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Recurrence and transience of reflecting Brownian motion in the quadrant

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1. *Introduction and preliminaries*

In this paper we obtain criteria for a reflecting Brownian motion in the first orthant $\mathbb{R}^2_+$ of the plane to reach an arbitrary open neighbourhood of the origin in finite time, or in finite mean time. The reflecting Brownian motion (RBM) is assumed to have a constant non-zero drift, and a constant non-singular covariance, and the directions of reflection on the two sides of $\mathbb{R}^2_+$ are constant along each side, but not necessarily normal. We explain why the criteria we find are to be expected.

The proof in the finite mean case is fairly unsophisticated, but the estimation required for the finite time but infinite mean case is more delicate. Of independent interest is a bound that we require on the amount of time a real Lévy process spends in an interval before first hitting zero.

Reflecting Brownian motion can be considered as a process $Z_t = (X_t, Y_t)$, where the coordinate processes satisfy (at least until the RBM first hits zero)

$$\left.\begin{aligned} X_t &= x+B_t+\mu t+L_t^X+\alpha L_t^Y, \\ Y_t &= y+W_t+\nu t+\beta L_t^X+L_t^Y. \end{aligned}\right\} \tag{1}$$

Here α, B, μ and ν are fixed reals, (B, W) is a Brownian motion in $\mathbb{R}^2$ with non-singular covariance Σ and L^X (respectively, L^Y) is the local time process at zero of X (respectively, Y). Thus, for example, until the time $\tau \equiv \inf\{u : X_u = 0\}$, we have

$$L_t^Y = \max_{s \leqslant t}(-y-W_s-\nu s)^+. \tag{2}$$

The construction of such a process from (B, W) is quite straightforward (at least until the first time Z hits 0) and we refer the reader to Varadhan and Williams[**14**] should further details seem necessary. We shall make the following assumption throughout:

Assumption 1. The drift vector $b \equiv (\mu, \nu)$ is non-zero.

Let $\mathcal{N}$ be a bounded neighbourhood of the origin. The main result of this paper is the following.

THEOREM 1·1. *Let* $T \equiv \inf\{u : Z_u \in \mathcal{N}\}$. *Then*

$$\mathbb{P}^z(T < \infty) = 1 \quad \textit{for all } z \in \mathbb{R}^2_+ \tag{3}$$

if and only if
$$\mu+\alpha\nu^- \leqslant 0, \quad \nu+\beta\mu^- \leqslant 0. \tag{4}$$
Moreover,
$$\mathbb{E}^z(T) < \infty \quad \textit{for all } z \in \mathbb{R}^2_+ \tag{5}$$
if and only if
$$\mu+\alpha\nu^- < 0, \quad \nu+\beta\mu^- < 0 \tag{6}$$
and then
$$\mathbb{E}^z(T) \leqslant C(1+|z|) \quad \textit{for all } z \in \mathbb{R}^2_+ \tag{7}$$
for some constant C.

Remarks. (i) The conditions (4) and (6) and conclusions (3) and (5) mirror those of Malyšev [**9**], who considered a two-dimensional random walk with state space $\mathbb{Z}^2_+$. Using methods anticipated by Kingman [**7**], Malyšev proved positive recurrence by constructing a Lyapunov function which could be approximated locally by a linear function and hence shown to satisfy Foster's criterion: see [**2**]. Specifically he used arcs of circles and straight lines to construct the Lyapunov function; his function was C^1 but not C^2. Rosenkrantz [**12**] considered a slightly more general Markov chain in $\mathbb{Z}^2_+$ and constructed a smooth Lyapunov function with which to determine positive recurrence. However, his method cannot be extended to the null recurrent situation.

Lyapunov functions are a natural and attractive approach to the problem we consider which is the continuous (time and state space) analogue of the random walk problem. The methods of Kingman and Malyšev could certainly be extended to our problem; however we use a completely different sample path approach. An additional benefit of our method is the estimate (7) of the mean time to hit a given neighbourhood, and Theorem 1·2 below on Lévy processes may be of independent interest.

(ii) The conditions of Theorem 1·1 do not involve the covariance Σ in any way. This can be understood easily, because if the drift b is non-zero, then the drift dominates the diffusion on long time and distance scales, and the recurrence or transience is determined by this large-scale behaviour. The case of zero drift is quite different, but easily reduced to known results. Indeed, the process $\Sigma^{-\frac{1}{2}}(X, Y)^T$ is again an RBM (in a wedge now, rather than $\mathbb{R}^2_+$) with identity covariance, and so questions of whether 0 can be reached are completely decided by the results of Varadhan and Williams [**14**] and Williams [**15**]. See also Rogers [**11**] for an excursion-theoretic proof.

(iii) Any wedge contained in the upper half plane $\mathbb{H}$ can be mapped onto the quadrant via a linear transformation. Thus our results extend to Brownian motion in a wedge with constant non-zero drift and constant angles of reflection over each face. See Section 3·1 for an application of this idea in reverse.

(iv) The spirit of (3) is that the process is neighbourhood recurrent, and the spirit of (5) is that the process is neighbourhood positive recurrent. There is need for some care in interpreting this, because of the possibility that the RBM may reach the corner and be trapped there. In the case of zero drift, the results of Varadhan and Williams decide exactly when this may happen, and, at first sight, the introduction of a constant drift cannot change anything here, because on the small scale which determines the behaviour at 0, the drift is dominated by the Brownian part. This reasoning is valid in cases where the RBM with zero drift can be obtained as a pathwise solution of the Skorohod equation (see Harrison and Reiman [**4**]), because then a Cameron–Martin change of measure allows the drift to be added painlessly. But in cases where the RBM with zero drift is not a semimartingale, this approach appears to break down. Can the inclusion of drift really change anything at 0?

