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$BM(\mathbb{R}^3)$ and its area integral $\int \beta\times d\beta$

by

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1. Let β be a $BM(\mathbb{R}^3)$, that is, a Brownian motion on $\mathbb{R}^3$. For the moment, regard β_0 as some fixed (deterministic) point of $\mathbb{R}^3$.

Let α denote the 'area integral' of β, defined by

$$(1.1) \qquad \alpha_t = \alpha_0 + \int_{(0,t]} \beta_s \times d\beta_s ,$$

where α_0 is some fixed point of $\mathbb{R}^3$, the $\times$ symbol signifies the vector product, and d signifies the Itô differential.

Since

$$d\langle\alpha^i,\alpha^j\rangle = -\beta^i\beta^j dt \qquad (i \neq j),$$

the path of α determines the path of β modulo a global (in t) sign change $\beta \mapsto -\beta$. For some remarkable examples of this kind of explicit construction of one process in terms of another, see Stroock and Yor [1].

We wish to investigate how much information the process $|\alpha|$ carries about β, but with a different interpretation of how this might be measured. In a sense we want to know how much freedom we have to 'perturb' β without changing $|\alpha|$. Now, let us be more precise.

(1.2) THEOREM. *Let $\tilde{\beta}$ be a Brownian motion relative to the augmented filtration determined by β. Let $\tilde{\alpha}_0$ be a fixed point of $\mathbb{R}^3$, and let*

$$\tilde{\alpha}_t = \tilde{\alpha}_0 + \int_{(0,t]} \tilde{\beta}_s \times d\tilde{\beta}_s .$$

Suppose that $|\tilde{\alpha}_t| = |\alpha_t|$, $\forall t$. *Then, on each component interval of the open set* $\{t : \alpha_t \cdot \beta_t \neq 0\}$, *the function* $\tilde{\beta}$ *is a constant orthogonal transformation of* β.

A much more complete description of the relation between β and $\tilde{\beta}$ will be given later.

Two of the results used in the proof of the theorem, Lemmas 1.3 and 1.4, have independent interest.

(1.3) LEMMA. *We have the following skew-product representation:*

$$|\alpha_t \times \beta_t| = r\left(\int_{(0,t]} \{|\alpha_s|^2 + |\beta_s|^4\} ds\right),$$

where r *is a BES(2) process.* *Thus,* $\alpha_t \times \beta_t$ *can never be zero at a positive time.*

Recall that a BES(2) process is a process identical in law to the radial part of 2-dimensional Brownian motion.

For the next lemma, we need some notation:

$O(3)$ denotes the group of orthogonal 3×3 matrices,

$o(3)$ denotes the Lie algebra of skew-symmetric 3×3 matrices,

a superscript T signifies transpose,

for a vector $\beta = (\beta^1, \beta^2, \beta^3)$ in $\mathbb{R}^3$, $V(\beta)$ denotes the element of $o(3)$ defined by

$$V(\beta) = \begin{pmatrix} 0 & -\beta^3 & \beta^2 \\ \beta^3 & 0 & -\beta^1 \\ -\beta^2 & \beta^1 & 0 \end{pmatrix},$$

so that $V(\beta)\gamma = \beta \times \gamma$, $\gamma \in \mathbb{R}^3$.

We let ∂ denote the Stratonovich differential.

(1.4) LEMMA. Let β and $\tilde{\beta}$ be two $BM(\mathbb{R}^3)$ processes. Suppose that

1.4 (i) $\tilde{\beta}$ is a Brownian motion relative to the augmented filtration generated by β,

1.4 (ii) $|\tilde{\beta}_t| = |\beta_t|$, $\forall t$.

Then there exists a previsible $O(3)$ valued process H such that

(1.5) $d\tilde{\beta} = Hd\beta$,

(1.6) $\tilde{\beta} = H\beta$.

Now make the extra assumption that H is a continuous semimartingale. Define a 3×3 matrix valued process A by

(1.7) $A_0 = 0$, $\partial A = H^{-1}\partial H$.

