Optimal Stopping and Embedding

Damien Lamberton
Université de Marne-la-Vallée
Equipe d'Analyse et de Mathématiques Appliquées
5 Boulevard Descartes
Cité Descartes, Champs-sur-Marne
77 454 Marne-la-Vallée CEDEX 2
France

Chris Rogers
School of Mathematical Sciences
University of Bath
Claverton Down
Bath BA2 7AY
UK

Abstract

We use embedding techniques to analyse the error of approximation of an optimal stopping problem along Brownian paths when Brownian motion is approximated by a random walk.

Keywords: Optimal stopping, Brownian motion, Skorohod embedding.
AMS 1991 sbject classification. 60G40, 90A09.

1 Introduction

The purpose of this paper is to estimate the rate of convergence of the approximation of an optimal stopping problem along Brownian paths, when Brownian motion is approximated by a normalized random walk. Our analysis of the approximation relies on embedding techniques *à la* Skorohod. In [4, 5], completely different methods have been used to tackle the same problem and we will compare our results below (see Remark 2.3).

Throughout the paper, we will use the following notations. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion and $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$ its (augmented) natural filtration. We denote by $\mathcal{T}$ the set of all $\mathbf{F}$-stopping times and set

$$\mathcal{T}_{0,1} = \{\tau \in \mathcal{T} \mid 0 \leq \tau \leq 1 \text{ a.s.}\}.$$

We denote by X a real valued random variable satisfying

$$\mathbf{E}\, X^2 = 1, \quad \mathbf{E}\, X = 0,$$

and by $(X_k)_{k\geq 1}$ a sequence of iid random variables with the same distribution as X. Let

$$S_0 = 0, \quad S_k = \sum_{j=1}^{k} X_j, \quad k \geq 1.$$

We denote by $\mathcal{T}^S$ the set of all stopping times with respect to the natural filtration of the sequence $S = (S_k)_{k\in\mathbf{N}}$ and set

$$\mathcal{T}^S_{0,n} = \{\nu \in \mathcal{T}^S \mid 0 \leq \nu \leq n \text{ a.s.}\}.$$

Let f be a continuous bounded function on $[0,1]\times\mathbf{R}$ and

$$P=\sup_{\tau\in\mathcal{T}_{0,1}}\mathbf{E}\,f(\tau,B_\tau).$$

Given an integer $n\geq 1$, we can approximate P by

$$P^{(n)}=\sup_{\nu\in\mathcal{T}^S_{0,n}}\mathbf{E}\,f\left(\frac{\nu}{n},\frac{S_\nu}{\sqrt{n}}\right).$$

It is well known (see [1, 2, 3]) that $\lim_{n\to\infty}P^{(n)}=P$. We will prove that, if $\mathbf{E}X^4<\infty$, and if f satisfies some regularity conditions, $P^{(n)}-P=O(1/\sqrt{n})$.

2 Embedding the random walk

We denote by T an *embedding* time for X, i.e. a stopping time $T\in\mathcal{T}$ such that

$$\mathbf{E}\,T=\mathbf{E}\,X^2=1,$$

and B_T has the same distribution as X. Various constructions of such a stopping time exist. One of them, due to Azéma and Yor, is very explicit in terms of the so-called barycentre function of X (see [7], chapter VI, section 5, or [8], chapter VI, section 51, for details). We note that the condition $\mathbf{E}\,X^4<\infty$ implies that T^2 is integrable. Given the embedding time T, we have the following result.

Theorem 2.1 *Assume $\mathbf{E}\,X^4<\infty$ and $f:\mathbf{R}^+\times\mathbf{R}\to\mathbf{R}$ is bounded and continuous and has bounded and continuous derivatives $\partial f/\partial t$ and $\partial^2 f/\partial x^2$. Let $Lf=(\partial f/\partial t)+(1/2)(\partial^2 f/\partial x^2)$ and $\sigma=\sqrt{Var\,T}$. We have*

$$P^{(n)}-P\leq\frac{\sigma}{\sqrt{n}}\left(||Lf||_\infty+||\partial f/\partial t||_\infty\right)$$

and

$$P-P^{(n)}\leq\frac{\sigma}{\sqrt{n}}\left(2||Lf||_\infty+||\partial f/\partial t||_\infty\right)+\frac{||Lf||_\infty}{n}.$$

It should be noted that the L^∞ norms here refer to suprema on the whole set $\mathbf{R}^+\times\mathbf{R}$ and not on $[0,1]\times\mathbf{R}$.