(v) Let us explain now intuitively why conditions (4) and (6) are what one would expect. If $\mu, \nu > 0$ then it is clear that with positive probability the process could escape to infinity without touching either axis (we substantiate this claim in Section 2). So suppose that $\nu < 0 < \mu$: it is possible for the process to reach a neighbourhood of 0 almost surely? At first sight, the positive drift in the x-direction seems to rule this out, but the y-co-ordinate has a downward drift of $\nu^- > 0$ and so (see (2)) L_t^Y is growing like $\nu^- t$. Thus until the first hit on the y-axis, the finite-variation part of X is going like $(\mu+\alpha\nu^-)\,t$, so if $\mu+\alpha\nu^- < 0$ then X will eventually reach 0. The point is that the effective drift in the x-direction is $\mu+\alpha\nu^-$, and not μ.

Study of this and related problems is motivated by a diffusion approximation from the theory of queuing networks. Harrison and Williams[5] outline how multi-dimensional RBM (in our sense with constant drift, covariance and reflection matrix) arises as the heavy traffic limit of an open queuing system with homogeneous customer populations. They also find necessary and sufficient conditions for a RBM which arises as such a limit to have a stationary distribution. However, the parameter values which can occur from the queuing approximation are not exhaustive and the proof uses, in an essential way, properties inherited from the underlying queuing model.

In Section 2 we shall prove that (6) implies (5) by considering successive crossings by the RBM from the x-axis to the y-axis and back. A relatively crude inequality allows us to conclude that there exists some x_0 such that the interval $[0, x_0]$ of the x-axis will be hit in finite mean time from any starting point. Then the equivalence of (6) and (5) follows quite easily.

As one would expect, the equivalence of (3) and (4) is considerably more delicate, and the estimation needed is given in Section 3. One result of note which we need on the way is the following estimate for the Green's function of a real Lévy process:

THEOREM 1·2. *Let $(V_t)_{t \geqslant 0}$ be a real Lévy process, and let $\tau \equiv \inf\{t: V_t \leqslant 0\}$. Then there exists some constant C such that, for all $x, a > 0$,*

$$\mathbb{E}^x\left[\int_0^\tau ds\, I_{[0,a]}(V_s)\right] \leqslant C(1+a)\,(1+(x \wedge a)). \tag{8}$$

In the case where V is a Brownian motion, we have for $a \leqslant x$ that the left-hand side of (8) is a^2, and for $x \leqslant a$ it is $2ax - x^2$. The case of a compound Poisson process shows that the left-hand side of (8) need not tend to zero as $x, a \downarrow 0$, so that the estimate is quite sharp.

2. *Positive recurrence*

In this section we shall prove part of theorem 1·1, namely that condition (6):

$$\mu+\alpha\nu^- < 0, \quad \nu+\beta\mu^- < 0,$$

implies that $\mathbb{E}^z(T) < \infty$ for all $z \in \mathbb{R}_+^2$. So until we reach Theorem 2·1 we shall assume condition (6).

The spirit of the proof is as follows. Suppose that the RBM starts at some distant point on the x-axis. Then we find estimates of the time and place of the first hit on the y-axis. The roles of the two axes can now be reversed to give estimates of the place of first return to the x-axis. It is then possible to define a jump process on each of the axes; ξ_n denoting the place of nth return to the x-axis where between each return the process has visited the y-axis. We show that this jump process returns to some large neighbourhood of the origin. Subsequently we extend this result to an arbitrary neighbourhood.

We begin by finding the estimates of the time and place of the first hit on the y-axis. It is convenient to consider two distinct cases.

PROPOSITION 2·1. *Suppose $Z_0 = (x, 0)$. Define $\tau = \inf\{t : X_t = 0\}$. Suppose that $\mu < 0$ and $\nu > 0$. Then there exist constants $(K_i)_{i=1,2,3,4}$ depending only on μ, ν and α such that*

$$\mathbb{E}(\tau) \leqslant K_1 x + K_2 \quad \textit{and} \quad \mathbb{E}(Y_\tau) \leqslant K_3 x + K_4.$$

Proof. First we show that $\mathbb{E}(\tau) < \infty$. For a continuous process U, null at 0, denote by U^* the maximum process, so that

$$U_t^* = \max_{0 \leqslant s \leqslant t} U_s.$$

For $t \leqslant \tau$,

$$X_t = x + B_t + \mu t + \alpha L_t^Y, \quad Y_t = W_t + \nu t + L_t^Y \tag{9}$$

and from (2)

$$0 \leqslant L_t^Y \leqslant (-W)_t^*; \tag{10}$$

hence

$$X_t \leqslant x + B_t + \mu t + |\alpha| \, (-W)_t^*.$$

Let $\tau' = \inf\{t : x + B_t + \mu t + |\alpha| \, (-W)_t^* \leqslant 0\}$. Then $\tau \leqslant \tau'$, and so it suffices to prove that $\mathbb{E}(\tau') < \infty$. We have

$$\begin{aligned}
\mathbb{E}(\tau') &= \int_0^\infty dt \, \mathbb{P}(\tau' > t) \\
&\leqslant \int_0^\infty dt \, \mathbb{P}(x + B_t + \mu t + |\alpha| \, (-W)_t^* > 0) \\
&\leqslant \int_0^\infty dt \, \mathbb{P}(\{x + B_t + \tfrac{1}{2}\mu t > 0\} \cup \{\tfrac{1}{2}\mu t + |\alpha| \, (-W)_t^* > 0\}) \\
&\leqslant \int_0^\infty dt \, \mathbb{P}(x + B_t + \tfrac{1}{2}\mu t > 0) + \int_0^\infty dt \, \mathbb{P}(\tfrac{1}{2}\mu t + |\alpha| \, (-W)_t^* > 0).
\end{aligned}$$

