau^\infty \frac{\tilde{\gamma}(t_0,u)}{\varphi_u}\left(1-\varphi_u\right)du}{\int_\tau^\infty \frac{\tilde{\gamma}(t_0,u)}{\varphi_u}du}\right) - \int_\tau^\infty \rho_J e^{-\rho_J(u-\tau)}\log\left(1-\varphi_u\right)du \\ > \; & \frac{\int_\tau^\infty \frac{\tilde{\gamma}(t_0,u)}{\varphi_u}\log\left(1-\varphi_u\right)du}{\int_\tau^\infty \frac{\tilde{\gamma}(t_0,u)}{\varphi_u}du} - \int_\tau^\infty \rho_J e^{-\rho_J(u-\tau)}\log\left(1-\varphi_u\right)du. \end{aligned} \tag{18}$$

Each of the two terms in (18) is an average of the function $\log(1 - \varphi_t)$. In the case of log utility the form of the optimal φ given by (16) simplifies to

$$\varphi_t \quad = \quad \frac{\sum_{i<J} y_i e^{-\rho_i t}}{e^{-\rho_J t} + \sum_{i<J} y_i e^{-\rho_i t}}. \tag{19}$$

We will look at the expression in (18) for large values of τ. The condition that $\rho_J < \rho_1 < \rho_2 < \ldots < \rho_{J-1}$ and (19) imply that $\log(1 - \varphi_t)$ is increasing for large t. We have

$$\begin{aligned} \frac{\tilde{\gamma}(t_0,u)}{\varphi_u} &= \frac{\left(\sum_{i<J} p_i(t_0)e^{-\rho_i u}\right)\left(e^{-\rho_J u} + \sum_{k<J} y_k e^{-\rho_k u}\right)}{\sum_{i<J} y_i e^{-\rho_i u}} \\ &= \frac{p_1(t_0)e^{-\rho_1 u}\left(1 + \frac{p_2(t_0)}{p_1(t_0)}e^{-(\rho_2-\rho_1)u} + \ldots\right)e^{-\rho_J u}\left(1 + \sum_{k<J} y_k e^{-(\rho_k-\rho_J)u}\right)}{y_1 e^{-\rho_1 u}\left(1 + \frac{y_2}{y_1}e^{-(\rho_2-\rho_1)u} + \ldots\right)} \\ &= \frac{p_1(t_0)}{y_1}e^{-\rho_J u}\left\{1 + \left(\frac{p_2(t_0)}{p_1(t_0)} - \frac{y_2}{y_1}\right)e^{-(\rho_2-\rho_1)u} + \ldots\right\} \end{aligned}$$

for large u, using the condition that $\rho_1 - \rho_J > \rho_2 - \rho_1$ for the final step. After being normalised, $\tilde{\gamma}(t_0, u)/\varphi_u$ will give a measure that tends to an average of two exponentials, one with rate ρ_J and the other with rate $\rho_J - \rho_1 + \rho_2$, as u tends to infinity. A comparison of the average of $\log(1-\varphi_t)$ under this measure with the average of $\log(1-\varphi_t)$ under an exponential measure with rate ρ_J depends on the sign of $p_2(t_0)/p_1(t_0) - y_2/y_1$. As $p_2(t_0)/p_1(t_0) - y_2/y_1$ is negative, the average of an increasing function under the measure generated by $\tilde{\gamma}(t_0, u)/\varphi_u$ will be greater than its average under an exponential measure of rate ρ_J. Therefore we conclude that in this case J will eventually prefer to break away. □

It remains to show that it is possible for the ρ_i, the $p_i(t_0)$ and the y_i to satisfy the conditions imposed on them for an optimal φ_t. We do this by presenting an example where the Lagrange multipliers have been numerically calculated. t_0 is taken to equal 0.