Remark 2.2 *The constants in the above estimate depend criticallly on the variance of the embedding time T. This variance may be different for different embedding times and we do not know of any construction of an embedding time with minimal variance. An upper bound for $\mathbf{E}\,T^2$ in terms of moments of X can be derived as follows. The process $(B_t^4-6tB_t^2+3t^2)_{t\geq 0}$ being a martingale (see for instance [7], chapter 4, proposition 3.8), we have*

$$\mathbf{E}T^2=2\mathbf{E}(TB_T^2)-\frac{1}{3}\mathbf{E}B_T^4.$$

Using the inequality $2TB_T^2\leq\varepsilon T^2+\frac{1}{\varepsilon}B_T^4$, for $\varepsilon>0$, we get, for $0<\varepsilon<1$,

$$\mathbf{E}\,T^2\leq\frac{3-\varepsilon}{3\varepsilon(1-\varepsilon)}\mathbf{E}\,X^4.$$

Therefore (by choosing $\varepsilon=3-\sqrt{6}$),

$$\mathbf{E}\,T^2\leq\frac{5+2\sqrt{6}}{3}\mathbf{E}\,X^4.$$

Remark 2.3 *The fact that $P - P^{(n)} = O(1/\sqrt{n})$ has been proved in [5] by completely different methods, under slightly different assumptions on f. More precisely, the pay-off functions considered in [5] are of the form $f(t,x) = e^{-rt} g(\mu t + x)$, where g is a continuous bounded function on the real line, with g' bounded and g'' bounded below. In particular, the results of [5] apply to standard American options. So far, we have not been able to capture non smooth pay-off functions using embedding techniques. The interest of our method lies in the simplicity of the proofs and in the explicit form of the constants involved.*

Another limitation of the embedding approach is that, with the additional condition $\mathbf{E}\, X^3 = 0$, one may expect a better rate of convergence (see [5], Theorems 1.2 and 5.1 for partial results in that direction). We have not been able to exploit this zero third moment condition to improve the estimates of Theorem 2.1 by embedding techniques.

The following Proposition, which follows easily from the scaling property of Brownian motion and the strong Markov property, shows how the approximating random walk can be embedded in the paths of Brownian motion.

Proposition 2.4 *Given an embedding time T and a positive integer n, there exists a sequence of stopping times $(T_k^{(n)})_{k \in \mathbf{N}}$ such that $T_0^{(n)} = 0$ and, for every $k \in \mathbf{N}$, $\left(T_{k+1}^{(n)} - T_k^{(n)}, B_{T_{k+1}^{(n)}} - B_{T_k^{(n)}}\right)$ is independent of $\mathcal{F}_{T_k^{(n)}}$ and has the same distribution as $\left(\frac{T}{n}, \frac{B_T}{\sqrt{n}}\right)$.*

3 Proof of the main result

Let $(T_k^{(n)})_{k \in \mathbf{N}}$ be a sequence of stopping times as in Proposition 2.4. Observe that the sequence $\left(B_{T_k^{(n)}}\right)_{k \in \mathbf{N}}$ has the same distribution as $(S_k/\sqrt{n})_{k \in \mathbf{N}}$. We denote by $\mathbf{F}^{(n)}$ the discrete time filtration $(\mathcal{F}_{T_k^{(n)}}, k \in \mathbf{N})$, and by $\mathcal{T}^{(n)}$ the set of all $\mathbf{F}^{(n)}$-stopping times. We also set

$$\mathcal{T}_{0,n}^{(n)} = \left\{ \nu \in \mathcal{T}^{(n)} \mid 0 \leq \nu \leq n \ \text{ a.s.} \right\}.$$

Our first Lemma relates $P^{(n)}$ to the embedded random walk.

