FX trading with transaction costs (24/5/05)

1) Suppose the price at time t of the foreign currency is S_t, $dS_t = S_t(\sigma dW_t + \mu dt)$ and at time t we hold X_t units of domestic, Y_t units of foreign, where

$$\begin{cases} dY_t = dL_t - dM_t \\ dX_t = rX_t\,dt - S_t(1+\varepsilon)dL_t + (1-\delta)S_t\,dM_t \end{cases}$$

and the agent wants to trade so as to obtain

$$V(x,y) \equiv \sup E\left[\int_0^\infty e^{-\rho t}(X_t + S_t Y_t)\,dt\right].$$

Usual story gives

$$Z_t = \int_0^t e^{-\rho u}(X_u + S_u Y_u)\,du + V(X_t, S_t Y_t)\,e^{-\rho t}$$

is a supermartingale, and a martingale etc. Let's note that $V(\lambda x, \lambda y) = \lambda V(x,y)$ for all $\lambda > 0$, which can simplify things.

Itô's formula:

$$e^{\rho t}\,dZ_t \doteq (X_t + S_t Y_t)dt - \rho V dt + V_x\, rX\,dt + V_y\,d(S_t Y_t) + \tfrac{1}{2}V_{yy}\,\sigma^2 S^2 Y^2\,dt$$

$$+ \{(1-\delta)S V_x - S V_y\}dM + \{S V_y - (1+\varepsilon)S V_x\}dL$$

$$\doteq \{X + SY - \rho V + rX V_x + YS\mu V_y + \tfrac{1}{2}\sigma^2 S^2 Y^2 V_{yy}\}dt$$

with the conditions

$$(1-\delta)V_x \leq V_y \leq (1+\varepsilon)V_x.$$

Writing $y = \theta x$, we get $V(x,y) = x\,v(\theta) \equiv x\,v(y/x)$, and $V_x = v(\theta) - \theta v'(\theta)$, $V_y = v'(\theta)$, leading to the equations

$$\boxed{\begin{gathered} 0 = 1 + \theta - \rho v(\theta) + r(v(\theta) - \theta v'(\theta)) + \mu\theta v'(\theta) + \tfrac{1}{2}\sigma^2\theta^2 v''(\theta) = 0 \\ (1-\delta)\{v(\theta) - \theta v'(\theta)\} \leq v'(\theta) \leq (1+\varepsilon)\{v(\theta) - \theta v'(\theta)\} \end{gathered}}$$

For a well-posed problem, must have $\rho > r$, $\rho > \mu$. Obvious arguments give for $a > 0$ that

$$V(x+a, y) \geq V(x,y) + \frac{a}{\rho - r}, \quad V(x, y+a) \geq V(x,y) + \frac{a}{\rho - \mu}$$

so $V_x \geq \frac{1}{\rho - r}$; $V_y \geq \frac{1}{\rho - \mu}$, so $v' \geq \frac{1}{\rho - \mu}$.

2) The differential equation has solutions of the form

$$A\,\theta^{-\alpha} + B\,\theta^{\beta} + \frac{\theta}{\rho-\mu} + \frac{1}{\rho-r} \qquad \text{when } \theta>0,$$

where $-\alpha<0<1<\beta$ are the roots of the associated quadratic. For $\theta<0$, the solution looks like

$$A\,|\theta|^{-\alpha} + B\,|\theta|^{\beta} + \frac{\theta}{\rho-\mu} + \frac{1}{\rho-r}$$

... but wouldn't we just invest as much as possible into the asset with the better mean rate of growth?!

Duality for utility-indifference prices (5/6/05)

(i) If we consider a one-step pricing operator defined by

$$\pi_{x,x+1}(k) = b$$

if $$\sum_{y \in x+1} \frac{\bar{p}_y}{\bar{p}_{x+1}} U(X_0 + k_y - b) = U(X_0)$$

where $\bar{p}_{x+1} = \sum_{y \geq x} p_y$, then this is the utility-indifference price (defined over one period) for the cash balance k to be received at $x+1$. Now these $\pi_{x,x+1}$ satisfy (C), (M), as well as (Ti), so we should be able to stick them together in the usual fashion ... what do we get?

(ii) We consider instead the equivalent but notationally easier problem

$$\sum_y p_y U(X_0 + K_y - \pi(K)) = U(X_0).$$

The dual of this is

$$\tilde{\pi}(\lambda) = \sup\{\pi(K) - \lambda \cdot K\}$$

$$= \sup\{b - \lambda \cdot K ; \quad \sum p_y U(X_0 + K_y - b) = U(X_0)\}$$

If we do this by Lagrangian

$$\sup \quad b - \lambda \cdot K + \eta\{\sum p_y U(X_0 + K_y - b) - U(X_0)\}$$

we shall have $$\left.\begin{aligned} 1 &= \eta \sum p_y U'(X_0 + K_y - b) \\ \lambda_y &= \eta \, p_y U'(X_0 + K_y - b) \end{aligned}\right\}$$

so that

$$\boxed{X_0 + K_y - b = I(\lambda_y / \eta p_y)}$$

(and λ is a probability). The dual problem will be to

$$\min_\eta \left[\sum_y \eta p_y \tilde{U}\left(\frac{\lambda_y}{\eta p_y}\right) - \eta U(X_0) \right]$$

and this is achieved when

$$\sum p_y U\left(I\left(\frac{\lambda_y}{\eta p_y}\right)\right) = U(X_0)$$

which is to say, when the solution $X_0 + K_y - b = I(\lambda_y / \eta p_y)$ is feasible. Therefore

$$\boxed{\tilde{\pi}(\lambda) = \min_\eta \sum_y \eta p_y \tilde{U}\left(\frac{\lambda_y}{\eta p_y}\right) - \eta U(X_0)}$$

Neither formulation seems easy to work with.

The analysis of 2), 3) doesn't seem to work out – go to 4) next.

Infinite-horizon examples from the study of executive stock options (6/6/05)

1) In the finite horizon problem, an agent seeks to

$$\max_{m} E\ U\left(x_0 + \int_0^T \varphi_s\, dm_s\right)$$

where m is nondecreasing, $m_0 = 0$, $m_T = A$, and φ_t is some reward process, taken to be $e^{-rt}(S_t - K)^+$, where S is a standard log-Brownian asset. The value function

$$V_T(t, y, x, a) = \sup E\left[U\left(x + \int_t^T e^{-ru}\tilde{\varphi}_u\, dm_u\right) \Big| Y_t = y,\ m_t = A - a \right]$$

where $Y_t \equiv \log S_t$, $\tilde{\varphi}_t = (e^{Y_t} - K)^+$ will satisfy

$$\max\left\{ \mathcal{G}V + V_t,\ -\frac{\partial V}{\partial a} + e^{-rt}(e^y - K)^+ \frac{\partial V}{\partial x} \right\} = 0$$

and as $T \uparrow \infty$ we get

$$V_T(t, y, x, a) \uparrow v(y, x, e^{-rt}a) \equiv \sup E\left[U\left(x + \int_0^\infty e^{-ru}\tilde{\varphi}_u\, d\tilde{m}_u\right) \Big| \tilde{m}_\infty = e^{-rt}a,\ Y_0 = y \right]$$

which solves

$$\boxed{\max\left\{ \mathcal{G}v - r\alpha v_\alpha,\ -v_\alpha + (e^y - K)^+ v_x \right\} = 0} \qquad (1)$$

2) <u>CARA example</u> When $U(x) = -e^{-\gamma x}$, we shall have $v(y, x, \alpha) = -e^{-\gamma x} f(y, \alpha)$, and the job is to find the function $f \geq 0$. We know $f(\cdot, 0) \equiv 1$, and that f should be decreasing in y, decreasing in α.

Try to solve this by discretising onto a grid $k\Delta\alpha$, $k \geq 0$. The dynamics of α is to be proxied by a Markov chain on $\Delta\alpha \cdot \mathbb{Z}^+$ which jumps down from $k\Delta\alpha$ to $(k-1)\Delta\alpha$ with intensity rk. The analogue of the gradient condition now must be

$$\boxed{f_k(y) \leq \exp\{-\gamma\Delta\alpha(e^y - K)^+\} f_{k-1}(y)} \qquad (2)$$

with exercise at equality. When there is no exercise, what governs things is

$$\boxed{\tfrac{1}{2}\sigma^2 f_k'' + \mu f_k' + rk(f_{k-1} - f_k) = 0} \qquad (3)$$

which gets solved recursively. The homogeneous solution for f_k is $f_k = A e^{\beta_k y}$, where $\beta_k \geq 0$ solves $\frac{1}{2}\sigma^2 x^2 + \mu x - rk = 0$, which is to say

$$\beta_k = \left(-\mu + \sqrt{\mu^2 + 2\sigma^2 rk}\right)/\sigma^2.$$

If we write $f_k(y) = \sum_{j=0}^{k} a_{kj} \exp(\beta_j y)$, then the differential equation gives for each $j = 0, \ldots, k-1$

$$0 = a_{kj}\left(\tfrac{1}{2}\sigma^2\beta_j^2 + \mu\beta_j - rk\right) + rk\, a_{k-1,j}$$

$$= a_{kj}\, r(j-k) + rk\, a_{k-1,j}$$

so that

$$\boxed{a_{kj} = \frac{k}{k-j}\, a_{k-1,j}} \qquad (j = 0,\dots,k-1)$$

What remains is to find the diagonal value a_{kk}. From the 'gradient' condition (2) we see that

$$a_{kk}\, e^{\beta_k y} \le \exp\{-\gamma\Delta x\,(e^y-K)^+\}\, f_{k-1}(y) - \sum_{j=0}^{k-1} a_{kj}\, e^{\beta_j y}$$

for all y below the exercise barrier, so this will give

$$\boxed{a_{kk} = \min\left\{ \exp\{-\gamma\Delta x\,(e^y-K)^+ - \beta_k y\}\, f_{k-1}(y) - \sum_{j=0}^{k-1} a_{kj}\, e^{(\beta_j-\beta_k)y}\right\}}$$

Numerically, we seem to be getting $(-1)^{j-1} a_{kj} \ge 0$; moreover, the numerics appear to be quite unstable.

One thing is a bit suspect about this approach; if you were to halve Δx, the expression for $f_k(y)$ is a linear combination of exactly the same exponentials as before, yet it represents the solution at an x-value which is half what it was before?!

3) Maybe it will be better to write

$$v(y,x,\alpha) = -e^{-\gamma x} g(y, \log\alpha) \equiv -e^{-\gamma x} g(y,z)$$

so that the equations are

$$\max\left\{ -\mathcal{G} g + r g_z\ ,\ \frac{1}{\alpha} g_z + \gamma(e^y-K)^+ g \right\} = 0.$$

The motion of z is just a downward drift of rate r, so if we discretise z onto $\Delta z\,\mathbb{Z}$, and let jumps down come at rate $\frac{r}{\Delta z}$, we get the equations

$$\begin{cases} \mathcal{G} g_k + \frac{r}{\Delta z}(g_{k-1} - g_k) = 0 & \text{while no exercise} \\[2ex] g_k \le \exp\{-\gamma(e^y-K)^+(e^{k\Delta z} - e^{(k-1)\Delta z})\}\, g_{k-1} & \text{, equal at exercise.} \end{cases}$$

We therefore could get started with $g_1 = f_1 = 1 + a_{11} e^{\beta_1 y}$, and then represent

$$g_k(y) = 1 + b_k e^{\beta_1 y} + c_k e^{b y} \qquad \text{where } \tfrac{1}{2}\sigma^2 b^2 + \mu b - r/\Delta z = 0.$$

It's easy to see that $b_k = b_{k-1}/(1-\Delta z)$, with c_k represented as a minimum. The question now is

NB. We don't want the solution to have any of the decreasing solution to the homogeneous equation - probably best to start at a large negative y with a small slope.

how we should get this started, and for this we turn to the first form, where the jumping inequality is,

$$f_1(y) = 1 + a_{11} e^{\beta_1 y} \leq \exp\{-\gamma \Delta\alpha (e^y - K)\}$$

so we find a_{11} by minimising $\exp\{-\gamma\Delta\alpha(e^y - K) - \beta_1 y\} - e^{-\beta_1 y}$. Calculus gives

$$(\beta_1 + \gamma \Delta\alpha e^y) \exp\{-\gamma\Delta\alpha(e^y - K)\} = \beta_1$$

to solve for y. If we write x for $\gamma \Delta\alpha e^y$, we have to solve

$$e^{x - \gamma\Delta\alpha K} = 1 + x/\beta_1$$

There are two cases:

(i) $\underline{\beta_1 > 1}$: in this case, $\frac{1}{2}\sigma^2 + \mu < r$, and $x \sim \frac{\beta_1 \gamma \Delta\alpha K}{\beta_1 - 1}$, so

$$\boxed{e^y \approx \frac{\beta_1 K}{\beta_1 - 1}}$$

(ii) $\underline{\beta_1 < 1}$; this time, we have that x converges to the unique positive root x^* to

$$e^x = 1 + x/\beta_1$$

and so

$$\boxed{e^y \approx x^* / \gamma\Delta\alpha}$$

(The second case corresponds to the stock appreciating faster than the riskless asset).

4) The numerical approach of the previous item doesn't appear to work too well. Let's see how we might alternatively proceed. Set down an increasing sequence $(z_k)_{k\geq 0}$ of z values, and set $\alpha_k = \exp(z_k)$, $\Delta\alpha_k = \alpha_k - \alpha_{k-1}$, $\alpha_{-1} = 0$. The idea is that we are forced to exercise immediately once we reach $z = z_0$. This gives

$$g_0(y) = \exp\{-\gamma (e^y - K)^+ \alpha_0\}$$

In general, $\mathcal{G} g_k - r \dfrac{g_k - g_{k-1}}{\Delta z_k} = 0$ in the continue region. To solve this, we first find $\underline{\text{some}}$ solution $g_k^0(\cdot)$ to the differential equation. The solution we seek is of the form

$$g_k(y) = g_k^0(y) + c_k e^{\beta_k y}$$

for some c_k, where β_k is positive root of $\frac{1}{2}\sigma^2 x^2 + \mu x - r/\Delta z_k = 0$. We require that up to the level η_k at which conversion happens we get $g_k(y) \leq \exp[-\gamma(e^y - K)^+ \Delta\alpha_k]\, g_{k-1}(y)$, so we go to get

$$c_k = \min_y \left[e^{-\beta_k y} \left(\exp(-\gamma(e^y - K)^+ \Delta\alpha_k)\, g_{k-1}(y) - g_k^0(y) \right) \right]$$, with η_k the minimising value.

We can then press on inductively.

5) For the CRRA example, we get

$$V_T(t, y, x, a) = x^{1-R}\, V_T(t, y, 1, a/x)$$

so that $\frac{x_t^{1-R}}{1-R}\, v(Y_t, \alpha_t/x_t)$ is a supermartingale etc. The equation to be satisfied is

$$\max\left\{ \mathcal{G} v - r\theta v_\theta ,\; -v_\theta + (e^y - K)^+ \left((1-R)v - \theta v_\theta \right) \right\} = 0$$

where we write $\theta \equiv \alpha/x$. Once again, it is helpful to take $z \equiv \log\theta$ as a new variable, and try a method-of-lines approach as before. We have $v(y, 0) = 1$, and if we discretise, and insist that immediately z reaches z_0 all remaining options get exercised, then

$$v_0(y) = \left(1 + \theta_0 (e^y - K)^+\right)^{1-R}$$

The differential equation for v_k is

$$\mathcal{G} v_k - \frac{r}{\Delta z_k}(v_k - v_{k-1}) = 0.$$

What happens when an exercise takes place? We're at x, $z_k \equiv \log\theta_k$, and we exercise $x\Delta\alpha$ options. Then

$$x \mapsto x\left(1 + \Delta\alpha\,(e^y - K)^+\right) \equiv x(1 + b\Delta\alpha) \qquad \text{for short}$$

$$\alpha \mapsto \alpha - x\Delta\alpha$$

$$\theta \mapsto \frac{\alpha - x\Delta\alpha}{x(1 + b\Delta\alpha)} = \frac{\theta_k - \Delta\alpha}{1 + b\Delta\alpha} = \theta_{k-1}$$

so that $\Delta\alpha = \dfrac{\theta_k - \theta_{k-1}}{1 + b\theta_{k-1}}$, and the new x is $\dfrac{1 + b\theta_k}{1 + b\theta_{k-1}} \cdot x$, so we shall require

$$\boxed{v_k(y) \geq \left(\frac{1 + b\theta_k}{1 + b\theta_{k-1}}\right)^{1-R} v_{k-1}(y)} \qquad (b \equiv (e^y - K)^+)$$

The solution method is now just as before.

6) How about the 2-slope example? The infinite-horizon problem will be ill posed here if $E\, e^{Y_t} > e^{rt}\, Y_0$, which is what you'd expect to be happening; so this example makes little sense...

7) Let's explore another approach. On each of the lines, the Y-process diffuses until the first-jump time τ_1 of a Poisson process with rate $r/\Delta z \equiv \lambda$, say. At that time, it jumps down to the previous z-level if it hasn't already been moved to the previous z-level by the agent choosing to exercise. If the agent exercises at level η, and $T_\eta = \inf\{t : Y_t = \eta\}$, then the value on the current z-slice is given by

$$g_k(y) = E^y\left[g_{k-1}(Y_{\tau_1}) ; \tau_1 < T_\eta \right] + E^y\left[g_{k-1}(\eta)\varphi_k(\eta) : T_\eta < \tau_1 \right]$$

$$\left[\varphi_k(y) \equiv \exp\{-\gamma(e^y-K)^+(e^{z_k}-e^{z_{k-1}})\} \right]$$

$$= E^y\left[g_{k-1}(Y_{\tau_1})\right] + E^y\left[g_{k-1}(\eta)\varphi_k(\eta) - \lambda R_\lambda g_{k-1}(\eta) : T_\eta < \tau_1 \right]$$

$$= \lambda R_\lambda g_{k-1}(y) + E^y\left[e^{-\lambda T_\eta}\right]\left\{ g_{k-1}(\eta)\varphi_k(\eta) - \lambda R_\lambda g_{k-1}(\eta) \right\}$$

for $y < \eta$, and $g_k(y) = \varphi_k(y)\, g_{k-1}(y)$ for $y \geq \eta$. If we can easily calculate $\lambda R_\lambda \varphi(\cdot)$ then we will be able to do this.