But

$$\int_0^\infty dt \, \mathbb{P}(\tfrac{1}{2}\mu t + |\alpha| (-W)_t^* > 0) = 2 \int_0^\infty dt \, \mathbb{P}(|\alpha| \, W_t > \tfrac{1}{2}\mu^- t)$$

by the reflection principle, and since the mean amount of time spent above zero by a downward drifting Brownian motion is finite, the conclusion follows.

Applying the Burkholder–Davis–Gundy inequality (BDG), with c as the universal constant, and Jensen's inequality, to (10) evaluated at the stopping time τ yields

$$\mathbb{E}(L_\tau^Y) \leqslant \mathbb{E}((-W)_\tau^*) \leqslant c\mathbb{E}(\tau^{\frac{1}{2}}) \leqslant c(\mathbb{E}(\tau))^{\frac{1}{2}}.$$

Now $\mathbb{E}(\tau)$ is finite so that $\mathbb{E}(B_\tau) = 0 = \mathbb{E}(W_\tau)$, and applying the Optional Sampling Theorem to (9) we have

$$\mathbb{E}(\tau) = \frac{x}{\mu^-} + \frac{\alpha \mathbb{E}(L_\tau^Y)}{\mu^-} \leqslant \frac{x}{\mu^-} + \frac{\alpha c}{\mu \mu^-} (\mathbb{E}(\tau))^{\frac{1}{2}} I_{(\alpha > 0)}$$

and

$$\mathbb{E}(Y_\tau) = \nu \mathbb{E}(\tau) + \mathbb{E}(L_\tau^Y) = \frac{\nu x}{\mu^-} + \left\{1 + \frac{\alpha \nu}{\mu^-}\right\} \mathbb{E}(L_\tau^Y) \leqslant \frac{\nu x}{\mu^-} + \frac{c\{\mu^- + \alpha \nu\}}{\mu^-} (\mathbb{E}(\tau))^{\frac{1}{2}} I_{(\mu^- + \alpha\nu > 0)}. \qquad \blacksquare$$

PROPOSITION 2·2. *Suppose that* $Z_0 = (x, 0)$. *Define* $\tau = \inf\{t : X_t = 0\}$. *Suppose that* $\nu \leqslant 0$ *and* $\mu + \alpha\nu^- < 0$. *Then there exist constants* $(K_i)_{i=1,2,3,4}$ *depending only on* μ, ν *and* α *such that*

$$\mathbb{E}^{(x,0)}(\tau) \leqslant K_1 x + K_2 \quad and \quad \mathbb{E}^{(x,0)}(Y_\tau) \leqslant K_3 x^{\frac{1}{2}} + K_4.$$

Proof. For $t \leqslant \tau$ we have again

$$X_t = x + B_t + \mu t + \alpha L_t^Y, \tag{11}$$

$$Y_t = W_t + \nu t + L_t^Y, \tag{12}$$

where $L_t^Y = \sup_{0 \leqslant s \leqslant t}\{-W_s - \nu s\}$. Then L_t^Y satisfies

$$-W_t \leqslant L_t^Y - \nu^- t \leqslant (-W)_t^* \tag{13}$$

and, applying BDG at the stopping time $\tau_n = \tau \wedge n$, we have

$$0 \leqslant \mathbb{E}(L_{\tau_n}^Y - \nu^- \tau_n) \leqslant \mathbb{E}((-W)_{\tau_n}^*) \leqslant c\mathbb{E}(\tau_n^{\frac{1}{2}}) \leqslant c(\mathbb{E}(\tau_n))^{\frac{1}{2}}.$$

Thus

$$\mathbb{E}(Y_{\tau_n}) = \mathbb{E}(L_{\tau_n}^Y - \nu^- \tau_n) \leqslant c(\mathbb{E}(\tau_n))^{\frac{1}{2}}, \tag{14}$$

and, taking expectations in (11) at the stopping time τ_n, we have

$$\begin{aligned} 0 &\leqslant x + (\mu + \alpha\nu^-)\,\mathbb{E}(\tau_n) + \alpha\mathbb{E}(L_{\tau_n}^Y - \nu^- \tau_n) \\ &\leqslant x + (\mu + \alpha\nu^-)\,\mathbb{E}(\tau_n) + c\alpha(\mathbb{E}(\tau_n))^{\frac{1}{2}} I_{(\alpha > 0)}. \end{aligned}$$

This quadratic yields an upper bound on $(\mathbb{E}(\tau_n))^{\frac{1}{2}}$ which is independent of n. The desired inequalities follow from the Monotone Convergence Theorem and the equivalent of (14) for the stopping time τ. ∎

Remark 2·1. In Proposition 3·1 the bound on the mean first hitting place is considerably improved. Indeed it is shown that, if $\nu < 0$ and $\mu + \alpha\nu^- \leqslant 0$ then there exists a global constant K such that $\mathbb{E}^{(x,0)}(Y_\tau) \leqslant K$, independently of the x-coordinate of the starting position.

