TIME-REVERSAL OF THE NOISY WIENER-HOPF FACTORISATION

by

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1. Introduction. Let $(X_t)_{t\geq 0}$ be a continuous-time irreducible Markov chain with finite statespace E and define the continuous additive functional

$$\varphi_t \equiv \varepsilon B_t + \int_0^t v(X_s)ds$$

of (X, B), where $(B_t)_{t\geq 0}$ is a Brownian motion independent of X, and $v : E \to R$. We define the time-changes

$$\tau_t^{\pm} \equiv \inf\{u : \pm\varphi_u > t\}$$

and the time-changed processes

$$Y_t^{\pm} \equiv X(\tau_t^{\pm}).$$

The processes $Y^{\pm}$ will again be Markov chains on the statespace E. The case of no noise ($\varepsilon = 0$) is the original case studied by Barlow, Rogers & Williams [2]; Y^+ (respectively, Y^-) will be a Markov chain in $E^+ \equiv \{i : v(i) > 0\}$ (respectively, $E^- \equiv \{i : v(i) < 0\}$) and the generators of $Y^{\pm}$ are characterised in [2] as the solution of a certain matrix equation, which can seldom be solved in closed form.

Latterly, however, the case $\varepsilon \neq 0$ has been investigated by Kennedy & Williams [4]; here, the generators $\Gamma_{\pm}$ of $Y^{\pm}$ are characterised as the unique solutions in $\mathbf{Q}(E) \equiv \{E \times E \quad \text{matrices } A : \sum_j a_{ij} \leq 0 \quad \forall i, a_{ij} \geq 0 \quad \forall i, j\}$ of the matrix equations

(1.1i) $$\tfrac{1}{2}\varepsilon^2\Gamma_+^2 - V\Gamma_+ + Q = 0$$

(1.1ii) $$\tfrac{1}{2}\varepsilon^2\Gamma_-^2 + V\Gamma_- + Q = 0$$

where $V = \text{diag}\,(v(i))$. It should be noticed that in the case $\varepsilon \neq 0$, the processes $Y^{\pm}$ can take values anywhere in E, not just in $E^{\pm}$

In a sequel to Barlow, Rogers & Williams [2], London, McKean, Rogers & Williams [6] obtained the Wiener-Hopf factorisation of the reversal of the original chain in the case $\varepsilon = 0$, by matrix manipulation of the fundamental Wiener-Hopf factorisation of [2]. *Our goal in this paper is to characterise the Wiener-Hopf factorisation of the reversal of X in* *the noisy case ($\varepsilon \neq 0$).* One would expect that this would be done by taking (1.1i-ii) and performing various matrix operations. However, though such a proof may well exist, no such proof has yet been found, and the proof given here is by way of excursion theory!

To state the result, we need a little notation. Let m denote the invariant distribution of X thought of as a row vector, and let $M \equiv \text{diag}\,(m_i)$. When we reverse X, we see a chain with generator $\hat{Q} \equiv M^{-1}Q^T M$, where Q is the generator of X. We shall also make the convention that the fluctuating additive functional $\hat{\varphi}$ by which we time-change $\hat{X}$ will be

$$\hat{\varphi}_t = \varepsilon \hat{B}_t - \int_0^t v(X_s)ds,$$

that is, we replace v by $-v$. The point of this is the following. If we make the assumption

$$\sum_i m_i v_i > 0, \tag{1.2}$$

then if we draw a sample-path of $(\varphi_t)_{t\in R}$, where we suppose that $(X_t)_{t\in R}$ is in equilibrium, and B is defined also for negative time by $B_t = W_{-t}$, $(t \leq 0)$ for some independent Brownian motion W, then $\varphi_0 = 0$, $\varphi_t \to \infty$ as $t \to \infty$, and *we can visualise the reversal in the same picture,* just by taking t decreasing; indeed, we have simply that

$$\hat{X}_t = X_{-t}, \qquad \hat{\varphi}_t = \varphi_{-t}.$$

The use of this picture is an essential ingredient of the proof. We shall make assumption (1.2) from now on. Nothing is changed if we were to assume instead that $mV1 \equiv \sum_i m_i v_i < 0$, the only problem is to rule out the balanced case $mV1 = 0$; we say a little more on this below. The main result is the following.