If $-\alpha < 0 < \beta$ are roots of $\frac{1}{2}\sigma^2 x^2 + \mu x - \lambda = 0$, then the resolvent density is

$$r_\lambda(x,y) = c(y)\, \exp\{\beta(x \wedge y) - \alpha(x \vee y)\}$$

where

$$c(y) \equiv \frac{2\exp\left(+2\mu y/\sigma^2\right)}{\sigma^2(\alpha+\beta)}$$

so that

$$\lambda r_\lambda(x,y) = \frac{\alpha\beta}{\alpha+\beta}\exp\left\{+\frac{2\mu y}{\sigma^2} + \beta(x\wedge y) - \alpha(x \vee y)\right\}$$

$$= \frac{\alpha\beta}{\alpha+\beta}\exp\left\{-\beta(y-x)^+ - \alpha(y-x)^-\right\}$$

$$\equiv q(y-x), \quad \text{say.}$$

We shall need to compute $\lambda R_\lambda g(y) = \int q(v-y)\, g(v)\, dv \equiv \int \tilde{q}(y-v)\, g(v)\, dv$, where $\tilde{q}(s) \equiv q(-s)$. The functions g will all tend to 1 at $-\infty$, and will drop very quickly to the right of zero, so it is probably worth multiplying g by $\exp(\varepsilon v)$ for some $\varepsilon > 0$; this should help the FFT that is the basis of a convolution algorithm. So

$$e^{\varepsilon y}\lambda R_\lambda g(y) = \int \tilde{q}(y-v)e^{\varepsilon(y-v)}\, e^{\varepsilon v} g(v)\, dv$$

$$= \int \frac{\alpha\beta}{\alpha+\beta}\exp\left\{\varepsilon(y-v) - \beta(y-v)^- - \alpha(y-v)^+\right\}\, e^{\varepsilon v} g(v)\, dv$$

is the calculation we aim to do numerically; taking $\varepsilon = \alpha/2$ seems a good choice. We then have

$$e^{\varepsilon y} g_k(y) = e^{\varepsilon y} \lambda R_\lambda g_{k-1}(y) + e^{-(\beta+\varepsilon)(\eta-y)} \left\{ e^{\varepsilon\eta} g_{k-1}(\eta)\, \varphi_k(\eta) - e^{\varepsilon\eta} \lambda R_\lambda g_{k-1}(\eta) \right\}$$

Accordingly, if we define $\tilde{g}_k(y) \equiv e^{\varepsilon y} g_k(y)$, $h(y) \equiv e^{\varepsilon y} \tilde{q}(y)$, we get

$$e^{\varepsilon y} \lambda R_\lambda g_{k-1}(y) = (h * \tilde{g}_{k-1})(y),$$

so

$$\tilde{g}_k(y) = (h * \tilde{g}_{k-1})(y) + e^{-(\beta+\varepsilon)(\eta-y)} \left\{ \tilde{g}_{k-1}(\eta)\, \varphi_k(\eta) - (h * \tilde{g}_{k-1})(\eta) \right\}$$

and the choice of η will be to take that η which minimises

$$e^{-(\beta+\varepsilon)\eta} \left\{ \tilde{g}_{k-1}(\eta)\, \varphi_k(\eta) - (h * \tilde{g}_{k-1})(\eta) \right\}$$

and then define $\tilde{g}_k(y) = \varphi_k(y)\, \tilde{g}_{k-1}(y)$ for $y \geq \eta$.

8) How does this get discretised? We suppose we have values h_j, $j = 1, \dots, N = 2^M$, where $h_j = h((j - N/2)\Delta y)$, and $\tilde{g}^j = \tilde{g}(\log K + (j - N/2)\Delta y)$. The approximation to $h * \tilde{g}$ at $\ell \Delta y$ is given by

$$(h * \tilde{g})(\ell \Delta y) \approx \Delta y \sum_j h_{\ell - j}\, \tilde{g}^j$$

9) <u>The 2-slope example</u> Take firstly the finite-horizon version, with horizon T, and value $V(t, y, x, a) = a\, V(t, y, x/a, 1)$.

Linking Scilab's fft and Fourier transforms. (18/6/05)

1) Given an n-vector a, Scilab will form an n-vector $\hat{a}$

$$\hat{a}_k \equiv fft(a,-1) = \frac{1}{n}\sum_{m=1}^{n} \exp\{2\pi i (k-1)(m-1)/n\}\, a_m \qquad (k=1,\dots,n)$$

which is then inverted by

$$a_k \equiv fft(\hat{a},1) \equiv \sum_{m=1}^{n} \exp\{-2\pi i (k-1)(m-1)/n\}\, \hat{a}_m \qquad (k=1,\dots,n)$$

Suppose we have some function $G: \mathbb{R} \to \mathbb{C}$; how do we use these Scilab functions to handle it?

2) Let's firstly note that having chosen n, Scilab is making the convention that $\Delta x = \Delta\theta = \sqrt{2\pi/n}$, or at least $\Delta x . \Delta\theta = 2\pi/n$. Given the G, we first restrict attention to $[a,b]$, and set $\Delta x = (b-a)/n$, so that $\Delta\theta = 2\pi/(b-a)$. When we use Scilab's $fft(\cdot,-1)$, we compute (using $g_k \equiv G(a+(k-1)\Delta x)$)

$$\begin{aligned}\hat{g}_k &\equiv \frac{1}{n}\sum_{m=1}^{n} \exp\{2\pi i (m-1)(k-1)/n\}\, g_m \\ &= \frac{1}{n}\sum_{m=1}^{n} \exp\{i(k-1)\Delta\theta.(m-1)\Delta x\}\, G(a+(m-1)\Delta x) \\ &\approx \frac{1}{n\Delta x}\int_a^b \exp\{i(k-1)\Delta\theta.\, v\}\, G(a+v)\,dv \\ &\approx \frac{1}{n\Delta x}\int \exp\{i(k-1)\Delta\theta(v-a)\}\, G(v)\,dv \\ &= \frac{1}{b-a}\hat{G}((k-1)\Delta\theta)\, e^{-i(k-1)a\Delta\theta}\end{aligned}$$

[! A better approximation to this integral is obtained by setting $g_1 = \frac{1}{2}(G(a)+G(b))$]

So our approximation is

$$\boxed{\hat{G}((k-1)\Delta\theta) \approx (b-a)\, e^{+i(k-1)a\Delta\theta}\, \hat{g}_k} \qquad (k=1,2,\dots,n)$$

3) We then do things in the frequency domain, and end up with $(\hat{f}_k)$, which approximates $\hat{f}((k-1)\Delta\theta)$. To go back now we have

$$\begin{aligned}f(a+(m-1)\Delta x) &= \int \frac{1}{2\pi} \exp\{-i\theta(a+(m-1)\Delta x)\}\, \hat{f}(\theta)\,d\theta \\ &\approx \frac{1}{2\pi}\Delta\theta \sum_{k=1}^{n} \exp\{-i\Delta\theta(k-1)(m-1)\Delta x\}\, \hat{f}((k-1)\Delta\theta)\, e^{-i(k-1)a\Delta\theta} \\ &= \frac{1}{b-a}\sum_{k=1}^{n} \hat{f}_k\, e^{-i(k-1)a\Delta\theta} \exp\{-2\pi i(k-1)(m-1)/n\}\end{aligned}$$

[! A better approximation to the inverse Fourier integral is obtained by halving the first and last elements of $(\hat{f}_k)$, and taking real part after inverting.]

4) How to make a good approximation to a Lévy process ~~transition~~ density? Suppose we aim to get hold of $f(k\Delta y)$, $k = -\frac{N}{2}+1, \dots, N/2$, a vector of length N. Here, Δy is a fixed spacing. The way to get this is to compute firstly $f(k\Delta y)$, $k = -N, \dots, N-1$, and then restrict. Thus we shall be using fft with vectors of length $2N \equiv n$. Now

$$f(x) = \int \exp\{-i\theta x + \psi(\theta)\} \frac{d\theta}{2\pi} \qquad \text{(assume wlog } t = 1)$$

so

$$f(k\Delta y) \approx \frac{\Delta\theta}{2\pi} \sum_j \exp\{-ij\,\Delta\theta\, k\Delta y + \psi(j\Delta\theta)\}$$

and we now restrict the j values so that we're only dealing with n of them. We could take these symmetrically about 0, but better is to use the fact that $[\Delta y \Delta\theta = 2\pi/n]$

$$f(k\Delta y) \approx \frac{\Delta\theta}{2\pi} \sum_{j=-n+1}^{n-1} \exp\{-ij\,k\,\tfrac{2\pi}{n}\}\exp\{\psi(j\Delta\theta)\}$$

$$= \frac{\Delta\theta}{\pi} \operatorname{Re}\left(\sum_{j=1}^{n} \exp(-i(j-1)k\,2\pi/n)\ \tilde{g}_j\right)$$

$$\text{where } \tilde{g}_j = \begin{cases} \exp\psi((j-1)\Delta\theta) & 1 < j < n \\ \tfrac{1}{2}\exp\psi((j-1)\Delta\theta) & j = 1, n \end{cases}$$

(the halving at the outer ends of the interval is in effect the trapezium rule, at $j=1$ is to avoid double counting the contribution in the middle). To relate back to Scilab's fft, we define

$$p_\ell \equiv f((-N+\ell-1)\Delta y) \qquad \text{for } \ell = 1, \dots, n$$

and observe then that

$$p_\ell \doteq \frac{\Delta\theta}{\pi} \operatorname{Re}\left(\sum_{j=1}^{n} \exp(-i(j-1)(-N+\ell-1)2\pi/n)\ \tilde{g}_j\right)$$

$$= \frac{\Delta\theta}{\pi} \operatorname{Re}\left(\sum_{j=1}^{n} \exp(-i(j-1)(\ell-1)\tfrac{2\pi}{n})\ \exp(i(j-1)\pi)\ \tilde{g}_j\right)$$

to allow us to use the Scilab fft(·, −1) routine.

Axiomatics of wealth-dependent valuation operators (20/6/05)

1) Looking at utility-indifference pricing, it seems we need to extend the theory of pricing operators to allow the valuation of a cash balance process to depend on an agent's current level of wealth - see p50 of WN XXIV. We still require properties which allow a recursive construction, and it seems that we should want

(C) $(k;a) \longmapsto \pi_{x,x+1}(k;a)$ is jointly concave;
(M) $(k;a) \longmapsto \pi_{x,x+1}(k;a)$ is increasing in both arguments;
(Si) $a \mapsto a - \pi_{x,x+1}(k;a)$ is strictly increasing for each k;
(ti) $\pi_{x,x+1}(k+b;a) = b + \pi_{x,x+1}(k;a)$

The claim then is that if we define valuation operators $\pi_x(K;a)$ by

$$\boxed{\pi_x(K;a) = \pi_{x,x+1}\left(K_x;\ \bar{\pi}_{x+1}(K;a-\pi_x(K,a));a\right)}$$

then (i) the operators are well defined;
(ii) Each $\pi_x(\cdot;\cdot)$ is jointly concave;
(iii) Each $\pi_x(\cdot;\cdot)$ is increasing in each argument;
(iv) (DC) holds for the $\pi_x(\cdot;\cdot)$;
(v) (TI) holds for the $\pi_x(\cdot;\cdot)$;
(vi) $a \mapsto a - \pi_x(K;a)$ is strictly increasing for each a
(vii) $\bar{\pi}_x$ is jointly concave, and increasing in each of its arguments.

! but in UIP example, $\bar{\pi}$ is not in general concave in K!

2) We now have to establish the desired properties inductively. In order to prove (i), we shall need

(vii) $a \mapsto \bar{\pi}_x(K;a)$ is increasing.

For this, suppose that $a_1 < a_2$, and $q_i = \bar{\pi}_x(K;a_i) = \pi_x(K;a_i+q_i)$. Now $v - \pi_x(K;v)$ is equal to a_i when $v = a_i + q_i$, so this implies that $a_1+q_1 < a_2+q_2$, using property (vi). But now using property (iii),

$$\pi_x(K;a_1+q_1) = q_1 \leq \pi_x(K;a_2+q_2) = q_2. \qquad \square$$

The valuation $\pi_x(K;a)$ is defined to be the solution p to

$$\boxed{p = \pi_{x,x+1}\left(K_x;\ \bar{\pi}_{x+1}(K;a-p);a\right)} \qquad (*)$$

Since the RHS is now known to be decreasing with p (property (vii) and axiom (M)), there is a unique root p, so $\pi_x(K;a)$ is well defined by this relation. Moreover, it follows from (M) and property (iii) for π_{x+1} that π_x is increasing in both arguments, so (iii) holds for π_x.
Next we prove property (vi), that $a \mapsto \psi_x(K;a) \equiv a - \pi_x(K;a)$ is strictly increasing in a. Indeed if we set $\psi_{x,x+1}(k;a) \equiv -\pi_{x,x+1}(k;a)+a$, then $q = \psi_x(K;a)$ solves

$$q = \psi_x(K;a) = \psi_{x,x+1}\left(K_x;\bar{\pi}_{x+1}(K;q);a\right)$$

Now the RHS here increases strictly with a, and $\psi_{x,x+1}(\cdot;\cdot)$ decreases in its first argument, so $\psi_x(k;a)$ must increase strictly with a.

The next task is the joint concavity of π_x. Let's set (with y denoting an element of $x+1$)

$$p_i = \pi_x(K^i; a^i), \qquad q_i = \bar\pi_y(K^i; a_i - p_i) \qquad i=1,2$$

and (taking $\theta_1 = 1-\theta_2 \in [0,1]$) consider

$$\bar p \equiv \theta_1 p_1 + \theta_2 p_2 \le \pi_{x,x+1}\left(\bar K_x, \theta_1 \bar\pi_{x+1}(K^1; a_1 - p_1) + \theta_2 \bar\pi_{x+1}(K^2; a_2 - p_2); \bar a\right) \quad \text{by (C)}$$

where $\bar K = \theta_1 K^1 + \theta_2 K^2$, of course. What we now need is that for each $y \in x+1$

$$\bar q \equiv \theta_1 q_1 + \theta_2 q_2 \le \bar\pi_y(\bar K; \bar a - \bar p).$$

Now the LHS $= \theta_1 \pi_y(K^1; a_1 - p_1 + q_1) + \theta_2 \pi_y(K^2; a_2 - p_2 + q_2) \le \pi_y(\bar K; \bar a - \bar p + \bar q)$. Considering the strictly increasing function $s \mapsto s - \pi_y(\bar K; \bar a - \bar p + s)$, we see that it is ≤ 0 at $s = \bar q$, so the zero $\tilde q$ of the function must be $\ge \bar q$. This tells us that

$$\bar\pi_y(\bar K; \bar a - \bar p) \ge \bar q$$

as required. Hence

$$\bar p \le \pi_{x,x+1}\left(\bar K_x, \bar\pi_{x+1}(\bar K; \bar a - \bar p); \bar a\right);$$

varying $\bar p$ to achieve equality here, it's clear that we don't decrease $\bar p$. But the value of $\bar p$ for which equality is attained is exactly $\pi_x(\bar K; \bar a)$, so we do have joint concavity.

To check (TI), we shall of course have to exploit (ti). It will be sufficient to prove that

$$\bar\pi_y(K+b; a-b) = b + \bar\pi_y(K; a) \qquad (y \in x+1)$$

for any a, b. So if $q = \bar\pi_y(K; a) = \pi_y(K; a+q)$, we have $q + b = \pi_y(K+b; a+q)$ so $\bar\pi_y(K+b; a-b) = q+b$ as required.

To prove (DC), it seems to be best to go back to p50 of WN $\underline{\text{XXIV}}$ and use the fact that the notion of dynamic consistency can best be expressed as (DC3)

$$\bar\pi_x(K; a) = \bar\pi_x\left(K I_{[x,\sigma)} + \bar\pi_\sigma(K; a) I_{[\sigma,\infty)}; a\right)$$

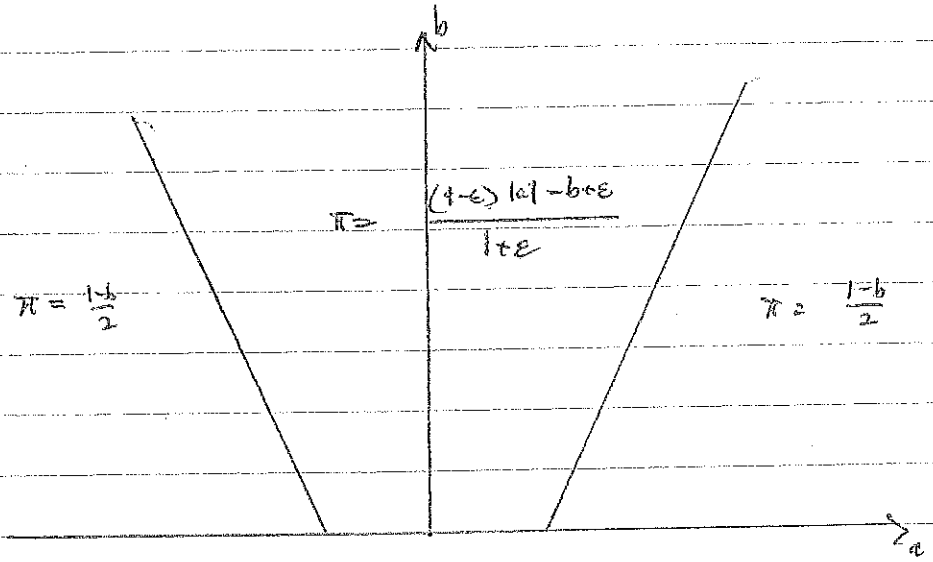
for this purpose.

Finally, property (vii) is quite easy to prove using the above ideas; for example if $q_i = \bar\pi_x(K^i; a_i) = \pi_x(K^i; a_i + q_i)$, then

$$\bar q \equiv \theta_1 q_1 + \theta_2 q_2 \le \pi_x(\bar K; \bar a + \bar q)$$

and to get equality here we would have to raise $\bar q$. Monotonicity is similarly easy.

This is all very well, but already on p.52 of WN $\underline{\text{XXIV}}$ we saw that utility-indifference pricing doesn't in general satisfy this — Doh!



b
a
$\pi = \frac{1-b}{2}$
$\pi = \frac{(1-\varepsilon)|a| - b + \varepsilon}{1+\varepsilon}$
$\pi = \frac{1-b}{2}$

3) I've done some numerics on UIPs in a single period setting where the RV takes values $-b<0$ and 1 with equal probability, and calculated with various utilities. An interesting one is

$$U(x) = \min\{x, \varepsilon x\}.$$

In this case, we get $\bar\pi$ satisfies (a is base wealth level)

$$U(a+\bar\pi) = \tfrac{1}{2}U(a+1) + \tfrac{1}{2}U(a-b)$$

so if $a > b$ we get $\bar\pi = \frac{1}{2}(1-b)$, as we do if $a \le -1$. If $a \in (-1, b)$, we get

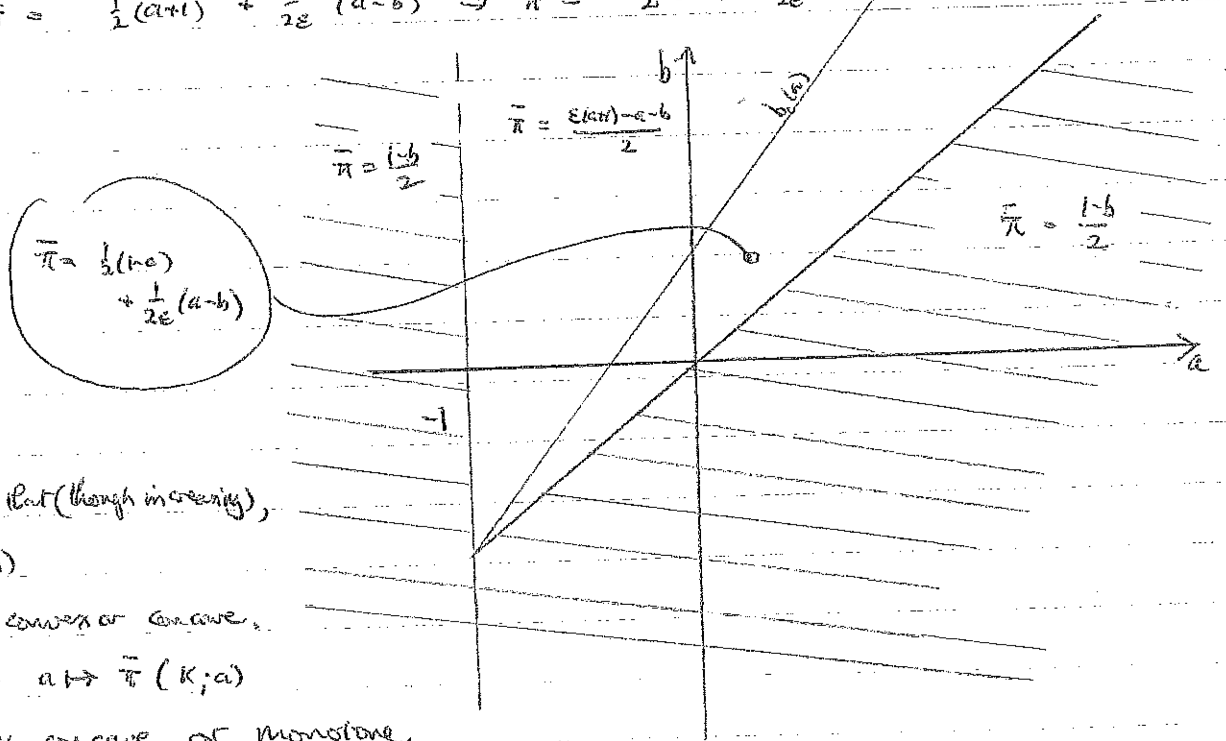
$$\tfrac{1}{2}U(a+1) + \tfrac{1}{2}U(a-b) = \tfrac{\varepsilon}{2}(a+1) + \tfrac{1}{2}(a-b)$$

and the critical case is when $b = b_c(a) \equiv a + \varepsilon(a+1)$. If $b > b_c(a)$, we shall have

$$a+\bar\pi = \tfrac{\varepsilon}{2}(a+1) + \tfrac{1}{2}(a-b) \Rightarrow \bar\pi = \tfrac{1}{2}\left\{-b-a+\varepsilon(a+1)\right\}$$

whereas if $b < b_c(a)$ we get

$$a+\bar\pi = \tfrac{1}{2}(a+1) + \tfrac{1}{2\varepsilon}(a-b) \Rightarrow \bar\pi = \tfrac{1}{2}(1-a) + \tfrac{1}{2\varepsilon}(a-b)$$



This example shows that (though increasing),

$$K \mapsto \bar\pi(K;a)$$

need not be either convex or concave.

