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TIME-SUBSTITUTION BASED ON FLUCTUATING ADDITIVE FUNCTIONALS

(WIENER-HOPF FACTORIZATION FOR INFINITESIMAL GENERATORS)

by

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1. This note is merely a first indication of how some of the ideas in the preceding paper [2] by Barlow, Rogers, and Williams (hereafter denoted by [BRW]), extend to Markov processes with 'continuous' state-space. We hope to publish a more detailed study soon. Unusual and interesting purely-analytic problems are posed by the work. However, our main purpose is to attempt to understand what is going on in the probabilistic aspects of the subject.

Our problem has considerable practical importance (but we can make no such claims for the results presented here!) Pure-mathematical technicalities are therefore avoided. We remark however that this work (though not today's examples) forces us to acknowledge the practical usefulness of branch-points, incursions, and other 'exotica' of the general theory. Vivent les hypothèses droites!

Here, we try to convey just a whiff of the flavour of things via two concrete examples. But, for the deepest concrete work done, and on a problem which is important, see McKean [5].

Note. We are aware that many of the results in the present paper may be obtained via the classical Wiener-Hopf methods described for example in Bingham [3]. That our methods are (in principle!) of much wider applicability is of course evident from [BRW].

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involves local times, and cases where φ is not of finite variation, are also of interest. For $t \geq 0$, set

$$\tau_t^+ \equiv \inf\{s : \varphi_s > t\}.$$

A standard argument based on the strong Markov property of X shows that $\tilde{X}^+$, where $\tilde{X}_t^+ \equiv X(\tau_t^+)$, is a (strong) Markov process. For $c \geq 0$, we wish to calculate the transition function $\{\tilde{P}_c^+(t)\}$, where

$$\tilde{P}_c^+(t)f(x) \equiv \underset{\sim}{E}^x[\exp(-c\tau_t^+)f \circ X(\tau_t^+)],$$

or, equivalently, the resolvent $\{\tilde{R}_c^+(\lambda)\}$, or 'natural' generator $\tilde{Q}^+$, of $\{\tilde{P}_c^+(t)\}$. When $c = 0$, we suppress c from the notation; but note that

$$\tilde{P}^+(t)f(x) \equiv \underset{\sim}{E}^x[f \circ X(\tau_t^+);\ \tau_t^+ < \infty].$$

Amongst interesting probabilistic problems posed by this work is the following: <u>what form of killing of</u> $\tilde{X}^+$ <u>is induced by killing</u> X <u>at rate</u> c ?

3. Let φ be of the form (2.1), and suppose that E^+ is closed, where $E^+ \equiv \{x \in E : V(x) \geq 0\}$. By right-continuity of paths, $\tilde{X}^+$ lives in E^+.

<u>Suppose first that</u> $c > 0$, <u>and regard</u> c <u>as fixed</u>. Keep [BRW] in mind, and hope for the best! So, <u>write</u> $g \in \mathcal{H}_{1,c}$ <u>if</u> $g \in D(Q)$ <u>and</u>

(3.1) $$Qg = \mu Vg + cg$$

<u>for some complex number</u> $\mu = \mu(g)$ <u>with</u> $\Re(\mu) < 0$. Then, $\exp(-\mu\varphi_t - ct)g(X_t)$ is a martingale (right-continuous under the right hypotheses) which is bounded on $[0, \tau_u^+]$ for every $u \geq 0$. Apply the optional-sampling theorem at time τ_t^+ to obtain

(3.2) $$\tilde{P}_c^+(t)g^+ = \underset{\sim}{E}^{\cdot}[\exp(-c\tau_t^+)g^+ \circ \tilde{X}^+(t)] = e^{\mu t}g^+$$ on E^+, where g^+ denotes the restriction of g to E^+. Note that the fact that $c > 0$ takes care of difficulties associated with the possibility that $\tau_t^+ = \infty$.

Let $c \downarrow\downarrow 0$ to obtain for $x \geq 0$, and with $\gamma \equiv (2\lambda)^{\frac{1}{2}} > 0$,

(4.3) $\quad \underset{\sim}{E}^x[\cos\gamma \tilde{X}_t^+ + K^{\frac{1}{2}}\sin\gamma\tilde{X}_t^+;\ \tau_t^+ < \infty] = \exp(-\frac{1}{2}\gamma^2 t)[\cos\gamma x + K^{\frac{1}{2}}\sin\gamma x].$

Now let $\gamma \downarrow\downarrow 0$ to obtain

$$\underset{\sim}{P}^x[\tau_t^+ < \infty] = 1, \qquad \forall t.$$

Assume for the moment that

(4.4) <u>the functions</u> $\{g_\gamma^+ : \gamma > 0\}$ <u>on</u> $[0,\infty)$, <u>where</u>

$$g_\gamma^+(x) \equiv \cos\gamma x + K^{\frac{1}{2}}\sin\gamma x , \qquad x \in [0,\infty),$$

<u>are full on</u> $[0,\infty)$.