COROLLARY 2·1. *With* τ *defined as above, define* $\sigma = \inf\{t \geqslant \tau : Y_t = 0\}$. *Then there exists a constant* $C = C(\mu, \nu, \alpha, \beta)$ *such that* $\mathbb{E}^{(x,0)}(\sigma) \leqslant C(1+x)$ *and* $\mathbb{E}^{(x,0)}(X_\sigma) \leqslant C(1+x^{\frac{1}{2}})$.

Proof. After transposition of μ and ν, and α and β, there are direct analogues of Proposition 2·1 and Proposition 2·2 for returns to the x-axis. The two parts to condition (6) ensure that the conclusions of one of these propositions must hold both for the first hit on the y-axis and the first return to the x-axis; moreover the tighter bound of Proposition 2·2 must pertain in at least one direction. ∎

We now proceed to define a jump process on the x-axis. For τ and σ as above let $\tau_1 = \tau$ and $\sigma_1 = \sigma$. Define inductively

$$\tau_n = \inf\{t > \sigma_{n-1} : X_t = 0\}, \quad \sigma_n = \inf\{t > \tau_n : Y_t = 0\}.$$

Write ξ_n for X_{σ_n}; then ξ_n is the place of the nth return to the x-axis.

From the above Corollary it follows that for $x \geqslant x_0$ (large enough)

$$\mathbb{E}^x(\xi_1) \leqslant x/2 \quad \text{and} \quad \mathbb{E}^x(\sigma_1) \leqslant 2Cx. \tag{15}$$

Since we are concerned here with a process on the x-axis the superscripts to our expectations are positive reals. If $N = \inf\{n : \xi_n \leqslant x_0\}$ then $2^{n\wedge N}\xi_{n\wedge N}$ is a non-negative supermartingale and

$$x \geqslant \mathbb{E}^x[2^{n\wedge N}\xi_{n\wedge N}] \geqslant 2^n x_0 \mathbb{P}^x(N > n),$$

then

$$\mathbb{P}^x(N > n) \leqslant \left(\frac{x}{x_0}\right)2^{-n}.$$

Also

$$2^n \mathbb{E}^x[\xi_n I_{\{N>n\}}] \leqslant \mathbb{E}^x[2^{n\wedge N}\xi_{n\wedge N}] \leqslant x$$

and then

$$\mathbb{E}^x\left[\sum_{n=0}^{N-1}\xi_n\right] = \mathbb{E}^x\left[\sum_{n\geqslant 0}\xi_n I_{\{N>n\}}\right] \leqslant \sum_{n\geqslant 0} 2^{-n}x = 2x.$$

Thus using (15) we have

$$\mathbb{E}^x[\sigma_N] \leqslant 4Cx, \tag{16}$$

and ξ_n hits $[0, x_0]$ almost surely. Moreover the mean time to hit this interval is finite.

Having proved that some large neighbourhood of O will be hit in finite mean time, it remains to prove that the given (small) neighbourhood $\mathscr{N}$ will be hit in finite mean time. Recall that $T \equiv \inf\{t : Z_t \in \mathscr{N}\}$.

Proposition 2·3. *There exists some constant C such that, for all $z \in \mathbb{R}^2_+$*

$$\mathbb{E}^z(T) \leqslant C(1+|z|). \tag{17}$$

Proof. Assume without loss of generality that $\mu < 0$.

Firstly we prove that

$$\mathbb{E}^z(T_1) \leqslant C(1+|z|), \tag{18}$$

where $T_1 = \inf\{t : Z_t = (x, 0) : x \in [0, x_0]\}$. Indeed, $T_0 = \inf\{t : X_t \leqslant 0 \text{ or } Y_t \leqslant 0\}$, then

$$0 \leqslant \mathbb{E}^z(X_{T_0}) = x + \mu\mathbb{E}^z(T_0) \leqslant x$$

and thus $\mathbb{E}^z(T_0) \leqslant x/\mu^-$ and

$$\mathbb{E}^z[Y(T_0)] = y + \nu\mathbb{E}^z(T_0) \leqslant y + \nu^+ x/\mu^- \leqslant C|z|.$$

We now estimate

$$\mathbb{E}^z(T_1 - T_0) = \mathbb{E}^z(T_1 - T_0; Y(T_0) = 0) + \mathbb{E}^z(T_1 - T_0; X(T_0) = 0). \tag{19}$$

The first term on the right-hand side of (19) is bounded by

$$\mathbb{E}^z[4CX(T_0); X(T_0) > x_0] \leqslant 4Cx,$$

using the estimate (16) above. The second term on the right-hand side of (19) is estimated similarly; if $T_0' = \inf\{t > T_0 : Y_t = 0\}$, then

$$\begin{aligned}
\mathbb{E}^z(T_1 - T_0; X(T_0) = 0) &= \mathbb{E}^z(\mathbb{E}^{Z(T_0)}(T_1); Y(T_0) > 0) \\
&\leqslant \mathbb{E}^z[K_1 + K_2 Y(T_0) + 4C\mathbb{E}^{Z(T_0)}(X(T_0')); Y(T_0) > 0)] \\
&\leqslant C\mathbb{E}^z[1 + Y(T_0); Y(T_0) > 0] \\
&\leqslant C(1+|z|),
\end{aligned}$$

where, as usual, the constant C varies from line to line. This establishes (18) as required.