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It also shows that $a \mapsto \pi(K;a)$ need not be convex, concave or monotone, so that's pretty conclusive – don't ask for <u>any</u> such properties. We have in fact:

$$\pi(K;a) = \begin{cases} \frac{1}{2}(1-b) & \text{if } |a| \ge \frac{1}{2}(1+b) \\ \frac{(1-\varepsilon)|a| - b + \varepsilon}{1+\varepsilon} & \text{if not.} \end{cases}$$

$$V_z = \gamma_*^{-R}\, U'(z-\Delta)\,(1-\Delta_z)$$

$$V_{zz} = \gamma_*^{-R}\left[U''(z-\Delta)(1-\Delta_z)^2 - U'(z-\Delta)\Delta_{zz} \right]$$

The Merton liquidity problem again (1/7/05)

1) In the study of the effects of liquidity on the Merton problem, we end up with the PDE

$$0 = \tilde{U}(V_z) - \tilde{\rho} V + \alpha (H-z) V_z + \tfrac{1}{2}\sigma^2 (H-z)^2 V_{zz} + \frac{V_H^2}{2\varepsilon V_z}$$

where $\alpha = \sigma^2 R(\pi_* - 1)$, $\tilde{\rho} = \rho - (1-R)(\mu - \frac{1}{2}\sigma^2 R)$. The Merton solution would be

$$V_M(z) = \gamma_*^{-R} \, U(z)$$

so we propose to look for a perturbation of this, of the form

$$V(z,H) = \gamma_*^{-R} \, U(z - \Delta(z,H))$$

for some small Δ. We have

$$V_z = (1-R) \frac{1-\Delta_z}{z-\Delta} V, \qquad V_H = -(1-R) \frac{\Delta_H}{z-\Delta} V$$

$$V_{zz} = \left\{ -R(1-R)\left(\frac{1-\Delta_z}{z-\Delta}\right)^2 - (1-R)\frac{\Delta_{zz}}{z-\Delta} \right\} V$$

Moreover, $\tilde{U}(V_z) = (1-\Delta_z)^{1-1/R} \gamma_* R V(z,H)$, so there are some simplifications. We obtain

$$0 = \gamma_* R (1-\Delta_z)^{1-1/R} - \tilde{\rho} + \alpha \frac{H-z}{z-\Delta}(1-R)(1-\Delta_z) - \frac{\sigma^2}{2}(1-R)(H-z)^2 \left\{ R\left(\frac{1-\Delta_z}{z-\Delta}\right)^2 + \frac{\Delta_{zz}}{z-\Delta} \right\}$$
$$+ \frac{(1-R)\Delta_H^2}{2\varepsilon(z-\Delta)(1-\Delta_z)}$$

Now we use the fact that

$$\gamma_* R - \tilde{\rho} + \alpha (1-R)(\pi_* - 1) - \frac{\sigma^2}{2}(1-R)R(\pi_*-1)^2 = 0,$$

and the notation $H = H_0(z) + y \equiv \pi_* z + y$ to derive

$$0 = \gamma_* R \left\{ (1-\Delta_z)^{1-1/R} - 1 \right\} + \alpha(1-R)\left\{ y \frac{1-\Delta_z}{z-\Delta} + \frac{\pi_*-1}{z-\Delta}(\Delta - z\Delta_z) \right\} + \frac{(1-R)\Delta_H^2}{2\varepsilon(z-\Delta)(1-\Delta_z)}$$

$$- \frac{\sigma^2}{2} R(1-R) \left\{ \frac{(y+(\pi_*-1)z)^2(1-\Delta_z)^2}{(z-\Delta)^2} - (\pi_*-1)^2 \right\} - \frac{\sigma^2}{2}(1-R)\left(y + (\pi_*-1)z\right)^2 \frac{\Delta_{zz}}{z-\Delta}.$$

Maybe things are more compactly expressed if we set

$$V(z,H) \equiv \gamma_*^{-R} \, U(\varphi(z,H)) \equiv \gamma_*^{-R} \, U(e^{\psi(z,H)})$$

so that in the case of no liquidity costs, $\varphi(z,H) = z$, $\psi(z,H) = \log z$. We then derive after some similar calculations the equation

$$0 = \gamma_* \frac{R}{1-R}\left(\varphi_z^{1-1/R} - 1\right) + \alpha\left\{y\psi_z + (\pi_*-1)(z\psi_z - 1)\right\} + \frac{\sigma^2}{2}\left(y^2 + 2yz(\pi_*-1)\right)\left(\psi_{zz} + (1-R)\psi_z^2\right)$$

$$+ \frac{\sigma^2}{2}(\pi_*-1)^2\left\{z^2\left(\psi_{zz} + (1-R)\psi_z^2\right) + R\right\} + \psi_H^2 / 2\varepsilon\psi_z$$

Again, if we set $V(z,H) = \gamma_*^{-R}\, U(z e^{-\Delta(z,H)})$, we have

$$0 = \Big[\gamma_* R\left(e^{-\Delta}(1 - z\Delta_z)\right)^{1-1/R} - \tilde{\rho} + \alpha(p-1)(1-R)(1 - z\Delta_z) + \tfrac{1}{2}\sigma^2(p-1)^2(1-R)\left\{-R + 2z(R-1)\Delta_z - (R-1)z^2\Delta_z^2 - z^2\Delta_{zz}\right\}$$

$$+ \frac{(1-R)\Delta_H^2}{2\varepsilon z(1 - z\Delta_z)}\Big]\, V(z,H)$$

2) How about the dual problem? If we change to dual variables $\lambda = V_z$, $\eta = V_H$, and define the dual value function

$$J(\lambda,\eta) = V(z,H) - \lambda z - \eta H \quad \left(= V(z(\lambda,\eta), H(\lambda,\eta)) - \lambda z(\lambda,\eta) - \eta H(\lambda,\eta)\right)$$

then we find that

$$J_\lambda = -z, \quad J_\eta = -H$$

$$\left.\begin{aligned} 1 &= \frac{\partial}{\partial\lambda}\lambda = \frac{\partial}{\partial\lambda}V_z = -V_{zz}J_{\lambda\lambda} - V_{zH}J_{\eta\lambda} \\ 0 &= \frac{\partial}{\partial\eta}\lambda = \frac{\partial}{\partial\eta}V_z = -V_{zz}J_{\lambda\eta} - V_{zH}J_{\eta\eta} \end{aligned}\right\} \Rightarrow V_{zz} = -\left\{J_{\lambda\lambda} - J_{\eta\lambda}^2/J_{\eta\eta}\right\}^{-1}$$

The dual form of the PDE is therefore

$$\boxed{0 = \tilde{U}(\lambda) - \rho(J - \lambda J_\lambda - \eta J_\eta) + \alpha(J_\lambda - J_\eta)\lambda - \frac{\frac{1}{2}\sigma^2(J_\lambda - J_\eta)^2 J_{\eta\eta}}{J_{\lambda\lambda}J_{\eta\eta} - J_{\eta\lambda}^2} + \frac{\eta^2}{2\varepsilon\lambda}}$$

3) Look at a simpler related problem? For our problem, we have dynamics

$$dz = \sigma(H-z)\,dW + \left\{\alpha(H-z) - c - \tfrac{1}{2}\varepsilon h^2\right\}dt, \qquad dH = h\,dt$$

but if we change to

$$dz = \sigma(H-z)\,dW + \left\{\alpha(H-z) - c - \tfrac{1}{2}\varepsilon\, h^2/z\right\}dt, \qquad dH = h\,dt$$

then the solution scales; $V(z,H) = z^{1-R}\, v(p)$ where $p \equiv H/z$. What equation does r solve? We get now

$$0 = \tilde{U}(V_z) - \tilde{\rho} V + \alpha z(p-1) V_z + \tfrac{1}{2}\sigma^2 z^2 (p-1) V_{zz} + \frac{z V_H^2}{2\varepsilon V_z}$$

and we have $V_H = z^{-R} v'$, $V_z = z^{-R}\big((1-R)v - pv'\big)$, $V_{zz} = z^{-1-R}\big(-R(1-R)v + 2Rpv' + p^2 v''\big)$,

so we get

$$\boxed{0 = \tilde{U}\big((1-R)v - pv'\big) - \tilde{\rho} v + \alpha(p-1)\big((1-R)v - pv'\big) + \tfrac{1}{2}\sigma^2 (p-1)^2\big(-R(1-R)v + 2Rpv' + p^2 v''\big) + \frac{(v')^2}{2\varepsilon\big((1-R)v - pv'\big)}}$$

Going into Maple for this one, we seem to be able to do a perfectly decent expansion in powers of $\varepsilon^{1/2} = \delta$, yielding

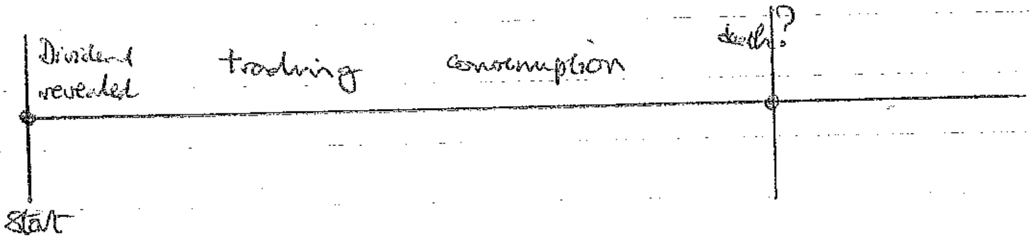
$$v(p) = \frac{\gamma_*^{-R}}{1-R} - \left\{\frac{\sigma^3}{2}\gamma_*^{1-R}\sqrt{R}\,\pi^2(1-\pi)^2 + \frac{\sigma\sqrt{R}}{2}\gamma_*^{-R}(p-\pi)^2\right\}\delta + O(\delta^2)$$

Overlapping generations model and the equity premium puzzle (27/8/05)

In the study of the EPP, the single representative agent assumption leads to various explanations that one struggles to work around. Here is a story that may help resolve the problem. We consider a discrete-time model with a large number of agents. In each period, the dividend of the single productive asset in that period is revealed, then agents trade, and finally consume. Just before the next period starts, each agent is either killed or allowed to continue into the next period; an agent who has been alive for j periods passes to the next period with probability $p_j \in (0,1)$, whereas an agent who is killed is immediately replaced by a new agent (age 0) who takes over the dead agent's shares. Assuming $\prod p_j = 0$, we find the invariant law in the form

$$\pi_j = \pi_0 \Big(\prod_{r=1}^{j} p_r\Big) \qquad (j = 0, 1, \dots)$$

Or we may alternatively suppose that we are given the strictly decreasing sequence $(\pi_r)_{r\ge 0}$, decreasing to 0, and express $p_j = \pi_j / \pi_{j-1}$, $j = 1, 2, \dots$.



(i) Let the output in period t be y_t, where $\xi_t = \log(y_t / y_{t-1})$ are IID. Suppose all agents are identical except in their times of birth and death. An agent values a consumption stream according to

$$E\Big[\sum_{r\ge 0} w_r\, U(c_r)\, I_{\{r<\tau\}}\Big]$$

where τ denotes the time the agent is killed. Let S_t be the equilibrium price process of the share. Let us consider an agent who starts period t aged j. He is going to attempt to invest and consume so as to

$$\max \; E_t\Big[\sum_{r=t}^{\infty} w_{r-t+j}\, U(c_r)\, I_{\{r<\tau\}}\Big]$$

where the budget constraint linking consumption and the numbers θ_t of shares at the start of period t will be

$$c_t = \theta_t (y_t + S_t) - \theta_{t+1} S_t \qquad \text{if } t < \tau$$

Introducing Lagrange multipliers λ_r we find the optimisation is

$$\max \; E_t\Big[\sum_{r\ge t}^{\infty} w_{r-t+j}\, U(c_r) I_{\{r<\tau\}} - \sum_{r\ge t}^{\infty} \lambda_r \big(c_r - \theta_r(y_r + S_r) + \theta_{r+1} S_r\big) I_{\{r<\tau\}}\Big]$$

$$= \max E_t\Big[\sum_{r=t}^{\infty}\big(w_{r-t+j}\, U(c_r) - \lambda_r c_r\big) I_{\{r<\tau\}} + \lambda_t\, \theta_t (y_t + S_t)$$
$$+ \sum_{r=t+1}^{\infty} \theta_r \Big(\lambda_r (y_r + S_r) I_{\{r<\tau\}} - \lambda_{r-1} S_{r-1}\, I_{\{r-1<\tau\}} \Big) \Big]$$

Usual story gives us

$$\boxed{w_{r-t+j}\, U'(c_r) = \lambda_r \qquad \text{for } r < \tau,}$$

$$E_{r-1}\Big[\lambda_r (y_r + S_r) I_{\{\tau > r\}} \Big] = \lambda_{r-1}\, S_{r-1}\, I_{\{\tau > r-1\}}$$

But because we are assuming killings happen independently of everything else, given that the agent has survived to $r-1$ the probability he makes it to the beginning of period r is p_{r-t+j}. Thus the (martingale) relation is

$$\boxed{\lambda_{r-1} S_{r-1} = p_{r-t+j}\; E_{r-1}\big[\lambda_r (y_r + S_r) \big] \qquad \text{if } \tau > r-1}$$

So for agents who entered period t aged j, the stateprice density process is (introducing the label $m \equiv t-j$ to distinguish the SPDs for different age cohorts)

$$\boxed{\zeta_r^{(m)} = \Big(\prod_{i \le r-m} p_i \Big) \lambda_r^{(m)}} \qquad (p_i \equiv 1 \;\; \forall i \le 0)$$

Assuming that in fact the SPD is common to all cohorts, we get $\lambda_r^{(m)} = \zeta_r \big/ \big(\prod_{i \le r-m} p_i\big)$
Accordingly, if we relate the SPD to the consumption, we get

$$w_{r-m} \big(c_r^{(m)}\big)^{-R} = \lambda_r^{(m)} = \zeta_r \Big/ \Big(\prod_{i \le r-m} p_i\Big)$$

assuming the CRRA utility. Hence

$$\boxed{c_r^{(m)} = \zeta_r^{-1/R} \Big(w_{r-m} \prod_{i \le r-m} p_i \Big)^{1/R}}$$

Market clearing:

$$\boxed{y_r = \sum_{k \ge 0} \pi_k\, c_r^{(r-k)} = \zeta_r^{-1/R} \sum_{k \ge 0} \Big(w_k \prod_{i \le k} p_i \Big)^{1/R} \pi_k \equiv A\, \zeta_r^{-1/R}}$$

where the constant A depends on the (p_j) and (w_j). Thus the SPD is $\propto y_t^{-R}$ ...
or is it?

) We could try to jump in one go to a solution where agents of age j hold b_j units of stock, and consume at rate $q_j y_t$, where y_t is the dividend in period t. If we suppose the stock price is $S_t = K y_t$, then we get $(m = t-j)$

$$c_t^{(m)} = q_j y_t = b_j (y_t + S_t) - b_{j+1} S_t$$

so that

$$\boxed{q_j = b_j (1+K) - b_{j+1} K}$$

so from the previous page,

$$w_j \left(c_t^{(m)}\right)^{-R} = \zeta_t / \prod_{i \le j} p_i = \zeta_t \pi_0 / \pi_j$$

so that

$$c_t^{(m)} = (w_j \pi_j)^{1/R} (\pi_0 \zeta_t)^{-1/R} = q_j y_t$$

Market clearing $(\sum q_j \pi_j = 1)$ together with $q_j \propto (w_j \pi_j)^{1/R}$ tells us that

$$\boxed{q_j = A (w_j \pi_j)^{1/R}, \qquad A^{-1} = \sum_{j \ge 0} \pi_j (w_j \pi_j)^{1/R}}$$

So there is no choice for the q_j, and if we set $\beta \equiv K/(1+K)$, we get

$$b_j = (1-\beta) \sum_{\ell \ge 0} \beta^\ell q_{j+\ell}$$

as the only bounded solution, but for this we don't have $\sum b_j = 1$. So this story doesn't work either.

Agents in cohort m have

$$w_{r-m}\, U'(c_r^{(m)}) = \lambda_r^{(m)} \propto Z_r/\pi_{r-m}$$

we get $\quad w_{r-m}\, U'(c_r^{(m)}) = a_m Z_r/\pi_{r-m}$ for some a_m which may be $\mathcal{F}_m$-meas.

ning this round, we have

$$c_r^{(m)} = \left(a_m Z_r / w_{r-m}\pi_{r-m}\right)^{-1/R}$$

d if we suppose that for some constant A we have $\forall m \quad c_m^{(m)} = A\, y_m$ we shall have

$$(a_m Z_m)^{-1/R} = A(w_0\pi_0)^{-1/R}\, y_m$$

he market-clearing condition gives

$$y_r = \sum_{k\geq 0} \pi_k\, c_r^{(r-k)} = \sum_{k\geq 0} \pi_k (w_k\pi_k)^{1/R} \left(a_{r-k} Z_r\right)^{-1/R}$$

$$y_r Z_r^{1/R} = \sum_{k\geq 0} \pi_k (w_k\pi_k)^{1/R} (w_0\pi_0)^{-1/R} A \left(y_{r-k} Z_{r-k}^{1/R}\right)$$

the process $\eta_r \equiv y_r Z_r^{1/R}$ solves the MA relation

$$\eta_r = \sum_{k\geq 0} \pi_k (w_k\pi_k)^{1/R} (w_0\pi_0)^{-1/R} A\, \eta_{r-k}.$$

ne solution would be to take $\eta_r = \beta^{r/R}$ for some $\beta > 0$, and this would lead to

$$\boxed{Z_r = \beta^r\, y_r^{-R}}$$

long with the condition $\quad 1 = A \sum_{k\geq 0} \pi_k (w_k\pi_k)^{1/R} (w_0\pi_0)^{-1/R} \beta^{-k/R}$

However, this still doesn't save us from the problem of explosions...

