Scaling in the SABR model (20/5/06)

1) Take the SABR model

$$dS = \sigma S^\beta dW, \qquad d\sigma = \eta\sigma dZ, \qquad dW dZ = \rho dt$$

and make the transformation $Y \equiv S^\gamma$, where $\gamma = 2(1-\beta) > 0$ to obtain the alternative more useful characterisation of the dynamics:

$$dY = \gamma\sqrt{Y}\sigma dW + \tfrac{1}{2}\gamma(\gamma-1)\sigma^2 dt, \qquad d\sigma = \eta\sigma dZ.$$

Now let's observe that if $(Y_t, \sigma_t)_{t\geq 0}$ is a solution to this SDE started from (y, s), then for any $c > 0$, $(cY_t, \sqrt{c}\sigma_t)_{t\geq 0}$ is a solution from $(cy, \sqrt{c}s)$. This allows possibilities for reducing the dimension.

2) Maybe it's better to work with the BES form $R_t = \sqrt{Y_t}$ rather than the BESQ form Y. Doing this change, we find

$$\begin{cases} dR = \dfrac{\gamma\sigma}{2} dW + \dfrac{\gamma\sigma^2}{8R}(\gamma-2)dt \\ d\sigma = \eta\sigma dZ \end{cases}$$

and (R_t, σ_t) starting from (r, s) scales to a solution $(cR_t, c\sigma_t)$ from (cr, cs).

Now let's set for $\lambda > 0$ fixed

$$F(t, r, s) \equiv E\left[R_t^\lambda \,\middle|\, R_0 = r, \sigma_0 = s\right]$$

and notice that by the scaling

$$F(t, cr, cs) = c^\lambda F(t, r, s) \Rightarrow F(t, r, s) = s^\lambda F(t, r/s, 1) \equiv s^\lambda f(t, r/s).$$

The PDE satisfied by F is

$$0 = -F_t + \tfrac{1}{2}\eta^2 s^2 F_{ss} + \tfrac{1}{2}\rho\eta\gamma\sigma^2 F_{sr} + \tfrac{1}{8}\gamma^2 s^2 F_{rr} + \frac{\gamma(\gamma-2)}{8r} s^2 F_r$$

and observing

$$F_r = s^{\lambda-1} f', \qquad F_{rr} = s^{\lambda-2} f'' \qquad [z \equiv r/s]$$

$$F_s = s^{\lambda-1}(\lambda f - z f'), \qquad F_{sr} = s^{\lambda-2}\left((\lambda-1)f' - z f''\right)$$

$$F_{ss} = s^{\lambda-2}\left(z^2 f'' - 2(\lambda-1) z f' + \lambda(\lambda-1) f\right)$$

We can re-express the PDE for F in terms of the simpler PDE for f:

$$\boxed{0 = -f_t + \tfrac{1}{2} f''\left(\eta^2 z^2 - \rho\eta\gamma z + \frac{\gamma^2}{4}\right) + f'\left(-\eta^2(\lambda-1)z + \frac{\rho\eta\gamma}{2}(\lambda-1) + \frac{\gamma(\gamma-2)}{8z}\right) + \tfrac{1}{2}\eta^2\lambda(\lambda-1)f}$$

with $f(0, z) = z^\lambda$. When you set $\rho = 0$, Maple gives a solution in terms of hypergeometrics —

$$\frac{\partial^2 F}{\partial K^2} = \frac{1}{K\sqrt{2\pi v}} \exp\{-(\log K + v/2)^2/2v\} = \frac{2}{K^2}\frac{\partial F}{\partial v}$$

$$\frac{\partial^2 F}{\partial K \partial v} = \frac{\exp\left(-(\log K + v/2)^2/2v\right)}{\sqrt{2\pi v}} \left\{ \frac{1}{4} - \frac{\log K}{2v} \right\} = \frac{1}{K}\frac{\partial F}{\partial v}\left(\frac{1}{2} - \frac{\log K}{v}\right)$$

$$\frac{\partial^2 F}{\partial v^2} = \frac{\partial F}{\partial v}\left(\frac{(\log K)^2}{2v^2} - \frac{1}{2v} - \frac{1}{8}\right)$$

From Abramowitz + Stegun 7.1.13 :

$$\frac{1}{x + \sqrt{x^2+2}} \leq e^{x^2}\int_x^\infty e^{-t^2}dt \leq \frac{1}{x + \sqrt{x^2 + 4/\pi}} \qquad \forall x \geq 0$$

whence

$$\boxed{\frac{2}{x + \sqrt{4+x^2}} \leq \sqrt{2\pi}\, e^{x^2/2}\bar{\Phi}(x) \leq \frac{2}{x + \sqrt{\frac{8}{\pi} + x^2}}} \qquad \forall x \geq 0$$

Some observations on implied volatility (21/5/06)

1) If we try to understand the implied vol surface, we may wlog suppose $r=0$, and then things get expressed in terms of

$$F(K,v) = \bar{\Phi}\left(\frac{\log K - v/2}{\sqrt{v}}\right) - K\,\bar{\Phi}\left(\frac{\log K + v/2}{\sqrt{v}}\right),$$

where $F(K,\sigma^2 T)$ is the BS call price for an option with expiry T, vol σ, strike K, and $S_0=1$

2) To start the story, suppose for now we fix T and consider a stock $S_0=1$ which has call function

$$C(K) \equiv E(S_T - K)^+ = F(K, \Sigma(K)^2 T).$$

Now observe that C is decreasing convex, and F is decreasing convex in K, increasing in v, so this means that $\Sigma(K)$ cannot increase too fast with K. Using the facts

$$\frac{\partial F}{\partial K} = -\bar{\Phi}\left(\frac{\log K + v/2}{\sqrt{v}}\right), \qquad \frac{\partial F}{\partial v} = \frac{1}{2\sqrt{v}}\;\frac{\exp(-(\log K - v/2)^2/2v)}{\sqrt{2\pi}},$$

we express the decrease of C as

$$0 \geq \frac{\partial C}{\partial K} = F_K + 2\Sigma(K)T\,\frac{\partial \Sigma}{\partial K}\,F_v$$

implying that

$$\sqrt{T}\;\frac{\partial \Sigma}{\partial K}\;\frac{\exp(-(\log K - v/2)^2/2v)}{\sqrt{2\pi}} \leq \bar{\Phi}\left(\frac{\log K + v/2}{\sqrt{v}}\right) \leq \frac{\sqrt{v}}{\log K + v/2}\;\frac{e^{-(\log K + v/2)^2/2v}}{\sqrt{2\pi}}$$

where $v \equiv \Sigma(K)^2 T$. Hence (provided $\log K + v/2 > 0$) we obtain

$$\boxed{\Sigma'(K) \leq \frac{1}{K}\;\frac{\Sigma(K)}{\log K + \frac{1}{2}\Sigma(K)^2 T}}$$

The upper bound we get from solving the ODE is

$$\boxed{\Sigma(K) \leq \frac{A + \sqrt{A^2 + 2T\log K}}{T}} \qquad \Sigma(1) = 2A/T.$$

So we see that implied vol cannot grow too rapidly with $K>1$, and that the rate of growth gets more gradual as T increases.

3) A useful observation of Mike Tehranchi: $1 - C(K) = E(S_T \wedge K)$, so for $K_1 < K_2$ we have

$$1 - C(K_1) \leq 1 - C(K_2) \leq \frac{K_2}{K_1}(1 - C(K_1)) = \frac{K_2}{K_1}\,E(S_T \wedge K_1)$$

so that $\dfrac{K_1}{K_2} \leq \dfrac{1 - C(K_1)}{1 - C(K_2)} \leq 1$.

We also have

$$1 - C(K) = \Phi\left(\frac{\log K - v/2}{\sqrt{v}}\right) + K\bar{\Phi}\left(\frac{\log K + v/2}{\sqrt{v}}\right).$$

Now let's make the hypothesis that

$$\boxed{\text{for some } K, \quad T\,\Sigma(K,T)^2 \longrightarrow \infty \quad (T\to\infty)}$$

This is equivalent (in view of Mike's observation) to the statement that

$$\text{for some } K \quad 1-C(K,T) \equiv E(S_T \wedge K) \longrightarrow 0 \quad (T\to\infty)$$

and hence also equivalent to

$$\text{for all } K, \quad 1-C(K,T) \longrightarrow 0 \quad (T\to\infty).$$

Now let's consider what happens for $K_1 < K_2$, writing $x_i \equiv (v_i/2 - \log K_i)/\sqrt{v_i}$, $y_i \equiv (v_i/2 + \log K_i)/\sqrt{v_i}$ $(i=1,2)$, where $v_i = \Sigma(K_i,T)^2\,T$ for short. Assuming T is so large that $v_i > 2|\log K_i|$ $(i=1,2)$, we can use the bound on $\bar{\Phi}$ from the previous page to establish

$$\frac{1-C(K_1,T)}{1-C(K_2,T)} \le \exp\left\{-\frac{(k_1-v_1/2)^2}{2v_1} + \frac{(k_2-v_2/2)^2}{2v_2}\right\}\cdot\frac{h_+(x_1)+h_+(y_1)}{h_-(x_2)+h_-(y_2)}$$

where

$$h_+(t) \equiv 2/\left(t+\sqrt{t^2+8/\pi}\right), \quad h_-(t) \equiv 2/\left(t+\sqrt{t^2+4}\right)$$

and there's a similar lower bound. The upper bound is asymptotic to

$$\exp\left\{-\frac{(k_1-v_1/2)^2}{2v_1} + \frac{(k_2-v_2/2)^2}{2v_2}\right\}\frac{1/x_1+1/y_1}{1/x_2+1/y_2} = \exp\left\{-\tfrac{1}{2}x_1^2+\tfrac{1}{2}x_2^2\right\}\frac{(x_1+y_1)}{(x_2+y_2)}\,\frac{x_2 y_2}{x_1 y_1}$$

$$\sim \exp\left\{-\tfrac{1}{2}x_1^2+\tfrac{1}{2}x_2^2\right\}\, x_2/x_1$$

and likewise for the lower bound. Thus we conclude that (at least asymptotically) $(k_i \equiv \log K_i)$

$$\log K_1/K_2 \le -\tfrac{1}{2}x_1^2+\tfrac{1}{2}x_2^2 = -\frac{v_1}{8}+\frac{v_2}{8}+\frac{k_1}{2}-\frac{k_2}{2}-\frac{k_1^2}{2v_1}+\frac{k_2^2}{2v_2} \le 0$$

This forces the difference $v_1 - v_2$ to remain bounded as $T\to\infty$. A little more precisely, given $\varepsilon>0$ there exists N such that if $x_i, y_i \ge N$ then

$$\log K_1/K_2 - \varepsilon \le \tfrac{1}{2}(x_2^2 - x_1^2) \le \varepsilon$$

Further, if $x_i, y_i \ge N$, and $k_i^2/v_i \le \varepsilon$, we deduce that

$$\tfrac{1}{2}(k_1-k_2) - 2\varepsilon \le \frac{v_2-v_1}{8} \le -\tfrac{1}{2}(k_1-k_2)+2\varepsilon$$

or again

$$\boxed{|v_1 - v_2| \le 8\left\{(k_2-k_1)/2 + 2\varepsilon\right\}}$$

This will force the implied volatilities $\Sigma(K_1,T)$ and $\Sigma(K_2,T)$ to be very close.

Variations on the SABR theme? (25/5/06)

1) We've seen that in the SABR model $dS = \sigma S^\beta dW$, $d\sigma = \eta\sigma dZ$, $dZ dW = \rho dt$, the process $Y = S^\gamma$, $\gamma \equiv 2(1-\beta)$ solves the time-changed BESQ SDE

$$dY = \gamma\sqrt{Y}\,\sigma dW + \tfrac{1}{2}\gamma(\gamma-1)\,dt\,\sigma^2$$

Suppose now we try writing $Y = y + \tilde{y}$, where

$$\begin{cases} dy = \gamma\sqrt{y}\,\sigma dW' + a\sigma^2 dt \\ d\tilde{y} = \gamma\sqrt{\tilde{y}}\,\sigma dZ + b\sigma^2 dt \end{cases}$$

for constants a, b which sum to $\frac{1}{2}\gamma(\gamma-1)$, with W' a BM independent of Z. The correlation $dY\,d\sigma = \eta\sigma^2\gamma\sqrt{\tilde{y}}$ instead of $\rho\gamma\sqrt{Y}\sigma^2\eta$ before, so that now the constant ρ changes to the variable $\sqrt{\tilde{y}/Y}$, but that's not obviously wrong; we can set some initial value for ρ by choice of $y_0, \tilde{y}_0$.

Now a particularly obliging choice of b is to take $\boxed{b = \gamma^2/4}$ for then

$$d\sqrt{\tilde{y}} = \frac{\gamma}{2}\sigma\,dZ = \frac{\gamma}{2\eta}\,d\sigma$$

which implies that for some constant $c \geq 0$, $\tilde{y} = (c + \gamma\sigma/2\eta)^2$. The corresponding choice of a will be $a = -\beta(1-\beta)$, and if $x \equiv y/\sigma^2$ we find that

$$dx = \left(\gamma\sqrt{x}\,dW' - 2\eta x\,dZ\right) + (a + 3\eta^2 x)\,dt$$

2) It's tempting to try writing instead $y = \sigma^2 x'$, $dx' = \gamma\sqrt{x'}\,dW' + (a + 3\eta^2 x')\,dt$, which is of course a different model. This would lead (taking $c=0$) to the model

$$Y_t = \sigma_t^2\left(\frac{\gamma}{2\eta} + x'_t\right)$$

where σ and x' are independent; however, we won't necessarily have $Y^{1/\gamma}$ a martingale. We therefore propose the representation $S_t = Y_t^{1/\gamma}$, where $Y_t = \sigma_t^2 z_t$, with z the diffusion

$$dz_t = 2\sqrt{z_t}\,dW_t + (a_1 - a_2 z)\,dt$$

where $a_1 = -2(1-\gamma)/\gamma < 0$, $a_2 = (2-\gamma)\eta^2/\gamma \equiv 2\beta\eta^2/\gamma > 0$. A few lines of Itô calculus shows that this thing makes $S_t \equiv Y_t^{1/\gamma}$ into a martingale,

$$\boxed{\frac{dS}{S} = \frac{2\eta}{\gamma}\,dZ + \frac{2}{\gamma\sqrt{z}}\,dW = \frac{2}{\gamma}\left(\eta^2 + \frac{1}{z}\right)^{\frac{1}{2}} d\tilde{W}}$$

In the SABR model, $S^{-1}dS = \sigma S^{\beta-1}dW$, whereas here $\sigma S^{\beta-1} = \sigma Y^{-\frac{1}{2}} = z^{-\frac{1}{2}}$, so the quadratic variation is quite similar for small z.

Another insurance example (9/6/06)

1) Let's think about a life insurance company which invests in the stock market, pays dividends at rate c_t to shareholders, and pays out at net rate $k \geq 0$ to policy holders. The objective is to obtain

$$V(w) = \sup E^w \left(\int_0^\tau e^{-\rho t} c_t \, dt - K e^{-\rho \tau} \right)$$

where τ is bankruptcy time, K some big penalty for going bust. As usual,

$$dw_t = r(w_t - \theta_t)\,dt + \theta_t \sigma (dW_t + \alpha dt) - c_t dt - k\,dt$$

so the HJB equation reads

$$\sup_{c,\theta} \left[-\rho V + \tfrac{1}{2}\sigma^2 \theta^2 V'' + \left(r(w-\theta) + \theta\sigma\alpha - c - k \right) V' + c \right] = 0 , \qquad V(0) = -K.$$

Solving, we see we must have $V' \geq 1$, and

$$-\rho V + (rw - k) V' - \tfrac{1}{2}\kappa^2 V'^2 / V'' = 0 \qquad \kappa \equiv (\sigma\alpha - r)/\sigma .$$

2) Passing to dual variables $z = V'(w)$, $J(z) = V(w) - wz$ transforms this to the linear ODE

$$\tfrac{1}{2}\kappa^2 z^2 J'' + (\rho - r) z J' - \rho J - kz = 0,$$

solved by

$$\boxed{J(z) = -\frac{kz}{r} + A z^{-a} + B z^{b}}$$

where $-a < 0 < 1 < b$ are the roots of the quadratic $\frac{1}{2}\kappa^2 t(t-1) + (\rho - r)t - \rho = 0$. We also have that for $z \geq z^* = V'(0)$ we shall have $J(z) = V(0) = -K$, and we expect to get a C^1 fit at z^*. Want to make J as large as possible, subject to $J'' \geq 0$ in $[1,\infty)$, so, in effect, we want to make z^* as large as possible. Solving for smooth fit at z^*, we obtain

$$\boxed{J(z) = -\frac{kz}{r} - \frac{K}{a+b}\left[b\left(\frac{z}{z^*}\right)^{-a} + a\left(\frac{z}{z^*}\right)^{b} \right] + \frac{kz^*}{r(a+b)}\left[(b-1)\left(\frac{z}{z^*}\right)^{-a} + (a+1)\left(\frac{z}{z^*}\right)^{b} \right]}$$

3) What should the correct boundary conditions be? By taking the case of CRRA utility with $R \in (0,1)$, letting $R \downarrow 0$, we can hopefully learn quite a bit. The dual form of HJB comes out at

$$\tilde{U}(z) - \rho J - kz + (\rho - r) z J' + \tfrac{1}{2}\kappa^2 z^2 J'' = 0$$

with bc $V(0) = \lim_{z \to \infty} J(z) = -K$. Moreover we know that as $w \to \infty$, the value $V(w)$ will look more & more like what you'd get without the loss on default.