Now let us consider what happens if Z starts at some point $(x,0)$ where $x\in[0,x_0]$, and we let the process run until the stopping time

$$R_1\equiv 1\wedge T\wedge \inf\{u:|Z_u|>2x_0\}.$$

It is a standard property of reflecting Brownian motion that there exists $\delta>0$ such that, for all $x\in[0,x_0]$,

$$\mathbb{P}^{(x,0)}(R_1=T)\geqslant\delta.$$

If $R_1'\equiv\inf\{t>R_1:Z_t\in[0,x_0]\}$ then in view of the estimate (18) we must have

$$\mathbb{E}^{(x,0)}(R_1')\leqslant C\quad\text{for all } x\in[0,x_0].$$

If $R_1'<T$ (and inductively if $R_n'<T$) define

$$R_{n+1}=R_1\circ\theta(R_n'),\quad R_{n+1}'=R_1'\circ\theta(R_n'),\quad (n\geqslant 1).$$

Then if $N=\inf\{n:Z(R_n)\in\mathcal{N}\}$, we have $\mathbb{P}^{(x,0)}(N>n)\leqslant(1-\delta)^n$ for all $x\in[0,x_0]$. Also

$$\sum_{j=1}^{n\wedge N}R_j'-(n\wedge N)\,C$$

is a supermartingale implying

$$\mathbb{E}^{(x,0)}(T)\leqslant\mathbb{E}^{(x,0)}\left(\sum_1^N R_j'\right)\leqslant C\mathbb{E}^{(x,0)}(N)\leqslant C/\delta.$$

(17) follows when we combine this result with (18). ∎

Thus we have shown that (6) implies (5). We close this section with a proof of (3) implies (4), and so, *a fortiori*, (5) $\Rightarrow$ (4). The implication (4) $\Rightarrow$ (3) is the aim of the next section, where we also complete the implication (5) $\Rightarrow$ (6).

THEOREM 2·1. *Suppose that either*

$$\mu+\alpha\nu^->0\quad\textit{or}\quad\nu+\beta\mu^->0.$$

Let $\mathcal{N}$ be a bounded, open, convex neighbourhood of 0 and suppose that the RBM Z is initially at some point in $\mathbb{R}^2_+\backslash\bar{\mathcal{N}}$. Define $T=\inf\{t:Z_t\in\mathcal{N}\}$. Then $\mathbb{P}(T<\infty)<1$.

Proof. Without loss of generality suppose that $\mu+a\nu^->0$. Suppose that $\mathcal{N}$ is the open disc of radius r centred at 0 and that the process starts at $(x,0)$ for some $x>x_0$, where x_0 is some large value.

Let $\tau=\inf\{t:X_t=r\}$. Then for $t\leqslant\tau$ we have

$$X_t=x+B_t+(\mu+\alpha\nu^-)\,t+\alpha(L_t^Y-\nu^-t),$$

and, applying (13),

$$X_t\geqslant x_0-(-B)_t^*+(\mu+\alpha\nu^-)\,t-(\alpha W)_t^*.$$

Provided that x_0 is chosen so large that

$$\mathbb{P}(U_t^*\leqslant\tfrac12(x_0-r)+\tfrac12(\mu+\alpha\nu^-)\,t,\text{ for all }t)>\tfrac12$$

for both $U_t=-B_t$ and $U_t=\alpha W_t$, then $\mathbb{P}(\tau<\infty)<1$.

Note also that conditioned on not hitting the y-axis, the process is transient to infinity.

In general, suppose that $\mathcal{N}$ is contained in a disc of radius r. Then, by standard properties of planar Brownian motion, for all starting points not in the closure of $\mathcal{N}$

there is a non-zero probability that Z hits the set $\{(x,0); x_0 < x < \infty\}$ before $\mathcal{N}$ and the above argument completes the result. The proof also extends to the case where $\mathbb{R}_2^+\backslash\mathcal{N}$ is connected. ▮

3. *Null recurrence*

The remaining case is when $(\mu+\alpha\nu^-)\vee(\nu+\beta\mu^-) = 0$. Note that if

$$(\mu+\alpha\nu^-) = 0 \geqslant (\nu+\beta\mu^-)$$

then $\nu < 0$ by our assumption of non-zero drift. This section is devoted to a proof that under such assumptions the RBM will almost surely reach any neighbourhood of the origin, but not in finite mean time. We begin by considering the location of the first hit on the y-axis.

Proposition 3·1. *Suppose that $\mu+\alpha\nu^- \leqslant 0$ and $\nu < 0$. Let $Z_0 = (x, 0)$ and define*

$$\tau = \inf\{t > 0 : X_t = 0\}.$$

Then $\tau < \infty$, almost surely, and there exists a constant K such that

$$\mathbb{E}^{(x,0)}(Y_\tau) \leqslant K \quad \textit{for all } x. \tag{20}$$

Remark 3·1. Neglecting the effect of oblique reflection from the y-axis (or alternatively for $t \leqslant \tau$), Y_t will be a positive recurrent process. Thus, under the assumption that $\tau < \infty$, a.s. the existence of a constant K independent of x for which

$$\mathbb{E}^{(x,0)}(Y_\tau) \leqslant K$$

is intuitively plausible.

This estimate improves on the bound $\mathbb{E}^{(x,0)}(Y_\tau) \leqslant K(1+x^{\frac{1}{2}})$ proved in Proposition 2·2 under the stronger assumption $\mu+\alpha\nu^- < 0$. However, the bound for $\mathbb{E}^{(x,0)}(\tau)$ proved there goes badly awry if we relax to $\mu+\alpha\nu^- = 0$; we prove below that under such circumstances $\mathbb{E}^{(x,0)}(\tau) = \infty$.

The proof of the estimate (20) is based on the following pair of Lemmas, the proofs of which are postponed to Sections 3·1 and 3·2.