) It seems that all we can hope to do is to model $\eta_t \equiv \log y_t$ as an AR(1) process:

$$(\eta_{t+1} - \mu) = \rho(\eta_t - \mu) + \varepsilon_{t+1},$$

where (ε_t) are IID $N(0, 1/\tau)$ and put a prior on $(\mu, \tau, \rho) \in \mathbb{R} \times (0,\infty) \times \{\rho_1, \ldots, \rho_n\}$, where the ρ_i are some finite set of values in $(-1,1)$. Let's take a prior density

$$\pi_0(\mu, \tau, \rho_i) \propto g(\tau)\, \tau^{\alpha-1} \exp\{-\tfrac{1}{2}\tau\mu^2 - b\tau\},$$

that after observing $\eta_0, \dots, \eta_t$ the posterior is

$$\pi_t(\mu, \tau, \rho_i) \propto g(\tau)\, \tau^{\alpha - 1 + t/2} \exp\left[-\tfrac{1}{2}\tau\mu^2 - b\tau - \tfrac{1}{2}\tau \sum_{j=1}^{t} \left((\eta_j - \mu) - \rho(\eta_{j-1} - \mu) \right)^2 \right].$$

he quadratic gets maximised when μ is

$$\hat{\mu}_t = \frac{(1-\rho) \sum_{j=1}^{t} (\eta_j - \rho \eta_{j-1})}{1 + t(1-\rho)^2},$$

owing us to re-express the posterior $[K \equiv 1 + t(1-\rho)^2]$

$$\propto g(\tau)\, \tau^{\alpha - 1 + t/2} \exp\left\{ -\tfrac{1}{2} K \tau (\mu - \hat{\mu}_t)^2 - b'\tau \right\}$$

ere

$$b' = \sum_{j=1}^{t} (\eta_j - \rho \eta_{j-1})^2 - \hat{\mu}^2 (1 + t(1-\rho)^2) + b.$$

so we have suppressed the dependence of ρ on i, but this isn't such a big deal to reinstate it.
e get a posterior which is proportional to

$$(K_i \tau)^{\frac{1}{2}} \exp\left[-\tfrac{1}{2}\tau K_i (\mu - \hat{\mu}_t^i)^2 \right] \cdot g(\tau)\, \tau^{\alpha + t/2 - 3/2} \exp(-b_i' \tau) \cdot \frac{1}{\sqrt{K_i}}.$$

o compute the value of the stock we find we need to calculate $E_t \exp\{(1-R)\eta_{t+j}\}$

$$= E_t \exp\left((1-R)\left\{ \rho^j \eta_t + (1-\rho^j)\mu + \sum_{l=0}^{j-1} \rho^l \varepsilon_{t+j-l} \right\} \right)$$

$$= \exp\left((1-R)\rho^j \eta_t\right) E_t \exp\left\{ (1-R)(1-\rho^j)\hat{\mu}_t + \frac{(1-\rho^j)^2 (1-R)^2}{2K\tau} + \frac{1-\rho^{2j}}{1-\rho^2} \frac{(1-R)^2}{2\tau} \right\}$$

nd all of this can be controlled by taking $g(\tau) = \exp(-c/\tau)$ for $c > \frac{1}{2} + \frac{1}{2(1-\rho^2)}$.
But beware; the values of ρ may be very close to 1, so the constant c will have to be enormous.
It seems much better to do the truncation at 0 by $\exp(-c/\tau^2)$, even though the integrals
e not so nice.

But there's a snag; the data shows that the consumption rate q is really growing, so modelling
it by a stationary process has to be wrong!

Interbank contagion (16/9/05)

1) Suppose we consider a single bank, whose assets at time t satisfy

$$dX_t = \sigma\, dW_t + (\mu - \delta(X_t))\,dt,$$

where for simplicity we assume μ, σ and the riskless rate r are all constants, and the rate $\delta(x)$ of withdrawal of dividend is to be chosen by the bank's management to maximise share value. If ever X hits 0, there is a loss of K. Let τ be the default time. Then the value fn

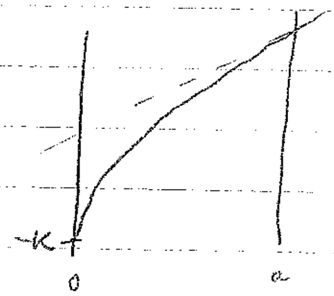
$$V(x) = \sup E\left[\int_0^\tau e^{-rt}\delta(X_t)\,dt - e^{-r\tau}K \;\middle|\; X_0 = x\right]$$

solves

$$\sup\left[\tfrac{1}{2}\sigma^2 V'' + (\mu - \delta)V' - rV + \delta\right] = 0$$

so we must have $V' \geq 1$, and V solves

$$\tfrac{1}{2}\sigma^2 V'' + \mu V' - rV = 0$$



at least while there is no jump. So we get a solution of bang-bang form, $V(x) = A\, e^{\alpha(x-a)} + B\, e^{-\beta(x-a)}$ for some a, where $V'(a) = 1$, $V''(a) = 0$, leading to

$$\left.\begin{array}{l} \alpha A - \beta B = 1 \\ \alpha^2 A + \beta^2 B = 0 \end{array}\right\} \quad \therefore\; A = \frac{\beta}{\alpha(\alpha+\beta)}, \quad B = -\frac{\alpha}{\beta(\alpha+\beta)}$$

and the position of a is chosen to match

$$-K = A\, e^{-\alpha a} + B\, e^{\beta a} = \frac{\beta^2 e^{-\alpha a} - \alpha^2 e^{\beta a}}{\alpha\beta(\alpha+\beta)}$$

2) Now that we understand that, we can decide for a single bank which has a current value of (x, μ) whether a proposed different value (x', μ') would be better. Thus if a single bank hits 0 ($X^i_t = 0$) we can decide whether a rescue package can be constructed, by transferring cash immediately from other banks in return for coupon payments from the distressed bank. However, the above valuation procedure does not take into account the fact that payments from the distressed bank might not all be made...

3) Jose suggests that we should treat this as a central planner problem... suggests also that there may be some benefit in jumping before X hits 0; or in using $\mu + \lambda X$ for the drift. This is a good suggestion. Suppose we now aim to

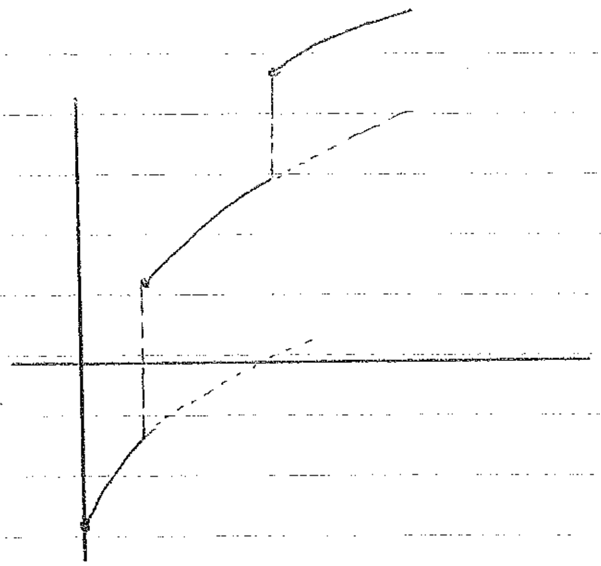
$$\max\; E\left[\sum_i \int_0^{\tau_i} e^{-rt}\,\delta_i(X^i_t)\,dt - e^{-r\tau_i}K_i\right]$$

then we can just add the X^i to form X, and add the $\delta_i(X^i_t)$ to form δ_t

where now

$$dX_t = \sum_i \left(\sigma_{ij} \, dW^j_t + \left(\mu_i - \delta_i(X^i_t)\right) dt \right).$$

We now get down to a 1-dimensional problem where we have to determine an order in which to throw out the individual banks, and a level for each of them.



If $u_1, \dots, u_k$ are IID $N(0, \Sigma)$ where Σ is $p \times p$ positive definite, then

$$S = \sum_{r=1}^{k} u_r u_r^T \sim W_p(k, \Sigma).$$

Wishart $W_p(k, \Sigma)$ has density

$$\propto \exp\left\{-\tfrac{1}{2} \operatorname{tr}(S\Sigma^{-1})\right\} (\det S)^{(k-p-1)/2}$$

Flight to quality? (8/10/05)

1) Suppose we have d assets, and in period t, asset i gives X_t^i, where $X_t \sim N(\mu, \Sigma) \equiv N(\mu, M^{-1})$ are IID. There's a single representative agent in the market with CARA felicity, but the agent is Bayesian. His state-price density is

$$\zeta_t = \beta^t \exp(-\gamma 1 \cdot X_t)$$

and he values the ex-div assets at time t according to

$$S_t = \zeta_t^{-1} E_t\Big[\sum_{j>t} \zeta_j X_j\Big] = \frac{\beta \zeta_t^{-1}}{1-\beta} E_t\Big[e^{-\gamma 1 \cdot \mu + \frac{1}{2}\gamma^2 1 \cdot \Sigma 1}(\mu - \gamma\Sigma 1)\Big]$$

by simple calculations. We now need to find the posterior law of (μ, M) given $(X_r ; r \le t)$.

2) Suppose we start with a Normal-Wishart prior for (μ, M)

$$\pi_0(\mu, M) = (\det M)^{\alpha} \exp\{-\tfrac{1}{2} tr(M V_0) - \tfrac{a}{2}\mu^T M \mu\} (\det M)^{+1/2}$$

and observe some X's. Then the posterior is

$$\pi_t(\mu, M) \propto \pi_0(\mu, M) \exp\Big\{-\tfrac{1}{2}\sum_{r=1}^{t}(X_r - \mu)\cdot M(X_r - \mu)\Big\} (\det M)^{t/2}$$

$$= \pi_0(\mu, M) \exp\Big\{-\tfrac{1}{2}\sum_{r=1}^{t}(X_r - \bar X)\cdot M(X_r - \bar X) - \tfrac{1}{2}t(\bar X - \mu)\cdot M(\bar X - \mu)\Big\}(\det M)^{t/2}$$

$$= \pi_0(\mu, M)\exp\Big\{-\tfrac{1}{2} tr(M S_{XX}(t)) - \tfrac{1}{2}t(\mu - \bar X)\cdot M(\mu - \bar X)\Big\}(\det M)^{t/2}$$

$$S_{XX}(t) \equiv \sum_{r=1}^{t}(X_r - \bar X)(X_r - \bar X)^T$$

$$\propto \exp\Big[-\tfrac{a}{2}\mu^T M \mu - \tfrac{1}{2}t(\mu - \bar X)\cdot M(\mu - \bar X) - \tfrac{1}{2} tr\big(M(V_0 + S_{XX}(t))\big)\Big](\det M)^{\alpha + (1+t)/2}$$

$$= \exp\Big[-\tfrac{1}{2}(a+t)\big(\mu - \tfrac{t}{a+t}\bar X\big)\cdot M\big(\mu - \tfrac{t}{a+t}\bar X\big) - \tfrac{a}{2}\bar X\cdot M\bar X + \tfrac{1}{2}\tfrac{a^2}{a+t}\bar X\cdot M\bar X - \tfrac{1}{2} tr\big(M(V_0 + S_{XX}(t))\big)\Big](\det M)^{\alpha + (1+t)/2}$$

$$= \exp\Big[-\tfrac{1}{2}(a+t)\big(\mu - \tfrac{t}{a+t}\bar X\big)\cdot M\big(\mu - \tfrac{t}{a+t}\bar X\big) - \tfrac{at}{2(a+t)}\bar X\cdot M\bar X - \tfrac{1}{2} tr\big(M(V_0 + S_{XX}(t))\big)\Big](\det M)^{\alpha + (1+t)/2}$$

$$\propto \exp\Big[-\tfrac{1}{2}(a+t)\big(\mu - \tfrac{t}{a+t}\bar X\big)\cdot M\big(\mu - \tfrac{t}{a+t}\bar X\big)\Big](\det M)^{\frac{1}{2}} \exp\Big[-\tfrac{1}{2} tr\Big\{M\big(V_0 + S_{XX}(t) + \tfrac{at}{a+t}\bar X \bar X^T\big)\Big\}\Big].$$
$$(\det M)^{\alpha + t/2}$$

so once again a Wishart-Normal law.

3) There's a very slick result in "Linear Statistical Inference and its Applications", C.R. Rao, who shows that if $M \sim W_p(k, \Sigma)$ [that is, we can represent

$M = \sum_{r=1}^{k} U_r U_r^T$, $U_r \sim N(0, \Sigma)$ where Σ is $p \times p$] and if $M \equiv S^{-1}$, then for any nonzero w we have

$$\frac{1}{w \cdot M^{-1} w} \sim \frac{1}{w \cdot \Sigma^{-1} w} \chi^2_{k-p+1}$$

(see (8b. 2.11), page 541).

So let's suppose that the prior for M is $W_d(k_0, V_0^{-1})$ $(k_0 - d - 1 \equiv 2\alpha)$, so that the posterior at time t is $W_d(k_0 + t, V_t^{-1})$,

$$V_t = V_0 + S_{XX}(t) + \frac{at}{a+t} \bar{X} \bar{X}^T.$$

4) We want to calculate

$$E_t \, e^{v \cdot X_t} = e^{v \cdot \mu_t} \, E_t \left[\exp\left\{ \tfrac{1}{2} \frac{a+t+1}{a+t} v \cdot M^{-1} v \right\} \right] \qquad \left(\mu_t \equiv \tfrac{t}{a+t} \bar{X} \right)$$

and then differentiate it so as to compute expectations. However, in view of the above-quoted result, this integral will explode, so we will need to stick some convergence factor, such as

$$\exp\{ - A \, tr(M^{-1})^2 \}$$

into the prior to guarantee that the integrals do converge. At this point, I think we are not going to be able to do much other than Monte Carlo. We have

$$E_t \left[X_t e^{v \cdot X_t} \right] = E_t \left[\left(\mu_t + \frac{a+t+1}{a+t} M^{-1} v \right) \exp\left\{ v \cdot \mu_t + \tfrac{1}{2} \frac{a+t+1}{a+t} v \cdot M^{-1} v \right\} \right]$$

and we need to evaluate that at $v = -\gamma 1$, with suitable prefactor.

5) <u>But somehow this can't be a good model</u>, because even assuming the posterior had pretty well stabilised, the statepriece density will still be fluctuating quite a lot, and so the price will fluctuate a lot, as will single-period interest rates...

So perhaps better is to let $X_t - X_{t-1} \equiv \xi_t$ be IID $N(\mu, M^{-1})$, and follow that through. Again, $\zeta_r = \beta^r \exp(v_0 \cdot X_r)$, where $v_0 = -\gamma 1$. To price the stock, we have to compute

$$\zeta_t^{-1} E_t \left[\sum_{r \geq t} \zeta_r X_r \right] = \zeta_t^{-1} E_t \left[\sum_{r \geq t} \beta^r \exp(v_0 \cdot X_r) X_r \right]$$

Now if we condition on (μ, M) we have to compute in effect $E \sum_{j \geq 0} \beta^j \exp(v_0 \cdot \sum_{r=1}^{j} \xi_r) X_j$

Careful! this isn't the correct analysis for the stock price – there's a term missing. See p.33.

$$= D\Big(\sum_{j\geq 0} \beta^j \exp(v\cdot \sum_{i=1}^{j} \xi_i)\Big)_{v=v_0} = D\Big(1/(1-\beta\exp(v\cdot\mu+\tfrac{1}{2}v\cdot M^{-1}v))\Big)_{v=v_0}$$

$$= (\mu + M^{-1}v_0)\,\frac{\beta e^{\mu\cdot v_0 + \frac{1}{2}v_0 M^{-1}v_0}}{(1-\beta\exp(\mu\cdot v_0+\frac{1}{2}v_0\cdot M^{-1}v_0))^2}$$

so this suggests we ought to use a prefactor

$$\varphi(\mu, M) = \Big[\big(1-\beta\exp\{\mu\cdot v_0 + \tfrac{1}{2}v_0 M^{-1}v_0\}\big)^+\Big]^2$$

to tame the integrals. Now conditional on $\mathcal{Y}_t$ we have $M \sim \text{Wishart}(k_0+t, V_t^{-1})$ and given $\mathcal{Y}_t$ and M, $\mu \sim N(\mu_t, (a+t)^{-1}M^{-1})$ where $V_t = V_0 + S_{\xi\xi}(t) + \frac{at}{a+t}\bar{\xi}\bar{\xi}^T$. We don't expect to be able to do the integral over M, but we ought to be able to do the integrals over μ conditional on M. All of these expectations are of the form of linear combinations of

$$E\Big[e^{w\cdot\mu}\varphi(\mu)\Big] = e^{w\cdot\mu_t + \frac{1}{2}\frac{w\cdot M^{-1}w}{a+t}}\,\tilde{E}\big[\varphi(\mu)\big]$$

where under $\tilde{P}$ we have $\mu \sim N(\mu_t + \frac{M^{-1}w}{a+t}, \; (a+t)^{-1}M^{-1})$, and where φ is the indicator of $\log\beta + \mu\cdot v_0 + \frac{1}{2}v_0 M^{-1}v_0 < 0$. Thus

$$E\Big[e^{w\cdot\mu}\varphi(\mu)\Big] = e^{w\cdot\mu_t + \frac{1}{2}\frac{w\cdot M^{-1}w}{a+t}}\;\Phi\left(\frac{\log\beta + v_0(\mu_t + \frac{M^{-1}w}{a+t}) + \frac{1}{2}v_0 M^{-1}v_0}{\{v_0 M^{-1}v_0/(a+t)\}^{1/2}}\right)$$

$$\equiv F(w; t, M, \mu_t)$$

say. When computing the stock price, we need a ratio, where the numerator is expectation of

$$\beta e^{\frac{1}{2}v_0 M^{-1}v_0}\Big[\nabla F(v_0; t, M, \mu_t) + M^{-1}v_0\, F(v_0; t, M, \mu_t)\Big]$$

and the denominator is expectation of

$$\Big[F(0; t, M, \mu_t) - 2\beta e^{\frac{1}{2}v_0 M^{-1}v_0}F(v_0; t, M, \mu_t) + \beta^2 e^{v_0 M^{-1}v_0}F(2v_0; t, M, \mu_t)\Big].$$

Should be OK to do the simulation over M.

There appears to be an issue here, which is that the extremely unlikely paths of the RWs which go to big negative values drag the price down. Maybe the thing to do is let a particle filter find good values ... I couldn't yet see an entirely convincing alternative model.

Many Bayesian agents (26/10/05)

1) Suppose we have J agents, each of whom values a consumption stream by the criterion $E\left[\sum_{t\geq 0} \beta^t U(c_t)\right]$ (same β, U) but have different priors over the parameters of the output process X_t of the single productive asset in the economy.

What this means in effect is that each agent has a likelihood-ratio process (Λ^j_t) relative to some reference probability P^*, and $\Lambda^j_t = \rho_j(\theta)\, L_t(X_{[0,t]};\theta)$ is the structural form, where these are in terms of the agents' different prior densities ρ_j. Agent j therefore seeks to max $E^*\left[\sum \beta^t \Lambda^j_t U(c_t)\right]$ so we expect to find

$$\beta^t \bar{\Lambda}^j_t U'(c^j_t) = \lambda_j \zeta_t \qquad \left[\bar{\Lambda}^j_t = \int \rho_j(\theta) L_t(\theta)\, d\theta\right]$$

as the recipe for the state-price density process. As usual, we then deduce by market clearing

$$X_t = \sum c^j_t = \sum I(\lambda_j \zeta_t / \beta^t \bar{\Lambda}^j_t)$$

and now assuming that U is CRRA, we shall have

$$X_t = \left(\zeta_t/\beta^t\right)^{-1/R} \sum_j \left(\lambda_j/\bar{\Lambda}^j_t\right)^{-1/R},$$

leading to

$$\boxed{\zeta_t = \beta^t X_t^{-R} \left\{\sum (\lambda_j/\bar{\Lambda}^j_t)^{-1/R}\right\}^R.}$$

One consequence is that once we know the (λ_j), we can deduce the consumption streams of individual agents, and in principle the price process too.