THEOREM. *Assume (1.2). The generators $\hat{\Gamma}_\pm$ of the noisy Wiener-Hopf factorisation of the reversed chain $\hat{X}$ are related to the generators $\Gamma_\pm$ of the Wiener-Hopf factors of the forward chain by*

$$\hat{\Gamma}_\pm = M^{-1}((\Gamma_+ + \Gamma_-)\Gamma_\mp(\Gamma_+ + \Gamma_-)^{-1})^T M. \tag{1.3}$$

Remarks (i) It is far from obvious that the expression on the right-hand side of (1.3) is a Q-matrix.

(ii) Under assumption (1.2), Γ_- is a transient Q-matrix, so $\Gamma_+ + \Gamma_-$ is also a transient Q-matrix. In the balanced case $mV1 = 0$, the Q-matrix $\Gamma_+ + \Gamma_-$ would not be invertible.

(iii) Abbreviating

$$\Gamma_+ + \Gamma_- \equiv 2J, \qquad \hat{\Gamma}_+ + \hat{\Gamma}_- \equiv 2\hat{J},$$

we see from (1.3) that $\hat{J} = M^{-1}J^T M$, and so it is immediate that the reversal of the reversal is the original.

(iv) The balanced case $mV1 = 0$ could be approached from the unbalanced case by approximation, but this seems unsatisfactory. It is worth remarking that the reversal in the noiseless case calculated in [6] was also obtained under the assumption (1.2).

The use of the Wiener-Hopf factorisation of Markov chains has important applications in the theory of fluid models of queues. Rogers [8] explains the connections, and Rogers & Shi [9] discuss efficient numerical methods. See also Asmussen [1] for the use of reversals in computing invariant measures for fluid models, and Gaver & Lehoczky [3], Lehoczky & Gaver [5] for examples where the noisy Wiener-Hopf factorisation is required.

2. Proof of the Theorem. The proof proceeds via four simple propositions, of independent interest. Recall that we assume $\sum_i m_i v_i > 0$.

PROPOSITION 1. *The invariant measure of* Γ_+ *is (proportional to)* mJ.

Proof. Left-multiply (1.1) by the row-vector m to learn that

$$\tfrac{1}{2}\varepsilon^2 m\Gamma_+^2 - mV\Gamma_+ = 0$$
$$\tfrac{1}{2}\varepsilon^2 m\Gamma_-^2 + mV\Gamma_- = 0.$$

Since Γ_- is invertible, one has then $-mV = \frac{1}{2}\varepsilon^2 m\Gamma_-$ and thence

$$m(\Gamma_+ + \Gamma_-)\Gamma_+ \equiv 2mJ\Gamma_+ = 0.$$

□

The process φ is a semimartingale, and a continuous additive functional of (X, B). It has a local time process $(L(t,a))_{t\geq 0}$ at each level $a \in R$ which we may and do assume to be jointly continuous, because of Trotter's theorem and the local equivalence of the laws of φ and εB. If $\tau(t,a) \equiv \inf\{u : L(u,a) > t\}$, we define for each a a process

$$\xi(t,a) \equiv X(\tau(t,a))$$

which it is not hard to see must be a Markov chain on $E \cup \{\partial\}$, where ∂ is a graveyard state to which ξ is sent once $\tau(\cdot, a)$ reaches ∞, as it eventually must.

PROPOSITION 2. *For each* $a \in R$, *the generator of* $\xi(\cdot, a)$ *is*

$$\begin{array}{cc} & \begin{array}{cc} E & \partial \end{array} \\ \begin{array}{c} E \\ \partial \end{array} & \begin{pmatrix} J & -J1 \\ 0 & 0 \end{pmatrix} \end{array}$$

where 1 *is the column vector of* 1*'s.*

Proof. In terms of local time at a, the rate at which excursions of φ happen, which start when $\varphi = a$ and $X = i$, rise from the level $\varphi = a$ and return to $\varphi = a$ when X is in state $j \neq i$ is

$$\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} P\big[Y_\varepsilon^- = j | Y_0^- = i\big] = \tfrac{1}{2}\Gamma_-(i,j). \tag{2.1}$$