Agent	1	2	3	J
ρ	1.2	1.3	1.9	0.1
$\theta(0)$	0.2	0.25	0.2	0.35
$p(0)$	1	1.298	1.316	
y	1.760	2.417	2.287	

In this case J initially prefers to continue with his original choice of φ_t. However, by time 6 he would benefit from holding onto his shares and consuming the dividend as he receives it.

Time	Payoff from original φ_t	Payoff from holding shares
0	**−2.241**	−3.732
3	**−0.1065**	−0.1088
6	−0.003403	**−0.003378**

φ_t decreases to zero and $\theta_J(t)$ increases to one. However, agent J is always consuming at a lower rate than he is receiving the dividend. Therefore consuming the dividend as he receives it results in an immediate increase in the consumption rate.

6 Conclusions

We have investigated the impact on a simple market of a large investor who does not act as a price taker. Traditional approaches to the effect of a large investor on price have assumed that price is determined by some instantaneous equalising of supply and demand, but, as Arrow & Kurz [1, p74] have made clear in a somewhat different context, '... we may say that it requires the future to determine the present resource allocation.' It is such an analysis we have conducted here, allowing the large investor to choose a future dividend flow consistent with the current division of the asset among market participants. This can equally be considered to be the problem facing XYZ plc in its attempts to take over ABC plc; the bidder must offer each of the existing shareholders a deal that would leave them no worse off in order to get the offer accepted. This can be compared with the solution that would obtain if the large agent simply entered the market, and allowed a global equilibrium to establish itself. Examples show that the large agent sometimes prefers one, sometimes the other.

The reason is that when the large agent sets up an agreed deal with the other agents, he must ensure that they are all no worse off, and even though he may configure the deal optimally for himself subject to this constraint, in a global equilibrium, it may turn out that some of the other agents do worse off than originally, and this may result in the large agent actually preferring the global equilibrium.

Having decided this, we investigated the viability of the large agent's optimal deal in circumstances when there was no enforceability of the deal. It may be that at all times after the deal is set up, all the agents prefer to continue with the deal than to go off in a subset and follow their own equilibrium in that subset. We have only partial results here; we have been able to show that no coalition of the original pool of agents would ever want to walk out on the deal that they agreed to, but that circumstances can arise where the large agent may wish to walk out with his current share holding and consume the output of that. The chief characteristic of the situation where we were able to show this 'walk-out' is that the large agent is very patient. As time increases, his share of the productive asset increases, as does his consumption stream, but it can be that his consumption stream at large time is less than the consumption stream that would accrue from his current holding of shares. This leads him to walk away.

The analysis of the unenforceable situation is still far from complete and appears to be difficult; without the rule of law, some of the deals that XYZ would propose would not be agreed by the ABC, because at some later stage XYZ would walk away from the deal. This would change the nature of the optimal solution proposed by XYZ in the first place.

Appendix

Proof of proposition 1

Proof. Assumption (A1) allows the expression for $\zeta(t_0,t)$ in (4) to be simplified:

$$\zeta(t_0,t) \quad = \quad p_j(t_0)e^{-\rho_j t}c_j(t)^{-R}. \tag{20}$$

Combining this with market clearing (1) gives

$$\zeta(t_0,t) \quad = \quad \tilde{\delta}_t^{-R}\tilde{\gamma}(t_0,t). \tag{21}$$

Substituting this expression for $\zeta(t_0,t)$ into (6) gives the form of the share price S_t in (9). The consumption stream of agent j, given by (12), is found by eliminating $\zeta(t_0,t)$ from (20) and (21). The wealth of agent j, given in (7), simplifies because of the expressions for $\zeta(t_0,t)$ in (21) and $c_j(t)$ in (12):

$$\begin{aligned} w_j(t) &= \frac{1}{\zeta(t_0,t)}E_t\int_t^\infty \tilde{\delta}_u^{1-R}\left(p_j(t_0)e^{-\rho_j u}\tilde{\gamma}(t_0,u)^{R-1}\right)^{1/R}du \\ &= \frac{\delta_t\varphi_t^R}{\tilde{\gamma}(t_0,t)}\int_t^\infty\left(p_j(t_0)e^{-\rho_j u}\tilde{\gamma}(t_0,u)^{R-1}\right)^{1/R}\varphi_u^{1-R}e^{\alpha(u-t)}du \\ &= \frac{S_t}{\psi(t_0,t)}\int_t^\infty\left(p_j(t_0)e^{-\rho_j u}\tilde{\gamma}(t_0,u)^{R-1}\right)^{1/R}\varphi_u^{1-R}e^{\alpha(u-t)}du \end{aligned}$$