The priors will need to have the convergence properties though, which may make it hard to calculate the $\bar{\Lambda}_j$ ... this looks a quite substantial technical obstacle.

Range-based estimation of correlation (2/11/05)

1) Suppose we see the extremes, open and close of two log Brownian assets on a given day – can we use that to estimate correlation? In more detail, if we have $X_t \equiv (X^1_t, X^2_t)$ is a 2-d BM with zero drift and covariance $\begin{pmatrix}1 & \rho\\ \rho & 1\end{pmatrix}$, and let $\bar{X}^i_T \equiv \sup_{0\le t\le T} X^i_t$, $\underline{X}^i_T = \inf_{0\le t\le T} X^i_t$, can we find (say) joint distⁿˢ of some of the variables?

2) A more promising approach might be to try to compute things like $E[\bar{X}^1_T \bar{X}^2_T]$, to see whether these depend linearly on the entries in the covariance matrix (this is what happened in one dimension, and was the basis of the Parkinson and R-S estimators). So suppose we set

$$f(h^1,h^2) \equiv E\left[\int_0^\infty \lambda e^{-\lambda t}\, \bar{X}^1_t \bar{X}^2_t\, dt \;\Big|\; X_0 = 0,\ \bar{X}_0 = h\right]. \qquad (h \ge 0)$$

so that

$$E\left[\int_0^\infty \lambda e^{-\lambda t}\, \bar{X}^1_t \bar{X}^2_t\, dt \;\Big|\; X_0 = x,\ \bar{X}_0 = h\right] \equiv F(x,h)$$

$$= E\left[\int_0^\infty \lambda e^{-\lambda t} (\bar{X}^1_t - x^1 + x^1)(\bar{X}^2_t - x^2 + x^2)\, dt \;\Big|\; X_0 = x,\ \bar{X}_0 = h\right]$$

$$= f(h-x) + x^1 E\left[\int_0^\infty \lambda e^{-\lambda t}(\bar{X}^2_t - x^2) \;\Big|\; X^2_0 = x^2,\ \bar{X}^2_0 = h^2\right] + x^2 E\left[\cdots \;\big|\; X^1_0 = x^1,\ \bar{X}^1_0 = h^1\right]$$
$$+\, x^1 x^2$$

$$= f(h-x) + x^1 \varphi(h^2 - x^2) + x^2 \varphi(h^1 - x^1) + x^1 x^2,$$

where
$$\varphi(b) = E\left[\int_0^\infty \lambda e^{-\lambda t}\, \bar{X}_t\, dt \;\Big|\; \bar{X}_0 = b\right]$$
$$= \int_0^\infty (x \vee b)\, \theta e^{-\theta x}\, dx$$
$$= \int_0^\infty (x \vee b)\; d(-e^{-\theta x})$$
$$= \Big[-(x \vee b) e^{-\theta x}\Big]_0^\infty + \int_b^\infty e^{-\theta x}\, dx$$
$$= b + e^{-\theta b}/\theta .$$

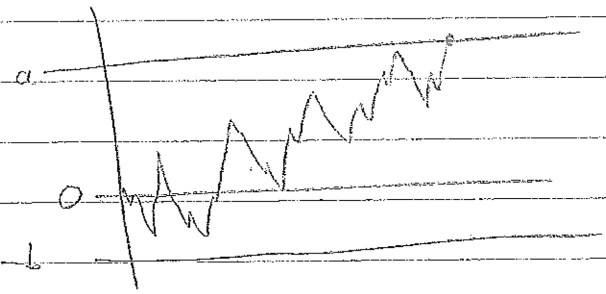
We have the characterising relation

$$\int_0^t \lambda e^{-\lambda s}\, \bar{X}^1_s \bar{X}^2_s\, ds + e^{-\lambda t} F(X_t, \bar{X}_t) \quad \text{is a martingale.}$$

Doing the Itô calculus leads us to

$$0 = \tfrac12 f_{11} + \rho f_{12} + \tfrac12 f_{22} - \lambda f + \lambda x_1 x_2 - \rho + \rho e^{-\theta x_1} + \rho e^{-\theta x_2}$$

with bcs $f_2(x_1, 0) = 0$, $f_1(0, x_2) = 0$. One solution to the inhomogeneous PDE is
$f = f^{(0)}(x_1,x_2) \equiv x_1 x_2 + \theta^{-1}(x_2 e^{-\theta x_1} + x_1 e^{-\theta x_2})$, but it doesn't do the BCs


a
0
b

3) Numerics suggest that $E[\bar{X}^1_T \bar{X}^2_T]$ is linear in ρ !! This would be pretty remarkable, but we can evaluate exactly for $\rho = -1, 0, 1$, and see. For $\rho = 1$, we are seeing $E[\bar{B}^2_T]$ $= E[B^2_T] = E[T] = 1/\lambda$. For $\rho = 0$, we are seeing independent BMs; now $E[\bar{B}_t] = E(|B_t|) = \sqrt{t} \int_0^\infty 2a\, e^{-a^2/2} \frac{da}{\sqrt{2\pi}} = \sqrt{2t/\pi}$, so in the case $\rho = 0$ we shall obtain

$$E[\bar{X}^1_T, \bar{X}^2_T] = \frac{2}{\pi} ET = \frac{2}{\pi} \cdot \frac{1}{\lambda} = 0.6366197722/\lambda .$$

For $\rho = -1$, the calculation is computing $E[\bar{B}_T \cdot \underline{B}_T]$, for which excursion theory seems the best technique? We have for $a, b > 0$

$$P^0[\ H_a < T,\ H_a < H_{-b}\ ] = \frac{\sinh \theta b}{\sinh \theta (a+b)},$$

so $$P^0[\ \underline{B}_T > -b,\ \bar{B}_T > a\ ] = \frac{\sinh \theta b}{\sinh \theta (a+b)} \left(1 - e^{-\theta(a+b)}\right).$$

Hence we get $$P[-\underline{B}_T > b,\ \bar{B}_T > a] = \frac{\sinh \theta b}{\sinh \theta (a+b)} \left(1 - e^{-\theta(a+b)}\right) + e^{-\theta a} = \psi(a,b)$$

and

$$E[-\underline{B}_T\ \bar{B}_T\ ] = \int_0^\infty da \int_0^\infty db\ \psi(a,b)$$

$$= \frac{1}{\lambda} \cdot (2 \log 2 - 1) = \frac{1}{\lambda} \cdot 0.386294361$$

Thus we have $$E[\bar{X}^1_t\ \bar{X}^2_t]/t = \begin{cases} 1 & (\rho = 1) \\ 2/\pi & (\rho = 0) \\ 2\log 2 - 1 & (\rho = -1) \end{cases}$$

which do not lie on a straight line. Nonetheless, they are not too far off collinear, and fitting a quadratic through these values should be a pretty good approximation.

4) The numerical simulations emphasise the need to make the appropriate continuity correction!

5) Again from numerics, we think that $E[\bar{X}^1_T X^2_T]$ is approximately linear in ρ. This time, simple calculations give us

$$E[\bar{X}^1_t X^2_t]/t = \begin{cases} \frac{1}{2} & (\rho = 1) \\ 0 & (\rho = 0) \\ -\frac{1}{2} & (\rho = -1) \end{cases}$$

A moment's thought shows that in this case the dependence is linear.

Some thoughts on contract design (8/11/05)

1) Suppose that a random vector Z of returns on d assets has $N(\mu, I)$ dist^n (more general covariance is irrelevant) and that the principal is going to offer an agent a wage $w(x)$, where x is the realised outcome. Outcome is $\theta \cdot Z$, where θ is the portfolio chosen by the agent. The agent makes effort a, and receives a signal $Y = Z + \varepsilon$, where $\varepsilon \sim N(0, a^{-1} I)$. Conditional on Y, we have

$$Z \sim N\left(\frac{\mu + aY}{1+a}, \frac{1}{1+a} I\right),$$

so the joint density of $(X, Y) \equiv (\theta \cdot Z, Y)$ is

$$\varphi(x, y; \theta, a) = \exp\left[-\frac{1+a}{2}\left(x - \frac{(\mu+ay)\cdot\theta}{1+a}\right)^2 |\theta|^{-2} - \frac{a}{2(1+a)}|y-\mu|^2\right] \frac{\sqrt{1+a}}{|\theta|}\left(\frac{a}{1+a}\right)^{d/2} (2\pi)^{-(d+1)/2}$$

2) Principal wants to pick $w(\cdot)$ so as to

$$\max E\, U(X - w(X)) \tag{1}$$

subject to participation constraint

$$\max_a E\, u(w(X)) - \psi(a) \geq \underline{u} \tag{2}$$

where ψ is convex increasing, the cost of effort, and the portfolio $\theta = \theta(Y)$ is chosen so as to

$$\max_\theta E\left[u(w(X)) \mid Y=y\right] \tag{3}$$

for every y. Now if we assume that $w(\cdot)$ is increasing, we see by considering φ as a function of θ, with $|\theta|$ fixed, the best thing is to align θ with $(\mu + ay)$. So the agent's optimal portfolio is $\boxed{\theta = h(y)(\mu+ay)}$ for some function $h(\cdot)$ which we have to discover.

The Lagrangian form of the problem is therefore

$$\max \iint \left\{ U(x-w(x)) + \gamma u(w(x)) \frac{\varphi_a}{\varphi} + \lambda(y) \frac{\varphi_h}{\varphi} u(w(x)) \right\} \varphi \, dx\, dy - \gamma \psi(a)$$

The optimisation over $w(x)$ gives us for each x

$$\boxed{U'(x-w(x)) \int \varphi\, dy = u'(w(x)) \int \{\gamma \varphi_a + \lambda(y)\varphi_h\}\, dy}$$

Optimising over a leads to

$$\boxed{\iint \left\{U(x-w(x))\varphi_a + u(w(x))\left(\gamma\varphi_{aa} + \lambda(y)\varphi_{ah}\right)\right\} dx\, dy = \gamma\psi'(a)}$$

and over $h(y)$ leads to

$$\boxed{\int \left\{U(x-w(x))\varphi_h + u(w(x))\left(\gamma\varphi_{ah} + \lambda(y)\varphi_{hh}\right)\right\} dx = 0 \quad \forall y}$$

See also Section 4.1.4 of Stole's notes, which takes example of CARA agent and risk neutral principal.

3) This is a bit clumsy in general, but if we suppose $\boxed{\mu=0}$ some of the expressions are a bit simpler:

$$\frac{\varphi_a}{\varphi} = -\frac{1}{2}|y|^2 - \frac{a+2-d}{2a(1+a)} + \frac{(a+2)x^2}{2a^3 h^2 |y|^2}$$

$$\frac{\varphi_h}{\varphi} = -\frac{1}{h} - \frac{x}{h^2} + \frac{(1+a)x^2}{a^2 h^3 |y|^2}$$

Notice that $h^{-1} x = a\ y\cdot(y-\varepsilon) = a\ y\cdot z$

FTQ again (15/11/05)

1) Let's return to the RW situation of Section (5), p 26, because the derivation of stockprice there was not correct. To price the cum-div stock at time t, we have to calculate

$$E_t\left[\sum_{r\geq t} \zeta_r X_r / \zeta_t\right] = E_t\left[\sum_{r\geq t} \beta^r e^{v_0\cdot X_r} X_r\right] / \zeta_t$$

$$= \frac{1}{\zeta_t} E_t\left[\sum_{r\geq t} \beta^r e^{(r-t)(v_0\cdot\mu+\frac{1}{2}v_0\cdot M^{-1}v_0)} X_r\, e^{v_0\cdot X_t}\right]$$

so we need to compute the gradient at $v=v_0$:

$$D\left(E_t\left[\sum_{r\geq t}\beta^r e^{v\cdot X_r}\right]\right)_{v=v_0} = D\left(\beta^t e^{v\cdot X_t} E_t\left(\frac{1}{1-\beta e^{v\cdot\mu+\frac{1}{2}v\cdot M^{-1}v}}\right)\right)_{v=v_0}$$

$$= \zeta_t\left\{X_t\, E_t\left(\frac{1}{1-\beta e^{v_0\cdot\mu+\frac{1}{2}v_0\cdot M^{-1}v_0}}\right) + E_t\, \frac{\beta e^{v_0\cdot\mu+\frac{1}{2}v_0\cdot M^{-1}v_0}(\mu+M^{-1}v_0)}{(1-\beta e^{v_0\cdot\mu+\frac{1}{2}v_0\cdot M^{-1}v_0})^2}\right\}.$$

So as before we take prefactor

$$\varphi_0(\mu,M) = \left\{\left(1-\beta e^{v_0\cdot\mu+\frac{1}{2}v_0\cdot M^{-1}v_0}\right)^+\right\}^2,$$

and we use the same function F as before. The cum-div stock price at time t is a ratio, where the numerator is

$$\boxed{\begin{aligned} &X_t\left\{F(0;t,M,\mu_t) - \beta e^{\frac{1}{2}v_0\cdot M^{-1}v_0} F(v_0;t,M,\mu_t)\right\}\\ &\quad + \beta e^{\frac{1}{2}v_0\cdot M^{-1}v_0}\left\{\nabla F(v_0;t,M,\mu_t) + M^{-1}v_0\, F(v_0;t,M,\mu_t)\right\}\end{aligned}}$$

and the denominator is (as before)

$$\boxed{F(0;t,M,\mu_t) - 2\beta e^{\frac{1}{2}v_0\cdot M^{-1}v_0} F(v_0;t,M,\mu_t) + \beta^2 e^{v_0 M^{-1}v_0} F(2v_0;t,M,\mu_t)}.$$

2) Maybe things get easier if we suppose that M is <u>known</u> and that our prior for μ is $N(0, K^{-1}M^{-1})$. Then the posterior for μ is $N\left(\frac{t\bar{\xi}(t)}{K+t}, ((K+t)M)^{-1}\right)$. The calculations above still stand, it's just that we don't need to mix over M, merely use its actual value!

Importance Sampling in particle filtering (30/11/05)

1) In particle filtering, if $p(x,x')$ is transition density, $f(\cdot|x')$ is density of observation given state x', then the simplest thing is just to take transitions of the particles according to p, then reweight by $f(y|\cdot)$. However if the observation y is the true value x' plus small noise, this is not such a good idea, because we will be generating x' value which are extremely unlikely given y. So perhaps better is to generate x' according to $q(x,x',y)$ and reweight according to $p(x,x')f(y|x')/q(x,x',y)$

2) It makes sense to try

$$q(x,x',y) \propto p(x,x')\, f(y|x')^{\theta}$$

for some $\theta \in [0,1]$. The case $\theta=0$ is the basic case; the case $\theta=1$ corresponds to picking x' according the law of $X_{t+1}|Y_{t+1}$. This seems in practice to lead to the survival of too many particles, so probably allowing $\theta \in (0,1)$ is beneficial.

3) How does this look when

$$X_{t+1}|y_t \sim N(\mu_X, V_X), \quad Y_{t+1}|X_{t+1} = N(X_{t+1}, V_Y)\,?$$

We shall have

$$p(x,x')f(y|x')^{\theta} \propto \exp\left[-\tfrac{1}{2}(x'-\xi)\bar{V}^{-1}(x'-\xi)\right](\det\bar{V})^{-\frac{1}{2}}\left\{\frac{\det\bar{V}}{\det V_X \det V_Y^{\theta}}\right\}^{\frac{1}{2}}\exp\left[-\tfrac{\theta}{2}(\mu_X-y)\tilde{V}^{-1}(\mu_X-y)\right]$$

where $\bar{V}^{-1} \equiv V_X^{-1}+\theta V_Y^{-1}$, $\tilde{V} = V_Y+\theta V_X$, $\xi = \bar{V}(V_X^{-1}\mu_X + \theta V_Y^{-1}y)$;

$$\equiv q(x,x',y)\left\{\frac{\det\bar{V}}{\det V_X \det V_Y^{\theta}}\right\}^{\frac{1}{2}}\exp\left\{-\tfrac{\theta}{2}(\mu_X-y)\tilde{V}^{-1}(\mu_X-y)\right\}$$

4) The reweighting factor (up to a power of 2π) will be

$$\frac{p(x,x')f(y|x')}{q(x,x',y)} \propto f(y|x)^{1-\theta}\left\{\frac{\det\bar{V}}{\det V_X \det V_Y^{\theta}}\right\}^{\frac{1}{2}}\exp\left\{-\tfrac{\theta}{2}(\mu_X-y)\tilde{V}^{-1}(\mu_X-y)\right\}$$

$$\propto \exp\left[-\tfrac{1-\theta}{2}(y-x')V_Y^{-1}(y-x') - \tfrac{\theta}{2}(\mu_X-y)\tilde{V}^{-1}(\mu_X-y)\right]\left\{\frac{\det\bar{V}}{\det V_X \det V_Y}\right.$$

Evidently, $V(w, \bar{w})$ will be decreasing in $\bar{w}$ for fixed w, so

$$(1-R)v(x) - xv'(x) \leq 0, \qquad \text{equal for } x \geq 1$$

Optimisation with draw-down constraints (1/12/05)

1) Suppose we take the conventional wealth dynamics

$$dw_t = r w_t\,dt + \theta_t\{\sigma dW_t + (\mu - r)dt\} - c_t\,dt$$

for an agent who wishes to maximise $E\left[\int_0^\infty e^{-\rho t} U(c_t)\,dt\right]$, where $U'(x) = x^{-R}$, but subject to the constraint that

$$w_t \geq b\,\bar{w}_t \equiv b \sup_{s \leq t} w_s \qquad \forall t,$$

where $b \in (0,1)$ is a fixed constant. What can be done with this?

2) The value function $V(w, \bar{w})$ clearly has homogeneity, so we can write

$$V(w,\bar{w}) = \bar{w}^{1-R}\, V(w/\bar{w}, 1) \equiv \bar{w}^{1-R}\, v(x)$$

where $x = w/\bar{w}$. We expect that $v(b) > U(0)$, but that $v(x)$ is $-\infty$ (if defined) for $x < b$; if we had reached some stage where $w/\bar{w} = b$, we would just come out of stock, consume modestly from the interest on wealth, and get clear of the constraint.

We shall have the HJB

$$\sup_{c,\theta}\left[U(c) - \rho V + \tfrac{1}{2}\sigma^2\theta^2 V_{ww} + (rw + \theta(\mu - r))V_w - c V_w\right] = 0$$

with $\dfrac{\partial V}{\partial \bar{w}} = 0$ when $w = \bar{w}$. Re-expressing this in terms of v, we shall have

$$\sup_{c,\theta}\left[U(c) - \rho \bar{w}^{1-R} v + \tfrac{1}{2}\sigma^2\theta^2\,\bar{w}^{-1-R} v'' + (rw + \theta(\mu-r))\,\bar{w}^{-R} v' - c\,\bar{w}^{-R} v'\right] = 0$$

$$(1-R)\,v(x) - x v'(x) = 0 \qquad \text{at } x = 1$$

or more simply

$$\boxed{\begin{gathered} \tilde{U}(v') - \rho v + r x v' - \frac{(\mu - r)^2}{2\sigma^2}\,\frac{(v')^2}{v''} = 0, \\ (1-R)\,v(1) = v'(1) \end{gathered}}$$

3) Do the dual variables trick; $z = v'(x)$, $J(z) = v(x) - xz$, $J_z = -x$, $J_{zz} = -1/v''$ to give the dual equations:

$$\boxed{\begin{gathered} \tilde{U}(z) - \rho J + (\rho - r) z J_z + \tfrac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 z^2 J_{zz} = 0 \\ -(1 - \tfrac{1}{R})\,J(z) + z J'(z) \leq 0, \text{ equal when } J'(z) \leq -1 \end{gathered}}$$

Condition for problem to be well posed is exactly that

$$Q(1-R') < 0$$

which is equivalent to $-\alpha < 1 - R'$

How is this solved? The ODE has a solution of the form

$$J(z) = Az^{-\alpha} + Bz^{\beta} + Kz^{1-R'} \qquad (*)$$

where $-\alpha, \beta$ are the roots of

$$Q(t) \equiv \tfrac{1}{2}\left(\tfrac{\mu-r}{\sigma}\right)^2 t(t-1) + (\rho-r)t - \rho = 0$$

and

$$K^{-1} = (1-R')\,Q(1-R') \qquad \text{with } R' \equiv 1/R.$$

The constants A, B are to be determined.