HOWEVER, there is an issue here of whether the problem is well posed; if we set

$Q(t) = \frac{1}{2}\kappa^2 t(t-1) + (\rho - r)t - \rho$, then the condition for the Merton problem to be well posed is that

$$0 > Q(1-\tfrac{1}{R}) = -\gamma \equiv -R^{-1}\{\rho + (R-1)(r + \tfrac{1}{2}\kappa^2/R)\} \qquad \text{(from Merton solution)}$$

so we will not be able to let $R \downarrow 0$ and still have a meaningful question. The solution for J takes the form

$$J(z) = \frac{z^{1-1/R}}{(1-1/R)Q(1-1/R)} - \frac{kz}{r} + c_1 z^{-a} + c_2 z^b$$

at least in the region where the ODE holds. For large z we expect that the ODE no longer holds, we just have $J(z) = -K$ for $z \geq z_*$, some z_*. We also expect to have $c_1 \geq 0$, because for small z it is the term in z^{-a} which dominates.
Now for very small z, corresponding to very large w, we expect to see something like the Merton solution, so this leads us to conclude that $\boxed{c_1 = 0}$ and

$$J(z) = -\frac{1}{\gamma}\frac{z^{1-1/R}}{1-1/R} - \frac{kz}{r} + c(z/z_*)^b$$

where z_* is the place where J is minimal. The equation $\underbrace{J'(z_*)}_{=0}$ gives us

$$\frac{1}{\gamma} z_*^{-1/R} + \frac{k}{r} = \frac{bc}{z_*}$$

so that

$$c = \frac{1}{b}\left[\frac{kz_*}{r} + \frac{1}{\gamma} z_*^{1-1/R}\right]$$

and the minimised value $J(z_*)$ is therefore equal to

$$-\frac{1}{\gamma}\frac{z_*^{1-1/R}}{1-1/R} - \frac{kz_*}{r} + \frac{1}{b}\left(\frac{kz_*}{r} + \frac{1}{\gamma} z_*^{1-1/R}\right)$$

$$= -\left(1-\frac{1}{b}\right)\frac{kz_*}{r} + \frac{z_*^{1-1/R}}{\gamma}\left\{\frac{1}{b} + \frac{R}{1-R}\right\}$$

Which we equate to the desired value $-K$.

Useful observation:

$$f(k,v) = 1 - e^k + e^k f(-k,v)$$

ie $$1 - f(k,v) = e^k(1 - f(-k,v))$$

Thence if $C(T,K)$ is the call price function, we can define

$$\tilde{C}(T,K) = 1 - K(1 - C(T, 1/K))$$

which is again increasing in T, decreasing convex in K, decreasing from 1 to 0.

Is implied vol locally Lipschitz? No; consider the example where the call function is $C(K) = (a-K)^+/a$ for some $a > 1$. Then the implied vol goes to 0 as $K \uparrow a$, and we can actually derive that

$$v \sim \frac{(\log a)^2}{-2\log(1-K/a)}$$

so the slope goes infinite just to the left of $K = a$.

Some bounds on the BS volatility surface 21/6/06

We know that

$$0 \le f(k,v) = \bar\Phi\left(\frac{k-v/2}{\sqrt{v}}\right) - e^k\,\bar\Phi\left(\frac{k+v/2}{\sqrt{v}}\right) = E\left(e^{\sqrt{v}Z - v/2} - e^k\right)^+ \le 1$$

is increasing in v, and decreasing in k. Suppose now we fix some $p \in (0,1)$ and ask about the behaviour of $k^*(p,v)$, the value of k at which the call price is equal to p, thinking particularly of the behaviour as $v \to \infty$. We have

$$p = E\left(e^{\sqrt{v}Z - v/2} - e^k\right)^+$$

$$= \int \frac{e^{-z^2/2}}{\sqrt{2\pi}} \left(e^{z\sqrt{v} - v/2} - e^k\right)^+ dz$$

$$= \int \left(e^{-(z-\sqrt{v})^2/2} - e^{k - z^2/2}\right)^+ \frac{dz}{\sqrt{2\pi}}$$

$$z = t + \sqrt{v}$$

$$= \int \left(e^{-t^2/2} - e^{+k - t^2/2 - t\sqrt{v} - v/2}\right)^+ \frac{dt}{\sqrt{2\pi}}$$

$$= \int \frac{e^{-t^2/2}}{\sqrt{2\pi}} \left(1 - e^{k - v/2 - t\sqrt{v}}\right)^+ dt$$

so if we write $k = v/2 + \theta\sqrt{v}$, we obtain

$$p = \int \frac{e^{-t^2/2}}{\sqrt{2\pi}} \left(1 - e^{(\theta - t)\sqrt{v}}\right)^+ dt$$

Using the fact that $\left(1 - e^{(\theta-t)\sqrt{v}}\right)^+ \le I_{\{t > \theta\}}$, we find the bound

$$p \le \bar\Phi(\theta) \equiv 1 - \Phi(\theta)$$

implying that $\theta \le \Phi^{-1}(1-p)$. On the other hand, for $\varepsilon > 0$ we have $\left(1 - e^{(\theta-t)\sqrt{v}}\right)^+ \ge \left(1 - e^{-\varepsilon\sqrt{v}}\right) I_{\{t \ge \theta + \varepsilon\}}$ so we obtain the bound

$$p \ge \left(1 - e^{-\varepsilon\sqrt{v}}\right) \bar\Phi(\theta + \varepsilon)$$

so

$$\Phi^{-1}\left(1 - \frac{p}{1 - e^{-\varepsilon\sqrt{v}}}\right) \le \theta + \varepsilon = \frac{k - v/2}{\sqrt{v}} + \varepsilon$$

$$= \Phi^{-1}\left(1 - p - \frac{p e^{-\varepsilon\sqrt{v}}}{1 - e^{-\varepsilon\sqrt{v}}}\right)$$

If we fix some $\lambda > \frac{1}{2}$ and take $\varepsilon = (\lambda \log v)/\sqrt{v}$, we obtain the bound

$$\boxed{k - \frac{v}{2} + \lambda \log v \ge \sqrt{v}\,\Phi^{-1}\left(1 - p - \frac{p}{v^{\lambda} - 1}\right)}$$

Putting these together, we get some fairly tight bounds on $k^*(p,v)$:

$$\boxed{\frac{v}{2} + \sqrt{v}\,\Phi^{-1}\!\left(1-p-\frac{p}{v^{\lambda}-1}\right) - \lambda \log v \;\le\; k^*(p,v) \;\le\; \frac{v}{2} + \sqrt{v}\,\Phi^{-1}(1-p)}$$

Back to the drawdown problem (23/6/06)

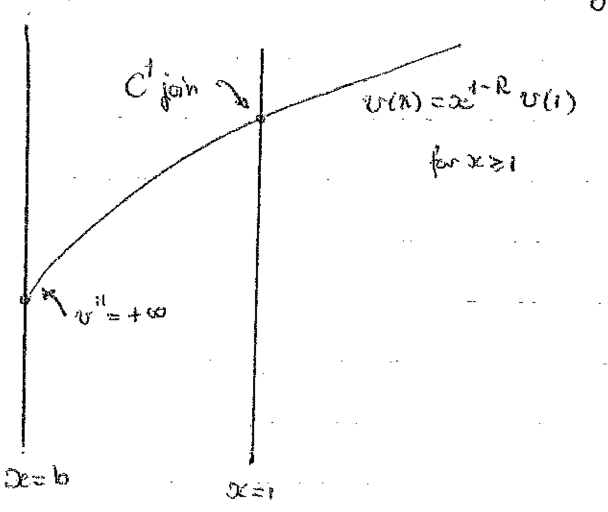
On pp 35-36, WN XXV, I made some notes on the problem of optimal control under a drawdown constraint. Transforming to dual variables, we get the ODE $[\kappa \equiv (\mu-r)/\sigma]$

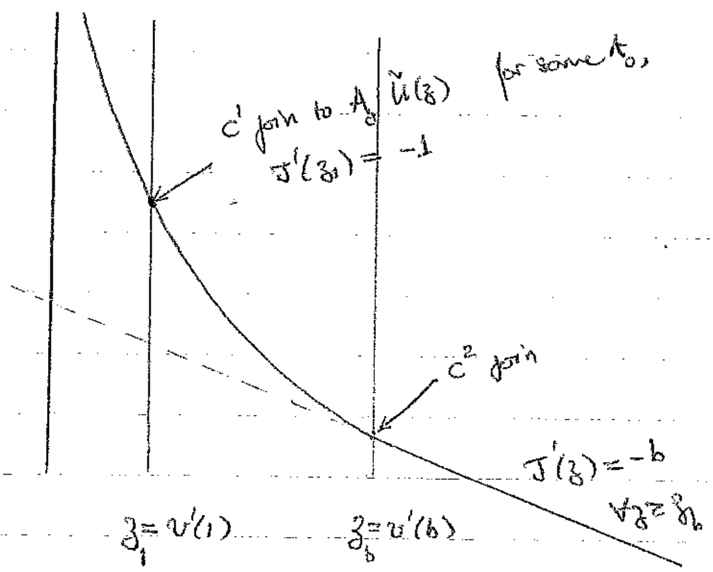
$$\tilde{U}(z) + \tfrac{1}{2}\kappa^2 z^2 J''(z) + (\rho - r) z J'(z) - \rho J(z) = 0$$

together with the condition

$$J(z) = \frac{R}{1-R} z \qquad \text{when } J_z' = -1$$

There is one other crucial observation to be made; if w decreases to $b\bar{w}$, then the holding of stock has to go to zero otherwise the bound $w \geq b\bar{w}$ will be violated. Since the optimal wealth in stock is $-\frac{\mu-r}{\sigma^2}\frac{V_w}{V_{ww}}$, we shall have to have $v'' = +\infty$ at $x=b$, and hence $J'' = 0$ at $z_b \equiv v'(b)$





Thus the solution J gets patched together in three pieces; below z_1, it's $A_0\tilde{U}(z)$ for some A_0; between z_1 and z_b, it's

$$J(z) = A_1 z^{-\alpha} + B_1 z^{\beta} + q\,\tilde{U}(z)$$

where $q = -1/Q(1-R^{-1}) > 0$ as the condition for well posed problem. The gradient condition at z_1 identifies $\boxed{z_1 = A_0^R}$ and the solution in (z_1, z_b) is now

$$\boxed{J(z) = -\frac{(q-A_0)\tilde{U}(z_1)}{\alpha+\beta}\left\{(\beta-1+R')\left(\frac{z}{z_1}\right)^{-\alpha} + (\alpha+1-R')\left(\frac{z}{z_1}\right)^{\beta}\right\} + q\,\tilde{U}(z)} \qquad \left(R' \equiv \frac{1}{R}\right)$$

As for the conditions which have to hold at z_b, we have $J'' = 0$, and $J' = -b$, which we can rework in terms of $t \equiv z_b / z_1$ to

This gives us the (non-linear) equations

$$\boxed{\begin{aligned} &-\frac{(q-A_0)\tilde{u}(z_1)}{(\alpha+\beta)z_1}\left\{-\alpha(\beta-1+R')t^{-\alpha-1}+\beta(\alpha+1-R)t^{\beta-1}\right\} \\ &\qquad + \frac{q(1-R')}{z_1}\frac{\tilde{u}(t)}{t} = -b \\ &-\frac{(q-A_0)\tilde{u}(z_1)}{\alpha+\beta}\left\{\alpha(\alpha+1)(\beta-1+R')t^{-\alpha}+\beta(\beta-1)(\alpha+1-R')t^{\beta}\right\} \\ &\qquad + q(1-R')(-R')\tilde{u}(t) = 0 \end{aligned}}$$

These two equations in (A_0, t) must now be solved (recall $z_1 = A_0^R$).
Notice alternatively that given b, once we select z_b we know that

$$\begin{cases} J(z_b) = \rho^{-1}\left[\tilde{u}(z_b) - b(\rho-r)z_b\right] \\ J'(z_b) = -b \end{cases}$$

So we can build the solution in (z_1, z_b) from this data; we obtain

$$J(z) = q\tilde{u}(z) + A_2\left(z/z_b\right)^{-\alpha} + B_2\left(z/z_b\right)^{\beta},$$

where

$$(\alpha+\beta)\begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = \begin{pmatrix} \beta & -1 \\ \alpha & 1 \end{pmatrix}\begin{pmatrix} v_0 \\ v_1 \end{pmatrix}, \quad v_0 = J(z_b) - q\tilde{u}(z_b), \; v_1 = -bz_b - q(1-R')\tilde{u}(z_b)$$

which determines A_2, B_2 as functions of z_b. Comparing the coefficients of $z^{-\alpha}, z^{\beta}$ in the two expressions for J, we learn that

$$\left.\begin{aligned} A_2 z_b^{\alpha} &= -\frac{(q-A_0)\tilde{u}(z_1)(\beta-1+R')z_1^{\alpha}}{\alpha+\beta} \\ B_2 z_b^{-\beta} &= -\frac{(q-A_0)\tilde{u}(z_1)(\alpha+1-R')z_1^{-\beta}}{\alpha+\beta} \end{aligned}\right\} \quad (*)$$

giving us $\boxed{\dfrac{\beta-1+R'}{\alpha+1-R'}\, z_1^{\alpha+\beta} = \dfrac{A_2}{B_2}\, z_b^{\alpha+\beta}}$

This allows us to discover z_1 as a function of z_b, and then vary z_b until (*) holds.

$$\gamma = R^{-1}\left\{\rho + (R-1)\left(r + \tfrac{1}{2}\kappa^2/R\right)\right\}$$

A simple treatment of tax (3/7/06)

There are many difficult issues in the treatment of tax, but here is a simple variant of the basic Merton consumption problem which incorporates some features of interest, and remains a little tractable. Suppose that there is an annual accounting; at times $h, 2h, 3h,$ you get charged (refunded) tax at rate τ on gains (losses) in the previous interval. Thus

$$w_{nh} = w_{nh-} - \tau(w_{nh-} - w_{nh-h}) = \tau w_{(n-1)h} + (1-\tau) w_{nh-}$$

With CRRA utility, it is clear that the value function at time 0,

$$V(w) \equiv \sup E\left[\int_0^\infty e^{-\rho t} U(\delta_t)\,dt\right]$$

is of the form $V(w) = A\,U(w)$ for some constant A to be discovered. Now we have

$$A U(w) = V(w) = \sup E\left[\int_0^h e^{-\rho t} U(\delta_t)\,dt + e^{-\rho h} A\, U(\tau w + (1-\tau) w_{h-})\right],$$

and must deduce A from this. If $\zeta_t^0 = \exp(-rt - \kappa W_t - \frac{1}{2}\kappa^2 t)$ is the SPD normalised by $\zeta_0^0 = 1$, then we shall have for some $\lambda > 0$

$$\left.\begin{aligned} e^{-\rho h} A (1-\tau)\, U'(\tau w + (1-\tau) w_h^*) &= \lambda \zeta_h^0 \\ e^{-\rho t} U'(\delta_t^*) &= \lambda \zeta_t^0 \end{aligned}\right\} \qquad 0 < t < h.$$

We relate λ to initial wealth in the usual way: (writing ζ in place of ζ^0 for short)

$$w = E\left(\int_0^h \zeta_u \delta_u^*\,du + \zeta_h w_h^*\right)$$

$$= E\left[\int_0^h \lambda^{-1/R} e^{-\rho t/R} \zeta_t^{1-1/R}\,dt + \zeta_h\left\{\frac{-\tau w}{1-\tau}\right\} + \frac{\zeta_h^{1-1/R} e^{-\rho h/R}}{1-\tau} \lambda^{-1/R} A^{1/R} (1-\tau)^{1/R}\right]$$

$$= \frac{-\tau w e^{-rh}}{1-\tau} + \lambda^{-1/R} \frac{1-e^{-\gamma h}}{\gamma} + \lambda^{-1/R} A^{1/R} (1-\tau)^{1/R-1} e^{-\gamma h},$$

so that

$$\boxed{w\left(1 + \frac{\tau e^{-rh}}{1-\tau}\right) = \lambda^{-1/R}\left[\frac{1-e^{-\gamma h}}{\gamma} + A^{1/R}(1-\tau)^{1/R-1} e^{-\gamma h}\right].}$$

We now need to compute the value so as to deduce what A will be. We get

$$A U(w) = E\left[\int_0^h e^{-\rho t} \frac{(\lambda e^{\rho t}\zeta_t)^{1-1/R}}{1-R}\,dt + \frac{A e^{-\rho h}}{1-R}\left(\frac{\lambda e^{\rho h}\zeta_h}{A(1-\tau)}\right)^{1-1/R}\right]$$

$$= \frac{\lambda^{1-1/R}}{1-R} \frac{1-e^{-\gamma h}}{\gamma} + \frac{A^{1/R}(1-\tau)^{1/R-1}}{1-R} \lambda^{1-1/R} e^{-\gamma h}$$

Now we have that $BW = \lambda^{-1/R} K$, with $B \equiv 1 + \frac{\tau e^{-rh}}{1-\tau}$, $K \equiv \frac{1-e^{-\gamma h}}{\gamma} + A^{1/R}(1-\tau)^{1/R-1} e^{-\gamma h}$,

so the preceding equation reads

$$A\, U(w) = \frac{\lambda^{1-1/R}}{1-R} K = U(\lambda^{-1/R})\, K$$

$$= U(w) \left(\frac{B}{K}\right)^{1-R} K$$

whence

$$A = B^{1-R} K^{R} \quad \therefore \quad A^{1/R} = B^{\frac{1}{R}-1} K$$

We can rearrange this to

$$A^{1/R} \left\{ 1 - e^{-\gamma h} (B(1-\tau))^{\frac{1}{R}-1} \right\} = B^{\frac{1}{R}-1} \frac{1-e^{-\gamma h}}{\gamma}$$

$$\Rightarrow \quad \boxed{A^{1/R} = \frac{B^{\frac{1}{R}-1}\left(1-e^{-\gamma h}\right)}{\gamma - \gamma e^{-\gamma h}(B(1-\tau))^{\frac{1}{R}-1}}}$$

which is certainly correct when $\tau = 0$.

Performance with constraints relative to a benchmark (6/7/06)

1) Suppose we want to do well relative to some benchmark asset/portfolio V_t, while generating (by time T) a good Sharpe ratio and not too much risk of drawdown... how? We could take w_t to be the wealth process, w_0 given, $V_0 = 1$, $\tilde{w}_t \equiv w_t / V_t$ and attempt to calculate

$$\varphi(a,b,k) = \sup E\left[\tilde{w}_T - \tfrac{1}{2} a \tilde{w}_T^2 - \tfrac{1}{2} b\left((-\tilde{w}_T + k)^+\right)^2 \right],$$

together with the associated optimal portfolio, $\pi^*_{a,b,k}(t)$, written $\pi^*_{a,b}(t)$ for short. Typically the parameter k is given (initial wealth, say, or max wealth in previous period), and the parameters a, b play the rôle of Lagrange multipliers. If we are able to solve this problem, we next calculate

$$M(a,b) \equiv E^{\pi^*_{a,b}} \tilde{w}_T, \quad Q(a,b) = E^{\pi^*_{a,b}} \tilde{w}_T^2, \quad D(a,b) = E^{\pi^*_{a,b}} \left((k - \tilde{w}_T)^+\right)^2.$$

Supposing wlog that $w_0 = 1$, our aim now is to

$$\max \frac{M(a,b)}{\{Q(a,b) - M(a,b)^2\}^{1/2}} \qquad \text{s.t.} \quad D(a,b) \le \text{const.}$$

Provided we can find these things reasonably explicitly, the numerical optimisation of this should be OK.