Lemma 3.1 *We have $P^{(n)} = \sup_{\nu \in \mathcal{T}_{0,n}^{(n)}} \mathbf{E}\, f\left(\nu/n, B_{T_\nu^{(n)}}\right)$.*

Proof: Let $\bar{P}^{(n)} = \sup_{\nu \in \mathcal{T}_{0,n}^{(n)}} \mathbf{E}\, f\left(\nu/n, B_{T_\nu^{(n)}}\right)$. Applying dynamic programming (see for instance [6], chapter VI), we have $\bar{P}^{(n)} = \bar{U}_0$, where the sequence $(\bar{U}_k)_{0 \leq k \leq n}$ is defined by the following backward recursive equations:

$$\begin{cases} \bar{U}_n &= f\left(1, B_{T_n^{(n)}}\right) \\ \bar{U}_k &= \max\left(f\left(\frac{k}{n}, B_{T_k^{(n)}}\right), \mathbf{E}\left(\bar{U}_{k+1} \mid \mathcal{F}_{T_k^{(n)}}\right)\right), \quad 0 \leq k \leq n-1. \end{cases}$$

On the other hand, $P^{(n)} = U(0,0)$, where the sequence $(U(k,\cdot))_{0 \leq k \leq n}$ is defined by

$$\begin{cases} U(n,\cdot) &= f(1,\cdot) \\ U(k,x) &= \max\left(f\left(\frac{k}{n}, x\right), \mathbf{E}\, U\left(k+1, x + \frac{X}{\sqrt{n}}\right)\right), \quad 0 \leq k \leq n-1, \quad x \in \mathbf{R}. \end{cases}$$

Now, let $U_k = U\left(k, B_{T_k^{(n)}}\right)$. We have $U_n = f\left(1, B_{T_n^{(n)}}\right)$ and

$$\begin{aligned}
\mathbf{E}\left(U_{k+1} \mid \mathcal{F}_{T_k^{(n)}}\right) &= \mathbf{E}\left(U\left(k+1, B_{T_{k+1}^{(n)}}\right) \mid \mathcal{F}_{T_k^{(n)}}\right) \\
&= \mathbf{E}\left(U\left(k+1, B_{T_k^{(n)}} + B_{T_{k+1}^{(n)}} - B_{T_k^{(n)}}\right) \mid F_{T_k^{(n)}}\right) \\
&= V\left(k+1, B_{T_k^{(n)}}\right),
\end{aligned}$$

where

$$V(k+1, x) = \mathbf{E}\, U\left(k+1, x + \frac{X}{\sqrt{n}}\right).$$

Here, we have used the fact that $B_{T_k^{(n)}}$ is $\mathcal{F}_{T_k^{(n)}}$-measurable and $B_{T_{k+1}^{(n)}} - B_{T_k^{(n)}}$ is independent of $\mathcal{F}_{T_k^{(n)}}$, with the same distribution as $X/\sqrt{n}$. It follows that $U_k = \bar{U}_k$. $\diamond$

Lemma 3.2 *Let* $\tilde{P}^{(n)} = \sup_{\nu \in \mathcal{T}_{0,n}^{(n)}} \mathbf{E}\, f\left(T_\nu^{(n)}, B_{T_\nu^{(n)}}\right)$. *We have*

$$\left|\tilde{P}^{(n)} - P^{(n)}\right| \leq \left|\left|\frac{\partial f}{\partial t}\right|\right|_\infty \frac{\sigma}{\sqrt{n}}.$$

Proof: For any $\nu \in \mathcal{T}_{0,n}^{(n)}$, we have

$$\left|f\left(T_\nu^{(n)}, B_{T_\nu^{(n)}}\right) - f\left(\frac{\nu}{n}, B_{T_\nu^{(n)}}\right)\right| \leq \left|\left|\frac{\partial f}{\partial t}\right|\right|_\infty \left|T_\nu^{(n)} - \frac{\nu}{n}\right|.$$

Hence, using Lemma 3.1,

$$\left|\tilde{P}^{(n)} - P^{(n)}\right| \leq \left|\left|\frac{\partial f}{\partial t}\right|\right|_\infty \sup_{\nu \in \mathcal{T}_{0,n}^{(n)}} \mathbf{E}\left|T_\nu^{(n)} - \frac{\nu}{n}\right|.$$

Now, $(T_k^{(n)} - (k/n))_{k \in \mathbf{N}}$ is an $\mathbf{F}^{(n)}$-martingale, so that

$$\begin{aligned}
\sup_{\nu \in \mathcal{T}_{0,n}^{(n)}} \mathbf{E}\left|T_\nu^{(n)} - \frac{\nu}{n}\right| &= \mathbf{E}\left|T_n^{(n)} - 1\right| \\
&\leq \left|\left|T_n^{(n)} - 1\right|\right|_{L^2} \\
&= \frac{\sigma}{\sqrt{n}}.
\end{aligned}$$

$\diamond$

Theorem 2.1 follows easily from Lemma 3.2 and from Lemma 3.3 and Lemma 3.4 below.