To find the share holding of agent j, we calculate the dynamics of the wealth process of that agent:

$$\begin{aligned}
dw_j(t) &= \frac{dS_t}{\psi(t_0,t)}\int_t^\infty \left(p_j(t_0)e^{-\rho_j u}\tilde{\gamma}(t_0,u)^{R-1}\right)^{1/R}\varphi_u^{1-R}e^{\alpha(u-t)}du \\
&\quad +\frac{S_t\tilde{\gamma}(t_0,t)\varphi_t^{-R}}{\psi(t_0,t)}\left(\int_t^\infty \left(p_j(t_0)e^{-\rho_j u}\tilde{\gamma}(t_0,u)^{R-1}\right)^{1/R}\varphi_u^{1-R}e^{\alpha(u-t)}du\right)dt \\
&\quad -\frac{S_t\varphi_t^{1-R}}{\psi(t_0,t)}\left(p_j(t_0)e^{-\rho_j t}\tilde{\gamma}(t_0,t)^{R-1}\right)^{1/R}dt \\
&= \frac{w_j(t)}{S_t}(dS_t+\delta_t dt)-c_j(t)dt
\end{aligned}$$

The expression for $\theta_j(t)$ given in (13) follows from this and (3). We also deduce that no agent holds any bonds at any time. □

Proof of lemma 1

We will prove this lemma in the case where the coefficient of relative risk aversion, R, is not equal to one. The log utility case, where $R=1$, requires a separate (but analogous) proof because of the different form of the utility function.

Proof. Assume $R \neq 1$. Suppose that A is a subset of the pool in which each agent prefers the subset equilibrium to the J-solution. The consumption path that agent j in this subset follows when A breaks away is

$$c'_j(t) = \frac{(p_j(\tau)e^{-\rho_j t})^{1/R}}{\tilde{\gamma}_A(\tau,t)^{1/R}}\delta_t\theta_A(\tau-), \qquad t\geq\tau \tag{22}$$

where

$$\begin{aligned}
\tilde{\gamma}_A(\tau,t) &= \left(\sum_{i\in A}(p_i(\tau)e^{-\rho_i t})^{1/R}\right)^R, \\
\theta_A(\tau-) &= \sum_{i\in A}\theta_i(\tau-).
\end{aligned} \tag{23}$$

The share holding process which leads to this consumption path is

$$\theta_j(t) = \frac{\theta_A(\tau-)^{1-R}}{\psi_A(\tau,t)}\int_t^\infty \left(p_j(\tau)e^{-\rho_j u}\right)^{1/R}\tilde{\gamma}_A(\tau,u)^{1-1/R}e^{\alpha(u-t)}du \tag{24}$$

where

$$\psi_A(\tau,t) = \int_t^\infty \tilde{\gamma}_A(\tau,u)\theta_A(\tau-)^{-R}e^{\alpha(u-t)}du.$$

The consumption path $c_j(t)$ and share holding process that agents in the pool follow under the original choice of φ_t are given by (12) and (13). For agent j in the subset A to prefer the alternative consumption path (22) to that given by (12) at time τ we need

$$E_\tau\int_\tau^\infty \frac{e^{-\rho_j u}}{1-R}c'_j(u)^{1-R}du \geq E_\tau\int_\tau^\infty \frac{e^{-\rho_j u}}{1-R}c_j(u)^{1-R}du$$

which can also be written as

$$\begin{aligned}
&\int_\tau^\infty \frac{e^{-\rho_j u/R+\alpha(u-\tau)}}{1-R}\left(\frac{p_j(\tau)^{1/R}}{\tilde\gamma(\tau,u)^{1/R}}\theta_A(\tau-)\right)^{1-R} du \\
\geq\ &\int_\tau^\infty \frac{e^{-\rho_j u/R+\alpha(u-\tau)}}{1-R}\left(\frac{p_j(t_0)^{1/R}}{\tilde\gamma(t_0,u)^{1/R}}\varphi_u\right)^{1-R} du.
\end{aligned}$$