2) If we suppose a complete market, and SPD ζ, then optimising subject to the budget constraint $E\,\zeta w_T = w_0$ leads to the Lagrangian solution

$$0 = -a\tilde{w}_T + b(k - \tilde{w}_T)^+ + 1 + \lambda \zeta V_T$$

$$\Rightarrow \quad \tilde{w}_T = f(1 + \lambda \zeta V_T), \qquad f(t) \equiv \frac{t}{a} + \frac{b}{(a+b)a}(ak - t)^+$$

(assuming $a > b$).

3) Let's now try to make this more explicit in the case where we have a driving BM (W_t), with stocks $dS^i_t = S^i_t(\sigma_{ij}\, dW^j_t + \mu^i\, dt)$, and where we suppose $V_T = \exp(\nu \cdot W_T + \beta)$. The SPD process is $\zeta_t = \exp(-rt - \kappa W_t - \tfrac{1}{2}|\kappa|^2 t)$, so we get $\zeta_T V_T = \exp\left((\nu - \kappa)\cdot W_T + (\beta - rT - \tfrac{1}{2}|\kappa|^2 T)\right)$
$= \exp\{\eta \cdot W_T + \gamma\}$ for short. The budget equation reads

$$w_0 = 1 = E\left[\zeta_T V_T\, \tilde{w}_T\right] = E\left[\zeta_T V_T\, f(1 + \lambda \zeta V_T)\right]$$

$$= a^{-1}\left[e^{\gamma + \frac{1}{2}\sigma^2} + \lambda e^{2\gamma + 2\sigma^2}\right] + \frac{\lambda b}{a(a+b)}\left[e^{c + \gamma + \sigma^2/2}\, \Phi\!\left(\frac{\gamma + \sigma^2}{\sigma}\right) - e^{2\gamma + 2\sigma^2}\, \Phi\!\left(\frac{\gamma + 2\sigma^2}{\sigma}\right)\right]$$

where $c = \log(ak-1)/\lambda$ ($-\infty$ if $ak < 1$), $\gamma = \beta - (r + \tfrac{1}{2}|\kappa|^2)T$, $\sigma^2 = |\nu - \kappa|^2 T$. Solving this numerically for λ isn't too hard. To compute wealth process, and therefore the portfolio, we need to do this calculation at intermediate times.

4). We shall suppose that the benchmark is a traded asset/portfolio, so we shall have

$$V_t = \exp\left[v \cdot W_t + \left(r + \tfrac{1}{2}|\kappa|^2 - \tfrac{1}{2}|\eta|^2 \right) t \right] \qquad (\eta \equiv v - \kappa)$$

if we normalise to $V_t = 1$ when $t = 0$. The martingale $\zeta_t V_t$ is a change-of-measure martingale, converting $\mathbb{P}$ to $\tilde{\mathbb{P}}$, where

$$W_t = \tilde{W}_t + t\eta$$

defines a $\tilde{\mathbb{P}}$-Brownian motion $\tilde{W}_t$. We shall then have $\zeta_t V_t = \exp\{\eta \cdot \tilde{W}_t + \tfrac{1}{2}|\eta|^2 t\}$,

$$\tilde{W}_t = \tilde{E}_t[\tilde{W}_T],$$

$$\tilde{E}_t[\zeta_T V_T] = \exp\{\eta \cdot \tilde{W}_t - \tfrac{1}{2}|\eta|^2 t\}\, e^{|\eta|^2 T},$$

$$\tilde{E}_t\left[(ak - 1 - \lambda \zeta_T V_T)^+ \right] = \tilde{E}_t\left[\left(ak - 1 - \lambda \zeta_t V_t\, e^{Z + \frac{1}{2}|\eta|^2 \tau} \right)^+ \right] \qquad \begin{matrix} \tau \equiv T - t \\ Z \sim N(0, |\eta|^2 \tau) \end{matrix}$$

$$= \theta \left\{ e^b\, \Phi(b/\sigma) - e^{\sigma^2/2}\, \Phi\!\left(\frac{b - \sigma^2}{\sigma} \right) \right\}$$

where $\theta \equiv \lambda \zeta_t V_t \exp(\tfrac{1}{2}|\eta|^2 \tau)$, $\sigma^2 = \tau |\eta|^2$, $b = \log\{(ak-1)/\theta\}$.

More on implied volatility (14/7/06)

1) We can prove that if $\Sigma(t,T,K) = \xi_t + \Sigma(T-t,K)$, then the process ξ must be non-decreasing. One way to try to prove that ξ cannot increase would be to look for some contingent claim which is initially worth almost nothing, but becomes worth something $O(1)$ on the event that ξ rises. The snag with this is that the initial valuation uses $v(T,\cdot)$, whereas the valuation at time t (when $\xi_t \geq \varepsilon$) uses $v(T-t,\cdot)$, and this can be <u>much</u> smaller than $v(T,\cdot)$ ...

2) Another approach is to try to argue that a rise in ξ would lead to a violation of the convexity of $C(T,K)$. Assuming a little differentiability, we have

$$\frac{\partial}{\partial K} C(T,K) = \frac{\partial}{\partial K} F(K, v(T,K)) = F_K + F_v v_K$$

$$0 \leq \frac{\partial^2 C}{\partial K^2} = F_{KK} + 2F_{Kv} v_K + F_{vv} v_K^2 + F_v v_{KK}$$

so dividing by F_v and using the relations on the reverse of p1 we conclude that

$$0 \leq \frac{2}{K^2} + \frac{2}{K}\left\{\frac{1}{2} - \frac{\log K}{v}\right\} v_K + \frac{1}{2} v_K^2 \left\{\left(\frac{\log K}{v}\right)^2 - \frac{1}{v} - \frac{1}{4}\right\} + v_{KK}$$

Let's now fix T, and write $v(T,K) = h(\log K)$, so $v_K = \frac{1}{K} h'$, $v_{KK} = \frac{1}{K^2}(h'' - h')$, leading to

$$0 \leq 2 + 2\left\{\frac{1}{2} - \frac{k}{h}\right\} h' + \frac{1}{2} h'^2 \left\{\left(\frac{k}{h}\right)^2 - \frac{1}{h} - \frac{1}{4}\right\} + h'' - h'$$

$$= 2 - \frac{2kh'}{h} + \frac{1}{2}\left(\frac{kh'}{h}\right)^2 - \frac{1}{2} h'^2 \left(\frac{1}{h} + \frac{1}{4}\right) + h''$$

$$= \frac{1}{2}\left(\frac{kh'}{h} - 2\right)^2 - \frac{1}{2} h'^2 \left(\frac{1}{h} + \frac{1}{4}\right) + h''$$

Now what we believe is that $h(k) = T(\xi + g(k))^2$, $g(k) = \Sigma(T, e^k)$, where the function g doesn't change (T fixed!) but ξ may vary, $\xi \geq 0$. Reworked in terms of this, the inequality reads

$$0 \leq \frac{1}{2}\left(\frac{2kg'}{\xi + g} - 2\right)^2 - \frac{1}{2} T^2 g'^2 (\xi + g)^2 + 2T g'' (\xi + g)$$

It is now evident that for each k, T fixed, since this expression is tending to $-\infty$ as $\xi \uparrow \infty$, <u>ξ must be bounded above</u>. Without some further knowledge about g, it seems hard to convert this into more precise bounds. However, we can conclude that there is some call price function $\tilde{C} \geq C$ such that

$$\boxed{S_t C(\tau, K/S_t) \leq C(t,T,K) \leq S_t \tilde{C}(\tau, K/S_t)} \qquad \tau \equiv T-t.$$

3) Another approach would be to use the fact that

$$C(t, T, K) = S_t \; F(Y_t, V_t)$$

is a martingale, where $Y_t \equiv K/S_t$, $V_t \equiv (T-t)\left(\xi_t + \Sigma(\tau, K/S_t)\right)^2$, $\tau \equiv T-t$.
To keep it simple, let's suppose that S is a continuous martingale,

$$dS_t = \sigma_t S_t \, dW_t \quad \Rightarrow \quad dY_t = Y_t\left(-\sigma_t \, dW_t + \sigma_t^2 \, dt\right).$$

Now we suppose

$$dV_t = a_t \, dW_t + b_t \, dt + \theta_t \, d\xi_t$$

and return to make this explicit later. We have

$$dF = F_K \, dY + F_V \, dV + \tfrac{1}{2}\sigma^2 Y^2 F_{KK} - \sigma a Y F_{KV} + \tfrac{1}{2} a^2 F_{VV}$$

$$= \left(aF_V - \sigma Y F_K\right) dW + \left[Y\sigma^2 F_K + b F_V + \tfrac{1}{2}\sigma^2 Y^2 F_{KK} - \sigma a Y F_{KV} + \tfrac{1}{2} a^2 F_{VV}\right] dt + \theta F_V \, d\xi$$

Hence

$$d(SF) = S\left[\, F\sigma \, dW + dF + \sigma\left(aF_V - \sigma Y F_K\right) dt \,\right]$$

and so

$$S^{-1} d(SF) \doteq \left[\, \sigma a F_V + bF_V + \tfrac{1}{2}\sigma^2 Y^2 F_{KK} - a\sigma Y F_{KV} + \tfrac{1}{2} a^2 F_{VV} \right] dt + \theta F_V \, d\xi$$

Divide by F_V and use the relations on the reverse of p 1 to discover

$$0 \doteq \theta \, d\xi + \left[\, \sigma a + b + \sigma^2 - \sigma a\left(\tfrac{1}{2} - \frac{\log Y}{V}\right) + \tfrac{1}{4} a^2\left(\left(\frac{\log Y}{V}\right)^2 - \frac{1}{V} - \frac{1}{4}\right)\right] dt$$

$$= \theta \, d\xi + \left[\, b + \frac{\sigma a}{2} + \frac{1}{4}\left(\frac{a \log Y}{V} + 2\sigma\right)^2 - \frac{1}{4} a^2\left(\frac{1}{V} + \frac{1}{4}\right)\right] dt.$$

Now we expand dV to make a, b, θ more explicit:

$$dV = -(\xi+\Sigma)^2 dt + \tau\left\{\, 2(\xi+\Sigma)\, d(\xi+\Sigma) + d\langle \Sigma \rangle \right\}$$

$$= -(\xi+\Sigma)^2 dt + 2\tau(\xi+\Sigma)\, d\xi + 2\tau(\xi+\Sigma)\left\{-\Sigma_t \, dt + \Sigma_K \, dY + \tfrac{1}{2}\Sigma_{KK} \, d\langle Y \rangle\right\} + \tau\sigma^2 Y^2 \Sigma_K^2 \, dt$$

$$= -2\tau(\xi+\Sigma)\Sigma_K \sigma Y \, dW + 2\tau(\xi+\Sigma)\, d\xi + \left[\tau\sigma^2 Y^2 \Sigma_K^2 - (\xi+\Sigma)^2 + 2\tau(\xi+\Sigma)\left\{\tfrac{1}{2}\sigma^2 Y^2 \Sigma_{KK} + \sigma^2 Y \Sigma_K - \Sigma_t\right\}\right] dt$$

$$\therefore \quad \begin{cases} a = -2\tau(\xi+\Sigma)\sigma\Sigma_K \\ b = \sigma^2 \tau Y^2 \Sigma_K^2 - (\xi+\Sigma)^2 + 2\tau(\xi+\Sigma)\left(\tfrac{1}{2}\sigma^2 Y^2 \Sigma_{KK} + \sigma^2 Y \Sigma_K - \Sigma_t\right) \\ \theta = 2\tau(\xi+\Sigma) \end{cases}$$

A stochastic optimal control problem from insurance (7/8/06)

1) In a paper of Emms, Haberman + Savoulli, the wealth of an insurance company w_t obeys the dynamics

$$(1) \qquad dx_t = x_t(\sigma dW_t + \mu dt)$$

$$(2) \qquad dq_t = aq_t(1 - p_t/x_t)dt$$

$$(3) \qquad dw_t = -\alpha w_t + q_t(p_t - \pi)dt$$

Dual:

$$(2') \qquad dy_1 = y_1 b_1 dt$$

$$(3') \qquad dy_2 = y_2 b_2 dt$$

where x_t is interpreted as the 'market average' premium, p_t (the control variable) is the premium charged by the company, π (a constant) is its break-even premium, and q_t is the quantity of business which it does. The constants α, a are positive. The firm's objective is

$$\max E\left[\int_0^T e^{-\beta t} w_t \, dt\right] = V(T, w_0, q_0, x_0).$$

Though EHS don't appear to realise it, this problem can be solved very completely.

2) Let us observe straight away two properties of the solution; it is clear from the linearity of the dynamics that for any $\lambda > 0$,

$$V(T, \lambda w_0, \lambda q_0, x_0) = \lambda V(T, w_0, q_0, x_0)$$

Moreover, since

$$w_t = e^{-\alpha t}\left\{w_0 + \int_0^t e^{\alpha s} q_s(p_s - \pi)ds\right\}$$

it follows that

$$V(T, w_0, q_0, x_0) = w_0 \frac{1 - e^{-(\alpha+\beta)T}}{\alpha + \beta} + V(T, 0, q_0, x_0)$$

$$= w_0 \frac{1 - e^{-(\alpha+\beta)T}}{\alpha + \beta} + q_0 V(T, 0, 1, x_0)$$

So what we have to do is get hold of $V(T, 0, 1, x_0)$ as explicitly as we can.

3) Let's do a duality approach to this problem, introducing the Lagrange multiplier processes (2'), (3'). Then the Lagrangian problem is

$$\max_{p \geq 0, q \geq 0, w} E\left[\int_0^T \left(e^{-\beta t} w_t - y_1 a q(1 - p/x) - y_2(-\alpha w + q(p - \pi))\right)dt + \Big[y_1 q + y_2 w\Big]_0^T - \int_0^T (q b_1 y_1 + w y_2 b_2)\, dt\right]$$

$$= \max_{\substack{p,q\ge 0\\ w}} E\Big[\, y_1(T)q_T + y_2(T)w_T - y_1(0)q_0 - y_2(0)w_0 + \int_0^T \Big\{ e^{-\beta t}w_t - y_1 a q(1-p/x) - y_2(-\alpha w + q(p-\pi)) - q y_1 b_1 - w y_2 b_2 \Big\}\, dt \,\Big]$$

Maxing over w, p, q gives respectively the dual-feasibility conditions

(4) $$e^{-\beta t} + \alpha y_2 - b_2 y_2 = 0$$

(5) $$\frac{a\, y_1(t)}{x_t} - y_2(t) \le 0$$

(6) $$-a y_1(t) + \pi y_2(t) - b_1(t)\, y_1(t) \le 0$$

and terminal conditions $y_2(T) = 0 \ge y_1(T)$. We expect $\overset{5}{(6)}$ to hold with equality at optimality while $q_t > 0$. We get the ODE $\alpha y_2 - \dot y_2 + e^{-\beta t} = 0$ from (4), which has solution

$$\boxed{y_2(t) = \frac{e^{\alpha t}}{(\alpha+\beta)}\Big\{ e^{-(\alpha+\beta)T} - e^{-(\alpha+\beta)t}\Big\} \le 0}$$

vanishing at $t=T$. Equation (6) gives

$$(\dot y_1 + a y_1)e^{at} = \frac{d}{dt}\big(e^{at} y_1(t)\big) \ge \pi y_2(t)\, e^{at}$$

with equality up til the first time S that q hits zero. We may as well solve with equality thereafter since it doesn't make the terminal condition $y_1(T) \le 0$ harder to attain, and it keeps y_1 as low as possible. This gives the solution

$$e^{at} y_1(t) = C + \frac{\pi}{\alpha+\beta}\Big\{ e^{-(\alpha+\beta)T}\,\frac{e^{(\alpha+a)t}}{\alpha+a} - \frac{e^{(a-\beta)t}}{a-\beta}\Big\}$$

whence

$$\boxed{y_1(t) = C e^{-at} + \frac{\pi e^{-\beta t}}{\alpha+\beta}\Big\{ \frac{e^{(\alpha+\beta)(t-T)}}{\alpha+a} - \frac{1}{a-\beta}\Big\}}$$

How is the constant C chosen? To satisfy (5), of course (which guarantees the terminal condition $y_1(T) \le 0$);

$$\boxed{C = \min_{0\le t\le T}\Big\{ \frac{x_t\, y_2(t)\, e^{at}}{a} - \frac{\pi e^{(a-\beta)t}}{\alpha+\beta}\Big(\frac{e^{(\alpha+\beta)(t-T)}}{\alpha+a} - \frac{1}{a-\beta}\Big)\Big\}}$$

Of course, this is a singular optimal control problem, where $\xi_t = \int_0^t p_s\, ds$ is the non-decreasing – –

Data-generated prediction (9/8/06)

Suppose we have some discrete-time vector-valued process $(X_t)_{t\geq 0}$ which we think might in some way be explanatory for a real-valued observation process $(y_t)_{t\geq 0}$. How could we start to explore any such relationship?

One idea is to fix some window length K, and consider the sequence

$X_{[t-K,t]}$, $t = K, \ldots, K+T$ as being explanatory for the values $y_{K+1}, \ldots y_{K+T+1}$ respectively; thus we are thinking of some association $\begin{pmatrix} X_{[t-K,t]} \\ y_{t+1} \end{pmatrix}$

If we rank the y_t, then for $p \in (0,1)$ we can split the population $\{X_{[t-K,t]} : t = K, \ldots, K+T\}$ according to whether y_{t+1} is bigger or less than the p^{th} order statistic of the y's. Now we can express the difference between the two subpopulations of X-paths in terms of the difference between the sample means of the two subpopulations; this vector gives the best linear discriminator between the two subpopulations. Renormalise this vector to $v(p)$, of length 1, and if necessary flip the sign so that the dot product with $v(0.5)$ is positive. A good discriminator would have the property that $v(p)$ would not change a lot with p, and this could then be used to predict the changes in y.