Now we can't have $J'(z) \to -b$ $(z \to \infty)$ if J is of the form $(*)$ all the way out to $z = \infty$, so what we have to have is that $J'(z) = -b$ for all $z \geq z_1$, where we have to determine z_1. Also, we expect that as $x \downarrow b$, the optimal investment

$$\theta^* = -\frac{\mu-r}{\sigma^2}\,\frac{v'}{v''}$$

will go to zero. So this leads to the conclusion that $v'(b+)$ is finite, $v''(b+) = -\infty$. So what must happen at z_1 is that J_z joins on to the value $-b$ with $J_{zz} = 0$. So the way to solve is as follows. Propose a value z_1, then solve

$$\left.\begin{aligned} J'(z_1) &= -b \\ J''(z_1) &= 0 \end{aligned}\right\}$$

for A, B. Having found these, we find the value of z_0 at which $J'(z_0) = -1$, and then vary z_1 until we've got

$$(1-\tfrac{1}{R})J(z_0) = z_0 J'(z_0) = -z_0.$$

Can only do this numerically, I guess, but should be OK.

4) But of course it's never quite as easy as you'd hope! The dominant term in J for z near zero is the term in $z^{-\alpha}$. To the left of z_1, we want J' to fall from $-b$ at z_1 to -1, and a sufficient condition for this will be that $A = A(z_1) > 0$.

5) Another way to look at this is to extend the definition to $w > \bar{w}$ by setting $V(w,\bar{w}) = V(w,w) = w^{1-R}v(1)$ for $w > \bar{w}$. This means that $v(x) = x^{1-R}v(1)$ for $x \geq 1$, and the condition at $x=1$ corresponds to v is C^1, whence J is C^1 at the low changeover, as well as the high changeover.

[Looks like this is done in Cvitanic + Karatzas "On portfolio optimisation under drawdown constraints" IMA Volumes on Math + its Applications 65 35–46 (2005)]

Trying to understand moves of nominal prices (19/12/05)

1) Suppose there are N countries, each one producing an output each period. A representative agent (central planner) enters period t holding θ_t^i units of the country-i asset, and ψ_t^i of bonds denominated in currency i, where both (θ_t^i) and (ψ_t^i) are previsible processes. Then the outputs δ_t^i, $i=1,\dots,N$, are revealed, generating $\mathcal{F}_t$, and the bonds mature, giving $\psi_t^i(1+r_t^i)$, where the rates of interest (r_t^i) are again previsible processes.

Suppose η_t^i is the amount of the consumption good bought by one unit of currency t (after dividends are revealed). Next there is trading of currencies; the agent selects $\tilde\psi_t^i$ units of currency i, subject to the budget constraint

$$\sum_i \eta_t^i \tilde\psi_t^i = \sum_i \eta_t^i \psi_t^i(1+r_t^i) + \sum_i \theta_t^i \delta_t^i - c_t ,$$

where $c = \sum_i c_t^i$ is the total consumption, and c_t^i is the amount consumed in period t by country i. We'll suppose (slightly restricting generality) that the central planner's objective is

$$\max E\left[\sum_{t\geq 0}\sum_{i=1}^N \beta_i^t U_i(c_t^i)\right].$$

After currency trading, the values of θ_{t+1}^i and ψ_{t+1}^i are chosen subject to

$$(\theta_{t+1}^i - \theta_t^i)S_t^i = \eta_t^i(\tilde\psi_t^i - \psi_{t+1}^i)$$

where S_t^i is the (ex-dividend) price of the stock in country i, measured in units of the single good. Notice what this is saying; you can only buy/sell asset i with the currency of country i. There is no cost associated with moving consumption from one country to another. (Though this could be incorporated)

2) The Lagrangian form of the problem is

$$\max E\Big[\sum_{t\geq 0}\sum_i \beta_i^t U_i(c_t^i) + \sum_{t\geq 0}\lambda_t\Big\{\sum \eta_t^i(\psi_t^i(1+r_t^i)-\tilde\psi_t^i) + \sum_i \theta_t^i\delta_t^i - c_t\Big\}$$

$$+\sum_{t\geq 0}\sum_i \mu_t^i\Big\{(\tilde\psi_t^i-\psi_{t+1}^i)\eta_t^i - (\theta_{t+1}^i-\theta_t^i)S_t^i\Big\}\Big]$$

$$=\max E\Big[\sum_{t\geq 0}\sum_i(\beta_i^t U_i(c_t^i)-\lambda_t c_t^i) + \sum_{t\geq 1}\sum_i \theta_t^i\{\lambda_t\delta_t^i + \mu_t^i S_t^i - \mu_{t-1}^i S_{t-1}^i\} + \sum_i \theta_0^i(\lambda_0\delta_0^i + \mu_0^i S_0^i)$$

$$+\sum_{t\geq 0}\sum_i \tilde{\psi}_t^i \eta_t^i(\mu_t^i-\lambda_t)+\sum_{t\geq 1}\sum_i \psi_t^i\left\{\lambda_t\eta_t^i(1+r_t^i)-\mu_{t-1}^i\eta_{t-1}^i\right\}+\psi_0^i(\lambda_0\eta_0^i-\mu_0^i\eta_0^i)\Bigg]$$

so the optimisation gives us

$$\boxed{\begin{aligned} \beta_i^t U_i'(c_t^i) &= \lambda_t \\ \mu_t^i &= \lambda_t \\ \lambda_{t-1} S_{t-1}^i &= E_{t-1}\left[\lambda_t(\delta_t^i+S_t^i)\right] \\ \lambda_{t-1}\eta_{t-1}^i &= E_{t-1}\left[\lambda_t\, \eta_t^i\,(1+r_t^i)\right] \end{aligned}}$$

Market clearing says that $\sum c_t^i = \sum \delta_t^i$, so if we take the processes (δ_t^i) as the given inputs, the pricing of stock and the equilibrium consumption allocations are exactly how they would be without any notion of cash – whether this is good or bad!!

3) How do we get a hold of the monetary entities ψ, η? Notice the individual countries' budget constraints;

$$\delta_t^i + \eta_t^i\tilde{\psi}_t^i = c_t^i + (1+r_t^i)\eta_t^i\psi_t^i.$$

In equilibrium, $\theta_t^i \equiv 1$ $\forall i$, $\forall t$, so we have that $\tilde{\psi}_t^i = \psi_{t+1}^i$, and now we re-express the national budget as

$$\eta_t^i\left\{(1+r_t^i)\psi_t^i - \psi_{t+1}^i\right\} = \delta_t^i - c_t^i.$$

Introduce the quantities $y_t^i \equiv \eta_t^i\psi_{t+1}^i$, the value (in consumption good) of the bonds issued by country i in period t. Then we have

$$\begin{aligned}\lambda_{t-1}y_{t-1}^i &= \lambda_{t-1}\eta_{t-1}^i\psi_t^i \\ &= E_{t-1}\left[\psi_t^i\lambda_t\eta_t^i(1+r_t^i)\right] \\ &= E_{t-1}\left[\lambda_t(\delta_t^i-c_t^i+y_t^i)\right]\end{aligned}$$

which leads (with an obvious transversality condition) to

$$\boxed{\lambda_{t-1}y_{t-1}^i = E_{t-1}\left[\sum_{s\geq t}\lambda_s(\delta_s^i-c_s^i)\right]}$$

The interpretation of this is clear! Now the national budget constraint reads alternatively as

$$\delta_t^i - c_t^i + y_t^i = (1+r_t^i)\eta_t^i \psi_t^i$$

and this is fixed by the equilibrium, as is $y_{t-1}^i = \eta_{t-1}^i \psi_t^i$ and $y_t^i = \eta_t^i \psi_{t+1}^i$.
Hence we have no choice about

$$\left.\begin{aligned} \frac{\delta_t^i - c_t^i + y_t^i}{y_{t-1}^i} &= \frac{(1+r_t^i)\,\eta_t^i}{\eta_{t-1}^i} \\ \frac{\delta_t^i - c_t^i + y_t^i}{y_t^i} &= \frac{(1+r_t^i)\,\psi_t^i}{\psi_{t+1}^i} \end{aligned}\right\}$$

If we selected some previsible processes (r_t^i), then everything would be determined by the recipes

$$\boxed{\frac{\psi_{t+1}^i}{\psi_0^i} = \prod_{n=0}^{t} \frac{(1+r_n^i)\,y_n^i}{\delta_n^i - c_n^i + y_n^i}}$$

but we cannot in general select a previsible process ψ and expect that the interest rate process corresponding will turn out previsible...

FTQ once again (7/2/06)

(i) We have $X_t - X_{t-1} = \xi_t$ are IID $N(\mu, V)$, and the SPD process is $\zeta_t \equiv \beta^t \exp(v \cdot X_t)$, where $v = -\gamma 1$. The cum-dividend stock price at time t is

$$E_t\Big[\sum_{j\geq t} \zeta_j X_j/\zeta_t\Big] = E_t\Big[\sum_{j\geq t} \beta^{j-t}\{X_t + (X_j - X_t)\} \exp\{v\cdot(X_j - X_t)\}\Big]$$

$$= E_t\Big[E\Big(\sum_{j\geq t} \beta^{j-t}\{X_t + (X_j - X_t)\} \exp(v\cdot(X_j - X_t))\Big)\Big|\mu, V\Big)\Big]$$

Now if $Z \sim N(a, V)$, then $E\big(Ze^{v\cdot Z}\big) = (a + Vv)\exp(a\cdot v + \frac{1}{2}v\cdot Vv)$, so we have the cum-dividend stock price is

$$E_t\Big[\sum_{j\geq t} \beta^{j-t}\{X_t + (j-t)(\mu + Vv)\} \exp(\mu\cdot v + \tfrac{1}{2}v\cdot Vv)(j-t)\Big]$$

$$= E_t\left[\frac{X_t}{1 - \beta\exp(\mu\cdot v + \frac{1}{2}v\cdot Vv)} + \frac{\beta\exp\{\mu v + \frac{1}{2}vVv\}\,(\mu + Vv)}{(1 - \beta\exp\{\mu v + \frac{1}{2}v\cdot Vv\})^2}\right]$$

(ii) Let's now suppose that we know V with reasonable certainty: $V = \tau^{-1}\Xi$, where Ξ is known, but τ has a Gamma prior, $\Gamma(\alpha_0, b_0)$, and that $\mu \sim N(0, (\tau\kappa_0)^{-1}\Xi)$ given τ. We'll also suppose the prior density has a prefactor $\varphi(\mu,\tau)$, where

$$\varphi(\mu,\tau) = \{(1 - \beta\exp(\mu\cdot v + \tfrac{1}{2}v\cdot\tau^{-1}\Xi v))^+\}^2 \exp\{-\theta/2\tau^2\}$$

Doing the usual prior/posterior analysis, the posterior given t observations will have density

$$\propto \tau^{\alpha_t - 1}\exp\big[-b_t\tau - \tfrac{1}{2}\tau(\mu - \mu_t)\cdot M(\mu - \mu_t)(t + \kappa_0)\big]\,\tau^{n/2}\,\varphi(\mu,\tau)$$

if we assume $M = \Xi^{-1}$, and where

$$\boxed{\begin{aligned} \mu_t &= t\bar{\xi}_t/(t+\kappa_0) \\ \alpha_t &= \alpha_0 + \tfrac{1}{2}nt \\ b_t &= b_0 + \tfrac{1}{2}S_{\xi\xi} + \frac{t\kappa_0}{2(t+\kappa_0)}\bar{\xi}_t\cdot M\bar{\xi}_t \\ S_{\xi\xi} &= \sum_1 (\xi_i - \bar{\xi}_t)\cdot M(\xi_i - \bar{\xi}_t) \end{aligned}}$$

(iii) Given observations to time t, and τ, the law of μ is $N(\mu_t, \tau^{-1}(t+\kappa_0)^{-1}M^{-1})$, together with the prefactor φ. The mean of μ given $v\cdot\mu$ is

$$E_t(\mu \mid v\cdot\mu) = \mu_t + \eta\,(v\cdot\mu - \mu_t), \qquad \eta \equiv \Sigma v / v\cdot\Sigma v \equiv M^{-1}v / v\cdot M^{-1}v.$$

We now find ourselves having to calculate things like

(a) $$P_t\left(\beta \exp(\mu\cdot v + \tfrac12 v\cdot \tau^{-1}\Sigma v) < 1 \right),$$

(b) $$E_t\left[\exp(\mu\cdot v + \tfrac12 v\cdot\tau^{-1}\Sigma v) \;;\; \beta\exp(\mu\cdot v + \tfrac12 v\cdot\tau^{-1}\Sigma v) < 1 \right],$$

(c) $$E_t\left[\mu\cdot v \, \exp(\mu\cdot v + \tfrac12 v\cdot\tau^{-1}\Sigma v) \;;\; \beta\exp(\mu\cdot v + \tfrac12 v\cdot\tau^{-1}\Sigma v) < 1 \right]$$

Let's write Z for $\mu\cdot v$; the law of Z conditional on $\mathcal{Y}_t$ and τ is $N\left(v\cdot\mu_t, \frac{v\cdot\Sigma v}{(t+K_0)\tau} \right)$ $\equiv N(a_t, c_t/\tau)$, say. In terms of this, we get

$$\left[\begin{array}{l} a_t \equiv v\cdot\mu_t \\ c_t \equiv v\cdot\Sigma v/(t+K_0) \end{array}\right]$$

(a) $$P_t\left[\beta\exp(\mu\cdot v + \tfrac12 v\cdot\tau^{-1}\Sigma v) < 1 \right]$$

$$\propto \int_0^\infty e^{-c/2\tau^2}\, \Phi\left(\frac{\log\beta + \frac12 v\cdot\Sigma v/\tau + a_t}{\sqrt{c_t/\tau}} \right) d\tau \,\, \tau^{\alpha_t+1} e^{-b_t\tau - c/2\tau^2}$$

For the other two, observe that if $Y \sim N(a, s)$ then

$$E\left[e^{\lambda Y} ; Y < \gamma \right] = \exp\left[\tfrac12\lambda(\lambda s + 2a) \right] \Phi\left(\frac{\gamma - a - s\lambda}{\sqrt{s}} \right)$$

whence

$$E\left[Y e^{\lambda Y} ; Y < \gamma \right] = \exp\left[\tfrac12\lambda(\lambda s + 2a) \right] \left\{ (\lambda s + a)\, \Phi\left(\frac{\gamma - a - s\lambda}{\sqrt{s}} \right) - \sqrt{s}\, \frac{e^{-(\gamma-a-s\lambda)^2/2s}}{\sqrt{2\pi}} \right\}$$

(in fact, even the case (a) can be obtained from this if we use $\lambda = 0$ ...)

The correlation of the maxima of two correlated BMs (17/2/06)

1) Suppose W^1, W^2 are two standard Brownian motions with constant correlation $\rho \in (-1,1)$, started from 0, and let $L^i_t = -\inf\{W^i_s : s \le t\}$ be the negative of the lowest level visited by W^i by time t. What is $E(L^1_t L^2_t)$? By scaling, it is evident that

$$E\left(L^1_t L^2_t\right) = c(\rho)\, t$$

but what is the constant of proportionality?

2) Various transformations of the problem may help. Firstly, let's work with the process

$$X_t = (X^1_t, X^2_t) \equiv \left(W^1_t + L^1_t,\ W^2_t + L^2_t\right)$$

which is a diffusion in $\mathbb{R}^2_+$ with generator

$$\mathcal{G} = \frac{1}{2}\frac{\partial^2}{\partial x_1^2} + \rho \frac{\partial^2}{\partial x_1 \partial x_2} + \frac{1}{2}\frac{\partial^2}{\partial x_2^2}$$

with orthogonal reflection on the axes; the processes L^i are the local times at 0 of the individual coordinates.

Now let's define (for $\lambda \equiv \frac{1}{2}\theta^2 > 0$)

$$f(x_1, x_2; \ell_1, \ell_2) = E\left[\int_0^\infty \lambda e^{-\lambda t} L^1_t L^2_t\, dt \,\Big|\, X^i_0 = x_i,\ L^i_0 = \ell_i\right]$$

$$= E\left[\int_0^\infty \lambda e^{-\lambda t} (L^1_t + \ell_1)(L^2_t + \ell_2)\, dt \,\Big|\, X^i_0 = x_i,\ L^i_0 = 0\right]$$

$$= E\left[\int_0^\infty \lambda e^{-\lambda t} L^1_t L^2_t\, dt \,\Big|\, X^i_0 = x_i\right] + \ell_1 \frac{e^{-\theta x_2}}{\theta} + \ell_2 \frac{e^{-\theta x_1}}{\theta} + \ell_1 \ell_2$$

$$= g(x_1, x_2) + \theta^{-1}\left(\ell_1 e^{-\theta x_2} + \ell_2 e^{-\theta x_1}\right) + \ell_1 \ell_2$$

$$\equiv E\left[L^1_T L^2_T \,\big|\, X^i_0 = x_i,\ L^i_0 = \ell_i\right]$$

where $T \sim \exp(\lambda)$ is independent of W. It's clear that

$$\int_0^t \lambda e^{-\lambda s} L^1_s L^2_s\, ds + e^{-\lambda t} f(X^1_t, X^2_t; L^1_t, L^2_t) \quad \text{is a martingale,}$$

so that

$$\lambda \ell_1 \ell_2 - \lambda f + \mathcal{G} f = 0$$

with boundary conditions

$$\frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial \ell_1} = 0 \text{ at } x_1 = 0, \qquad \frac{\partial f}{\partial x_2} + \frac{\partial f}{\partial \ell_2} = 0 \text{ at } x_2 = 0.$$

Useful facts:

$$\int_{-\infty}^{\infty} \frac{e^{zt}}{\cosh t}\, dt = \frac{\pi}{\cosh(i\pi z/2)} \qquad \text{for } |\operatorname{Re}(z)| < 1$$

$$\int_{-\infty}^{\infty} \frac{\sinh at}{\sinh t}\, e^{zt}\, dt = \frac{\pi e^{i\pi z/2}}{2i} \left\{ \frac{e^{ia\pi/2}}{\cos \pi(a+z)/2} - \frac{e^{-ia\pi/2}}{\cos \pi(z-a)/2} \right\}$$

So in terms of g we get

$$(\lambda - \mathcal{G})\, g = 0, \qquad \frac{\partial g}{\partial x_1} + \frac{1}{\theta} e^{-\theta x_2} = 0 \quad \text{at } x_1 = 0$$

$$\frac{\partial g}{\partial x_2} + \frac{1}{\theta} e^{-\theta x_1} = 0 \qquad \text{at } x_2 = 0.$$

Now clearly

$$g(x_1, x_2) = E\left[L^1_T L^2_T \mid X^i_0 = x_i \right] = E\left[(L^1_T - x_1)^+ (L^2_T - x_2)^+ \mid X^i_0 = 0 \right]$$

is non-negative, decreasing in each component, and bounded above by $\lambda^{-1} c(\rho)$. As $\lambda \to 0$, we get $\lambda g(x_1, x_2) \to c(\rho)$ (Brownian scaling). This suggests we try to define

$$\tilde{g}(x_1, x_2) \equiv \lambda g(x_1/\theta, x_2/\theta)$$

It is now simple to verify that

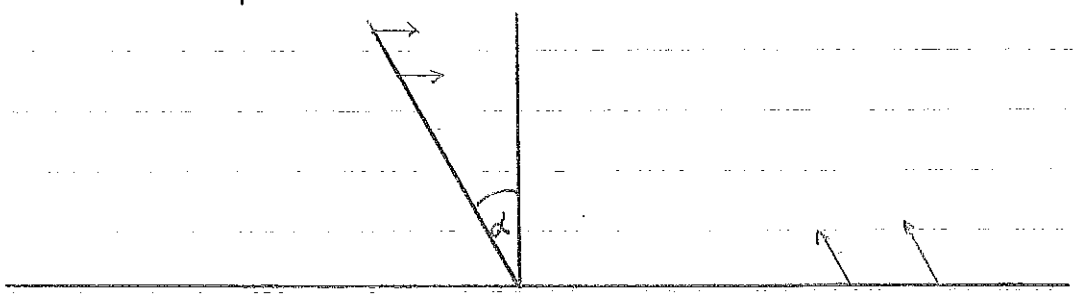
$$(\tfrac{1}{2} - \mathcal{G})\tilde{g} = 0, \quad \frac{\partial \tilde{g}}{\partial x_1} + \tfrac{1}{2} e^{-x_2} = 0 \ (x_1 = 0), \quad \frac{\partial \tilde{g}}{\partial x_2} + \tfrac{1}{2} e^{-x_1} = 0 \ (x_2 = 0),$$

which is in effect the original set of equations with θ fixed at 1.