Our aim now is to show that this inequality leads to a contradiction. The vector $p(\tau)$ is chosen so that $\theta(t)$ is continuous at $t=\tau$. This means that (13) and (24) must be equal when $t=\tau$. Using this, our condition is equivalent to

$$\frac{p_j(t_0)}{(1-R)\psi(t_0,\tau)} \quad\geq\quad \frac{p_j(\tau)}{(1-R)\psi_A(\tau,\tau)}$$

or again

$$\begin{aligned}
&\frac{p_j(t_0)}{1-R}\psi(t_0,\tau)^{-1} \\
\geq\ &\frac{p_j(\tau)}{1-R}\left(\int_\tau^\infty \tilde\gamma_A(\tau,u)\theta_A(\tau)^{-R}e^{\alpha(u-\tau)}du\right)^{-1}. \qquad (25)
\end{aligned}$$

An expression for $\theta_A(\tau)$ can be found from the expressions for $\theta_j(t)$ in (13) and $\tilde\gamma_A(\tau,t)$ in (23):

$$\theta_A(\tau) \quad=\quad \frac{1}{\psi(t_0,\tau)}\int_\tau^\infty \tilde\gamma_A(t_0,u)^{1/R}\tilde\gamma(t_0,u)^{1-1/R}\varphi_u^{1-R}e^{\alpha(u-\tau)}du.$$

We can take $\theta_A(\tau)$ outside the integral in (25) and use the expression above to obtain

$$\begin{aligned}
&\frac{p_j(t_0)}{1-R}\int_\tau^\infty \tilde\gamma_A(\tau,u)e^{\alpha(u-\tau)}du \\
\geq\ &\frac{p_j(\tau)}{(1-R)\psi(t_0,\tau)^{R-1}}\left(\int_\tau^\infty \tilde\gamma_A(t_0,u)^{1/R}\tilde\gamma(t_0,u)^{1-1/R}\varphi_u^{1-R}e^{\alpha(u-\tau)}du\right)^R.
\end{aligned}$$

As this inequality holds for all $j\in A$, it follows that we must have

$$\begin{aligned}
&\frac{\left(\sum_{j\in A}(p_j(t_0)e^{-\rho_j s})^{1/R}\right)^R e^{\alpha(s-\tau)}}{1-R}\int_\tau^\infty \tilde\gamma_A(\tau,u)e^{\alpha(u-\tau)}du \\
\geq\ &\frac{\left(\sum_{j\in A}(p_j(\tau)e^{-\rho_j s})^{1/R}\right)^R e^{\alpha(s-\tau)}}{(1-R)\psi(t_0,\tau)^{R-1}}\left(\int_\tau^\infty \tilde\gamma_A(t_0,u)^{1/R}\tilde\gamma(t_0,u)^{1-1/R}\varphi_u^{1-R}e^{\alpha(u-\tau)}du\right)^R
\end{aligned}$$

for all $s\geq\tau$. Using the definition of $\tilde\gamma_A$ in (23), and integrating with respect to s on the interval $[\tau,\infty)$, we deduce that

$$\begin{aligned}
&\frac{1}{1-R}\int_\tau^\infty \tilde\gamma_A(t_0,s)e^{\alpha(s-\tau)}ds \\
\geq\ &\frac{1}{(1-R)\psi(t_0,\tau)^{R-1}}\left(\int_\tau^\infty \tilde\gamma_A(t_0,u)^{1/R}\tilde\gamma(t_0,u)^{1-1/R}\varphi_u^{1-R}e^{\alpha(u-\tau)}du\right)^R (26)
\end{aligned}$$