Lemma 3.3 *We have* $\tilde{P}^{(n)} - P \leq ||Lf||_\infty \frac{\sigma}{\sqrt{n}}$.

Lemma 3.4 *We have* $P - \tilde{P}^{(n)} \leq ||Lf||_\infty \left(\frac{1}{n} + \frac{2\sigma}{\sqrt{n}}\right)$.

Proof of Lemma 3.3: Fix $\nu \in \mathcal{T}_{0,n}^{(n)}$. We observe that $T_\nu^{(n)} \in \mathcal{T}$. Indeed, for $t \geq 0$, we have

$$\left\{T_\nu^{(n)} \leq t\right\} = \bigcup_{k=0}^{n} \left\{T_k^{(n)} \leq t\right\} \cap \{\nu = k\}.$$

Since ν is an $\mathbf{F}^{(n)}$-stopping time, $\{\nu = k\} \in \mathcal{F}_{T_k^{(n)}}$, for $k = 0, \ldots, n$. Hence $\left\{T_\nu^{(n)} \leq t\right\} \in \mathcal{F}_t$.

Using the definition of P and Ito's formula, we have

$$\begin{aligned}
\mathbf{E}\, f\left(T_\nu^{(n)}, B_{T_\nu^{(n)}}\right) &= \mathbf{E}\, f\left(T_\nu^{(n)} \wedge 1, B_{T_\nu^{(n)} \wedge 1}\right) + \mathbf{E}\, f\left(T_\nu^{(n)}, B_{T_\nu^{(n)}}\right) \\
&\quad -\mathbf{E}\, f\left(T_\nu^{(n)} \wedge 1, B_{T_\nu^{(n)} \wedge 1}\right) \\
&\leq P + \mathbf{E} \int_{T_\nu^{(n)} \wedge 1}^{T_\nu^{(n)}} Lf(s, B_s) ds \\
&\leq P + ||Lf||_\infty \mathbf{E}\, \left(T_\nu^{(n)} - 1\right)_+ \\
&\leq P + ||Lf||_\infty \left|\left|T_n^{(n)} - 1\right|\right|_2 \\
&= P + \frac{\sigma}{\sqrt{n}} ||Lf||_\infty.
\end{aligned}$$

◇

For the proof of Lemma 3.4, we need the following result, which will be proved at the end.

Lemma 3.5 *Let $(Z_k)_{k \in \mathbf{N}}$ be a sequence of iid square integrable random variables. We have*

$$\mathbf{E} \sup_{0 \leq k \leq n} Z_k \leq \mathbf{E}\, Z_0 + \sqrt{(n+1) \mathrm{Var}\, Z_0}.$$

Proof of Lemma 3.4: Fix $\tau \in \mathcal{T}_{0,1}$ and let

$$\nu = \inf\left\{k \in \mathbf{N} \mid T_k^{(n)} \geq \tau\right\}.$$

We observe that $\nu \in \mathcal{T}^{(n)}$. Indeed,

$$\{\nu \leq k\} = \{T_k^{(n)} \geq \tau\} \in \mathcal{F}_{T_k^{(n)}}.$$

We have

$$\begin{aligned}
\mathbf{E}\, f(\tau, B_\tau) &= \mathbf{E}\, f\left(T_{\nu \wedge n}^{(n)}, B_{T_{\nu \wedge n}^{(n)}}\right) + \mathbf{E}\, f(\tau, B_\tau) - \mathbf{E}\, f\left(T_{\nu \wedge n}^{(n)}, B_{T_{\nu \wedge n}^{(n)}}\right) \\
&\leq \tilde{P}^{(n)} + \mathbf{E}\, f(\tau, B_\tau) - \mathbf{E}\, f\left(T_{\nu \wedge n}^{(n)}, B_{T_{\nu \wedge n}^{(n)}}\right).
\end{aligned}$$

Now, let

$$A_1 = \mathbf{E}\, \left(f(\tau, B_\tau) - f\left(T_{\nu \wedge n}^{(n)}, B_{T_{\nu \wedge n}^{(n)}}\right)\right) \mathbf{1}_{\{\nu > n\}}$$

and

$$A_2 = \mathbf{E}\, \left(f(\tau, B_\tau) - f\left(T_{\nu \wedge n}^{(n)}, B_{T_{\nu \wedge n}^{(n)}}\right)\right) \mathbf{1}_{\{\nu \leq n\}}.$$