Then A is $o(3)$ valued, and

(1.8) $\partial H = H\partial A$.

Moreover, A solves an Itô equation

(1.9) $dA = V(\beta)dx + V(\lambda)dt$,

where x is a 1-dimensional semimartingale with canonical decomposition

(1.10) $dx = \lambda.d\beta + df$,

where λ is a previsible $\mathbb{R}^3$ valued process, and f is a continuous (adapted) process of finite variation.

The switching between Itô and Stratonovich is a little annoying. However, (1.5) and (1.10) must be Itô equations, while the Stratonovich form of (1.7) and (1.8) best brings out their meaning. In Stratonovich form, equation (1.9) reads:

$$\partial A = V(\beta)\partial x + \frac{1}{2}V(\lambda)\partial t .$$

We emphasize that the 'converse' to Lemma 1.4 holds. Thus, take an arbitrary previsible $\mathbb{R}^3$ valued process λ, and an arbitrary continuous adapted process f of finite variation. Define x via (1.10), and A (with $A_0 = 0$) via (1.9). Next define H via (1.8) with H_0 an arbitrary element of $O(3)$. Finally, define $\tilde{\beta}$ via (1.6). Then (1.5) holds, so that $\tilde{\beta}$ is a $BM(\mathbb{R}^3)$ satisfying 1.4(i); and, of course, 1.4(ii) follows from (1.6).

Notation. We continue to use :
Greek letters for processes with values in $\mathbb{R}^3$;
capital Roman letters for 3×3 matrix valued processes;
small Roman letters for real valued processes.

For continuous semimartingales x and y, we write Itô's formula for the derivative of a product as

$$d(xy) = xdy + (dx)y + dxdy.$$

so that $dxdy = d\langle x,y\rangle$. This extends to 3×3 matrix valued continuous semimartingales as

$$d(XY) = XdY + (dX)Y + dXdY,$$

where, with X^i_j denoting the (i,j) th component of X,

$$(dXdY)^i_k = \sum_j d\langle X^i_j, X^j_k\rangle .$$

We make much use of the standard formulae:

$$(\alpha\times\beta)\times\gamma = (\alpha.\gamma)\beta - (\beta.\gamma)\alpha, \qquad (\alpha\times\beta).\gamma = \alpha.(\beta\times\gamma),$$

$$(\alpha\times\beta).(\gamma\times\delta) = (\alpha.\gamma)(\beta.\delta) - (\alpha.\delta)(\beta.\gamma),$$

etc..

2. Proof of Lemma 1.3. Let β be a $BM(\mathbb{R}^3)$, and let α be its area integral. Define

$$a = |\beta|^2, \qquad b = (\alpha.\beta), \qquad c = |\alpha|^2.$$

It is intuitively clear that the triple (a,b,c) is Markovian, and this is easily confirmed from the following calculations:

(2.1) $da = 2\beta.d\beta + d\beta.d\beta = 2\beta.d\beta + 3dt,$

(2.2) $db = \alpha.d\beta + (d\alpha).\beta + (d\alpha).(d\beta) = \alpha.d\beta,$

(2.3) $dc = 2\alpha.d\alpha + d\alpha.d\alpha = 2\alpha.(\beta\times d\beta) + (\beta\times d\beta).(\beta\times d\beta)$

$$= 2(\alpha\times\beta).d\beta + 2adt.$$

What clinches the Markov property is of course that

(2.4) $u \equiv |\alpha\times\beta|^2 = |\alpha|^2|\beta|^2 - (\alpha.\beta)^2 = ac - b^2.$

Thus the diffusion process (a,b,c) has drift $(3,0,2a)$, and diffusion matrix

$$\begin{pmatrix} 4a & 2b & 0 \\ 2b & c & 0 \\ 0 & 0 & 4u \end{pmatrix}.$$

We do not actually use here the Markovian nature of (a,b,c), but it did suggest the skew-product formula.