From (10), $\psi(t_0, \tau)$ can be written as

$$\psi(t_0, \tau) \quad = \quad \int_\tau^\infty \tilde{\gamma}_A(t_0, u)^{1/R} \tilde{\gamma}(t_0, u)^{1-1/R} \varphi_u^{1-R} e^{\alpha(u-\tau)} \left(\frac{\tilde{\gamma}(t_0, u)^{1/R}}{\tilde{\gamma}_A(t_0, u)^{1/R} \varphi_u} \right) du.$$

Substituting this expression for $\psi(t_0, \tau)$ into (26) and rearranging gives

$$\begin{aligned} & \frac{1}{1-R} \frac{\int_\tau^\infty \tilde{\gamma}_A(t_0, u)^{1/R} \tilde{\gamma}(t_0, u)^{1-1/R} \varphi_u^{1-R} e^{\alpha(u-\tau)} \left(\frac{\tilde{\gamma}(t_0,u)^{1/R}}{\tilde{\gamma}_A(t_0,u)^{1/R}\varphi_u} \right)^{1-R} du}{\int_\tau^\infty \tilde{\gamma}_A(t_0, u)^{1/R} \tilde{\gamma}(t_0, u)^{1-1/R} \varphi_u^{1-R} e^{\alpha(u-\tau)} du} \\ \geq \quad & \frac{1}{1-R} \left(\frac{\int_\tau^\infty \tilde{\gamma}_A(t_0, u)^{1/R} \tilde{\gamma}(t_0, u)^{1-1/R} \varphi_u^{1-R} e^{\alpha(u-\tau)} \left(\frac{\tilde{\gamma}(t_0,u)^{1/R}}{\tilde{\gamma}_A(t_0,u)^{1/R}\varphi_u} \right) du}{\int_\tau^\infty \tilde{\gamma}_A(t_0, u)^{1/R} \tilde{\gamma}(t_0, u)^{1-1/R} \varphi_u^{1-R} e^{\alpha(u-\tau)} du} \right)^{1-R}. \end{aligned}$$

Jensen's inequality tells us that the reverse inequality is also true, and so we must in fact have equality. This is only possible when

$$\begin{aligned} \varphi_t &= k \frac{\tilde{\gamma}(t_0, t)^{1/R}}{\tilde{\gamma}_A(t_0, t)^{1/R}} \\ &= k \frac{\sum_{i<J} p_i(t_0)^{1/R} e^{-\rho_i t/R}}{\sum_{i \in A} p_i(t_0)^{1/R} e^{-\rho_i t/R}} \qquad (27) \end{aligned}$$

for some positive constant k. For φ_t to lie in the range $[0, 1]$ it is necessary that $k < 1$. We are able to show that this form of φ_t contradicts that given in (16) by looking at the behaviour when t tends to infinity. If agent 1 is not in the subset A then (27) implies that φ_t tends to infinity as t increases, which contradicts (16) where φ_t is always in the range $[0, 1]$. If agent 1 is in subset A then according to (16), φ_t tends to k and so

$$\left(\frac{\varphi_t}{1-\varphi_t} \right)^R \quad \longrightarrow \quad \left(\frac{k}{1-k} \right)^R.$$

By rearranging (16) we find that

$$\begin{aligned} \left(\frac{\varphi_t}{1-\varphi_t} \right)^R &= \frac{\sum_{i<J} y_i e^{-\rho_i t/R} p_i(t_0)^{(1-R)/R} e^{\rho_J t}}{\left(\sum_{i<J} p_i(t_0)^{1/R} e^{-\rho_i t/R} \right)^{1-R}} \\ &\longrightarrow y_1 e^{(\rho_J - \rho_1)t} \end{aligned}$$

and as $\rho_J \neq \rho_1$ we have a contradiction. □

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