Some thoughts on law-invariant utilities (23/8/06)

1) Let's suppose we have probability space $(\Omega, \mathcal{F}, P)$ rich enough to support a $U[0,1]$ random variable (so we could suppose $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}([0,1])$, $P = \text{Leb}$ if we wanted to) and take (for now) $H = L^2(\Omega, \mathcal{F}, P)$. A law-invariant utility U is a map $U : H \to \mathbb{R}$ with the properties

(i) U is monotone : $X \le Y \Rightarrow U(X) \le U(Y)$

(ii) U is concave

(iii) if $X, X' \in H$ have the same law, then $U(X) = U(X')$.

<u>Remarks</u>: Certainly there exist LIUs; for example, if $u : \mathbb{R} \to \mathbb{R}$ is concave increasing, then $U(X) = E\, u(X)$ is a LIU. Moreover, if $F : \mathbb{R}^N \to \mathbb{R}$ is concave and increasing, and $U_1, \dots, U_N$ are LIUs, then so is

$$X \longrightarrow F(U_1(X), \dots, U_N(X))$$

2) What about the duals? If U is a LIU, we define for $Y \in H$

$$\tilde{U}(Y) = \sup_{X \in H} \{ U(X) - \langle X, Y \rangle \}$$

If we assume that U is <u>relevant</u> in the sense that $X' \ge X$, $P(X' > X) > 0 \Rightarrow U(X') > U(X)$, then $\tilde{U}(Y)$ is infinite if $P(Y < 0) > 0$.

Can we say more about $\tilde{U}$? Well, if $\mathcal{D}$ denotes the set of finite-variance distribution functions, we have

$$\tilde{U}(Y) = \sup_{F \in \mathcal{D}} \sup_{X \sim F} \{ U(X) - \langle X, Y \rangle \}$$

$$= \sup_{F \in \mathcal{D}} \left[U(X^F) - \inf_{X \sim F} \langle X, Y \rangle \right]$$

where X^F is any random variable with law F. Now if q_X denotes the quantile function of random variable X, writing also q_F for quantile f$^{\underline{n}}$ of CDF F, we have

$$\inf_{X \sim F} E(XY) = \int_0^1 q_F(t)\, q_Y(1-t)\, dt$$

so we see that $\tilde{U}$ is also law-invariant.

<u>Can we characterise LIUs?</u>

3) For the simple case $U(X) = E\, u(X)$, we get

$$\tilde{U}(Y) = \sup_{X \in H} E\big(u(X) - XY\big) = E\, \tilde{u}(Y)$$

(assuming $(u')^{-1}(Y)$ is in L^2 ... if not, we'll need to do some approximation/truncation, which will require conditions.

Dual of $\Phi(U_1(X), \dots, U_N(X))$ doesn't look so easy... is there some other condition which a LIU ought to satisfy??

[This is also in the appendix to Ingersoll, Skelton, Weil, JFQA **13**, 627-650, 1978]

Making arbitrage when the yield curve makes parallel shifts (1/9/06)

1) It is not impossible that the yield curve moves by parallel jumps. If we suppose the riskless rate is a subordinator with characteristic exponent ψ, then

$$P(t,T) = E_t \exp\{-\int_t^T r_s\,ds\}$$

$$= \exp\{-(T-t)r_t\}.\ E\left[\exp\{-\int_0^{T-t} r_u\,du\}\ \Big|\ r_0=0\right]$$

$$= \exp\left[-(T-t)r_t + \int_0^{T-t} \psi(u-T+t)\,du\right]$$

so that
$$y(t,T) = r_t - \frac{1}{T-t}\int_0^{T-t} \psi(-s)\,ds$$

is the yield curve, which evidently does move by parallel shifts. When r is a Poisson process of rate λ, $\psi(s) = \lambda(e^s-1)$, and $\lim_{T\to\infty} y(t,T) = r_t + \lambda$ is the long rate, increasing as we know it must. (Mike Tehranchi has this argument too)

2) Can we make arbitrage in the case where the yield curve makes parallel shifts, if we impose some simple condition? If we let $P_k \equiv P(0,k)$ be initial discount curve, and we choose a portfolio of w_k bonds of maturity $k = 1,2,\ldots$, then the value of this portfolio at time 1 will be

$$H(z) \equiv \sum_{k\geq 1} w_k P_{k-1} z^{k-1} - \Big(\sum_{k\geq 1} w_k P_k\Big)/P_1$$

(portfolio plus the cost of paying back the initial investment. Here, $z = e^{\eta}$, where η is the shift that occurred in the yield curve). We want H to be minimised at $z=0$, to value 0. If we write $a_k = w_{k+1} P_k$ $(k=0,1,\ldots)$ then

$$H(z) = \sum_{k\geq 0} a_k z^k - \sum_{k\geq 0} a_k P_{k+1}/P_k P_1$$

Since the $k=0$ terms cancel in any case, we may as well suppose $a_0=0$. If we want $H(1)=0$, $H(z)\geq H(1)$, we must have

$$\sum_{k\geq 1} a_k d_k \equiv \sum_{k\geq 1} a_k\left(1 - P_{k+1}/P_k P_1\right) = 0$$

$$\sum_{k\geq 1} k a_k = 0, \quad \sum_{k\geq 1} k^2 a_k > 0$$

Not clear how to do this... Mike has a result that if (i) $\limsup_{T\to\infty} y_0(T) < \infty$ (ii) $\sup E|r_t| < \infty$ then there cannot be parallel shifting.

Forward utilities (4/9/06)

(i) Mike reported that there was an interesting talk of Thaleia in Tokyo on the following problem. Suppose we have a single risky asset $dS_t = S_t(\sigma dW_t + \mu dt)$ and a riskless bank account bearing interest at constant rate r. When can we have some map $u : [0,\infty) \times \mathbb{R}^{++} \to \mathbb{R}$ such that for all $0 \le t < T$

$$u(t,x) = \sup E\left[u(T, w_T) \,\middle|\, w_t = x \right] ?$$

Clearly, the example where $u(t,x) = U(x) \exp\{-(1-R)(r + \frac{1}{2}\sigma^2 R \pi^2)t\}$ is the familiar Merton CRRA example $[U(x) = x^{(1-R)}/(1-R), \quad \pi = (\mu - r)/\sigma^2 R]$ but are there other examples? Can we characterise them?

(ii) If we take the HJB equation, we shall have

$$\sup_\theta \left[u_t + (rx + (\mu - r)\theta) u_x + \frac{\theta^2}{2}\sigma^2 u_{xx} \right] = 0$$

$$= u_t + rx u_x - \frac{1}{2}\kappa^2 u_x^2 / u_{xx} \qquad \kappa = \frac{\mu - r}{\sigma}$$

Thus if we do the usual change to dual variables $y = u_x(t,x)$, $J(t,y) = u(t,x) - xy$, we have $J_t = u_t$, $J_y = -x$, $J_{yy} = -1/u_{xx}$, and the HJB equation linearises to

$$J_t - ry J_y + \frac{1}{2}\kappa^2 y^2 J_{yy} = 0.$$

It's clear that for each t, the function $y \mapsto J(t,y)$ is convex decreasing. If we change variables (t,y) to (t,z) where $z \equiv \log y + (r + \frac{1}{2}\kappa^2)t$, then write $J(t,y) = g(t,z)$, then the equation simplifies to

$$g_t + \frac{1}{2}\kappa^2 g_{zz} = 0,$$

the heat equation. Now we know that g is decreasing in z, so $-g_z$ is a non-negative function solving the heat equation; therefore for some (non-negative) measure m we have the representation

$$-g_z(t,z) = \int_{\mathbb{R}} m(dc) \exp\{ cz - \frac{1}{2}c^2\kappa^2 t \}$$

Also, $J_y = e^{-z + (r + \frac{1}{2}\kappa^2)t} g_z$ is increasing in z, so this will imply that m has to be concentrated on $(-\infty, 1]$. Integrating the above expression for $-g_z$ leads us to the conclusion that

$$g(t,z) = \varphi(t) - \int_{-\infty}^{1} \frac{m(dc)}{c} \exp\{cz - \tfrac{1}{2}c^2 k^2 t\}$$

which has to satisfy the heat equation (so φ is constant) and must be decreasing in z. For the CRRA Merton example, we find after a few calculations that $\varphi = 0$, and m is a unit mass at $1 - 1/R < 1$, so these models are in some sense the extremal examples.

(iii) Parameter uncertainty? If posterior for $\alpha \equiv \mu/\sigma$ at time t is $N(\hat{\alpha}_t, \tau_t^{-1})$ then the dynamics we have is that

$$\begin{cases} d\tau = dt \\ d\hat{\alpha} = d\hat{W}/\tau \\ dw = rw\,dt + \theta\left(\sigma(d\hat{W} + \hat{\alpha}\,dt) - r\,dt\right) \end{cases}$$

HJB and duality goes much as before to deliver ($J(t,y,\alpha,\tau)$ is the function)

$$J_t - ry J_y + J_\tau + \tfrac{1}{2}\left\{(\alpha - r/\sigma)^2 y^2 J_{yy} - 2(\alpha - r/\sigma) y J_{y\alpha}/\tau + J_{\alpha\alpha}/\tau^2\right\} = 0$$

$\tilde{F}^2(k, v) \equiv E\left(S_v \wedge e^k\right)$ if $dS_t = S_t \, dW_t$ $\longrightarrow$

Implied vol: Steve Ross' conjecture again (5/9/06)

1) Suppose we were to have a parallel shifting of the IV surface; we know that this has to be upward, so for any $0<\theta<1$ we can consider

$$\int_{-\infty}^{\infty} e^{-\theta k}\, E_t\left(S_T \wedge e^k\right) dk = \frac{1}{\theta(1-\theta)}\, E_t\left(S_T^{1-\theta}\right) \qquad \text{by usual sort of things}$$

$$= \int e^{-\theta k}\, S_t\, E_t\left(\frac{S_T}{S_t} \wedge \frac{e^k}{S_t}\right) dk$$

$$= S_t \int e^{-\theta k}\, \tilde{F}\left(k-\log S_t,\ (T-t)\left(\Sigma(T-t, k-\log S_t)+\xi_t\right)^2\right) dk$$

$$= S_t^{1-\theta} \int e^{-\theta z}\, \tilde{F}\left(z,\ (T-t)\left(\Sigma(T-t, z)+\xi_t\right)^2\right) dz \qquad k = z+\log S_t$$

$$\equiv S_t^{1-\theta}\, g(T-t, \xi_t)$$

where the function g is decreasing in both arguments. We may transform also in time to arrive at a martingale

$$M_t = E_t \int_0^{\infty} \frac{e^{-\lambda T}}{\theta(1-\theta)}\, S_T^{1-\theta}\, dT$$

$$= \int_0^t \frac{e^{-\lambda u}}{\theta(1-\theta)}\, S_u^{1-\theta}\, du + e^{-\lambda t}\, S_t^{1-\theta}\, f(\xi_t),$$

where $f(x) \equiv \int_0^{\infty} e^{-\lambda u} g(u,x)\, du$ is decreasing, positive. Any use??

2) We can also similarly consider for $\alpha>0$

$$\int e^{\alpha k}\, E_t\left(S_T - e^k\right)^+ dk = \frac{1}{\alpha(\alpha+1)}\, E_t\, S_T^{1+\alpha}$$

$$= S_t^{1+\alpha}\, h(T-t, \xi_t)$$

where h increases in both arguments;

$$h(\tau,\xi) \equiv \int e^{\alpha k}\, F\left(k,\ \tau\left(\Sigma(\tau,k)+\xi\right)^2\right) dk$$

?? Need $a = (r + \frac{1}{2}|\kappa|^2)T + \log(\lambda)$

Taking $I(y) = b y^{-1/R_1} + (1-b) y^{-1/R_2}$, we get (for $Z \sim N(0,1)$)

$$E\, I(e^{a+vZ}) = b \exp\left(-\frac{a}{R_1} + \frac{v^2}{2R_1^2}\right) + (1-b)\exp\left(-\frac{a}{R_2} + \frac{v^2}{2R_2^2}\right)$$

$$E\, I(e^{a+vZ})^2 = b^2 \exp\left\{-\frac{2a}{R_1} + \frac{2v^2}{R_1^2}\right\} + 2b(1-b)\exp\left\{-\frac{a}{R_1} - \frac{a}{R_2} + \frac{v^2(R_1+R_2)^2}{2R_1^2R_2^2}\right\}$$

$$+ (1-b)^2 \exp\left\{-\frac{2a}{R_2} + \frac{2v^2}{R_2^2}\right\}$$

which allows us to compute the Sharpe ratio.

Dist^n of w_T? We write $w_T \equiv I(\lambda \zeta_T) = h(W_T/\sqrt{T})$, where $h(z) = I(\lambda \exp(a - vz))$ with $Z \sim N(0,1)$, $a = -(r + \frac{1}{2}|\kappa|^2)T$, $v = |\kappa|\sqrt{T}$. Then

$P[w_T \leq x] = P[h(Z) \leq x] = P[Z \leq h^{-1}(x)]$

$\varphi(z) = e^{-z^2/2}/\sqrt{2\pi}$

$\Rightarrow P[w_T \in dx]/dx = \varphi(h^{-1}(x)) / h'(h^{-1}(x))$

But $h'(z) = v\left\{a y^{-1/R_1}/R_1 + \theta y^{-1/R_2}/R_2\right\}$, $y \equiv \lambda \exp(a - vz)$,

so we could plot the density by choosing some vector u, setting $x = h(u)$, and plotting $\varphi(u)/h'(u)$ against x.

The question I should have got Hanping to work on (6/9/06)

We try to do optimal investment to horizon T, so we get

$$w_T = I(\lambda \zeta_T)$$

but the thing to do is to choose the utility so that

$$I(y) = y^{-1/R_1} + \theta\, y^{-1/R_2}$$

$$\left[\zeta_t = e^{-rt - \frac{1}{2}|\kappa|^2 t - \kappa \cdot W_t}\right. \qquad \kappa \equiv \sigma^{-1}(\mu - r)$$

Then the budget equation reads

$$w_0 = E\,\zeta_T w_T = E\,\zeta_T I(\lambda\zeta_T)$$

$$= \lambda^{-1/R_1} E\,\zeta_T^{1-1/R_1} + \theta \lambda^{-1/R_2} E\,\zeta_T^{1-1/R_2}$$

$$= \lambda^{-1/R_1} \exp\left((\tfrac{1}{R_1} - 1) T (r + \tfrac{1}{2}|\kappa|^2/R_1) \right) + \theta \lambda^{-1/R_2} \exp\left((\tfrac{1}{R_2} - 1) T (r + \tfrac{1}{2}|\kappa|^2/R_2) \right)$$

The wealth at intermediate times is

$$w_t = \frac{1}{\zeta_t} E_t\left[\zeta_T w_T\right]$$

$$= \frac{1}{\zeta_t}\left\{ \lambda^{-1/R_1} E_t\left[\zeta_T^{1-1/R_1}\right] + \theta\lambda^{-1/R_2} E_t\left[\zeta_T^{1-1/R_2}\right] \right\}$$

$$= \lambda^{-1/R_1} \zeta_t^{-1/R_1} E_t\left[\left(\frac{\zeta_T}{\zeta_t}\right)^{1-1/R_1} \right] + \theta \lambda^{-1/R_2} \zeta_t^{-1/R_2} E_t\left[\left(\frac{\zeta_T}{\zeta_t}\right)^{1-1/R_2} \right]$$

$$= \lambda^{-1/R_1} \zeta_t^{-1/R_1} \exp\{ (\tfrac{1}{R_1} - 1)\tau (r + \tfrac{1}{2}|\kappa|^2/R_1) \}$$

$$+ \theta \lambda^{-1/R_2} \zeta_t^{-1/R_2} \exp\{ (\tfrac{1}{R_2} - 1)\tau (r + \tfrac{1}{2}|\kappa|^2/R_2) \} \qquad (\tau \equiv T - t)$$

$$\equiv F(t, W_t).$$

We can find the portfolio from the martingale parts of the two sides; $\theta_t = (\sigma^T)^{-1} \nabla F(t, W_t)$.

$$\theta_t = \left[\frac{\lambda^{-1/R_1}}{R_1} \zeta_t^{-1/R_1} \exp\{ (\tfrac{1}{R_1} - 1)\tau (r + \tfrac{1}{2}|\kappa|^2/R_1) \} + \theta \frac{\lambda^{-1/R_2}}{R_2} \zeta_t^{-1/R_2} \exp\{ (\tfrac{1}{R_2} - 1)\tau (r + \tfrac{1}{2}|\kappa|^2/R_2) \} \right].$$

$$\cdot\, (\sigma\sigma^T)^{-1}(\mu - r)$$

which is a weighted average of the Merton portfolios for R_1, R_2.

Could now work out $E\,W_T$, $\mathrm{var}(W_T)$ + find Sharpe ratio. To use the analysis in practice, if we see S, we know W and therefore ζ, and whenever we see our w we work out what λ must be (most likely changing with t). Then we take that value of λ in calculating the portfolio to use.

Estimating $(\sigma\sigma^T)^{-1}$ (11/9/06)

1) When it comes to computing the Merton proportions, we need $(\sigma\sigma^T)^{-1}(\mu - r)$, and the question arises how to estimate not $\sigma\sigma^T$, but rather $(\sigma\sigma^T)^{-1}$, hopefully unbiasedly.