In more detail, the rate of Brownian excursions which rise at least ε from that level is $(2\varepsilon)^{-1}$ (Rogers & Williams [7] VI.51.2) and for a Brownian motion with constant drift, the rate of excursions which rise at least ε is asymptotically equivalent to $(2\varepsilon)^{-1}$ ([7], VI.55). Once the excursion of φ has risen ε, the probability that φ gets back down to the starting level but with X having changed state to j is $\varepsilon\Gamma_-(i,j) + o(\varepsilon)$, from which (2.1) follows. Note that we may ignore the possibility that X changes in the time taken for the excursion to rise to ε, since this time is $O(\varepsilon^2)$, because the initial part of the Brownian excursion is a BES(3) process ([7], VI.55.11). The downward excursions of φ are treated similarly, and the result follows. □

The next step is to consider what happens when we time-change the process (φ, X) by $L(\cdot, 0) + L(\cdot, a)$, where $a > 0$. We once again see a Markov chain, on the statespace $E_0 \cup E_a \cup \{\partial\}$, where $E_b \equiv \{b\} \times E$, $b = 0, a$; the process spends a finite time in $E_0 \cup E_a$ and then jumps to the graveyard state ∂ where it remains for ever.

PROPOSITION 3. *The generator of the chain on* $E_0 \cup E_a \cup \{\partial\}$ *is*

$$\begin{array}{cc} & \begin{array}{ccc} E_0 & E_a & \partial \end{array} \\ \begin{array}{c} E_0 \\ E_a \\ \partial \end{array} & \begin{pmatrix} JK_+(a) & -JK_+(a)e^{a\Gamma_+} & 0 \\ -JK_-(a)e^{a\Gamma_-} & JK_-(a) & -J1 \\ 0 & 0 & 0 \end{pmatrix} \end{array} \tag{2.2}$$

where

$$K_\pm(a) \equiv (I - \exp(a\Gamma_\pm)\exp(a\Gamma_\mp))^{-1}.$$

Proof. The chain viewed only in $E_0 \cup E_a$ is transient, and has Green's function given by the matrix

$$\begin{pmatrix} I & e^{a\Gamma_+} \\ e^{a\Gamma_-} & I \end{pmatrix} \begin{pmatrix} -J^{-1} & 0 \\ 0 & -J^{-1} \end{pmatrix}.$$

Thus the restriction of the generator to $E_0 \cup E_a$ is simply the negative of the inverse of this matrix. Routine calculations yield the appropriate submatrices of (2.2). Once the chain reaches ∂, it never leaves, which is why the bottom row of (2.2) is zero. Since φ drifts upwards, it is impossible to jump from E_0 to ∂. In order to make the row sums in the E_a part of the matrix zero, we must make the $E_a \times \partial$ submatrix equal to

$$JK_-(a)e^{a\Gamma_-}1 - JK_-(a)1 = JK_-(a)(e^{a\Gamma_-}e^{a\Gamma_+} - I) = -J1,$$

since $\exp(a\Gamma_+)$ is stochastic. □

PROPOSITION 4. *For each $a \in R$, define*

$$\sigma_a \equiv \sup\{t : \varphi_t = a\}.$$

Then

$$P(X(\sigma_0) = j, X(\sigma_a) = k | X_0 = i, \varphi_0 = 0) = -J^{-1}(i,j)(Je^{a\Gamma_+}J^{-1})(j,k)(-J1)_k. \tag{2.3}$$

Proof. Abbreviating the Q-matrix (2.2) to

$$\begin{array}{c} \\ E_0 \\ E_a \\ \partial \end{array} \begin{array}{c} \begin{array}{ccc} E_0 & E_a & \partial \end{array} \\ \begin{pmatrix} Z_{00} & Z_{0a} & 0 \\ Z_{a0} & Z_{aa} & Z_{a\partial} \\ 0 & 0 & 0 \end{pmatrix} \end{array},$$