From (2.4),

(2.5) $du = adc + cda + dadc - 2bdb - dbdb$

$$= 2\{a(\alpha\times\beta) + c\beta - b\alpha\}.d\beta + (2a^2 + 3c - c)dt$$

$$= 2\{|\beta|^2(\alpha\times\beta) + (\alpha\times\beta)\times\alpha\}.d\beta + 2(|\beta|^4 + |\alpha|^2)dt.$$

Thus

$$du - 2(|\beta|^4 + |\alpha|^2)dt = d(\text{local martingale}),$$

$$dudu = 4u(|\beta|^4 + |\alpha|^2)dt.$$

It is well known that these properties imply Lemma 1.3.

3. Proof of Lemma 1.4. Let β and $\tilde{\beta}$ be two $BM(\mathbb{R}^3)$ processes, with $\tilde{\beta}$ a Brownian motion relative to the augmented filtration determined by β. Then the martingale representation theorem guarantees that there exists a previsible $O(3)$ valued process H such that

(3.1) $d\tilde{\beta} = Hd\beta$.

Suppose further that $|\tilde{\beta}_t| = |\beta_t|$, $\forall t$. Then

$$d(\tilde{\beta}.\tilde{\beta}) = 2\tilde{\beta}.d\tilde{\beta} + 3dt = d(\beta.\beta) = 2\beta.d\beta + 3dt.$$

Hence

$$\tilde{\beta}.d\tilde{\beta} = (H^T\tilde{\beta}).d\beta = \beta.d\beta$$

and so $H^T\tilde{\beta} = \beta$, equivalently, $\tilde{\beta} = H\beta$, for almost all t. If we modify H on a set of measure zero, we do not affect (3.1). Hence, we can assume that

(3.2) $\tilde{\beta} = H\beta$ (for all t).

Now, we assume that H is a continuous semimartingale. Taking the Itô derivative of (3.2), and comparing with (3.1), we see that

(3.3) $(dH)\beta + dHd\beta = 0$.

It will be convenient for a moment to work with Stratonovich derivatives. From

$$HH^T = I,$$

it follows that

$$(\partial H)H^T + H\partial H^T = 0, \quad \text{so} \quad H^{-1}\partial H = -(\partial H^T)(H^T)^{-1}.$$

Let

$$A_0 = 0, \quad \partial A = H^{-1}\partial H = -\partial A^T.$$

Then, obviously, A is $o(3)$ valued, and

(3.4) $$\partial H = H\partial A.$$

The Itô form of (3.4) reads

$$dH = HdA + \frac{1}{2}dHdA.$$

Thus (3.3) now yields

(3.5) $$(HdA)\beta + \frac{1}{2}(dHdA)\beta + dHd\beta = 0.$$

Let M be the martingale part of the 3×3 matrix valued process A, and let F be the continuous finite-variation part: $A = M + F$. On looking at the martingale part of (3.5), we see that

$$(HdM)\beta = 0, \quad \text{so} \quad (dM)\beta = 0.$$

It is easy to deduce, using the fact that M is skew-symmetric, that

$$dM = dmV(\beta),$$

where m is a 1-dimensional martingale. Necessarily, we have

$$dm = \lambda.d\beta$$

for some previsible $\mathbb{R}^3$ valued process λ.

We now have

$$dH = HV(\beta)dm + d(\text{finite variation}),$$

so

$$dHdA = HdAdA = HdMdM = H|\lambda|^2 V(\beta)^2 dt,$$

and

$$(dHdA)\beta = 0.$$

Moreover,

$$dHd\beta = HV(\beta)dmd\beta = HV(\beta)\lambda dt.$$

Substitution in (3.5) now gives

$$(HdF)\beta + HV(\beta)\lambda dt = 0,$$

so that

$$(dF)\beta + V(\beta)\lambda dt = (dF)\beta - V(\lambda)\beta dt = 0.$$

Since F is skew-symmetric, we must have

$$dF = V(\beta)df + V(\lambda)dt,$$

where f is a 1-dimensional continuous finite-variation process.

Lemma 1.4 is proved.