2) If we define

$$X_t = \sec\alpha . X^1_t - \tan\alpha\, X^2_t, \qquad Y_t = X^2_t \qquad (\rho = \sin\alpha)$$

then we make a linear transformation which takes (X^1, X^2) to a 2-dimensional BM (X, Y) in the wedge with reflections as shown:



$$dX_t = dB^1_t + \sec\alpha\, dL^1_t - \tan\alpha\, dL^2_t$$
$$dY_t = dB^2_t + dL^2_t$$

3) Another observation:

$$d(X^1 X^2) = (W^1 + L^1)\, d(W^2 + dL^2) + (W^2 + L^2)\, d(W^1 + L^1) + \rho\, dt$$

$$\doteq W^1 dL^2 + W^2 dL^1 + d(L^1 L^2) + \rho\, dt$$

Now what can we say of the terms $W^1 dL^2$? This time, we write $W^1 = \rho W^2 + \beta$

where β is indept of W^2 and so

$$E\int_0^t W_s^1\, dL_s^2 = E\int_0^t (\rho W_s^2 + \beta_s)\, dL_s^2$$

$$= E\int_0^t \rho W_s^2\, dL_s^2 = -\frac{\rho}{2} E\left[(L_t^2)^2\right] = -\rho t/2.$$

Hence $\boxed{E\left[X_t^1 X_t^2\right] = E\left[L_t^1 L_t^2\right]}$.

We can now try to solve

$$f(x_1,x_2) = E^{(x_1,x_2)}\left[\int_0^\infty \lambda e^{-\lambda t} X_t^1 X_t^2\, dt\right]$$

which satisfies

$$(\lambda - \mathcal{G}) f = \lambda x_1 x_2, \qquad \frac{\partial f}{\partial x_i} = 0 \text{ at } x_i = 0.$$

One solution to the PDE is

$$f(x_1,x_2) = x_1 x_2 + \rho/\lambda,$$

but it doesn't do the boundary conditions...

Some observations on asymptotics of implied volatility (17/2/06)

1) Assume $r=0$, $T=1$ and $S_0=1$, with strike $K=e^k$. If $f(\cdot)$ is the risk-neutral density of $\log S_1$, then the call prices are

$$C(e^k) = \int_k^\infty (e^x - e^k) f(x)\,dx,$$

and Black-Scholes call prices for vol σ will be

$$C_{BS}(\sigma,k) = \bar{\Phi}\left(\frac{k-\frac{1}{2}\sigma^2}{\sigma}\right) - e^k\,\bar{\Phi}\left(\frac{k+\frac{1}{2}\sigma^2}{\sigma}\right)$$

and we choose $\sigma(k)$ so as to ensure $C_{BS}(\sigma(k),k) = C(e^k)$. It seems to be important to consider the function

$$\boxed{d(k) = \left(k - \tfrac{1}{2}\sigma(k)^2\right)/\sigma(k)}$$

For one thing, we can express the implied vol $\sigma(k)$ in terms of it:

$$\sigma(k) = \sqrt{d^2+2k} - d$$

so that

$$C_{BS}(\sigma,k) = \bar{\Phi}(d) - e^k\,\bar{\Phi}\left(\sqrt{d^2+2k}\right) = \bar{\Phi}(d) - e^k\,\bar{\Phi}(d+\sigma).$$

Another observation is that d is increasing: to see this, differentiate $C(e^k) = C_{BS}(\sigma(k),k)$ with respect to k to obtain

$$0 > \frac{d}{dk}C(e^k) = -d'\,\frac{e^{-d^2/2}}{\sqrt{2\pi}} - e^k\,\bar{\Phi}\left(\sqrt{d^2+2k}\right) + \frac{dd'+1}{\sqrt{d^2+2k}}\,e^k\,\frac{e^{-(d^2+2k)/2}}{\sqrt{2\pi}}$$

$$> \frac{e^{-d^2/2}}{\sqrt{2\pi}}\left[-d' - \frac{1}{\sqrt{d^2+2k}} + \frac{dd'+1}{\sqrt{d^2+2k}}\right]$$

$$= -\frac{e^{-d^2/2}}{\sqrt{2\pi}}\,d'\left\{1 - \frac{d}{\sqrt{d^2+2k}}\right\}$$

which implies $d'>0$. Since $C(e^k)\downarrow 0$, we must also have $d(k)\uparrow\infty$.

2) Let's now do the asymptotics of $C_{BS}(\sigma,k)$ in a more manageable form. Using the elementary inequalities

$$\frac{e^{-x^2/2}}{x+x^{-1}} \le \sqrt{2\pi}\,\bar{\Phi}(x) \le \frac{e^{-x^2/2}}{x} \qquad \forall x>0$$

we can bound $\sqrt{2\pi}\,C_{BS}(\sigma,k)$ above by

$$e^{-d^2/2}\left[\frac{1}{d} - \frac{\sqrt{d^2+2k}}{d^2+2k+1}\right] = e^{-d^2/2}\,\frac{\sqrt{d^2+2k}\left(\sqrt{d^2+2k}-d\right)-1}{d\,(d^2+2k+1)} \le \frac{\sqrt{d^2+2k}-d}{d\,\sqrt{d^2+2k}}$$

The other way, we bound $\sqrt{2\pi}\, C_{BS}(\sigma,k)$ below by

$$e^{-d^2/2}\left[\frac{d}{1+d^2}-\frac{1}{\sqrt{d^2+2k}}\right] = e^{-d^2/2}\,\frac{d\left(\sqrt{d^2+2k}-d\right)-1}{(1+d^2)\sqrt{d^2+2k}} \sim e^{-d^2/2}\,\frac{\left(\sqrt{d^2+2k}-d\right)}{d\,\sqrt{d^2+2k}}$$

provided $\sigma(k)$ doesn't go to zero as $k\to\infty$. Thus given this proviso, we have

$$\boxed{\sqrt{2\pi}\, C_{BS}(\sigma,k) \sim e^{-d^2/2}\,\frac{\sqrt{d^2+2k}-d}{d\,\sqrt{d^2+2k}}}$$

3) Now let's suppose that

$$\boxed{\sqrt{2\pi}\; C(e^k) \sim e^{-\beta k}\, R(k)}$$

where R varies regularly at infinity with exponent α. From this, the function

$$\tilde{R}(k) \equiv \exp\left(\beta k-\tfrac12 d^2\right)\frac{\sqrt{d^2+2k}-d}{d\,\sqrt{d^2+2k}} = e^{\beta k-\frac12 d^2}\,\frac{2k}{d\sqrt{d^2+2k}\left(\sqrt{d^2+2k}+d\right)}$$

varies regularly at infinity with exponent α; so

$$\tilde{R}(k) = c_0(k)\; k^{\alpha}\;\exp\left\{\int_1^k \varepsilon(u)\,du/u\right\}$$

for some function c_0 converging to a positive finite limit, and $\varepsilon\to 0$. (see BGT)
Now if $\limsup \frac12 d(k)^2/k > \beta$, then $\exists\; k_n\to\infty$ s.t.* $\frac12 d(k_n)^2 > (\beta+\eta)\,k_n$ and down this sequence $\tilde{R}(k_n)$ goes to zero exponentially fast ※. Similarly, if $\liminf \frac12 d(k)^2/k < \beta$ we can find $k_n\to\infty$ s.t. $\frac12 d(k_n)^2 < (\beta-\eta)\,k_n$, and down this sequence $\tilde{R}(k_n)$ grows exponentially; therefore

$$\tfrac12 d(k)^2 = \beta k + \varphi(k)$$

where $k^{-1}\varphi(k)\to 0$. Thus we obtain

$$\tilde{R}(k) \sim \frac{1}{\sqrt{k}}\exp\{-\varphi(k)\}\;\frac{2}{\sqrt{2\beta}\;\sqrt{2\beta+2}\;\left\{\sqrt{2\beta+2}+\sqrt{2\beta}\right\}}$$

So $\exp\{-\varphi(k)\}$ varies regularly at infinity with exponent $\alpha+\frac12$

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* $\eta>0$ given

Back to the deterministic stochastic optimal control (26/2/06)

We have in the discrete-time controlled Markov process situation that

$$V_0(x) = \inf_{\{h_j\}} E\Big[\sup_a \sum_{j=0}^{T-1} \Lambda_j(a)\{f(X_j,a) + Ph_{j+1}(X_j,a) - \varphi(X_j,X_{j+1};a)\,h_{j+1}(X_{j+1})\} + \Lambda_T(a)\,F(X_T) \Big]$$

$$= \inf_{(h_j)} E \sup_a \Big\{ h_0(x) + \sum_{j=0}^{T-1} \Lambda_j(a)\{f(X_j,a) + Ph_{j+1}(X_j,a) - h_j(X_j)\} \Big\}$$

$$\leq \inf_{(h_j)} E\Big[h_0(x) + \sum_{j=0}^{T-1} \sup_a \Lambda_j(a)\{f(X_j,a) + Ph_{j+1}(X_j,a) - h_j(X_j)\} \Big].$$

However, taking $h_j = V_j$ and using the Bellman equation we see that in fact the inf is attained there, and

$$V_0(x) = \inf_{(h_j)} \Big[h_0(x) + \sum_{j=0}^{T-1} E \sup_a \Lambda_j(a)\{f(X_j,a) + Ph_{j+1}(X_j,a) - h_j(X_j)\} \Big]$$

This promises to be <u>much</u> simpler to work with...!

Flight to quality again (28/2/06)

(i) The previous attempt on this took the moral high ground of a general equilibrium solution. Here we propose something less lofty, but hopefully easier to do. Take our n assets to be

$$dS^i_t = S^i_t \, \sigma_{ij} \, dX^j_t \qquad \text{where } dX^j_t = dW^j_t + \alpha_j \, dt$$

and suppose that σ is known, but the α vector is not; we'll suppose that α has a prior

$$\alpha \sim N(\alpha_0, M_0^{-1}).$$

Now doing a Bayesian analysis, after observing to time t the vector α has posterior

$$\propto \exp\left\{ -\tfrac{1}{2}(\alpha-\alpha_0) M_0 (\alpha - \alpha_0) + \alpha \cdot X_t - \tfrac{1}{2}|\alpha|^2 t \right\}$$

$$\propto \exp\left\{ -\tfrac{1}{2}(\alpha - \hat{\alpha}_t) M_t (\alpha - \hat{\alpha}_t) \right\}$$

where

$$\hat{\alpha}_t \equiv M_t^{-1}(M_0 \alpha_0 + X_t), \qquad M_t \equiv M_0 + tI$$

In the observation filtration, we have

$$dX_t = d\hat{W}_t + \hat{\alpha}_t \, dt$$

and from this and the expression for $\hat{\alpha}$ we learn that

$$d\hat{\alpha}_t = M_t^{-1} \, d\hat{W}_t$$

(ii) We now must work out what the SPD is for this problem. Set $\underset{\sim}{r} \equiv r\mathbf{1}$, so

$$\zeta_t^{-1} d\zeta_t = -r\,dt + (\sigma^{-1}\underset{\sim}{r} - \hat{\alpha}_t)\, d\hat{W}_t$$

$$= -r\,dt + \sigma^{-1}\underset{\sim}{r}\, d\hat{W}_t - \hat{\alpha}_t M_t \, d\hat{\alpha}_t$$

$$= -r\,dt + \sigma^{-1}\underset{\sim}{r}\, d\hat{W}_t - d\left(\tfrac{1}{2}\hat{\alpha}_t M_t \hat{\alpha}_t\right) + \tfrac{1}{2}|\hat{\alpha}_t|^2 dt + \tfrac{1}{2}\,\mathrm{tr}\left(M_t \, d\hat{\alpha}\, d\hat{\alpha}^T\right)$$

$$= -r\,dt + \sigma^{-1}\underset{\sim}{r}\, d\hat{W}_t - d\left(\tfrac{1}{2}\hat{\alpha}_t M_t \hat{\alpha}_t\right) + \tfrac{1}{2}|\hat{\alpha}_t|^2 dt + \tfrac{1}{2}\,\mathrm{tr}(M_t^{-1})\, dt$$

Thus

$$\log \zeta_t = -rt + \sigma^{-1}\underset{\sim}{r}\, \hat{W}_t - \tfrac{1}{2}\cancel{\int_0^t} \hat{\alpha}_t M_t \hat{\alpha}_t \cancel{ds} + \tfrac{1}{2}\int_0^t |\hat{\alpha}_s|^2 ds + \tfrac{1}{2}\int_0^t \mathrm{tr}\, M_s^{-1}\, ds$$
$$- \tfrac{1}{2}\int_0^t |\sigma^{-1}\underset{\sim}{r} - \hat{\alpha}_s|^2 ds + \tfrac{1}{2}\alpha_0 M_0 \alpha_0$$

$$= -rt + \sigma^{-1}\underset{\sim}{r}\, X_t - \tfrac{1}{2}\hat{\alpha}_t M_t \hat{\alpha}_t - \tfrac{1}{2} t |\sigma^{-1}\underset{\sim}{r}|^2 + \tfrac{1}{2}\log\det M_t - \tfrac{1}{2}\log\det M_0 + \tfrac{1}{2}\alpha_0 M_0 \alpha_0$$

$$= -rt + \tfrac{1}{2}\log\frac{\det M_t}{\det M_0} + \sigma^{-1}\underset{\sim}{r}\,(M_t\hat{\alpha}_t - M_0\alpha_0) - \tfrac{1}{2}\hat{\alpha}_t M_t \hat{\alpha}_t - \tfrac{1}{2} t |\sigma^{-1} r|^2 + \tfrac{1}{2}\alpha_0 M_0 \alpha_0$$

$$= -rt + \tfrac{1}{2}\log\frac{\det M_t}{\det M_0} - \tfrac{1}{2}(\hat{\alpha}_t - \sigma^{-1}\underset{\sim}{r}) M_t (\hat{\alpha}_t - \sigma^{-1} r) + \tfrac{1}{2}(\alpha_0 - \sigma^{-1}\underset{\sim}{r}) M_0 (\alpha_0 - \sigma^{-1} r).$$

(iii) The next step is to find the wealth process

$$w_t = E_t\left[\int_t^\infty \zeta_s\, c_s\, ds\right] / \zeta_t$$

$$\propto \zeta_t^{-1}\, E_t\left[\int_t^\infty e^{-\rho s/R}\, \zeta_s^{1-1/R}\, ds\right]$$

$$\equiv \Psi(t, \hat{z}_t)$$

(which will probably need to be done numerically). Once we know Ψ, we shall be able to calculate

$$dw_t = \nabla\Psi\, d\hat{z} + \dots = \nabla\Psi\, M_t^{-1}\, d\hat{W}_t + \dots = \nabla\Psi\, M_t^{-1}\, dX_t + \dots$$

$$= \nabla\Psi\, M_t^{-1}\, \sigma^{-1}\, (S^{-1} dS)$$

which identifies the portfolio weights (i.e. relative values of the holdings in the different stocks) quite simply as

$$\sigma^{-1}\, M_t^{-1}\, \nabla\Psi$$

Modelling the cashflows of a life insurance business (23/3/06)

1) Suppose we have a stock with MM dynamics

$$dS_t = S_t \{ \sigma(\xi_t)\, dW_t + b(\xi_t)\, dt \}$$

as the risky asset, and riskless rate $r_t = r(\xi_t)$. [We might instead consider MM eg Brownian dividend process like Pepper & I did, but for this application the jumps in stock price could be rather inconvenient...]

At time t, there are N_t policyholders. Each exits the system at rate $\mu(\xi_t)$, taking away Z_t when they do so. Each pays premia at rate $\pi(\xi_t)\,dt$. We'll assume everyone sees ξ. Let's suppose that new policyholders join at rate $f(\pi(\xi_t))\,dt$. If we just consider the insurance account, this has dynamics

$$dy_t = N_t \left(\pi(\xi_t) - Z_t \mu(\xi_t) \right) dt + dM_t$$

where M is a jump martingale, $\langle M \rangle_t = \int_0^t Z_s^2 N_s \mu(\xi_s)\, ds$. Let's straight away approximate this by a Brownian motion, so the dynamics of y become

$$dy_t = N_t \{ \pi(\xi_t) - Z_t \mu(\xi_t) \} dt + Z_t \sqrt{N_t \mu(\xi_t)}\, dW_t'$$

where W' is independent of W. Let's also do a fluid approximation for N_t, so

$$dN_t = \{ -N\mu(\xi_t) + f(\pi(\xi_t)) \} dt .$$

The dynamics of the wealth of the insurance company therefore become

$$dw_t = \theta_t S_t^{-1} dS_t + r_t (w_t - \theta_t) dt + dy_t - c_t\, dt$$

2) Various forms of Z, f etc need to be considered if we are to get further in optimising the objective

$$\max E \left[\int_0^\tau \exp(-\rho t)\, c_t\, dt - e^{-\rho\tau} K \right]$$

where $K >> 0$ is some bankruptcy penalty.

As a first attempt, let's suppose that $Z_t \equiv a$, for simplicity. We then want to find

$$V(w, \xi, N) = \sup E \left[\int_0^\tau e^{-\rho t} c_t\, dt - K e^{-\rho\tau} \right].$$

HJB is

$$\tfrac{1}{2} a^2 N \mu V_{ww} + \{ rw + N(\pi - a\mu) \} V_w - \frac{(b-r)^2}{2\sigma^2 V_{ww}} V_w^2 + QV - \rho V + (f(\pi) - N\mu) V_N = 0$$

Looks pretty tough.

Implied correlation of an index (3/4/06)

(i) Suppose we have N assets with (Black-Scholes) dynamics

$$dS_i(t) = S_i(t)\left[\sigma_{ij}\, dW^j_t + r\, dt\right]$$

and the corresponding index $J_t = \sum_i^N w_i S_i(t)$, where $w_i > 0$, $\sum w_i = 1$. Then the dynamics of J are

$$dJ_t = \sum w_i S_i(t)\{\sigma_{ij}\, dW^j_t + r\, dt\} = \sum w_i S_i \sigma_{ij}\, dW^j + rJ\, dt,$$

motivating the definition of the <u>instantaneous volatility</u> $\sigma_I(t)$ as

(1) $$\boxed{J_t^2\, \sigma_I(t)^2 = \sum w_i S_i(t) v_{ij} w_j S_j(t), \qquad v \equiv \sigma\sigma^T}$$

Then $dJ_t = \{\sigma_I(t)\, dB_t + r\, dt\}\, J_t$,

which is a stochastic volatility description of the index.