Then the transition function $\{\tilde{P}^+(\cdot)\}$ is uniquely determined by the fact that its resolvent $\{\tilde{R}^+(\cdot)\}$ satisfies:

$$2\lambda\tilde{R}^+(\lambda)g_\gamma^+ = g_\gamma^+ \qquad (\lambda = \tfrac{1}{2}\gamma^2).$$

Let us make an intelligent guess about $\{\tilde{R}^+(\cdot)\}$. <u>Let</u> $\tilde{Y}^+$ <u>be the Markov</u> process on $[0,\infty)$ <u>which behaves like Brownian motion away from</u> 0, <u>never 'exits</u> 0 <u>continuously'</u>, <u>and jumps from</u> 0 <u>according to the Lévy measure</u>

(4.5) $\quad J(dx) = \text{constant}.\ x^{-(1+\alpha)}dx, \quad 0 < \alpha < 1, \quad \tan\frac{1}{2}\pi\alpha = K^{-\frac{1}{2}}.$

Let $\{{}_0R^+(\cdot)\}$ be the resolvent of Brownian motion on $(0,\infty)$ killed at 0. Then the resolvent $\{\tilde{U}^+(\cdot)\}$ of Y is given by

$$\tilde{U}^+(\lambda)h^+(x) = {}_0R^+(\lambda)h^+(x) + e^{-\gamma x}J({}_0R^+(\lambda)h^+)/\lambda J({}_0R^+(\lambda)I_{(0,\infty)}),$$

where h^+ denotes an arbitrary bounded function on $[0,\infty)$, and, as always, $\gamma \equiv (2\lambda)^{\frac{1}{2}}$. It is easily checked that

(4.6) $\quad {}_0R^+(\lambda)g_\gamma^+(x) = (2\lambda)^{-1}[\cos\gamma x + K^{\frac{1}{2}}\sin\gamma x - e^{-\gamma x}],$

$$ {}_0R^+(\lambda)I_{(0,\infty)}(x) = \lambda^{-1}[1 - e^{-\gamma x}].$$

The essential fact is that for J as at (4.5),

(4.7) $\quad \int_{(0,\infty)} (\cos\gamma x + K^{\frac{1}{2}}\sin\gamma x - 1)\ J(dx) = 0, \qquad \forall\gamma > 0.$

Then f is analytic in $\{\mathscr{I}(\gamma) > 0\}$, and continuous on $\{\mathscr{I}(\gamma) \geq 0\}$. Moreover, $\mathscr{R}(f(\gamma)) \geq 0$ (with equality at $\gamma = 0$ and perhaps at multiples of a purely real θ). Now, f is real and positive on the upper imaginary axis $\{\mathscr{R}(\gamma) = 0,\ \mathscr{I}(\gamma) > 0\}$; and, since (4.7) holds, $(1 - K^{\frac{1}{2}}i)f$ is imaginary on the right half $\{\mathscr{R}(\gamma) > 0,\ \mathscr{I}(\gamma) = 0\}$ of the real axis. Hence, in the first quadrant the harmonic function φ, where

$$\varphi(\gamma) \equiv \arg f(\gamma) = (\log f(\gamma)),$$

stays bounded between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$, and has boundary values as shown in Figure 1.

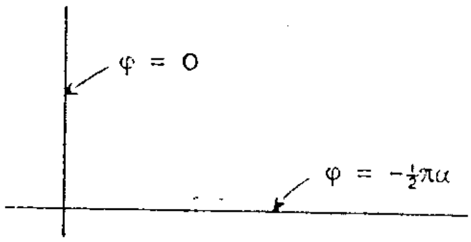

Figure 1

Hence, $\varphi(\gamma) = \alpha \arg(-i\gamma)$. Thus $\log f(\gamma)$ is determined up to an additive constant, and

$$f(\gamma) = \text{constant.}(-i\gamma)^{\alpha}.$$

In particular, for real $\theta > 0$,

$$\int_{(0,\infty)} (1 - e^{-\theta x}) J(dx) = \theta^{\alpha}$$

and so J is determined.

7. Now, of course, there is much more to study in connection with the above example. In particular, the question mentioned earlier about how killing X at rate c induces a killing of $\tilde{X}^+$, is rather interesting. It is clear that $\tilde{X}^+$ is killed according to a discontinuous multiplicative functional which takes into account the jumps of $\tilde{X}^+$ from 0. But we are not going to become involved with the analytic complexities of that problem now.

Instead, we end with an example of a very different type.

8. Example. Let $\{B_t ; t \geq 0\}$ be a Brownian motion on $\mathbb{R}$, starting at 0, with drift $\mu > 0$, so that the law of $\{B_t - \mu t ; t \geq 0\}$ is Wiener measure. Define :

$$V_t \equiv M_t - B_t \quad , \qquad \varphi_t \equiv 2M_t - B_t = V_t + M_t$$

$$\tau_t^+ \equiv \inf\{s ; \varphi_s > t\}, \qquad \tilde{V}_t^+ \equiv V(\tau_t^+) \quad . \qquad \left[M_t = \sup_{u \leq t} B_u\right]$$

Now V is a time-homogeneous strong Markov process, and M is local time at 0 for V. Thus φ is a fluctuating continuous additive functional for V. Obviously, $P[\tau_t^+ < \infty] = 1$.

The results of Rogers and Pitman [7] make it plain that the transition semigroup $\{\tilde{P}_t^+\}$ of $\tilde{V}^+$ is given by the following formulae:

(8.1.i) $$\tilde{P}_t^+(0,dy) = 2\mu\, e^{-2\mu y}(1 - e^{-2\mu t})^{-1}\, dy \quad \text{on} \quad [0,t]$$

and, for $x > 0$,

(8.1.ii) $$\tilde{P}_t^+(x,\{x+t\}) = e^{-2\mu t}(1 - e^{-2\mu x})(1 - e^{-2\mu(x+t)})^{-1}$$

(8.1.iii) $$\tilde{P}_t^+(x,dy) = 2\mu e^{-2\mu y}(1 - e^{-2\mu t})(1 - e^{-2\mu(x+t)})^{-2}\, dy \quad \text{on} \quad [0,x+t)$$

(8.1.iv) $$\tilde{P}_t^+(x,(x+t,\infty)) = 0$$

Here is a martingale proof in the spirit of the remainder of this paper. Begin by observing that for $\theta \geq 0$,

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