4. <u>Proof of Theorem 1.2.</u> Let β be a $BM(\mathbb{R}^3)$. Let $\tilde{\beta}$ be another $BM(\mathbb{R}^3)$ relative to the augmented filtration generated by β. We assume equality of the moduli of the area integrals:

$$|\tilde{\alpha}_t|^2 = |\alpha_t|^2, \quad \forall t.$$

By equation (2.3),

$$(4.1) \qquad 2(\tilde{\alpha}\times\tilde{\beta}).d\tilde{\beta} + 2|\tilde{\beta}|^2 dt = 2(\alpha\times\beta).d\beta + 2|\beta|^2 dt.$$

Equating the finite-variation parts gives:

$$(4.2) \qquad |\tilde{\beta}_t| = |\beta_t|, \quad \forall t.$$

We can now apply the trivial first part of Lemma 1.4 to show that, for some previsible $O(3)$ valued process H,

$$(4.3) \qquad d\tilde{\beta} = H d\beta,$$

$$(4.4) \qquad \tilde{\beta} = H\beta .$$

On equating martingale parts at (4.1), we obtain

$$(\tilde{\alpha}\times\tilde{\beta}).d\tilde{\beta} = (\alpha\times\beta).d\beta,$$

whence (compare the argument leading to (3.2))

$$(4.5) \qquad \tilde{\alpha}\times\tilde{\beta} = H(\alpha\times\beta),$$

for almost all t, and it can be assumed that (4.5) holds for all t.

It is obvious from (4.5) that

$$|\tilde{\alpha}\times\tilde{\beta}|^2 = |\alpha\times\beta|^2.$$

Take Itô derivatives using (2.5) to see that (again via the argument leading to (3.2))

$$|\tilde{\beta}|^2(\tilde{\alpha}\times\tilde{\beta}) + (\tilde{\alpha}\times\tilde{\beta})\times\tilde{\alpha} = |\beta|^2 H(\alpha\times\beta) + H\{(\alpha\times\beta)\times\alpha\} ,$$

so that, from (4.2) and (4.5),

$$|\tilde{\alpha}|^2\tilde{\beta} - (\tilde{\alpha}.\tilde{\beta})\tilde{\alpha} = |\alpha|^2 H\beta - (\alpha.\beta)H\alpha.$$

Thus, because of (4.4) and the given fact that $|\tilde{\alpha}| = |\alpha|$, we have

$$(4.6) \qquad (\tilde{\alpha}.\tilde{\beta})\tilde{\alpha} = (\alpha.\beta)H\alpha.$$

Take the scalar product of (4.6) with $\tilde{\beta} = H\beta$, and recall that H preserves scalar products, to find that

$$(\tilde{\alpha}.\tilde{\beta})^2 = (\alpha.\beta)^2.$$

For $\alpha.\beta \neq 0$, define $e_t = (\tilde{\alpha}.\tilde{\beta})_t/(\alpha.\beta)_t = \pm 1$. Then (4.6) implies that

$$\tilde{\alpha} = eH\alpha. \tag{4.7}$$

But

$$(\tilde{\alpha}\times\tilde{\beta}) = H(\alpha\times\beta) = (\det H)(H\alpha)\times(H\beta) = (\det H)e(\tilde{\alpha}\times\tilde{\beta}),$$

and, for $\alpha.\beta \neq 0$,

$$e_t = \det H_t .$$

It is obvious from the definition of e that e is continuous, and therefore constant either at 1 or at -1, on component intervals of the set $\{t : \alpha_t.\beta_t = 0\}$.

The reader will be able to see that, to finish the proof, we need only show that <u>if</u> $\alpha_0 \times \beta_0 \neq 0$, <u>and</u> e <u>is globally constant</u> $(e_t = e_0, \forall t)$, <u>then</u> $H_t = H_0$, $\forall t$.