Lemma 3·1. *Suppose that $\nu < 0$. Define $T_0 = \inf\{t : X_t \leqslant 0$ or $Y_t \leqslant 0\}$. Then there exist constants A, $B > 0$ such that*

$$\lim_{\epsilon\downarrow 0} \frac{1}{\epsilon}\mathbb{E}^{(x,\epsilon)}(Y_{T_0}) \leqslant A\, e^{-Bx}. \tag{21}$$

Lemma 3·2. *Let $(V_s)_{s\geqslant 0}$ be a real Lévy process, and let $\sigma_V \equiv \inf\{s : V_s \leqslant 0\}$. Then there exists some constant C such that, for all x, $a > 0$,*

$$\mathbb{E}^x\left[\int_0^{\sigma_V} ds\, I_{[0,a]}(V_s)\right] \leqslant C(1+a)\,(1+(x\wedge a)). \tag{22}$$

Proof of Proposition 3·1. Consider a reflecting Brownian motion in the upper half-plane with drift (μ,ν) and reflection at angle $\tan^{-1}\alpha$ from the normal to the x-axis where μ, ν and α are the same as those for the RBM in the quadrant. Let V be the Lévy process obtained by time changing the x-component of this RBM by the local time of the y-component at 0. Then by the Rogozin trichotomy [1] we have

$$\liminf V_s = -\infty \quad \text{and} \quad \sigma_V \equiv \inf\{s : V_s \leqslant 0\} < \infty, \quad \text{a.s.}$$

But if σ_V is finite then so is τ for our RBM in the quadrant. Let $\sigma = L_\tau^Y$ and define V^τ by

$$V_s^\tau = \begin{cases} V_s & \text{for } s \leqslant \sigma \\ 0 & \text{for } s > \sigma, \end{cases}$$

so that V^τ is the process obtained by time changing the x-component of our RBM run until X_t first hits 0. Define

$$n(x) = \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \mathbb{E}^{(x,\epsilon)}(Y_{T_0}).$$

Then

$$I_{\{t \geqslant \sigma\}} Y_\tau - \int_0^{t \wedge \sigma} n(V_s^\tau)\, ds$$

is a martingale (since the integrand exactly compensates for the tendency of the first term to jump). This is similar to several computations in [**10**]. Thus

$$\mathbb{E}^{(x,0)}(Y_\tau) = \mathbb{E}\left(\int_0^\sigma n(V_s^\tau)\, ds\right) = \int_{\mathbb{R}^+} G(x, dz)\, n(z)$$

$$\leqslant \int_0^\infty G(x, dz)\, A\, e^{-Bz} \leqslant AB \int_0^\infty dz\, e^{-Bz} G(x, z),$$

where $G(x,z)$ is the expected time spent in $(0,z]$ by the process V^τ started at x. But $G(x,z)$ can be bounded by a comparison with the Green's function of the Lévy process V and we can apply Lemma 3·2 to obtain that

$$\mathbb{E}^{(x,0)}(Y_\tau) \leqslant ABC \int_0^\infty e^{-Bz}(1+z)^2\, dz = K. \quad \blacksquare$$

THEOREM 3·1. *Suppose that* $(\mu + \alpha\nu^-) \vee (\nu + \beta\mu^-) = 0$. *Then if* $T = \inf\{u : Z_u \in \mathcal{N}\}$,

$$\mathbb{P}^z(T < \infty) = 1 \quad \textit{for all } z \in \mathbb{R}_+^2.$$

Moreover if $\mathbb{R}_+^2 \backslash \mathcal{N}$ *has a single component*

$$\mathbb{E}^z(T) = \infty \quad \textit{for all } z \in \mathbb{R}_+^2 \backslash \mathcal{N}.$$

Proof. Without loss of generality $\mu + \alpha\nu^- = 0 > \nu$. Let K be the constant from Proposition 3·1. By a similar argument to that of Proposition 2·3, there exists $\delta > 0$ such that for all $y \in [0, 2K]$ we have

$$\mathbb{P}^{(0,y)}(T \leqslant 1) \geqslant 2\delta.$$

Now consider the RBM started from $(x, 0)$. By Proposition 3·1 we have

$$\mathbb{P}^{(x,0)}(Y_\tau \leqslant 2K) > \tfrac{1}{2}$$

and then

$$\mathbb{P}^{(x,0)}(T \leqslant \tau + 1 < \infty) > \delta.$$

If $T > \tau + 1$ then we can wait until the first subsequent return to the x-axis and repeat the argument. That there is (almost surely) such a return can be shown by methods similar to those at the start of Proposition 3·1. The RBM will enter $\mathcal{N}$ after at most a geometric number of such trials. Thus

$$\mathbb{P}^{(x,0)}(T < \infty) = 1$$

and the extension to a general starting position is immediate.

For the final part of the argument we show firstly that there exists z such that $\mathbb{E}^z(T) = \infty$ and then that this must hold for all z not in the closure of $\mathcal{N}$. Here we use the fact that $\mathbb{R}^2_+ \backslash \mathcal{N}$ has a single component.