and decomposing according to the number of crossings from 0 to a, we see that

$$\begin{aligned}
&P(X(\sigma_0) = j, X(\sigma_a) = k | X_0 = i, \varphi_0 = 0) \\
&= \sum_{n \geq 0} (-Z_{00}^{-1} Z_{0a}(-Z_{aa})^{-1} Z_{a0})^n (-Z_{00})^{-1}(i,j) Z_{0a}(-Z_{aa})^{-1}(j,k) Z_{a\partial}(k) \\
&= -(I - Z_{00}^{-1} Z_{0a} Z_{aa}^{-1} Z_{a0}) Z_{00}^{-1}(i,j) Z_{0a}(-Z_{aa})^{-1}(j,k) Z_{a\partial}(k) \\
&= -(I - e^{a\Gamma_+}e^{a\Gamma_-})^{-1} Z_{00}^{-1}(i,j)(Je^{a\Gamma_+}J^{-1})(j,k)(-J1)_k \\
&= -J^{-1}(i,j)(Je^{a\Gamma_+}J^{-1})(j,k)(-J1)_k.
\end{aligned}$$

This is what we wanted. □

We may now assemble all this, by mixing over the state i of X_0 in (2.3), according to the invariant law of Γ_+, which is $\nu \equiv cmJ$, from Proposition 1 (here, $c \equiv (mJ1)^{-1}$.) This gives

$$P(X(\sigma_0) = j, X(\sigma_a) = k) = -cm_j(Je^{a\Gamma_+}J^{-1})(j,k)(-J1)_k.$$

Summing over j yields the useful information that

$$P(X(\sigma_a) = k) = P(\hat{Y}_0^- = k) = cm_k(J1)_k,$$

and from this

$$\begin{aligned}&P(X(\sigma_0)=j|X(\sigma_a)=k)\\&=P(\hat{Y}_a^-=j|\hat{Y}_0^-=k)\\&=m_j(Je^{a\Gamma_+}J_{-1})(j,k)/m_k.\end{aligned}$$

From this, the asserted form of $\hat{\Gamma}_-$ follows immediately.

Finally, we need to find the reversal $\hat{\Gamma}_+$. The proof is structurally similar, except that now if we set $\gamma_a \equiv \sup\{t < \sigma_0 : \varphi_t = a\}$, for any $a > 0$, we have to calculate

$$\begin{aligned}P(X(\sigma_0)=k,X(\gamma_a)=j|X_0=i,\varphi_0=0)&\\=\sum_{n\geq 0}(Z_{00}^{-1}Z_{0a}Z_{aa}^{-1}Z_{a0})^n(Z_{00}^{-1}Z_{0a}Z_{aa}^{-1})(i,j)(-Z_{a0}Z_{00}^{-1})(j,k)Z_{0a}(-Z_{aa}^{-1})Z_{a\partial}(k)&\\=(I-Z_{00}^{-1}Z_{0a}Z_{aa}^{-1}Z_{a0})^{-1}Z_{00}^{-1}Z_{0a}Z_{aa}^{-1}(i,j)(Z_{a0}Z_{00}^{-1})(j,k)Z_{0a}Z_{aa}^{-1}Z_{a\partial}(k)&\\=(-e^{a\Gamma_+}J^{-1})(i,j)(-Je^{a\Gamma_-}J^{-1})(j,k)(J1)(k).&\end{aligned}$$

Mixing now over i with law ν yields

$$P(X(\sigma_0)=k,X(\gamma_a)=j)=cm_j(Je^{a\Gamma_-}J^{-1})(j,k)(J1)_k.$$

Hence finally

$$P(X(\gamma_a)=j|X(\sigma_0)=k)=m_j(Je^{a\Gamma_-}J^{-1})_{jk}/m_k,$$

as claimed.

Remarks. (i) We now have an entirely probabilistic proof of this result, but, since it is a statement phrased entirely in terms of finite matrices, one suspects that there must be a proof entirely in terms of finite matrices. None is yet known. Indeed, even if one could somehow verify that $\hat{\Gamma}_\pm$ defined by (1.3) satisfied (the reversed form of) (1.1), it is far from obvious that $\hat{\Gamma}_\pm \in \mathbf{Q}(E)$, so the identification of these as the reversed generators is not complete.

(ii) We see here an example of the addition of two generators. There seems to be no natural probabilistic interpretation of this operation.

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