Suppose we have $X_1, \ldots, X_N$ IID $N(\mu, \Sigma)$, p-vectors, and we form

$$V = \sum_1^N (X_i - \bar{X})(X_i - \bar{X})^T$$

which has density given by $(n \equiv N-1)$

$$|V|^{(n-p-1)/2} \exp\left\{-\tfrac{1}{2} \operatorname{tr} \Sigma^{-1} V\right\} \Big/ 2^{np/2} \pi^{p(p-1)/4} |\Sigma|^{n/2} \prod_{i=1}^{p} \Gamma\left(\tfrac{1}{2}(n+1-i)\right)$$

the Wishart density. From this we deduce that the e-values of V are IID with density proportional to

$$x^{(n-p-1)/2} e^{-x/2}$$

which is to say, λ_i are χ^2_{N-p} distributed (in the case where $\Sigma = I$, of course)

2) It is clear that when $\Sigma = I$, the law of V^{-1} is rotation invariant, and therefore

$$E V^{-1} \text{ is a multiple of } I, \quad E V^{-1} = cI, \text{ say.}$$

By considering the trace, we see that

$$pc = p\, E(\lambda_1^{-1}) = p \int_0^\infty \frac{1}{x} \left(\frac{x}{2}\right)^{\frac{N-p}{2}-1} e^{-x/2} \frac{dx/2}{\Gamma((N-p)/2)}$$

$$\Rightarrow \boxed{c = 1/(N-p-2)} \qquad \text{and we need } N \geq p+3$$

3) For general covariance Σ, it is now easy to deduce that

$$E V^{-1} = \frac{1}{N-p-2} \cdot \Sigma^{-1}$$

For big N, it shouldn't really make a great deal of difference

[cf WN XXV, p 28]

(Stock price? Shares of consumption)

Many Bayesian agents (13/9/06)

1) Let's go back to the idea of multiple Bayesian agents, all with the same observations, but with different priors. Suppose that the output return sequence

$$\xi_t = \log(y_t / y_{t-1}), \quad t = 1, 2, \ldots$$

are IID, and for simplicity that they are $N(\mu, \tau^{-1})$, where all agents know the value of τ, but agent j has a $N(\hat{\mu}_j(0), \tau_j(0)^{-1})$ prior over μ. The reference probability we take to be IID $N(0,\tau)$ for the ξ_t, and with respect to this the LR process for agent j will be (after some calculations)

$$\Lambda^j_T = \exp\left[-\frac{\tau_j(0)}{2}(\mu - \hat{\mu}_j(0))^2 - \frac{\tau}{2}\sum_{t=1}^T (\xi_t - \mu)^2 + \frac{\tau}{2}\sum_1^T \xi_t^2 \right] \Big/ \sqrt{2\pi/\tau_j(0)}$$

$$= \exp\left[-\frac{\tau_j(T)}{2}(\mu - \hat{\mu}_j(T))^2 - \frac{1}{2}\frac{\tau T \tau_j(0)}{\tau_j(T)}(\bar{\xi}_T - \hat{\mu}_j(0))^2 + \frac{\tau T}{2}\bar{\xi}_T^2 \right] \sqrt{\tau_j(0)/2\pi}$$

where $\tau_j(T) \equiv \tau_j(0) + T\tau$, $\hat{\mu}_j(T) = (\tau_j(0)\hat{\mu}_j(0) + \tau T \bar{\xi}_T)/\tau_j(T)$. Integrating out the parameter μ, we shall get

$$\tilde{\Lambda}^j_T = \left(\frac{\tau_j(0)}{\tau_j(T)}\right)^{1/2} \exp\left[-\frac{\tau T \tau_j(0)}{2\tau_j(T)}(\bar{\xi}_T - \hat{\mu}_j(0))^2 + \frac{\tau T}{2}\bar{\xi}_T^2 \right]$$

2) Combining these is probably only going to work for CARA utilities, so let's suppose that $U_j'(x) = \exp(-\gamma_j x)$, with discount factor β_j for agent j. Then the FOC will be

$$\beta_j^t \tilde{\Lambda}^j_t \exp(-\gamma_j c^j_t) = \lambda_j \zeta_t$$

If $\bar{c}_t \equiv \sum_j c^j_t$ is aggregate consumption, $\Gamma^{-1} = \sum \gamma_j^{-1}$, $\log L_t \equiv \Gamma \sum \gamma_j^{-1} \log \tilde{\Lambda}^j_t$ then we get (to within a constant multiple)

$$\zeta_t = \bar{\beta}^t L_t \exp(-\Gamma \bar{c}_t) \qquad \left(\log \bar{\beta} \equiv \Gamma \sum \gamma_j^{-1} \log \beta_j\right)$$

(in this context, probably makes more sense to take $\xi_t = y_t - y_{t-1}$, $\bar{c}_t \equiv y_t$.)

3) Just as a first notion, suppose the agents have identical preferences: $\beta_j = \beta$, $\gamma_j = \gamma$, but differ only in their prior information. Then $\Gamma = \gamma/N$, $\bar{\beta} = \beta$,

$$\log L_T = \frac{1}{N}\sum \log \tilde{\Lambda}^j_T = \frac{\tau T}{2}\bar{\xi}_T^2 + \frac{1}{2N}\sum \log(\tau_j(0)/\tau_j(T)) - \frac{\tau T}{2N}\sum_1^N \frac{\tau_j(0)}{\tau_j(T)}(\bar{\xi}_T - \hat{\mu}_j(0))^2$$

This leads to an expression for the consumption stream of agent j,

$$\boxed{\begin{aligned}\gamma c^j_T = \ & \frac{1}{2}\log\left(\frac{\tau_j(0)}{\tau_j(T)}\right) - \frac{\tau T}{2}\,\frac{\tau_j(0)}{\tau_j(T)}\left(\bar{\xi}_T - \hat{\mu}_j(0)\right)^2 - \log\lambda_j \\ & -\frac{1}{2N}\sum_{i=1}^{N}\log \tau_i(0)/\tau_i(T) + \frac{\tau T}{2N}\sum_{i=1}^{N}\frac{\tau_i(0)}{\tau_i(T)}\left(\bar{\xi}_T - \hat{\mu}_j(0)\right)^2 + \frac{\gamma}{N}\sum_{i=1}^{N} c^i_T\end{aligned}}$$

where $\sum \log \lambda_j = 0$ is required. Notice that the agent with $\hat{\mu}_j(0)$ closest to $\bar{\xi}_T$ is doing well at time T. The scaling $\tau_j(0)/\tau_j(T)$ increases to 1 with $\tau_j(0)$. If everyone had the same prior precision, then the only difference comes from the differences in $\hat{\mu}_j(0)$, and the one who predicts nearest $\bar{\xi}_T$ does the best. As T increases, this advantage gets bigger and bigger.

If all agents have the same $\hat{\mu}_j(0)$, then increasing $\tau_j(0)$ helps up to a point $\left(\tau_j(0) = \tau T/\{\tau T(\bar{\xi} - \hat{\mu}(0))^2 - 1\}\right.$, if $\left.\tau T(\bar{\xi} - \hat{\mu}(0))^2 > 1\right)$ but then helps less

Infinite mean wealth with bounded drawdown! (22/9/06)

1) Suppose we have the usual log-Brownian market, $dS = S(\sigma dW + \mu dt)$ with just one risky asset, constant interest rate r, $\kappa \equiv (\mu - r)/\sigma$, and SPD $\zeta_t = \exp(-rt - \frac{1}{2}\kappa^2 t - \kappa W_t)$. At time T, we could make any wealth process $w_T = g(W_T)$ for which

$$E\left[\zeta_T \, g(W_T)\right] = w_0$$

If we know $\kappa > 0$, then we may try using $g(W_T) = \exp(\frac{1}{2} W_T^2/T) \, I_{\{W_T > 0\}}$. The wealth at time $t \in [0, T)$ will be

$$w_t = E_t\left(g(W_T)\, \zeta_T/\zeta_t\right)$$

$$= e^{-r(T-t)} \, E\left[g(W_t + W'_\tau)\, \exp\left(-\tfrac{1}{2}\kappa^2 \tau - \kappa W'_\tau\right)\right] \qquad (\tau \equiv T-t)$$

$$= e^{-r\tau} \, E \, g(W_t + W'_\tau - \kappa\tau)$$

$$= e^{-r\tau} \int_b^\infty \exp\left\{\frac{1}{2T}(W_t + W'_\tau - \kappa\tau)^2 - \frac{1}{2\tau} W_\tau'^2\right\} \frac{dW'_\tau}{\sqrt{2\pi\tau}} \qquad b \equiv \kappa\tau - W_t$$

$$= e^{-r\tau} \int_b^\infty \exp\left\{-\frac{x^2}{2\tau} + \frac{1}{2T}(x-b)^2\right\} \frac{dx}{\sqrt{2\pi\tau}}$$

$$= e^{-r\tau} \int_b^\infty \exp\left\{-\frac{\tau+T}{2\tau T}\left(x - \frac{b\tau}{\tau+T}\right)^2 + \frac{b^2}{2T}\,\frac{2\tau+T}{\tau+T}\right\} \frac{dx}{\sqrt{2\pi\tau}}$$

$$= e^{-r\tau} \exp\left\{\frac{b^2}{2T}\,\frac{2\tau+T}{\tau+T}\right\} \bar{\Phi}\left(\frac{bT}{\tau+T}\sqrt{\frac{\tau+T}{\tau T}}\right)\sqrt{\frac{T}{\tau+T}}$$

$$\equiv g(t, W_t), \text{ say.}$$

Differentiation wrt W_t gives the portfolio $\theta_t = \sigma^{-1} g'(t, W_t)$ of money to be invested in the stock at time t. Explicitly,

$$g'(t, W_t) = \sqrt{\frac{T}{\tau+T}} \exp\left\{-r\tau + \frac{b^2}{2T}\,\frac{2\tau+T}{\tau+T}\right\}\left[\sqrt{\frac{T}{\tau(\tau+T)}}\,\frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\,\frac{b^2 T}{\tau(\tau+T)}\right\} - \frac{b}{T}\,\frac{2\tau+T}{\tau+T}\,\bar{\Phi}\left(b\sqrt{\frac{T}{\tau(\tau+T)}}\right)\right].$$

But does any of this survive uncertainty in the drift μ??

[WORK/CCP/IDE13/BigMean.mw   for the calculation]

2) If $dS = \sigma S(dW + \alpha\, dt)$ where α has a $N(\hat{\alpha}_0, \tau_0^{-1})$ prior, then we observe the process $X_t \equiv W_t + \alpha t = \hat{W}_t + \int_0^t \hat{\alpha}_s ds$, so we shall have that the posterior for α given $\mathcal{F}_t$ has $N(\hat{\alpha}_t, \tau_t^{-1})$ law, $\tau_t \equiv \tau_0 + t$, $\hat{\alpha}_t = (\tau_0 \hat{\alpha}_0 + X_t)/(t + \tau_0)$.

The SPD process this time is

$$\zeta_t = \sqrt{\frac{\tau_t}{\tau_0}} \exp\left[-rt - \tfrac{1}{2} m_t^2 \tau_t + \tfrac{1}{2} m_0^2 \tau_0 \right], \qquad m_t \equiv \hat{\alpha}_t - \sigma^{-1} r.$$

$$\equiv H(t, \hat{\alpha}_t) \quad \text{, say.}$$

The variance of $\hat{\alpha}_t$ is $t/\tau_0 \tau_t \to 1/\tau_0$ as $t \to \infty$, as it should. We could then try to make a terminal wealth

$$g(\hat{\alpha}_T) = \exp\{ \tfrac{1}{2}(\hat{\alpha}_T - \hat{\alpha}_0)^2 \tau_0 \tau_T / T \}$$

which clearly has infinite expectation, but $E\, \zeta_T\, g(\hat{\alpha}_T) < \infty$. At intermediate times, we calculate the wealth by computing

$$e^{-\frac{1}{2} m_0^2 \tau_0} e^{rT} E_t\left[\zeta_T w_T\right] = \sqrt{\frac{\tau_T}{\tau_0}}\, E_t\left[\exp\left\{ -\tfrac{1}{2} m_T^2 \tau_T + \tfrac{1}{2}(\hat{\alpha}_T - \hat{\alpha}_0)^2 \tau_0 \tau_T / T \right\} \right].$$

Now given $\mathcal{F}_t$ we have $\hat{\alpha}_T \sim N\left(\hat{\alpha}_t, \frac{T-t}{\tau_t \tau_T}\right)$, so the RHS here becomes (after some calculation)

$$\left\{ \frac{T \tau_t \tau_T}{\tau_0 (T^2 + t\tau_0)} \right\}^{\frac{1}{2}} \exp\left[\frac{\tau_t \tau_T (\tau_0 - T)}{2(T^2 + t\tau_0)} \left(m_t - \frac{m_0 \tau_0}{\tau_0 - T} \right)^2 - \frac{\tau_0 \tau_T m_0^2}{2(\tau_0 - T)} \right],$$

and from this we deduce that

$$\boxed{\begin{aligned} w_t &= \frac{1}{\zeta_t} E_t\left[\zeta_T w_T\right] \\ &= \left(\frac{T \tau_t \tau_T}{T^2 \tau_T + \tau_0 t \tau_T} \right)^{\frac{1}{2}} e^{-r(T-t)} \exp\left\{ \frac{\tau_0 (\tau_T m_0 - \tau_t m_t)^2}{2(T^2 + t\tau_0)} \right\} \end{aligned}}$$

Optimal investment in discrete-time MM situation (27/9/06)

Suppose we have a hidden Markov chain (ξ_t) which controls the laws of the returns via $g(\cdot|\xi)$, and the chain jumps as $p(\xi,\xi')$. We now wish to max $E\,U(w_T)$, where U is CRRA. Then the value function

$$V_t(w,\pi) = \sup E\left[U(w_T) \mid w_t = w,\ \pi_t = \pi\right]$$

depends on the conditional law of ξ_t given $\mathcal{F}_t$, the observation filtration. We know from scaling that $V_t(w,\pi) = U(w)\,f_t(\pi)$, and

$$V_t(w,\pi) = \sup_\theta \left[\sum_\xi \pi(\xi) \int g(x|\xi)\, V_{t+1}\big((w-\theta+\theta x), \tilde\pi(\xi,x)\big)\, dx \right]$$

$$\text{where } \tilde\pi(\xi,x) \propto \sum_{\xi'} \pi(\xi')\, g(x|\xi')\, p(\xi',\xi)$$

$$= \sup_\theta U(w) \left[\sum_\xi \pi(\xi) \int g(x|\xi)\, (1-\theta+\theta x)^{1-R} f_{t+1}\big(\tilde\pi(\xi,x)\big)\, dx \right].$$

Can we do any good examples? Probably it's numerical - even if there was no MM dynamics, we still have to do numerical stuff.

In fact, the additional complexity introduced by the $f_{t+1}(\pi(\xi,x))$ is a big burden - ignoring that would probably also give quite a good rule.

If we had $\gamma < 0$, then

$$H_t(\lambda) = \exp\{\lambda m_t + \lambda^2/2K_t\tau\}\ \Phi\left((a - m_t - \lambda/K_t\tau)\sqrt{K_t\tau}\right)$$

NB

Equity premium puzzle without uncertainty on τ (1/10/06)

1) Bayesian agent with preferences (β, R) takes $N(0,(K_0\tau)^{-1})$ prior over the mean μ of the IID $\xi_t \sim N(\mu, \tau^{-1})$, where τ is known to the Bayesian agent. Now as before

$y_t / y_{t-1} = \exp \xi_t$, $\zeta_t = \beta^t y_t^{-R}$, and $(\nu \equiv R-1)$

$$S_t = y_t \, E_t\left[\sum_{n\geq t} \frac{\zeta_n y_n}{\zeta_t y_t} \right] = y_t \, E_t\left[\sum_{n \geq t} \beta^{n-t} \exp\{-\nu \sum_{j=t+1}^{n} \xi_j \} \right]$$

$$= y_t \, E_t \sum_{n\geq t} \beta^{n-t} \exp\{ (n-t)(-\nu\mu + \tfrac{1}{2}\nu^2/\tau) \}$$

$$= y_t \, E_t\left[\frac{1}{1-\beta \exp(-\nu\mu + \nu^2/2\tau)} \right]$$

If we take P^0 to be the law under which μ has $N(0,(K_0\tau)^{-1})$ prior, P to be the same law but with prefactor

$$\varphi(\mu) \propto \left(1 - \beta \exp(-\nu\mu + \nu^2/2\tau)\right)^+ ,$$

then

$$S_t = y_t \; \frac{P^0\left[\mu > (\log\beta + \nu^2/2\tau)/\nu \;\middle|\; \mathcal{F}_t \right]}{E^0_t\left[\{1-\beta \exp(-\nu\mu + \nu^2/2\tau)\}^+ \right]}$$

Now the law of μ given $\mathcal{F}_t$ has density

$$\propto \varphi(\mu) \exp\left[-\tfrac{1}{2} K_t \tau (\mu - m_t)^2 \right] \qquad m_t \equiv t\bar{\xi}_t / K_t, \quad K_t \equiv K_0 + t$$

so the numerator in the expression for the stock is

$$\bar{\Phi}\left(\frac{\log\beta + \nu^2/2\tau - \nu m_t}{\nu} \sqrt{K_t \tau} \right).$$

The denominator is $(a \equiv (\log\beta + \nu^2/2\tau)/\nu)$

$$\bar{\Phi}\left((a-m_t)\sqrt{K_t\tau}\right) - \beta \exp\left\{ +\frac{\nu^2}{2\tau} - \nu m_t + \frac{\nu^2 (K_t\tau)^{-1}}{2} \right\} \bar{\Phi}\left(\left(a - m_t + \frac{\nu}{K_t\tau}\right)\sqrt{K_t\tau} \right).$$

2) If we note that in general for $\lambda \in \mathbb{R}$

$$E^0_t\left[e^{\lambda\mu} : \nu\mu > \log\beta + \nu^2/2\tau \right]$$

$$= \exp\{ \lambda m_t + \lambda^2/2K_t\tau \} \; \bar{\Phi}\left((a - m_t - \lambda/K_t\tau)\sqrt{K_t\tau} \right)$$

$$\equiv H_t(\lambda), \quad \text{say},$$

then we may express

$$S_t = y_t \frac{H_t(0)}{H_t(0) - \beta e^{v^2/2\tau} H_t(-v)}$$

$$B_t = \frac{\beta e^{R^2/2\tau} \{H_t(-R) - \beta e^{v^2/2\tau} H_t(-R-v)\}}{H_t(0) - \beta e^{v^2/2\tau} H_t(-v)}$$

Optimal investment in a GARCH asset (3/10/06)

Suppose that $\log S_t \equiv X_t$ evolves as

$$X_{t+1} = X_t + \sqrt{h_{t+1}}\ \varepsilon_{t+1} \qquad \varepsilon \sim N(0,\sigma^2)$$

where

$$h_{t+1} = \omega + \alpha \varepsilon_t^2 + \beta h_t$$

and we have the objective of maximising ($U'(x) = x^{-R}$, let's say)

$$E\ U(w_T)$$

when we get $w_{t+1} = r w_t + \theta_t \{ S_{t+1} - r S_t \}$. By scaling, the value function

$$V_t(w, h) \equiv \sup E\left[U(w_T) \,\middle|\, w_t = w,\ h_{t+1} = h \right]$$

$$= U(w)\ f_t(h)$$

and the Bellman equations give us

$$V_t(w,h) = \sup_\theta E\left[V_{t+1}\left(rw + \theta S_t (e^{\sqrt{h}\varepsilon} - r),\ \omega + \alpha\varepsilon^2 + \beta h \right) \right]$$

$$= \sup_\theta E\left[w^{1-R}\, U(r + \theta(e^{\sqrt{h}\varepsilon} - r)) \cdot f_{t+1}(\omega + \alpha\varepsilon^2 + \beta h) \right]$$

$$= \sup_\theta U(w)\ E\left[(r + \theta(e^{\varepsilon\sqrt{h}} - r))^{1-R}\ f_{t+1}(\omega + \alpha\varepsilon^2 + \beta h) \right].$$

Only hope is probably going to be numerical, which is all we could do even without the GARCH effects...

