Reflections on modelling, arbitrage, and equilibrium

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1) The Fundamental Error of Financial Modelling

2) APT and equilibrium pricing compared

3) APT: issues and examples

4) EPT: representative agent and terminal wealth

5) EPT: many agents, terminal wealth.

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- The BS model of a stock takes the stock price as fundamental, whereas the fundamental is the dividend process.

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Equilibrium prices satisfy more properties than just absence of arbitrage; once we recognise this, the problems of APT vanish.

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We would *like* to have

$$S_t = E_t^Q[S_T],$$

but that's not what we get.

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$$V \equiv \{\eta \in L^0(\mathcal{F}_T) : \text{for some } \varepsilon > 0, \quad U(\Delta + t\eta) \in L^1 \quad \forall |t| \leq \varepsilon\}$$

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Note we do *not* require $U'(\Delta) \in L^1$; if $\mathbf{1} \in V$, then certainly $U'(\Delta) \in L^1$, but this is not assumed.

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The agent generates a wealth process w by investing in the assets according to some self-financing portfolio process θ: (η_t is holding of numeraire at time t)

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It is elementary to prove that $N_t w_t$ is a martingale; and if we start from $w_0 = q \cdot S_0$, we can easily prove that the portfolio process $\theta_t \equiv q$ is optimal.

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$$\begin{aligned} E[\, U(\nu w_T)\,] &\leq E[\, U(\Delta) + U'(\Delta)(\nu w_T - \Delta)\,] \\ &= E[\, U(\Delta) + U'(\Delta)\nu(w_T - q \cdot S_T)\,] \\ &= E[\, U(\Delta) + N_T w_T - N_T q \cdot S_T\,] \\ &= E[\, U(\Delta)\,]. \end{aligned}$$

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Theorem. The following are equivalent:

(i) Prices $(S_t)_{0 \leq t \leq T}$ are equilibrium prices for a representative agent economy with utility from terminal consumption, satisfying (A);

(ii) There exists a positive martingale N such that

$N_t S_t$ is a martingale.

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The equation $N_T S_T = \Delta U'(\Delta)$ has a unique solution Δ, which has the same sign as S_T. Define

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Because U is Lipschitz, we have

$$|U(\Delta + \varepsilon\nu) - U(\Delta)| \ \leq \ 2\varepsilon|\nu| \ \leq \ 2\varepsilon|N_T| \in L^1$$

and so $\nu \in V$. Similarly, $\Delta \in V$.

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Given N and S, seek $\nu > 0$, Δ and U such that

$$N_T = U'(\Delta)\nu, \quad N_T S_T = \Delta U'(\Delta).$$

This implies that $S_T = \Delta/\nu$. Now choose a concave C^2 function U such that $1 \le U' \le 2$ and such that $x \mapsto xU'(x)$ is strictly increasing.

The equation $N_T S_T = \Delta U'(\Delta)$ has a unique solution Δ, which has the same sign as S_T. Define

$$\nu = \begin{cases} \Delta/S_T & (S_T \neq 0) \\ 1 & (S_T = 0) \end{cases}$$

Because U is Lipschitz, we have

$$|U(\Delta + \varepsilon\nu) - U(\Delta)| \ \le \ 2\varepsilon|\nu| \ \le \ 2\varepsilon|N_T| \in L^1$$

and so $\nu \in V$. Similarly, $\Delta \in V$.

REMARK: a similar analysis works for finite horizon, with intermediate consumption.

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Is $N^j w^j$ a martingale? If it is, then we get optimality just as before.

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How would this look as we thicken up the time grid?

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