(ii) If it comes to pricing an option on the index, we have that $e^{-rt} J_t$ is a martingale, and if we do a log-Brownian approximation, the thing that matters is the conditional variance of J_T given $\mathcal{F}_t$. In more detail, if $X_i(t) \equiv \log S_i(t)/S_i(0)$ $= \sigma_{ij} W^j(t) + (r - \frac{1}{2} v_{ii})t$, we compute $(\tau \equiv T-t)$

$$E_t\left[\left(e^{-r\tau} J_T/J_t\right)^2\right] = J_t^{-2}\, E_t \sum w_i w_j\, S_i(T) S_j(T)\, e^{-2r\tau}$$

$$= J_t^{-2} \sum w_i w_j \exp\left[\tau v_{ij}\right] S_i(t) S_j(t)$$

to be compared with a simple B-S asset, where $E_t\left[(e^{-r\tau} S_T/S_t)^2\right] = e^{\sigma^2\tau}$. Thus if we were trying to price options with expiry $T = t+\tau$, using the log-Brownian approximation to J, we would use the BS formula with volatility

(2) $$\bar{\sigma}(t,T)^2 = \frac{1}{\tau}\log\left[\frac{\sum w_i w_j S_i(t) S_j(t)\, e^{\tau v_{ij}}}{J_t^2}\right] \geq \frac{\sum w_i w_j S_i(t) S_j(t) v_{ij}}{J_t^2} \equiv \sigma_I(t)^2$$

Notice that as $\tau \downarrow 0$, $\bar{\sigma}(t,T) \to \sigma_I(t)^2$, and as $\tau \to \infty$, $\bar{\sigma}(t,T) \to \max v_{ii}$, the dependence on τ being monotone.

(iii) Now in the project that Iain Mathieson is doing, the modelling assumption is that

$$v_{ij} = \sigma_i^2 \qquad (i=j)$$
$$= \rho\sigma_i\sigma_j \qquad (i \neq j)$$

for some $-1 \leq \rho \leq 1$, $\sigma_i > 0$; the correlations are constant. If we then look the definition of σ_I^2, this would specialise to

$$(3) \quad \sigma_I(t)^2 = \left[\sum_i w_i^2 S_i(t)^2 \sigma_i^2 + 2\rho \sum_{i<j}\sum w_i w_j \sigma_i \sigma_j S_i(t) S_j(t) \right] / J_t^2 .$$

The problem of estimating ρ (assuming the model) is approachable in various ways –

(a) Do historical estimate of the covariance of the returns based on daily prices;

(b) Estimate instantaneous vols of the stocks and index, and work out ρ from (3);

(c) Use the implied vol of the index together with (2) and estimates of individual σ_i (either historical or implied) to deduce ρ.

What it appears the industry does is to stick implied vols into (3) everywhere and then backout ρ. In view of the fact that $\bar{\sigma}(t,T) > \sigma_I(t)^2$, the effect of this is to bias the estimate of ρ upwards.

So as expected from the beginning, what is really happening here is a convexity bias !!

Odds and ends on valuations (19/4/06)

(i) If we have some valuation $(\pi_\tau)_{\tau\in\Theta}$ in the finite-tree context of the paper with Armand, it is of interest to consider valuation relative to some baseline cash balance process K^*; thus we shall define

$$\pi^*_\tau(K) \equiv \pi_\tau(K+K^*) - \pi_\tau(K^*).$$

Does the collection (π^*_τ) satisfy the axioms? It's immediate that (C), (M), (TI), (Z), (CL) all hold – what about the others?

(ii) Proof of (L)

$$\begin{aligned}
I_A\,\pi^*_\tau(K) &= \pi_\tau\big(I_A I_{[\tau,T]}(K+K^*)\big) - \pi_\tau(I_A K^*)\\
&= \pi_\tau\big(I_A I_{[\tau,T]}K^* + I_A I_{[\tau,T]}K\big) - \pi_\tau(I_A K^*)\\
&= \pi_\tau\big(I_A I_{[\tau,T]}(K^* + I_A I_{[\tau,T]}K)\big) - \pi_\tau(I_A K^*)\\
&= I_A\big\{\pi_\tau(K^* + I_A I_{[\tau,T]}K) - \pi_\tau(K^*)\big\}\\
&= I_A\,\pi^*_\tau(I_A I_{[\tau,T]}K).
\end{aligned}$$

We also have $I_{A^c}\,\pi^*_\tau(I_A I_{[\tau,T]}K) = \pi_\tau\big(I_{A^c}\{K^* + I_A I_{[\tau,T]}K\}\big) - \pi_\tau(I_{A^c}K^*) = 0$, so the conclusion is

$$I_A\pi^*_\tau(K) = \pi^*_\tau(I_A I_{[\tau,T]}K),$$

as required.

(iii) Proof of (DC)

$$\begin{aligned}
&\pi^*_\tau\big(K I_{[\tau,\sigma)} + \pi^*_\sigma(K) I_{[\sigma,T]}\big)\\
&\quad= \pi_\tau\big(K^* + K I_{[\tau,\sigma)} + \pi^*_\sigma(K) I_{[\sigma,T]}\big) - \pi_\tau(K^*)\\
&\quad= \pi_\tau\big((K+K^*) I_{[\tau,\sigma)} + (K^* + \pi^*_\sigma(K)) I_{[\sigma,T]}\big) - \pi_\tau(K^*)\\
&\quad= \pi_\tau\big((K+K^*) I_{[\tau,\sigma)} + \underbrace{\pi_\sigma(K^* + \pi^*_\sigma(K))}\, I_{[\sigma,T]}\big) - \pi_\tau(K^*) \qquad \text{by (DC)}
\end{aligned}$$

$$\begin{aligned}
&= \pi_\sigma(K^*) + \pi^*_\sigma(K) \qquad \text{by (TI)}\\
&= \pi_\sigma(K+K^*)
\end{aligned}$$

$$\begin{aligned}
&= \pi_\tau\big((K+K^*) I_{[\tau,\sigma)} + \pi_\sigma(K+K^*) I_{[\sigma,T]}\big) - \pi_\tau(K^*)\\
&= \pi_\tau(K+K^*) - \pi_\tau(K^*) = \pi^*_\tau(K),
\end{aligned}$$

as required.

Life insurance business again. (20/4/06)

(i) Let's return to the model on p 50, but now let's ignore the variation in N, and in effect just suppose N is constant, and that the dynamics of y are

$$dy_t = \alpha(\xi_t)\, dW'_t + \beta(\xi_t)\, dt$$

As then

$$dw_t = \theta_t \left(\sigma(\xi_t) dW_t + b(\xi_t) dt\right) + r_t (w_t - \theta_t) dt + dy_t - c_t\, dt$$

and now the HJB equation will be

$$-\rho V + QV + \tfrac{1}{2}\alpha^2 V'' + (rw + \beta) V' - \tfrac{1}{2}\left(\frac{b-r}{\sigma}\right)^2 \frac{(V')^2}{V''} = 0$$

$$V' \geq 1,$$

with $V(0,\xi) = -K$ for each ξ. This looks like it should have a piecewise quadratic solution.

(ii) In more detail, what we expect to find is that for each ξ there will be some critical level $k(\xi)$ such that $V'(\xi, w) = 1$ for $w \geq k(\xi)$, and below $k(\xi)$ the function $V(\xi, \cdot)$ is piecewise quadratic, C^1. For the lowest levels of w, we propose that $V(\xi, w) = \frac{1}{2} A(\xi) w^2 + B(\xi) w + C(\xi)$, and by taking HJB together with BC $V(\xi, 0) = -K$ we get

$$\boxed{\begin{aligned} &\left(Q - \rho + 2r - \left(\frac{b-r}{\sigma}\right)^2\right) A = 0 \\ &(Q-\rho) B + (rB + \beta A) - \left(\frac{b-r}{\sigma}\right)^2 B = 0 \\ &-K(Q-\rho)1 + \tfrac{1}{2}\alpha^2 A + \beta B - \tfrac{1}{2}\left(\frac{b-r}{\sigma}\right)^2 B/A = 0 \end{aligned}}$$

... but this can't work, because in general the solution to the first equation is $A = 0$ ... and if we consider the case where there is just one state the quadratic form looks impossible.

(iii) But suppose we assume $\alpha = 0$ (the income from life business may reasonably be approximated by a drift if we have huge numbers of policyholders). Doing the usual dual variables trick $z = V'$, $J = V - zw$, we get the linear system $(\kappa \equiv (b-r)/\sigma)$

$$\boxed{\tfrac{1}{2} z^2 \kappa^2 J'' + (\beta - rJ') z + (Q - \rho)(J - zJ') = 0}$$

with $J(\xi, z) = +\infty$ for $z < 1$, and for each ξ there is some z_ξ such that $J'(\xi, z_\xi) = 0$, $J(\xi, z) = -K \quad \forall\, z > z_\xi$ ... <u>or is it?</u>

Actually, if $\beta(\xi) \geq 0$, then you would <u>never</u> go bust while in state ξ (obviously!) So the story is rather more complicated. This is non trivial, even in the simple case where there are just two states $\beta(\xi_0) < 0 < \beta(\xi_1)$

[illegible], [illegible] the usual arguments

This was erroneous because I mixed up some terms in (*)

Flight to quality in continuous time (10/5/06)

1) This is developing the continuous-time analogue of the model developed earlier. We suppose that dividend (vector) process δ satisfies

$$d\delta_t = \sigma dX_t \equiv \sigma(dW_t + \alpha\, dt)$$

where α is not known, but has a $N(\hat\alpha_0, \tau_0^{-1})$ prior. The solution to the filtering problem (see p48) leads to

$$d\hat\alpha_t = \tau_t^{-1} d\hat W_t, \qquad dX_t = d\hat W_t + \hat\alpha_t dt, \qquad \tau_t = \tau_0 + tI, \; \hat\alpha_t = \tau_t^{-1}(\tau_0\hat\alpha_0 + X_t)$$

as before.

2) The SPD is $\exp(-\gamma 1^T\delta_t - \rho t) = \exp(-\rho t - \gamma 1^T\sigma X_t) \equiv e^{-\rho t}\exp(v\cdot X_t)$, and prices we derive are

$$S_t = \hat E_t\left[\int_t^\infty \frac{\zeta_u}{\zeta_t}\,\delta_u\, du\right] \qquad [v \equiv -\gamma\,\sigma^T 1]$$

$$= \hat E_t\left[\int_t^\infty e^{-\rho(u-t)}\exp\{v(X_u - X_t)\}\,\sigma(X_t + X_u - X_t)\,du\right]$$

If we condition on the value of α, this will give us

$$\int_t^\infty e^{-\rho(u-t)}\exp\left((\tfrac12|v|^2 + \alpha\cdot v)(u-t)\right)\{\sigma X_t + \sigma(\alpha+v)(u-t)\}\,du$$

$$= \frac{\sigma X_t}{\rho - (\frac12|v|^2 + \alpha\cdot v)} + \frac{\sigma(\alpha+v)}{(\rho - (\frac12|v|^2+\alpha\cdot v))^2}$$

We now have to average over α using the $N(\hat\alpha_t, \tau_t^{-1})$ posterior, but with a convergence factor $(\rho - (\frac12|v|^2+\alpha\cdot v))^2$. If $A \equiv \{\alpha : \frac12|v|^2 + \alpha\cdot v < \rho\}$, we shall have

$$S_t = \frac{\int_A \exp\{-\frac12(\alpha-\hat\alpha_t)\cdot\tau_t(\alpha-\hat\alpha_t)\}\left(\sigma(\alpha+v) + \sigma X_t\;(\rho - \frac12|v|^2 - \alpha\cdot v)\right)d\alpha}{\int_A \exp\{-\frac12(\alpha-\hat\alpha_t)\cdot\tau_t(\alpha-\hat\alpha_t)\}\;(\rho - \frac12|v|^2 - \alpha\cdot v)^2\; d\alpha} \qquad (*)$$

Conditional on $\alpha\cdot v \equiv \eta$, the law of α is $N\left(\frac{\tau_t^{-1}v}{v\cdot\tau_t^{-1}v}(\eta - \hat\alpha_t\cdot v),\; \tau_t^{-1} - \frac{\tau_t^{-1}vv^T\tau_t^{-1}}{v\cdot\tau_t^{-1}v}\right) + \hat\alpha_t$

that is, $\alpha - \hat\alpha_t \mid (\alpha-\hat\alpha_t)\cdot v = y \sim N(\theta y, M)$, where $\theta \equiv \tau_t^{-1}v / v\cdot\tau_t^{-1}v$, $M = \tau_t^{-1} - \frac{\tau_t^{-1}vv^T\tau_t^{-1}}{v\cdot\tau_t^{-1}v}$.

We compute

$$E\left[e^{\lambda\cdot(\alpha-\hat\alpha_t)} : A\right] = E\left[E\left(e^{\lambda\cdot(\alpha-\hat\alpha_t)}\,|\,Y\right) : A\right] \qquad \left(Y \equiv (\alpha-\hat\alpha_t)\cdot v\right)$$

$$= E\left[\exp\left(\tfrac12\lambda\cdot M\lambda + \lambda\cdot\theta\, Y\right) : A\right]$$

$$= e^{\frac12\lambda\cdot M\lambda}\int_{-\infty}^b e^{-y^2/2s + \lambda\cdot\theta y}\,\frac{dy}{\sqrt{2\pi s}} \qquad \begin{bmatrix} s \equiv v\cdot\tau_t^{-1}v \\ b \equiv \rho - \frac12|v|^2 - \hat\alpha_t\cdot v\end{bmatrix}$$

$$= \exp\left\{\tfrac12\lambda\cdot M\lambda + \tfrac12(\theta\cdot\lambda)^2 s\right\}\,\Phi\left(\frac{b - s(\theta\cdot\lambda)}{\sqrt s}\right) \equiv H(\lambda), \text{ say.}$$

This gives us

$$P(A) = H(0) = \Phi(b/\sqrt{s})$$

$$E\left[(\alpha-\hat{\alpha}_t) : A\right] = \nabla H(0) = -\tau_t^{-1} v \frac{e^{-b^2/2s}}{\sqrt{2\pi s}}$$

Differentiating one more time gives us

$$E\left[\left(v\cdot(\alpha-\hat{\alpha}_t)\right)^2 : A\right] = (v\cdot Mv + s)\,\Phi\left(\frac{b}{\sqrt{s}}\right) + \frac{bs\,e^{-b^2/2s}}{\sqrt{2\pi s}}.$$

These things in various combinations are all we need to compute an explicit form for the stock price. The quadratic variation of S will allow us to find out something (everything?) about σ.

Factoids about the SABR model (11/05/06)

The SABR model is specified via

$$dS = \sigma S^\beta \, dW, \qquad d\sigma = \eta \sigma \, dZ, \qquad dW\,dZ = \rho\, dt$$

for constant $\eta > 0$, $\beta \in (0,1)$. Writing $Y = S^\gamma$, we get for $\gamma = 2(1-\beta)$

$$dY = \gamma \sqrt{Y}\, \sigma \, dW + \tfrac{1}{2}\gamma(\gamma-1)\sigma^2 \, dt$$

a time-changed BESQ. If we write $y \equiv Y/\sigma^2$, and develop the Itô expansion we find

$$dy = \left\{ 2(1-\beta)\sqrt{y}\, dW - 2\eta y \, dZ \right\} + \left[(1-2\beta)(1-\beta) - 4\eta\rho(1-\beta)\sqrt{y} + 3\eta^2 y \right] dt$$

so that y is an autonomous diffusion.

If we assume $\rho = 0$, then Y is an independent timechange of a BESQ process, and we may be able to do things. If we use

$$dY = \gamma \sqrt{Y}\, dW + b\, dt$$

then

$$E^y \exp(-\lambda Y_t) = \left(1 + \tfrac{1}{2}\lambda\gamma^2 t\right)^{-2b/\gamma^2} \exp\left[- \frac{\lambda y}{1 + \frac{1}{2}\lambda \gamma^2 t} \right]$$

An incorrect conjecture (20/5/06)

I had imagined that if M_t is a martingale, then $E_t |M_T - M_t|$ is a supermartingale, but this is false as we see by the following simple 2-period example: $M_0 = 0$, $M_1 = \pm 1$ with equal probability. Given $M_1 = 1$, then $P(M_2 = 1/\varepsilon) = \varepsilon$, $P(M_2 = 0) = 1 - \varepsilon$, with a symmetric definition if $M_1 = -1$. Then

$$E\, |M_0 - M_2| = 1$$

$$E\, |M_1 - M_2| = \varepsilon\left(\tfrac{1}{\varepsilon} - 1\right) + (1-\varepsilon) = 2(1-\varepsilon).$$

Interesting questions observations

1) Commenting on structural models for default, Jose comments that the tax rebate argument of Leland no longer applies. Litterman in a GS internal report on municipal bonds has many of the ideas on credit

2) Philip Protter asks: suppose you are trying to hedge a derivative in a market where short selling is forbidden; what can you do?

3) If we have something like the valuations paper with Amaral, where reduction of regulatory capital occurs when subsidiaries pool risk, how should the benefits be shared? (Chuck Lucas)

4) A question from Steve Ross: can you show that it's impossible to have an implied vol surface whose only possible moves are parallel shifts?

5) Another question from Steve Ross: can one make some kind of theory for [illegible] in financial markets, in the sense that costs to be paid for in the future don't vary much in price...?

6) And another: Can we tell a story about the endogenous capital structure of a firm? Firms try to maximize the value of their marketed assets, unmarketed assets are not so important

7) Compensation structures of a firm influencing what it does (eg, in Goldman Sachs there are some very highly paid people at the top, in JP Morgan the salaries don't rise anything like as steeply)

Mike That's a nice result on the limiting implied vol as $T \to \infty$. It seems that this and the DIR result follows from a simple proposition:

Proposition: Let $M_{t,T}$ be a family of positive martingales with the property that

$$-\frac{1}{T} \log M_{t,T} \xrightarrow{a.s.} \lambda_t \in \mathbb{R}$$

for each $t \geq 0$. Then the process λ_t is non-decreasing.

Proof Fix $0 \leq s < t$, and set $A = \{ \lambda_t \leq \lambda_s - \varepsilon \}$. Suppose that (if possible) $P(A) > 0$. Now

$$A \subseteq \bigcup_n \left\{ -\frac{1}{T} \log M_{t,T} \leq -\frac{1}{T} \log M_{s,T} - \varepsilon/2 \quad \forall T \geq n \right\}$$

so there is some n such that

$$P\left[M_{t,T} \geq M_{s,T}\, e^{\varepsilon T/2} \quad \forall T \geq n \right] = \eta > 0.$$

However,

$$1 = E\left[\frac{M_{t,T}}{M_{s,T}} \right] \geq \eta\, e^{\varepsilon T/2}$$

holds for each $T \geq n$, a contradiction. $\square$

Remarks: The DIR result uses $M_{t,T} = e^{-\int_0^t r_s ds} P(t,T)$, and your result uses $S_t - C_{t,T}(K)$ ... because we have

$$\Sigma^2_{t,\infty}(K) = \lim_{T \to \infty} \left\{ -\frac{8}{T} \log\left(S_t - C_{t,T}(K) \right) \right\}$$

An interesting side remark is that this is decreasing in K ...

There may well be further examples ... possibly under some mild conditions one can show that λ_t is constant ...

It may well be that this is known in some form ... ISI/google on DIR before writing anything up!

Chris

15/3/06