Many Bayesian agents, CARA structure (25/10/06)

1) Suppose there's a single asset producing a dividend stream δ_t,

$$\delta_t - \delta_0 = \sigma X_t \equiv \sigma (W_t + \alpha t)$$

where α is unknown, with $N(\alpha_0, \tau_0^{-1})$ prior. With respect to Wiener measure on path space, the joint law of $(\alpha, (X_s)_{0\le s\le t})$ has density

$$\sqrt{\tau_0/2\pi}\ \exp\left[-\frac{\tau_t}{2}(\alpha - \alpha_t)^2 - \frac{\alpha_0^2 \tau_0}{2} + \alpha_t^2 \tau_t/2 \right]$$

with $\tau_t = \tau_0 + t$, $\alpha_t = (\alpha_0 \tau_0 + X_t)/\tau_t$. Integrating out the α variable leaves the density for the path in the form

$$(\tau_0/\tau_t)^{\frac{1}{2}} \exp\left\{ \tfrac{1}{2}\alpha_t^2 \tau_t - \tfrac{1}{2}\alpha_0^2 \tau_0 \right\} \equiv \Lambda_t$$

which puts a drift α_t onto the BM.

2) The usual story for equilibrium pricing gives $(U(x) = -\gamma^{-1}\exp(-\gamma x))$

$$\Lambda_t e^{-\rho t} U'(c_t) = \exp(-\rho t - \gamma c_t)\Lambda_t \propto \zeta_t$$

and now introducing index j for agent j we see

$$c_t^j = \frac{1}{\gamma_j}\left\{ -\rho_j t + \log \Lambda_t^j - \log \zeta_t \right\} + \text{const}$$

Hence by market clearing

$$\delta_t = \sum_j c_t^j = -\frac{\tilde{\rho} t}{\Gamma} + \sum \frac{1}{\gamma_j} \log \Lambda_t^j - \frac{1}{\Gamma}\log \zeta_t + \text{const}$$

where $\Gamma^{-1} = \sum \gamma_j^{-1}$, $\tilde{\rho}/\Gamma = \sum \gamma_j^{-1}\rho_j$. Thus

$$\boxed{\log \zeta_t + \text{const} = -\tilde{\rho} t - \Gamma \delta_t + \sum_j \frac{\Gamma}{\gamma_j}\log \Lambda_t^j}$$

All of the action is in the final piece, which can be developed to

$$\eta_t \equiv \sum_j \frac{\Gamma}{\gamma_j}\left\{ \frac{1}{2}\log\frac{\tau_0(j)}{\tau_t(j)} - \frac{1}{2}\alpha_0(j)^2 t\, \frac{\tau_0(j)}{\tau_0(j)+t} + \frac{\alpha_0(j)\tau_0(j)}{\tau_0(j)+t} X_t + \frac{1}{2}X_t^2/\tau_t(j) \right\}$$

Notice that if the precisions all tend to infinity, we return to the classical (?) situation

$$\eta_t = -\tfrac{1}{2} t \sum \Gamma \alpha_0(j)^2/\gamma_j + X_t \sum \Gamma \alpha_0(j)/\gamma_j .$$

This is the situation which obtains when all agents are certain of the drift, though they are certain of different constants. In this instance, if we set

$$a \equiv \sum_j \Gamma \alpha_j(0)/\gamma_j \quad , \quad b = \sum \Gamma \alpha_j(0)^2/\gamma_j \, ,$$

we shall have $a^2 \leq b$, and quite simply

$$\zeta_t \propto \exp\{-\tilde{\rho}t - \Gamma\delta_t + aX_t - bt/2\},$$

which leads to an expression for the stock price

$$\boxed{S_0 = \frac{\delta_0}{m} + \frac{\sigma(a-\Gamma\sigma)}{m^2}} \qquad \left(m \equiv \tilde{\rho} + \frac{b}{2} - \frac{(a-\Gamma\sigma)^2}{2} \right)$$

if we assume that $\boxed{m > 0}$ (otherwise the stock price is infinite).

3) More generally, the stock price is

$$S_0 = E^* \int_0^\infty \zeta_t \delta_t/\zeta_0 \, dt = E^* \int_0^\infty \exp\{-\tilde{\rho}t - \lambda X_t + \eta_t\} \, dt \, \delta_t \qquad \text{when } \lambda = \sigma\Gamma.$$

After prolonged calculations, I find that (assuming $X_0 = 0$)

$$E^* \int_0^\infty e^{-\tilde{\rho}t - \lambda X_t + \eta_t} \, dt$$

$$= \int_0^\infty \exp\{-\tilde{\rho}t - \tfrac{1}{2}\sum_j p_j \alpha_0(j)^2 \tau_0(j) t/\tau_t(j) + \tfrac{1}{2} m_t^2 V_t\} \, e^{\frac{1}{2}\sum p_j \log\{\tau_0(j)/\tau_t(j)\}} \, dt$$

$$\text{where } p_j \equiv \gamma_j^{-1}\Gamma, \quad m_t \equiv \sum p_j \frac{\alpha_0(j)\tau_0(j)}{\tau_t(j)} - \lambda$$

$$V_t^{-1} = \sum p_j \tau_0(j)/t\,\tau_t(j) .$$

Differentiation w.r.t. λ gives us most of what we require, but it's far from explicit.
If all agents have same τ_0 value, then there are some simplifications. With $a = \sum p_j \alpha_j(0)$, $b = \sum p_j \alpha_j(0)^2$, we have $m_t = a\tau_0/\tau_t - \lambda$, $V_t = t\tau_t/\tau_0$, and the integral becomes

$$\int_0^\infty \exp\left[-\tilde{\rho}t - \tfrac{1}{2} b \frac{t\,\tau_0}{\tau_t} + \tfrac{1}{2}\left(a\frac{\tau_0}{\tau_t} - \lambda\right)^2 \frac{t\,\tau_t}{\tau_0}\right] \left(\frac{\tau_0}{\tau_t}\right)^{J/2} dt$$

but even so we will have to be doing things numerically.

4) If all agents have the same value of γ, and τ_0, we can inspect the consumption

$$c_t(j) = \frac{1}{\gamma}\left\{ -\rho t + \frac{1}{2}\log\frac{\tau_0}{\tau_t} + \frac{1}{2}\tau_t \alpha_t(j)^2 - \frac{1}{2}\tau_0 \alpha_0(j)^2 - \log \zeta_t \right\} + \text{const.}$$

If we look at the dependence on $\alpha_0(j)$ we see that (ignoring the appearances of $\alpha_0(j)$ in $\log \zeta$) the best value for $\alpha_0(j)$ would be X_t/t at time t, which makes a lot of sense.

5) Suppose we now allow the number of agents to get huge, so that sums become integrals, but we also suppose that agents of age t die at rate $\mu(t)$ to be replaced by agents of age ε. The equilibrium profile of age has density

$$\varphi(t) \propto \exp\left\{ -\int_0^t \mu(s)\,ds \right\}$$

by renewal theory if you want. The finite-sum average which happens in η now approaches (assuming that death rate doesn't vary with $\alpha(0)$)

$$\int \frac{1}{2}\log\left(\frac{\varepsilon}{\varepsilon+t}\right)\varphi(t)\,dt - \frac{1}{2}\langle\alpha_0\rangle^2 \int \frac{t\varepsilon}{\varepsilon+t}\varphi(t)\,dt + X_t \int \frac{\langle\alpha_0\rangle\varepsilon}{\varepsilon+t}\varphi(t)\,dt$$

$$+ \frac{1}{2}X_t^2 \int \frac{\varphi(t)}{\varepsilon+t}\,dt$$

$$\equiv A_0 + A_1 X_t + \frac{1}{2}A_2 X_t^2 \quad , \text{say.}$$

The state-price density is therefore $\propto \exp\{-\tilde{\rho}t + (A_1 - \sigma^{-1})X_t + \frac{1}{2}A_2 X_t^2\}$ ... but this will be no good if t is too large - it will not be integrable !! So we will it seems need to change the model for X_t... perhaps this is for the best.

6) So let's suppose that the observed process X is actually an OU process

$$dX_t = dW_t + \lambda(a - X_t)\,dt$$

where $\lambda > 0$ is known, but we have a $N(\alpha_0, \tau^{-1})$ prior for a. The likelihood of X wrt Wiener measure is

$$\exp\left[\lambda \int_0^t (a - X_s)\,dX_s - \frac{1}{2}\int_0^t \lambda^2 (a - X_s)^2\,ds\right]$$

$$= \exp\left[\lambda a(X_t - X_0) - \frac{1}{2}\lambda(X_t^2 - X_0^2 - t) - \frac{1}{2}\lambda^2\left(a^2 t - 2a\int_0^t X_s\,ds + \int_0^t X_s^2\,ds\right)\right]$$

So posterior is

$$\propto \exp\left[-\frac{\tau+\lambda^2 t}{2}\left(a - \frac{\alpha_0\tau + \lambda(X_t - X_0) + \lambda^2\int_0^t X_s\,ds}{\tau+\lambda^2 t}\right)^2\right]$$

$K_p = \lambda(Y_G - Y_\theta)$ in Angui notation.

Thus the posterior for a given $\mathcal{F}_t$ is $N\left(\dfrac{\alpha_0\tau + \lambda(X_t - X_0) + \lambda^2\int_0^t X_s ds}{\tau + \lambda^2 t}, (\tau+\lambda^2 t)^{-1} \right)$

and $\hat{\alpha}_t = \left\{ \alpha_0\tau + \lambda(X_t - X_0) + \lambda^2 \int_0^t X_s ds \right\} / (\tau + \lambda^2 t)$.

Integrating over α to compute Λ_t leads to

$$\Lambda_t = \left(\frac{\tau}{\tau+\lambda^2 t} \right)^{\frac{1}{2}} \exp\left[- \frac{\alpha_0^2 \lambda^2 \tau t - 2\alpha_0 \tau K_t - K_t^2}{2(\tau + \lambda^2 t)} - \tfrac{1}{2}\lambda(X_t^2 - X_0^2 - t) - \tfrac{1}{2}\lambda^2 \int_0^t X_s^2 ds \right]$$

where $K_t \equiv \lambda(X_t - X_0) + \lambda^2 \int_0^t X_s ds$. If we write the likelihood relative to the base OU law, with initial distribution which is the invariant $N(0, 1/2\lambda)$, we end up with

$$\Lambda_t = \left(\frac{\tau}{\tau+\lambda^2 t} \right)^{\frac{1}{2}} \exp\left\{ - \frac{\alpha_0^2 \lambda^2 \tau t - 2\alpha_0 \tau K_t - K_t^2}{2(\tau+\lambda^2 t)} \right\}.$$

7) If we now take the idea of a rolling population, an agent who enters at time t_j with prior mean α_j and precision $\lambda^2 \varepsilon_j$ will have LR martingale Λ_t^j for $t \geq t_j$ given by

$$\Lambda_t^j = \left(\frac{\varepsilon_j}{\varepsilon_j + t - t_j} \right)^{\frac{1}{2}} \exp\left\{ - \frac{\alpha_j^2 \lambda^2 \varepsilon_j (t - t_j) - 2\alpha_j \varepsilon_j K_{t_j, t} - (K_{t_j,t}/\lambda)^2}{2(\varepsilon_j + t - t_j)} \right\}$$

where $K_{s,t} \equiv \lambda(X_t - X_s) + \lambda^2 \int_s^t X_u du$. If we repeat the idea of the age profile $\varphi(t)$, independent choices of α_j, common value of ε, we are looking at

$$\sum p_j \log \Lambda_t^j \approx \frac{1}{2} \int_0^\infty \log\left(\frac{\varepsilon}{\varepsilon+u}\right) \varphi(u) du - \tfrac{1}{2}\langle\alpha^2\rangle \lambda^2 \varepsilon \int_0^\infty \frac{u}{\varepsilon+u} \varphi(u) du$$

$$+ \varepsilon \langle\alpha\rangle \int_0^\infty \frac{1}{\varepsilon+u}\left(\int_{t-u}^t \lambda^2 X_v dv + \lambda(X_t - X_{t-u}) \right) \varphi(u) du$$

$$+ \frac{1}{2} \int_0^\infty \frac{1}{\varepsilon+u} \underbrace{\left(X_t - X_{t-u} + \lambda \int_{t-u}^t X_v dv \right)^2}_{W_t - W_{t-u}} \varphi(u) du$$

If we want to make a model where $(r_t^i - r_t^j)$ is an autonomous diffusion for all pairs i, j, then we shall have to have

$$dr_t = \sigma \, dW_t - (\lambda I + 1\gamma^T) r_t \, dt + \mu \, dt$$

for some vectors γ, μ. Should be able to do MLE of this reasonably OK.

Possible ways to model carry trades (1/11/06)

Let's write Y^j_t for the price at time t (measured in country-0 currency) of one unit of country-j currency. Let r^j_t denote the riskless rate of interest in country j at time t, with $r_t \equiv (r^0_t, r^1_t, \dots, r^n_t)^T$ for the whole lot. We know that in the pricing measure

$$Y^j_t \exp\left(\int_0^t (r^j_s - r^0_s)\,ds\right) \quad \text{is a martingale, so this}$$

suggests:

Model 1

$$dr_t = \sigma_r\, dW + B(\bar{r} - r_t)\,dt$$

$$dY^j_t = Y^j_t \left\{ (r^0_t - r^j_t + \lambda_j)\,dt + \sigma^j_y\, dW \right\}$$

where σ_r, σ_y, B are constant matrices, λ_j are constant risk premia. Writing $y^j_t \equiv \log Y^j_t$ turns the second of these into

$$dy^j_t = \left\{ r^0_t - r^j_t + \lambda_j - \tfrac{1}{2}|\sigma^j_y|^2 \right\} dt + \sigma^j_y\, dW_t$$

Discretising this would give us (time step $= \Delta t$):

$$\begin{cases} r_{t+\Delta t} = r_t + B(\bar{r} - r_t)\,\Delta t + \sigma_r\, \Delta W_{t+\Delta t} \\ y^j_{t+\Delta t} = (r^0_t - r^j_t + \lambda_j - \tfrac{1}{2}|\sigma^j_y|^2)\,\Delta t + \sigma^j_y\, \Delta W_{t+\Delta t} + y^j_t \end{cases}$$

Unknown parameters here: <u>matrices</u> B, σ_r, σ_y, vectors $\bar{r}, \lambda$. If we were working with more than a few currencies, this would likely be too clumsy. We could make the model simpler by assuming (i) independent noise terms (ii) B is diagonal (iii) $B = \mathrm{diag}(\beta)\,(I - \epsilon 11^T)$, to incorporate the idea that high rates in other currencies tend to raise rates in our own.

We could make the model more flexible by letting the noise terms be non-Gaussian. There's a symmetry observation to be made here, namely that if we look at exchange rates expressed in some other base currency we should be talking about the same model. This means that we ought to express the noise term in the dynamics of y^j as $\eta^j - \eta^0$ where the η^j should ideally be independent (possibly related to the noises in the dynamics of the r^j)

In this model, the r^j and y^j are observables, there are no hidden Markov variables but the parameters are of course unknown, and take on the rôle of hidden Markov variables.

Optimisation within this model simplifies somewhat, because the effects of interest are already accounted for in the dynamics. If we use a CRRA utility, and just go for a one-period meiopic optimisation, the objective is

$$\max\ E\, U\left(\sum_{i=1}^{n} \theta_i \left(Y_t^i \exp\left(\int_0^t r_s^i\, ds\right) / Y_0^i \right) + \left(1 - \sum \theta_i\right) \exp\left(\int_0^t r_s^0\, ds\right) \right)$$

$$\simeq \exp((1-R) r_0^0 t) \max E\, U\left(\sum_{i=1}^{n} \theta_i \left\{ Y_t^i \exp\left(\int_0^t (r_s^i - r_s^0)\, ds\right) / Y_0^i - 1 \right\} + 1 \right)$$

since t will typically be so small that rates don't change much. Thus the optimisation will be to

$$\max\ E\, U\left(1 + \sum_{i}^{n} \theta_i \left(\exp(a_i + \eta_i - \eta_0) - 1 \right) \right)$$

where $a_i = (\lambda_i - \frac{1}{2}|\sigma_Y^i|^2)t$. If we assume the argument of the utility is close to 1, and do a Taylor expansion, we shall aim to

$$\max\ E\left[\sum \theta_i \left(\exp(a_i + \eta_i - \eta_0) - 1 \right) - \tfrac{1}{2} R \left(\sum \theta_i \left(\exp(a_i + \eta_i - \eta_0) - 1 \right) \right)^2 \right]$$

which is just a standard Markowitz-type optimisation.

Model 2 This time, we just focus on the exchange rates. We suppose that the y^j will satisfy

$$dy_t^j = \sigma_Y^j\, dW + \lambda\, (b_j - y_t^j)\, dt.$$

The same value of λ is used for all j, because we want the structure of $y^i - y^j$ to be the same. This time we just need the joint law of the noises, and to estimate the parameters λ and b_j. Observables required will be the FX rates, and (for the optimisation only) the riskless rates.

Notice that this is an example to show how important it is to refer the likelihood to the correct density; Model 1 models (r_t^j, y_t^j), this one only models (y_t^j).

Optimisation is similar to Model 1.