So <u>assume that</u> $\alpha_0 \times \beta_0 \neq 0$, <u>and</u> $e_t = e_0$, $\forall t$. Recall from Lemma 1.3 that then $\alpha_t \times \beta_t \neq 0$, $\forall t$. Then H_t is uniquely determined by the fact that it maps the orthogonal triple

$$(\beta_t,\ \alpha_t \times \beta_t,\ \beta_t \times (\alpha_t \times \beta_t))$$

into the triple

$$(\tilde{\beta}_t,\ \tilde{\alpha}_t \times \tilde{\beta}_t,\ e_0\tilde{\beta}_t \times (\tilde{\alpha}_t \times \tilde{\beta}_t)).$$

Hence H <u>is a continuous semimartingale</u>, and all the results of Lemma 1.4 apply. We use the notation of that Lemma.

From (4.7),

$$\tilde{\alpha} = e_0 H\alpha,$$

so that

$$d\tilde{\alpha} = e_0 H d\alpha + e_0 (dH)\alpha + e_0 dH d\alpha.$$

But

$$d\tilde{\alpha} = \tilde{\beta} \times d\tilde{\beta} = (H\beta)\times(Hd\beta) = (\det H)Hd\alpha = e_0 H d\alpha,$$

so that

$$(dH)\alpha + dHd\alpha = 0.$$

Thus,

$$(4.8) \qquad (HdA + \tfrac{1}{2}dHdA)\alpha + dHd\alpha = 0.$$

Looking at the martingale-differential part of (4.8), we see that

$$HV(\beta)\alpha\, dm = H(\beta\times\alpha)dm = 0, \quad \text{where} \quad dm = \lambda.d\beta.$$

Since $\beta\times\alpha$ is never zero, it follows that $dm = 0$. Thus, (4.8) reduces to the statement

$$HV(\beta)\alpha df = 0 = H(\beta\times\alpha)df,$$

and, again because $\beta\times\alpha$ is never zero, we have $df = 0$. Thus, $dA = 0$, and $H_t = H_0$, $\forall t$.

5. Example. The proof of Theorem 1.2 shows clearly how to construct an example to show what can 'go wrong' when $\alpha.\beta = 0$.

Let β be a $BM(\mathbb{R}^3)$ with $\beta_0 = 0$, and let $\alpha = \int \beta x d\beta$. Let

$$\tau = \inf\{t > 1 : \alpha_t.\beta_t = 0\}.$$

Let

$$H_t = \begin{cases} I, & t < \tau, \\ J, & t \geq \tau, \end{cases}$$

where J is specified by

$$J(\beta_\tau) = \beta_\tau, \quad J(\alpha_\tau \times \beta_\tau) = \alpha_\tau \times \beta_\tau, \quad J(\gamma_\tau) = -\gamma_\tau,$$

where

$$\gamma_\tau = \beta_\tau \times (\alpha_\tau \times \beta_\tau) = |\beta_\tau|^2 \alpha_\tau ,$$

since $\alpha_\tau . \beta_\tau = 0$. Note that

$$J(\alpha_\tau) = -\alpha_\tau .$$

Set $\tilde{\beta}_0 = 0$,

$$\tilde{\beta}_t = \int H_s d\beta_s .$$

Then

$$\tilde{\beta}_t = \begin{cases} \beta_t, & t < \tau, \\ \beta_\tau + J(\beta_t - \beta_\tau) = J\beta_t, & t \geq \tau . \end{cases}$$

Define $\tilde{\alpha} = \int \tilde{\beta} \times d\tilde{\beta}$. Then, since $\det I = 1$ and $\det J = -1$,

$$d\tilde{\alpha}_t = \begin{cases} d\alpha_t, & t < \tau , \\ -Jd\alpha_t, & t \geq \tau . \end{cases}$$

Thus

$$\tilde{\alpha}_t = \begin{cases} \alpha_t, & t < \tau, \\ \alpha_\tau - J(\alpha_t - \alpha_\tau) = -J\alpha_t, & t \geq \tau . \end{cases}$$

Finally, $$|\tilde{\alpha}_t| = |\alpha_t|, \ \forall t .$$

REFERENCE

[1] D.W. Stroock and M. Yor, Some remarkable martingales, Séminaire de Probabilités, XV Springer Lecture Notes in Math., 850, 1981.

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