Suppose that $\mathbb{E}^{(x',0)}(\tau) < \infty$; then $\mathbb{E}^{(x',0)}(Y_\tau) = -x'/\alpha$. But this contradicts (20) at least for sufficiently large x', and so there exists x_0 such that

$$\mathbb{E}^{(x,0)}(\tau) = \infty \quad \text{for all } x \geqslant x_0.$$

If

$$x_1 = \sup\{x : (x,y) \in \mathcal{N} \text{ for some } y\}$$

then by shift invariance away from the y-axis

$$\mathbb{E}^{(x,0)}(T) = \infty \quad \text{for all } x \geqslant x_0 + x_1.$$

Let $D = \{(x,0) : x \geqslant x_0 + x_1\}$ and let $T_D = \inf\{t : Z_t \in D\}$. Then for any $z \in \mathbb{R}^2_+ \backslash \mathcal{N}$ we have $\mathbb{P}^z(T_D < T) > 0$ and hence $\mathbb{E}^z(T) = \infty$. ∎

3·1. *Exponential bounds*

The purpose of this section is to provide a proof of Lemma 3·1. Since we are only interested in the RBM up to the first hit of either axis we can assume that reflection is normal. However, the drift parameters remain as before. To simplify the calculations consider the process (U, Y) with $U = X + \eta Y$, where $\eta = -\Sigma_{12}/\Sigma_{22}$. This process is an RBM in a wedge, with co-ordinate processes driven by independent Brownian motions. Choose $\lambda > \eta^+$. For the process (U, Y) started at (u, ϵ) define the stopping times $T_0 \equiv \inf\{t : X_t \leqslant 0 \text{ or } Y_t \leqslant 0\}$, $T_Y \equiv \inf\{t : Y_t \leqslant 0\}$, $T_V \equiv \inf\{t : U_t \leqslant \tfrac{1}{2}u\}$ and $T_H \equiv \inf\{t : Y_t \geqslant u/(2\lambda)\}$.

Proof of Lemma 3·1. Clearly (21) will hold if and only if

$$\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \mathbb{E}^{(U=u,\, Y=\epsilon)}(Y_{T_0}) \leqslant A\, e^{-Bu}. \tag{23}$$

We prove (23). Since $\mathbb{E}(T_0) < \infty$, we have by the Optional Sampling Theorem

$$\mathbb{E}^{(u,\epsilon)}(Y_{T_0}) = \epsilon + \nu \mathbb{E}^{(u,\epsilon)}(T_0)$$

and hence, since $\nu < 0$,

$$\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \mathbb{E}^{(u,\epsilon)}(Y_{T_0}) \leqslant 1. \tag{24}$$

By our assumption on the reflection matrix, $\mathbb{E}^{(u,\epsilon)}(T_Y) = \epsilon/\nu^-$ and we have

$$\mathbb{E}^{(u,\epsilon)}(Y_{T_0}) = \nu^- \mathbb{E}^{(u,\epsilon)}(T_Y - T_0).$$

But $T_0 \geqslant \min\{T_Y, T_V, T_H\}$, so that

$$\mathbb{E}^{(u,\epsilon)}(T_Y - T_0) \leqslant \mathbb{E}(T_Y - T_H \wedge T_Y) + \mathbb{E}(T_Y - T_V \wedge T_Y). \tag{25}$$

The first expression on the right can be rewritten as

$$\mathbb{P}(T_H < T_Y)\, \mathbb{E}^{(\cdot,\, u/(2\lambda))}(T_Y) = \left\{\frac{\exp(2\nu^- \epsilon/\Sigma_{22}) - 1}{\exp(\nu^- u/\lambda\Sigma_{22}) - 1}\right\} \frac{u}{2\lambda\nu^-}. \tag{26}$$

For the second, for $\delta > 0$,

$$\mathbb{E}(T_Y - T_V \wedge T_Y) = \mathbb{E}(T_Y - T_V \wedge T_Y ; T_Y < \delta u) + \mathbb{E}(T_Y - T_V \wedge T_Y ; T_Y \geqslant \delta u)$$
$$\leqslant \mathbb{E}(T_Y ; T_V < \delta u) + \mathbb{E}(T_Y ; T_Y \geqslant \delta u).$$

In particular, if $\rho = \mu + \eta\nu$ is the drift of U, take $\delta = \infty$ if $\rho > 0$ and $\delta = (1/4\rho^-) \wedge 1$ if $\rho \leqslant 0$. If $H_{a,b}$ is the time of the first hit by a one-dimensional Brownian motion with variance σ^2 on the sloping line $a + bt$ then it is well known that $H_{a,b}$ has density

$$\frac{|a|}{\sigma\sqrt{(2\pi t^3)}} \exp\left(-\frac{(a+bt)^2}{2\sigma^2 t}\right).$$

If $ab > 0$ then $\mathbb{P}(H_{a,b} < \infty) = e^{-2ab}$. Then, for $\rho > 0$,

$$\mathbb{E}(T_Y ; T_V < \infty) = \mathbb{E}(T_Y)\,\mathbb{P}(H_{u/2,\rho} < \infty) = \frac{\epsilon}{\nu^-} \exp\{-u\rho\}. \tag{27}$$

Similarly, for $\rho \leqslant 0$,

$$\mathbb{P}(T_V < \delta u) = \int_0^{\delta u} dt \frac{u}{2\sigma\sqrt{(2\pi t^3)}} \exp\left(-\frac{(u/2+\rho t)^2}{2\sigma^2 t}\right) \leqslant \int_0^u dt \frac{u}{2\sigma\sqrt{(2\pi t^3)}} \exp\left(-\frac{u^2}{32\sigma^2 t}\right),$$

where σ^2 is the variance of the Brownian motion which drives U. If

$$J(x) = \int_0^x dt \frac{\gamma}{\sqrt{(2\pi t^3)}} e^{-\gamma^2/(2t)} - e^{-\gamma^2/(2x)}$$

then $J(0) = J(\infty) = 0$ and by inspection of the derivative of J, $J(x) \leqslant 0$; thus

$$\mathbb{E}(T_Y ; T_V \leqslant \delta u) \leqslant \frac{2\epsilon}{\nu^-} e^{-u/(32\sigma^2)} \tag{28}$$