Model 3 This is like Model 1, except you don't suppose the individual riskless rates are moving; they are supposed to be constants, but they will be unobservables, while the market interest rates are noisy observations of the hidden constants. The constants move a bit by shaking, and the posterior weights will jiggle around too

Another way to see particle filtering (10/11/06)

1) The basic paradigm for particle filtering is some Markov process $(X_t)_{t\geq 0}$ and some observation process $(Y_t)_{t\geq 0}$ whose density given X_t, $f(y|x)$, is easy to specify. We do the filtering by approximating the posterior $\pi_t(dx) = \sum_i w_t^i \delta_{\xi_t^i}$, then move the Markov process forward by simulation, and reweight the weights by $f(y_{t+1}|x)$.

There can be problems with this, particularly if some part of the Markov state is observed with no error; then the law of $Y_{t+1}|X_{t+1}$ will typically not have a density wrto any reference measure, and this approach just doesn't work.

2) It seems that what we have to do is consider the density $\varphi(y|x)$ of Y_{t+1} given X_t (and this will typically be OK - if some component of Y_{t+1} were known with certainty once we knew X_t, we would augment the Markov variable to $\binom{X_t}{X_{t-1}}$ and drop the component(s) of Y which is/are known). We then need to have the rcd for X_{t+1} given X_t, Y_{t+1}, $\kappa(\cdot|X_t, Y_{t+1})$. The updating equations for the posterior are

$$\pi_{t+1}(dx) = \int \pi_t(dx') \int \varphi(y_{t+1}|x')\ \kappa(dx|x', y_{t+1})\ m(dy_{t+1})$$

so in the particle approximation we have to

(i) reweight particle i with $\varphi(Y_{t+1}|x_t^i)$
(ii) generate x_{t+1}^i according to $\kappa(\cdot|x_t^i, Y_{t+1})$.

Notice that this is quite different from the usual approach, which first generates the particle, then reweights.

3) Let's take an example. Suppose (ε_t) are IID $N(0,1)$, and $\Delta t>0$ is fixed

$$Y_{t+1} - Y_t = \sigma\sqrt{\Delta t}\,\varepsilon_{t+1} + (a - b(Y_t - \bar{Y}_t))\Delta t,$$

where $\bar{Y}_t = p\bar{Y}_{t-1} + (1-p)Y_t$ is an EWMA of past values. The parameters (a,b,σ) would be unknown, and the Markov variable would be the parameter vector stacked over $(Y_t, \bar{Y}_t)$:

$$X_t = \begin{pmatrix} a \\ b \\ \tau \\ Y_t \\ \bar{Y}_t \end{pmatrix}, \qquad \tau = \sigma^{-2}.$$

The particle reweighting is done using the density of the Gaussian law, so that's easy. For the move of X, we suppose there is a little shaking of the parameters $a, b, \tau = 1/\sigma^2$: we add a small Gaussian (with precision τ_a, τ_b) respectively to a, b, and we replace τ by a Gamma $(\lambda\tau, \lambda)$ for some big λ.

Writing $\Delta Y \equiv Y_{t+1} - Y_t$, $\mu \equiv Y_t - \bar{Y}_t$, we have the joint density proportional to

$$\sqrt{\tau} \quad \exp\Big[- \frac{\tau}{2\Delta t}\big(\Delta Y - (a - b\mu)\Delta t\big)^2 - \frac{\tau_a}{2}(a-a_0)^2 - \frac{\tau_b}{2}(b-b_0)^2 - \lambda\tau + (\lambda\tau_0 - 1)\log(\lambda\tau) \Big]$$

if we first shake the parameters (a, b, τ) before we move Y. This may be worth doing, but it will require a more complicated expression for $\varphi(Y_{t+1} \mid x_t)$. Perhaps simpler for a first trick is simply to shake (a, b, τ) after the move of Y. We expect that the precisions τ_a, τ_b of the shaking of (a, b) will be pretty large, so the dominant term in the likelihood will probably be the terms in $\tau_a(a-a_0)^2$, $\tau_b(b-b_0)^2$. Things else become more complicated, probably for no good reason.

4) Suppose we take another example of an AR(1) process observed with noise

$$x_{t+1} = \beta x_t + \varepsilon_{t+1}$$

$$Y_{t+1} = x_{t+1} + \eta_{t+1}.$$

The precision of the ε's and the η's is not known. Markov variable is $X_t = (x_t, \beta, \tau_\varepsilon, \tau_\eta)^T$.

We have $\mathcal{L}(Y_{t+1} \mid x_t)$ is $N(\beta x_t, \sigma_\varepsilon^2 + \sigma_\eta^2)$, which gives us the density φ. As for the conditional law of x_{t+1} given Y_{t+1}, we have that it's Gaussian, with mean

$$\beta x_t + \frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2 + \sigma_\eta^2}\left(Y_{t+1} - \beta x_t\right),$$

and variance

$$\frac{\sigma_\varepsilon^2 \sigma_\eta^2}{\sigma_\varepsilon^2 + \sigma_\eta^2}.$$

This will be easy to do.

MLE for a simple model of interest rates (14/11/06)

Let's look at a model for the co-movement of some vector $r_t = (r_t^1, \dots, r_t^N)$ of interest rates, as presented being p40. We want a diffusion model, where any interest rate differential is an autonomous diffusion. If we go for something that's OU-like we'll be looking at

$$dr_t = \sigma \, dW_t + \mu \, dt - (\lambda I + 1 z^T) r_t \, dt.$$

Suppose we see this at times $t_0 < t_1 < \dots < t_N$; then $-2 \log L$ will be

$$\sum_{j=0}^{N-1} \xi_j \cdot V^{-1} \xi_j \, (t_{j+1} - t_j)^{-1} + N \log \det V + \text{const},$$

where $V \equiv \sigma\sigma^T$, and

$$\xi_j = r_{j+1} - r_j - \{\mu - \lambda r_j - z^T r_j\} (t_{j+1} - t_j)$$

with r_{t_j} abbreviated to r_j. The vectors μ, z and the scalar λ are not known, to be obtained by ML. Once we have fixed μ, z and λ, the best V is just

$$\frac{1}{N} \sum_{j=0}^{N-1} \xi_j \xi_j^T / (t_{j+1} - t_j)$$

so this should be comparatively simple. [It seems to work, but the predictive power is negligible]

Equilibrium pricing to explain transmission of price effects (24/11/06)

(i) We can try an equilibrium story where the dividend process (n-vector) evolves as

$$\delta_{t+1} - \delta_t = -B(\delta_t - a) + \varepsilon_{t+1} \qquad [\varepsilon_t \text{ IID } N(0, V)]$$

for fixed vector a, fixed $n\times n$ matrix B. If we set $x_t \equiv \delta_t - a$, $A \equiv I - B$, then we have a simple AR(1) model for x:

$$x_{t+1} = A x_t + \varepsilon_{t+1}.$$

Thus

$$x_{t+n} = A^n x_t + \sum_{r=0}^{n-1} A^r \varepsilon_{t+n-r} \sim N(A^n x_t, \Sigma_n),$$

where

$$\Sigma_n = \sum_{r=0}^{n-1} A^r V (A^T)^r$$

(ii) The usual situation of J CARA agents, agent j trying to

$$\max E \sum_{t\geq 0} \exp\{-\rho t - \gamma c_t^j\}$$

leads to a SPD process $\left(\Gamma^{-1} = \sum \gamma_j^{-1}, \quad \tilde{\rho} = \sum \frac{\Gamma}{\gamma_j} \rho_j\right)$

$$\zeta_t = \exp\{-\tilde{\rho} t - \Gamma 1\cdot\delta_t\}$$

and therefore the time-0 value of the stocks shall be

$$E\left[\sum_{t\geq 0} \zeta_t \delta_t / \zeta_0 \;\middle|\; \delta_0\right]$$

$$= E\left[\sum_{t\geq 0} \exp\{-\tilde{\rho} t + v\cdot\delta_t\}\, \delta_t\, e^{-v\cdot\delta_0} \;\middle|\; \delta_0\right] \qquad (v \equiv -\Gamma 1)$$

$$= e^{-v\cdot\delta_0} \sum_{t\geq 0} \exp(-\tilde{\rho} t)\, E\left[e^{v\cdot\delta_t}\, \delta_t\right]$$

$$= \exp(-v\cdot\delta_0) \sum_{t\geq 0} \exp\{-\tilde{\rho} t + v\cdot\mu_t + \tfrac{1}{2} v\cdot\Sigma_t v\} \quad (\mu_t + \Sigma_t v)$$

where $\mu_t = E(\delta_t | \delta_0) = a + A^t(\delta_0 - a)$. There will inevitably be price shock transmission. The obvious trick is to compute $A^t(\delta_0 - a)$, Σ_t recursively, until the sum is done

Some thoughts on non-collision of UAVs (28/11/06)

The guys from Smith Institute see this as a stochastic optimal control problem, but the stochastic element needs to be pushed to the background. If our vehicle is moving in a straight line with speed V, another vehicle has speed $\leq U$, then if we are going to look ahead to time T then the danger zone is $\bigcup_{0 \leq t \leq T} S(x_t, tU)$ where $S(x,r) = \{y : |x-y| \leq r\}$ – any vehicle in that set could in principle collide with us by time t. Let the danger set be denoted D_T (or perhaps $D_t(T)$ if we are considering the danger set at time t to later time horizon $t+T$). If some other aircraft is observed at $y(t)$ in the danger zone D_t, we need to estimate its velocity (and perhaps acceleration) from a sequence of nearby (radar) observed positions. Then we compute some penalty function

$$F(x_t, \dot{x}_t, y_t, \dot{y}_t) = \int_t^\infty |x_s - y_s|^{-2} e^{-\gamma(s-t)} ds$$

Any. Looking ahead Δt, we can see where we might get to, and where the other aircraft would be (assuming their dynamics continue thus). Should be able to optimise over the controls to be used for the next step, perhaps with a rather slow search. Probably also need to compute a 'clear-skies' value function before the flight (objective appears to be time to destination), and keep an eye on this while we're watching other aircraft in the danger zone

If We have $A = [-1, I]$, $\Sigma = \left(\begin{array}{c|c} v_1 & 0 \\ \hline 0 & V \end{array}\right)$ is diagonal

$$A\Sigma A^T = (V - v_1 1 1^T) = V(I - q 1^T) \qquad (q \equiv v_1 V^{-1} 1)$$

$$(A\Sigma A^T)^{-1} = (I - q 1^T)^{-1} V^{-1}$$

$$= (I + \omega q 1^T) V^{-1} \qquad \omega \equiv \frac{1}{1 - v_1 1^T V^{-1} 1}$$

$$\Sigma A^T = \begin{pmatrix} -v_1 1^T \\ V \end{pmatrix} \qquad = \frac{1}{1 - 1^T q}$$

Thus

$$\Sigma A^T (A\Sigma A^T)^{-1} = \begin{pmatrix} \frac{-v_1 1^T}{1 - 1^T q} \\ \\ V + \omega V q 1^T \end{pmatrix} V^{-1}$$

We want conditional variance of x^1_{t+1} , so we look at

$$e_1 \cdot \left(\Sigma - \Sigma A^T (A\Sigma A^T)^{-1} A \Sigma\right) e_1$$

$$= v_1 - v_1^2 \; 1^T (A\Sigma A^T)^{-1} 1$$

$$= v_1 - v_1^2 \; (1 + \omega \, 1 \cdot q) \; 1 \cdot V^{-1} 1$$

$$= v_1 \left\{ 1 - (1 + \omega \, 1 \cdot q) \; 1 \cdot q \right\} = \frac{v_1}{1 - 1 \cdot q} \left\{ (1 - 1 \cdot q)^2 - (1 \cdot q)^2 \right\}$$

$$= \frac{v_1}{1 - 1 \cdot q} \left\{ \quad \right\} = \frac{v_1}{1 - 1 \cdot q} \left\{ 1 - 2 \, (1 \cdot q) \right\}$$

Modelling FX rates symmetrically (6/12/2006)

Suppose we want to model a group of N currencies. Then let's suppose that there is some N-vector AR(1) process x_t

$$x_{t+1} = a + B(x_t - a) + \varepsilon_{t+1} \qquad \varepsilon_t \sim N(0, \Sigma)$$

such that the log-exchange rate of currency j in terms of currency i is $x_t^j - x_t^i$. At time t, we observe

$$y_t \equiv A x_t = (x_t^2 - x_t^1, \dots, x_t^N - x_t^1),$$

the log-exchange rates of currencies $2, \dots, N$ w/to currency 1.

(i) Using a particle filtering approach, an individual particle will know its x_t, and parameters a, B, Σ; given this, the law of y_{t+1} will be normal with mean

$$A(a + B(x_t - a)) \quad \text{and covariance} \quad A \Sigma A^T.$$

This permits us to do the particle reweighting given the new y_{t+1}. Conditional on y_{t+1},

$$x_{t+1} \sim N\left(m + (\Sigma A^T)(A\Sigma A^T)^{-1}(y_{t+1} - Am),\ \Sigma - \Sigma A^T (A\Sigma A^T)^{-1} A\Sigma \right)$$

where $m \equiv a + B(x_t - a)$, so we can reweight the particle, then simulate x_{t+1} from the conditional law.

(ii) We could Kalman filter (x_t). Similar calculations give

$$(x_t \mid Y_t) \sim N(\mu_t, V_t)$$

where
$$\mu_{t+1} = m_{t+1} + \Sigma_t A^T (A\Sigma_t A^T)^{-1} (y_{t+1} - A m_{t+1})$$
$$m_{t+1} = a + B(\mu_t - a)$$
$$\Sigma_t = \Sigma + B V_t B^T$$
$$V_{t+1} = \Sigma_t - \Sigma_t A^T (A\Sigma_t A^T)^{-1} A \Sigma_t$$

is the recursion, but this requires knowledge of A, B, Σ, and a, which we don't have in our story.

(iii) Suppose $\Sigma = \left(\begin{array}{c|c} v & 0 \\ \hline 0 & V \end{array}\right)$; how does this look? $A = [-1, I]$, and so

$$A\Sigma A^T = V + v 1 1^T \quad \Rightarrow \quad V^{-\frac{1}{2}}(A\Sigma A^T)V^{-\frac{1}{2}} = I + v a a^T, \quad a \equiv V^{-\frac{1}{2}} 1$$

To invert this, we do the usual sort of trick:

[PTO]

$$(I + v\, aa^T)^{-1} = I - \omega\, aa^T \qquad \omega = \frac{v}{1+v|a|^2}$$

$$\Rightarrow \quad (A\Sigma A^T)^{-1} = V^{-1} - \omega\, bb^T, \qquad b \equiv V^{-1}1.$$

This leads to

$$\Sigma A^T (A\Sigma A^T)^{-1} = \begin{pmatrix} -v1^T \\ V \end{pmatrix} (V^{-1} - \omega\, bb^T)$$

$$= \begin{pmatrix} -\frac{v b^T}{1+v|a|^2} \\ I - \omega\, 1b^T \end{pmatrix}$$

and since

$$e_1^T(\Sigma A^T) = -v\, 1^T$$

we deduce that

$$e_1^T\left(\Sigma - \Sigma A^T (A\Sigma A^T)^{-1} A\Sigma\right) e_1$$

$$= v - v^2\, 1^T (A\Sigma A^T)^{-1} 1$$

$$= \frac{v}{1+v|a|^2}$$

Developing multiple Bayesian agents a bit further. (9/12/06)

(1) Returning to the situation on p 39, where $X_t - X_{t-u} + \lambda \int_{t-u}^{t} X_s\,ds = W_t - W_{t-u}$, if we define

$$\xi_t \equiv \int_0^\infty \lambda e^{-\lambda u}(W_t - W_{t-u})\,du, \quad y_t \equiv \int_0^\infty \lambda e^{-\lambda u}(W_t - W_{t-u})^2\,du,$$

we derive $\xi_t = W_t - e^{-\lambda t}\int_{-\infty}^{t} \lambda e^{\lambda s} W_s\,ds$, so that

$$d\xi_t = dW_t + \lambda(W_t - \xi_t)dt - \lambda W_t\,dt = dW_t - \lambda \xi_t\,dt \qquad (1)$$

and hence $\xi_t = X_t$ (the arbitrary constant being identified as zero since $E\xi_t = 0$).
Similarly, $y_t = W_t^2 - 2W_t(W_t - \xi_t) + \lambda e^{-\lambda t}\int_{-\infty}^{t} e^{\lambda s} W_s^2\,ds$, and hence

$$\begin{aligned} dy_t &= 2(W d\xi + \xi dW) + 2dt - 2W dW - dt - \lambda(y - 2\xi W + \cancel{W^2})dt + \cancel{\lambda W^2}dt \\ &= 2\xi\,dW + dt - \lambda y\,dt \qquad (2) \\ &= d(\xi^2) + 2\lambda\xi^2\,dt - \lambda y\,dt. \qquad (3) \end{aligned}$$

(2) This suggests we should try the density

$$\varphi(u) = A(\varepsilon + u)\lambda e^{-\lambda u} \quad \text{where } A \equiv \lambda/(1+\varepsilon\lambda), \text{ for then the}$$

form of the state-price density ζ becomes [NB because of renewal thy interpretation, we must have φ decreasing, so $\varepsilon\lambda \geq 1$]

$$\text{const} + \log \zeta_t = -\tilde{\rho} t - \sigma\Gamma X_t + \varepsilon A\langle\alpha\rangle X_t + \tfrac{1}{2} A y_t$$

$$\therefore\ d(\log \zeta_t) = (\varepsilon A\langle\alpha\rangle - \sigma\Gamma + AX)dW - \{\tilde{\rho} - \lambda\sigma^{-1}\Gamma X_t + \lambda\varepsilon A\langle\alpha\rangle X_t - \tfrac{1}{2}A(1 - \lambda y_t)\}dt$$

$$\equiv (b_1 + AX)dW - \{\tilde{\rho} + \lambda b_1 X_t - \tfrac{1}{2}A(1 - \lambda y_t)\}dt$$

with $b_1 \equiv \varepsilon A\langle\alpha\rangle - \sigma\Gamma$;

$$= \left\{(b_1 + AX)dW - \tfrac{1}{2}(b_1 + AX)^2 dt\right\} + \left[\tfrac{1}{2}(b_1 + AX)^2 - \tilde{\rho} - \lambda b_1 X + \tfrac{1}{2}A(1 - \lambda y_t)\right]dt.$$

The first bit $\{\ldots\}$ can be interpreted as a change of measure transforming the SDE for X to

$$dX = d\tilde{W} + (b_1 + AX)dt - \lambda X\,dt = d\tilde{W} + (\lambda - A)\left(\frac{b_1}{\lambda - A} - X\right)dt$$

where $\lambda - A = \varepsilon\lambda^2/(1+\varepsilon\lambda) > 0$. A further deduction we can make from (3) is that

$$y_t - X_t^2 = e^{-\lambda t} \int_{-\infty}^{t} \lambda e^{\lambda s} X_s^2 \, ds \qquad (+\text{const?})$$

or more generally for $t_0 < t$

$$y_t - X_t^2 = e^{-\lambda(t-t_0)} \left\{ y_{t_0} - X_{t_0}^2 \right\} + \int_{t_0}^{t} \lambda e^{\lambda(s-t)} X_s^2 \, ds$$

Hence

$$\int_{t_0}^{t} y_s \, ds = \frac{1-e^{-\lambda(t-t_0)}}{\lambda} \left(y_{t_0} - X_{t_0}^2 \right) + \int_{t_0}^{t} X_s^2 \left(1 - e^{-\lambda(t-s)} \right) ds + \int_{t_0}^{t} X_s^2 \, ds .$$

We thus have

$$\zeta_t \propto Z_t \, \exp\{ \Psi_t \}$$

where $dZ_t = Z_t \{ b_1 + A X_t \} dW_t$ is the change-of-measure martingale, and

$$\boxed{\Psi_t = \left(\tfrac{1}{2} b_1^2 - \tilde{\rho} + \tfrac{1}{2} A\right)(t-t_0) - b_1(\lambda - A) \int_{t_0}^{t} X_s \, ds - \frac{A}{2}\left(1 - e^{-\lambda(t-t_0)}\right)\left(y_{t_0} - X_{t_0}^2\right) - \tfrac{1}{2} A \int_{t_0}^{t} X_s^2 \left(\lambda - A + \lambda\left(1 - e^{-\lambda(t-s)}\right) \right) ds ,}$$

Crackers; we dont need to integrate y again!