We have

$$\begin{aligned}
A_1 &= \mathbf{E}\,\left(f(\tau, B_\tau) - f\left(T^{(n)}_{\nu\wedge n}, B_{T^{(n)}_{\nu\wedge n}}\right)\right)\mathbf{1}_{\{T^{(n)}_n<\tau\}}\\
&= \mathbf{E}\,\int_{T^{(n)}_n}^{\tau} Lf(B_s)ds\,\mathbf{1}_{\{T^{(n)}_n<\tau\}}\\
&\leq ||Lf||_\infty \mathbf{E}\,(\tau - T^{(n)}_n)_+\\
&\leq ||Lf||_\infty \mathbf{E}\,(1 - T^{(n)}_n)_+ \leq ||Lf||_\infty \frac{\sigma}{\sqrt{n}}.
\end{aligned}$$

We now estimate A_2.

$$\begin{aligned}
A_2 &= \mathbf{E}\,\left(f(\tau, B_\tau) - f\left(T^{(n)}_{\nu}, B_{T^{(n)}_{\nu}}\right)\right)\mathbf{1}_{\{\nu\leq n\}}\\
&= \mathbf{E}\,\left(f(\tau, B_\tau) - f\left(T^{(n)}_{\nu}, B_{T^{(n)}_{\nu}}\right)\right)\mathbf{1}_{\{T^{(n)}_\nu\geq\tau\}\cap\{\nu\leq n\}}\\
&= -\mathbf{E}\,\int_{\tau}^{T^{(n)}_\nu} Lf(s,B_s)ds\,\mathbf{1}_{\{T^{(n)}_\nu\geq\tau\}\cap\{\nu\leq n\}}\\
&= -\mathbf{E}\,\int_{\tau}^{T^{(n)}_\nu} Lf(s,B_s)ds\,\mathbf{1}_{\{1\leq\nu\leq n\}}\\
&\leq ||Lf||_\infty \mathbf{E}\,\left(T^{(n)}_\nu - T^{(n)}_{\nu-1}\right)\mathbf{1}_{\{1\leq\nu\leq n\}}\\
&\leq ||Lf||_\infty \mathbf{E}\,\sup_{0\leq k\leq n-1}\left(T^{(n)}_{k+1} - T^{(n)}_{k}\right)
\end{aligned}$$

Applying Lemma 3.5 with

$$Z_k = T^{(n)}_{k+1} - T^{(n)}_k,$$

we obtain

$$A_2 \leq ||Lf||_\infty \left(\frac{1}{n} + \frac{\sigma}{\sqrt{n}}\right)$$

◇

Proof of Lemma 3.5: We may assume without loss of generality that $\mathbf{E}\,Z_0 = 0$. In that case, we have

$$\begin{aligned}
\mathbf{E}\,\sup_{0\leq k\leq n} Z_k &\leq \sqrt{\mathbf{E}\,\sup_{0\leq k\leq n} Z_k^2}\\
&\leq \sqrt{\mathbf{E}\,\sum_{k=0}^{n} Z_k^2}\\
&= \sqrt{(n+1)\mathrm{Var}\ Z_0}.
\end{aligned}$$

◇

Aknowledgement We are grateful to C. Stricker for pointing out the argument of Lemma 3.5, which is simpler than our original proof and gives a sharper bound.

References

[1] Aldous, D. (1979). *Extended weak convergence*, Unpublished manuscript.

[2] Kushner, H.J. (1977). *Probability Methods for Approximations in Stochastic Control and for Elliptic Equations.* Academic Press, New York.

[3] Lamberton, D. and Pagès, G. (1990). Sur l'approximation des réduites. *Ann. Inst. H. Poincaré, Probab. et Statistiques* **26**, 331-355.

[4] Lamberton, D. (1998). Error Estimates for the Binomial Approximation of American Put Options. *Annals of Applied Probability* **8** n^o 1, 206-233.

[5] Lamberton, D. (1999) Brownian optimal stopping and random walks. Revised version. *Preprint.*

[6] Neveu, J. (1975): *Discrete-parameter Martingales.* North Holland, Amsterdam.

[7] Revuz, A., and M. Yor (1994): *Continuous Martingales and Brownian Motion*, Springer.

[8] Rogers, L.C.G., and D. Williams (1994): *Diffusions, Markov processes and martingales*, Volume 2, John Wiley and Sons.