Also
$$\mathbb{E}(T_Y ; T_Y \geqslant \delta u) = \int_{\delta u}^\infty dt \frac{\epsilon t}{\sqrt{(2\pi t^3)}} \exp\left(-\frac{(\epsilon+\nu t)^2}{2\sigma^2 t}\right)$$
$$\leqslant e^{-\epsilon\nu/(2\sigma^2)} \frac{\epsilon}{\sqrt{(2\pi\delta u)}} \int_{\delta u}^\infty dt\, e^{-\nu^2 t/(2\sigma^2)}, \tag{29}$$

where now σ refers to the Brownian motion driving Y. Combining (26) with (27) or (28) and (29) yields, for all values of ρ

$$\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \mathbb{E}^{(u,\epsilon)}(Y_{T_0}) \leqslant A\, e^{-Bu}(u + u^{-\frac{1}{2}}). \tag{30}$$

The inequality (23) follows in view of (24), subject to the renaming of constants. ∎

3·2. *The Green's function for a Lévy process*

Let $(V_t)_{t \geqslant 0}$ be a real Lévy process, and define $\tau = \inf\{t > 0 : V_t \leqslant 0\}$. The proof of Proposition 3·1 required a bound on the amount of time the Lévy process started at $x > 0$ spends in the interval $[0, a]$ before time τ: we prove an estimate of the form

$$\mathbb{E}^x\left[\int_0^\tau ds I_{[0,a]}(V_s)\right] \leqslant C(1+a)\,(1+(x \wedge a)) \quad \text{for all } x, a > 0 \tag{31}$$

for some constant C.

Replacing V by $V+\epsilon W$ if necessary, we may assume without loss of generality that 0 is regular for $(-\infty, 0)$ and $(0, \infty)$. Define $\tilde{V} = \inf\{V_s : s \leqslant t\}$; the process $V-\tilde{V}$ has a local time L at zero with inverse A. Let $R_t \equiv \tilde{V}(A_t)$; then $(A_t, -R_t)$ is a bivariate subordinator (see Fristedt[3]).

Let $G(x, \cdot)$ and $g(\cdot)$ be the Green's functions of the (decreasing) ladder process R and the excursion process of $V-\tilde{V}$ respectively, with the former also parametrized by the initial position x. (Thus

$$G(x, [0, a]) = \mathbb{E}^x\left(\int_0^\infty I_{[0,a]}(R_t)\,dt\right),$$

and, in the notation of Rogers[10],

$$\int_0^\infty g(dy)\,h(y) = \int v(d\rho)\int_0^\zeta h(\rho_s)\,ds,$$

where v is the excursion measure, and ζ is the lifetime of an excursion.)

Then

$$\mathbb{E}^x\left[\int_0^\tau ds\, I_{[0,a]}(V_s)\right] = \mathbb{E}^x\left[\int_0^\infty I_{[0,x\wedge a]}(R_t)\,g([0, a-R_t])\,dt\right]$$
$$\leqslant G(x, [0, x\wedge a])\,g([0,a]).$$

It remains to estimate the Green's functions G, g and it is convenient to exploit the following result of Tauberian type.

LEMMA 3·3. *Suppose that $U:\mathbb{R}^+\to\mathbb{R}^+$ is increasing and right continuous, with $U(0)=0$. Write $U(dz)$ for the measure induced by U. Then*

$$\limsup_{\lambda\downarrow 0} \lambda\int_0^\infty e^{-\lambda z}\,U(dz) \leqslant C \quad \textit{implies} \quad \limsup_{z\uparrow\infty}\frac{U(z)}{z} \leqslant Ce.$$

Proof. Fixing $z>0$, and setting $\lambda = 1/z$, we have

$$\frac{U(z)}{ez} = \lambda U(z)\,e^{-\lambda z} \leqslant \int_z^\infty \lambda^2 U(y)\,e^{-\lambda y}\,dy \leqslant \lambda\int_0^\infty e^{-\lambda y}U(dy),$$

this last step following by first increasing the range of integration and then integrating by parts. ∎

From the definition of G as the Green's function of the ladder process R we have

$$\int_{-\infty}^0 G(0, dy)\,e^{\lambda y} = \mathbb{E}^0\int_{-0}^\infty e^{\lambda R_s}\,ds = \frac{1}{\phi(0,\lambda)},$$

where $\phi(\eta, \lambda)$ is the Laplace exponent of $(A, -R)$. But a Laplace exponent satisfies $\lambda^{-1}\phi(0,\lambda)\uparrow$ as $\lambda\downarrow 0$, whence, for some constant c,

$$\limsup_{\lambda\downarrow 0}\lambda\int_{-\infty}^0 e^{\lambda y}G(0, dy) \leqslant c/e$$

and $G(0, [-a, 0]) \leqslant c(1+a)$ by Lemma 3·3. Finally

$$G(x, [0,a]) \leqslant G(0, [-(x\wedge a), 0]) \leqslant c(1+(x\wedge a)) \quad \text{for all } x, a>0$$

since R is a decreasing process.

Similarly, an identity of Silverstein [13], namely

$$\int \nu(d\rho) \int_0^\zeta e^{-\lambda\rho_s} ds = \frac{1}{\phi(0,\lambda)},$$

can be re-interpreted for our purposes as

$$\int_0^\infty g(dy)\, e^{-\lambda y} = \frac{1}{\phi(0,\lambda)}.$$

A bound of the form

$$g([0,a]) \leqslant c(1+a)$$

is obtained as above.

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