Particle story: choosing the portfolio (13/12/06)

Suppose particle i has prob p_i, and predicts that the log-return

$$\Delta X_t \equiv X_{t+\Delta t} - X_t \sim N(\mu_i \Delta t, V_i \Delta t)$$

Agent currently has wealth w, which he splits in proportions θ among the risk assets, with $(1-\theta\cdot 1)w$ put into riskless asset

$$\max_\theta E\, U\left((1-\theta\cdot 1) w e^{r\Delta t} + \theta w \cdot e^{\Delta X_t} \right)$$

$$= \max_\theta E\, U\left(w + w(e^{r\Delta t}-1) + w\,\theta\cdot\{e^{\Delta X_t} - 1 e^{r\Delta t}\} \right)$$

$$\approx \max_\theta U(w) + wU'(w)\, EZ + \frac{w^2}{2} U''(w)\, EZ^2,$$

where $Z = (e^{r\Delta t}-1) + \theta\cdot(e^{\Delta X_t} - 1 e^{r\Delta t})$

$$\approx r\Delta t + \theta\cdot(\Delta X_t - r\Delta t) = r\Delta t + \theta\cdot(\mu - r)\Delta t + \theta\cdot(\Delta X_t - \mu \Delta t)$$

Now

$$EZ = \sum_i p_i \{ r + \theta\cdot(\mu_i - r1) \} \Delta t$$

$$EZ^2 = \sum_i p_i \left\{ (r + \theta\cdot(\mu_i - r1))^2 \Delta t^2 + \Delta t\ \theta\cdot V_i \theta \right\}$$

The optimality equation for θ is therefore

$$\boxed{ -\frac{wU'(w)}{w^2 U''(w)} \cdot \sum p_i (\mu_i - r1) = \theta\cdot \sum_i p_i \left(V_i + \Delta t (\mu_i - r1)(\mu_i - r1)^T \right) }$$

- Allow different p/f rules on same model
- Allow 'reserved' weight on some models
- Allow new particles to appear

Multiple Bayesian agents: summarising the story so far (5/1/07)

1) When we form the equilibrium for a collection of CARA agent upwith is

$$\log \zeta_t + \text{const} = -\tilde{\rho} t - \Gamma \delta_t + \sum p_j \log \Lambda_t^j$$

where $\Gamma^{-1} = \sum \gamma_j^{-1}$, $p_j = \gamma_j^{-1}\Gamma$, $\delta_t = \sigma X_t$, $dX_t = dW_t + \lambda(a - X_t)\,dt$, and the all agents know everything except a, for which they have prior Gaussian distribution at time t_0. We suppose that $X_{t_0} = W_{t_0} = 0$, and are concerned with times $t \geq t_0$.

The reference prob'y is where $a=0$, so X is a centred OU process. Relative to this, the LR martingale will be

$$\exp\{\ \lambda a(W_t - W_{t_0}) - \tfrac{1}{2}(\lambda a)^2(t-t_0)\}$$

so if we mix this over a $N(\alpha, \varepsilon^{-1})$ prior for λa we obtain

$$\Lambda_t = \left(\frac{\varepsilon}{\varepsilon+\Delta t}\right)^{\frac{1}{2}} \exp\left[\frac{(\Delta W)^2 + 2\alpha\varepsilon\,\Delta W - \varepsilon\alpha^2\Delta t}{2(\varepsilon+\Delta t)}\right] \qquad \left[\begin{matrix}\Delta t \equiv t - t_0 \\ \Delta W \equiv W_t - W_{t_0}\end{matrix}\right]$$

2) Now arguing as before, we let the population of agents thicken up to a distribution, so that

$$\sum p_j \log \Lambda_t^j \simeq \text{const} + \varepsilon\langle\alpha\rangle \int_0^\infty \frac{W_t - W((t-u)\vee t_0)}{\varepsilon+u}\,\varphi(u)\,du$$

$$+ \frac{1}{2}\int_0^\infty \frac{(W_t - W((t-u)\vee t_0))^2}{\varepsilon+u}\,\varphi(u)\,du$$

where we implicitly assume that agents alive before time t_0 only see the evolution of X from time t_0 onwards. Because of the renewal theory interpretation of φ, it has to be that φ is <u>decreasing</u>; thus if we make the natural choice

$$\boxed{\varphi(u) = A(\varepsilon+u)\,\lambda e^{-\lambda u},} \qquad A \equiv \lambda/(1+\varepsilon\lambda)$$

then we have the condition

$$\boxed{\lambda \geq 1/\varepsilon}$$

3) Next we must develop the processes

$$\xi_t \equiv \int_0^\infty \lambda e^{-\lambda u}\{W_t - W((t-u)\vee t_0)\}\,du,$$

$$y_t \equiv \int_0^\infty \lambda e^{-\lambda u}\{W_t - W((t-u)\vee t_0)\}^2\,du.$$

Notice that

$$\xi_t = W_t - \int_0^{t-t_0} \lambda e^{-\lambda u} W_{t-u}\, du$$

$$= W_t - \int_{t_0}^t \lambda e^{-\lambda(t-s)} W_s\, ds$$

so that

$$d(e^{\lambda t}\xi_t) = e^{\lambda t}(d\xi_t + \lambda \xi_t\, dt) = dW_t \cdot e^{\lambda t}$$

from which we see that

$$\boxed{\xi_t = X_t}$$

For y_t, we have

$$y_t = W_t^2 - 2W_t \int_0^{t-t_0} \lambda e^{-\lambda u} W_{t-u}\, du + \int_0^{t-t_0} \lambda e^{-\lambda u} W_{t-u}^2\, du$$

$$= W_t^2 - 2W_t(W_t - X_t) + \int_{t_0}^t \lambda e^{-\lambda t} e^{\lambda s} W_s^2\, ds$$

$$= W_t(2X_t - W_t) + e^{-\lambda t}\int_{t_0}^t e^{\lambda s} W_s^2\, ds.$$

Hence

$$d(e^{\lambda t} y_t) = d(e^{\lambda t} X_t^2) + \lambda e^{\lambda t} X_t^2\, dt$$

and

$$\boxed{y_t = X_t^2 + e^{-\lambda t}\int_{t_0}^t \lambda e^{\lambda s} X_s^2\, ds}$$

This now gives us (to within irrelevant constant)

$$\boxed{\begin{aligned} \log \zeta_t &= -\tilde{\rho} t - \sigma\Gamma X_t + \varepsilon\langle\alpha\rangle A\, X_t + \frac{A}{2} y_t \\ &= -\tilde{\rho} t + B X_t + \frac{A}{2}\left\{X_t^2 + e^{-\lambda t}\int_{t_0}^t \lambda e^{\lambda s} X_s^2\, ds\right\}, \end{aligned}}$$

with $B \equiv \varepsilon\langle\alpha\rangle A - \sigma\Gamma$.

4) When it comes to calculating an expression for the price of the stock, it will therefore be helpful to be able to compute (for general θ, for $t_0 \le t \le T$)

$$E\left[\exp\left\{\theta X_T + \frac{A}{2} X_T^2 + e^{-\lambda T}\int_{t_0}^T \lambda \frac{A}{2} e^{\lambda s} X_s^2\, ds\right\} \,\middle|\, \mathcal{F}_t\right]$$

$$= \exp\left[e^{-\lambda T}\int_{t_0}^t \frac{\lambda A}{2} e^{\lambda s} X_s^2\, ds + \frac{1}{2} a(t) X_t^2 + b(t) X_t + c(t)\right] \equiv e^{Y_t},$$

we surmise.

There's a Bessel function identity

$$Y_1(x) - \frac{x}{2} Y_2(x) = \frac{x}{2} Y_0(x)$$

which simplifies the solution for g. Ignoring irrelevant multiplicative constants, we shall have

$$e^{\lambda t} g(t) = J_0(\kappa)\, Y_2(\kappa e^{-\lambda t/2}) - Y_0(\kappa)\, J_2(\kappa e^{-\lambda t/2}), \qquad \kappa \equiv 2\sqrt{A/\lambda}.$$

Numerics suggest that if

$$\alpha(\tau) \equiv a(T-\ ), \qquad \beta(\tau) \equiv b_1(T-\tau), \qquad \gamma(\tau) \equiv c(T-\tau)$$

$$= -\dot{g}(\tau)/g(\tau) \qquad = \frac{g(0)}{e^{\lambda t} g(t)} \qquad = \frac{1}{2}\log\left\{\frac{g(\tau)}{g(0)}\right\} + \frac{1}{2}\int_0^\tau \theta^2 b_1(s)^2\,ds$$

then

$$\begin{cases} \alpha \geq 0, & \alpha \text{ decreases} \\ \beta \geq 0, & \beta \text{ decreases} \\ g(\tau)/g(0) & \text{decreases to a positive finite limit} \end{cases}$$

The usual Itô analysis leads us to

$$\boxed{\begin{aligned} 0 &= \tfrac{1}{2}a^2 + \frac{\lambda A}{2}\, e^{-\lambda(T-t)} + \tfrac{1}{2}\dot{a} - \lambda a \\ 0 &= ab + \dot{b} - \lambda b \\ 0 &= \dot{c} + \tfrac{1}{2}b^2 + \tfrac{1}{2}a \end{aligned}}$$

with bcs $a(T) = A$, $b(T) = \theta$, $c(T) = 0$. Solve the first one by Riccati trick: $a = \dot{\varphi}/\varphi$

So we get

$$0 = \tfrac{1}{2}\ddot{\varphi} - \lambda\dot{\varphi} + \tfrac{1}{2}\lambda A\, e^{-\lambda(T-t)}\varphi.$$

If we set $g(u) \equiv \varphi(T-u)$, we can solve explicitly for g, using Maple:

$$\boxed{\begin{aligned} e^{\lambda t} g(t) = {} & \left\{\sqrt{A\lambda}\, Y_1\left(2\sqrt{A/\lambda}\right) - AY_2\left(2\sqrt{A/\lambda}\right)\right\} J_2\left(2e^{-\lambda t/2}\sqrt{A/\lambda}\right) \\ & - \left\{\sqrt{A\lambda}\, J_1\left(2\sqrt{A/\lambda}\right) - AJ_2\left(2\sqrt{A/\lambda}\right)\right\} Y_2\left(2e^{-\lambda t/2}\sqrt{A/\lambda}\right) \end{aligned}}$$

Similarly, if $h(u) \equiv b(T-u)$, we get from the second DE that

$$0 = -\frac{\dot{g}}{g}h - \dot{h} - \lambda h \quad \Rightarrow \quad \boxed{h(t) = \frac{\theta g(0)}{e^{\lambda t} g(t)}}$$

It seems the expression for c will not simplify.

Nonetheless, the differentiation wrto θ (which is what we require to calculate the stock price) is not too hard (differentiating exponential of a quadratic in θ), so we should be able to do something numerically.

We have $(\tau \equiv T-t)$

$$\begin{cases} a(t) = \dot{\varphi}(\tau)/\varphi(\tau) \\ b(t) = \theta\, b_1(\tau) \equiv \theta\, g(0) / e^{\lambda\tau} g(\tau) \\ c(t) = \tfrac{1}{2}\log\dfrac{g(\tau)}{g(0)} + \tfrac{1}{2}\theta^2 \displaystyle\int_0^\tau b_1(s)^2\, ds \end{cases}$$

FX order-flow stuff again (8/1/07)

(i) Suppose that Z^i_t is the time-t price of one unit of currency i, measured in units of currency 0 (possibly in units of currency 0 discounted back to time 0). Suppose that by time t, $K_{ij}(t)$ units of currency i have been converted into currency j by clients. Suppose that by time t, $A_{ij}(t)$ units of currency i have been converted into currency j by the bank. The bank charges proportional transaction cost $\varepsilon_{ij}(t)$ on moves from currency i to currency j, which it can in principle vary with time, and must itself pay transaction cost $\varepsilon'_{ij}(t)$. If the bank sets transaction cost ε, it will get proportion $\varphi(\varepsilon)$ of the total client order flow (it may be that φ depends on i, j).

If X^i_t denotes the total number of units of currency i held at time t by the bank, then the dynamics are given by

$$dX^i_t = \sum_{j\neq i} \left\{ \varphi(\varepsilon_{ij}(t))\, dK_{ij}(t) - (1-\varepsilon_{ji}(t))\,\varphi(\varepsilon_{ji}(t))\, dK_{ji}(t)\, Z^j_t / Z^i_t \right\}$$

$$- \sum_{j\neq i} \left\{ dA_{ij}(t) - (1-\varepsilon'_{ji}(t))\, Z^j_t\, dA_{ji}(t) / Z^i_t \right\}$$

$$+ r^i_t X^i_t\, dt.$$

The bank controls A_{ij}, ε_{ij}. It is probably helpful to work with $x^i_t \equiv X^i_t Z^i_t$, the value (in currency 0) of the holding of currency i at time t. If we set

$$dZ^i_t = Z^i_t\, dz^i_t$$

then the dynamics for x^i are

$$\boxed{\begin{aligned} dx^i_t &= x^i_t(dz^i_t + r^i_t\, dt) + \sum_{j\neq i} \left\{ \varphi(\varepsilon_{ij})\, dk_{ij}(t) - (1-\varepsilon_{ji})\,\varphi(\varepsilon_{ji})\, dk_{ji} \right\} \\ &\quad - \sum_{j\neq i} \left\{ da_{ij} - (1-\varepsilon'_{ji})\, da_{ji} \right\} \end{aligned}}$$

where $dk_{ij} \equiv Z^i_t\, dK_{ij}(t)$, $da_{ij} \equiv Z^i_t\, dA_{ij}(t)$.

Let's suppose that the bank's objective is to

$$\max E\left[\int_0^\infty e^{-\beta t} F(x_t)\, dt\right]$$

for some function F which we'll come to later. Let's also suppose that we need to keep the X^i non-negative (so another interpretation is that we have to keep the holdings of currency above some given position limits).

(ii) Dual formulation. Introduce multiplier processes $\tilde{y}^i_t = e^{-\beta t} y^i_t$, one for each currency, and do the usual Lagrangian/Pontryagin trick:

$$\max \; E \int_0^\infty e^{-\beta t}\Big[F(x_t)\,dt + \sum_{i\neq j}\sum \{ y^i \varphi(\varepsilon_{ij}) - y^j(1-\varepsilon_{ij})\varphi(\varepsilon_{ij}) \}\, dk_{ij}$$

$$- \sum_{i\neq j}\sum \{ y^i - y^j(1-\varepsilon'_{ij}) \}\, da_{ij} + \sum_i y^i x^i (dz^i + r^i dt)$$

$$+ \sum_i x^i (dy^i - \beta y^i dt) + \sum_i dx^i dy^i \Big] - \Big[x_t \cdot \tilde{y}_t \Big]_0^\infty$$

From this we can learn quite a lot:

$$\boxed{ y^i_t \geq y^j_t \left(1 - \varepsilon'_{ij}(t)\right) \quad \forall t, \; \forall i \neq j \quad , \text{ equal when } da_{ij} > 0 }$$

Assuming that k_{ij} is always increasing, a perfectly reasonable assumption, we find that the setting of $\varepsilon_{ij}(t)$ is determined by the y^i_t, to maximise

$$y^i \varphi(\varepsilon) - y^j (1-\varepsilon)\varphi(\varepsilon).$$

Examples: If $\varphi(\varepsilon) = e^{-\lambda\varepsilon}$, then best ε is $\varepsilon = 1 + 1/\lambda - y^i/y^j$.

If $\varphi(\varepsilon) = e^{-\lambda\varepsilon^2/2}$, then best ε is

$$\varepsilon = \frac{y_j - y_i + \{ (y_j - y_i)^2 + 4\lambda y_j^2 \}^{1/2}}{2\lambda y_j}.$$

However the form of φ is chosen, we always have $\max_\varepsilon \, y^i\varphi(\varepsilon_{ij}) - y^j(1-\varepsilon_{ij})\varphi(\varepsilon_{ij})$
$= y^i \psi(y^j/y^i)$ for some function ψ which can be made explicit in some examples.
This brings the Lagrangian problem to

$$\max \Big[x_0 \cdot y_0 + E \int_0^\infty e^{-\beta t} \Big[F(x_t)\,dt + \sum_{i\neq j}\sum y^i \psi(y^j/y^i)\, dk_{ij}$$

$$+ \sum_i y_i x^i (dz^i + r^i dt) + \sum x^i (dy^i - \beta y^i dt) + \sum x^i dy^i dz^i \Big]\Big]$$

We could go a bit further by assuming z is a drifting BM, and y has exponential form, but the inequality $y^i \geq y^j(1-\varepsilon'_{ij})$ is going to get in the way of an explicit solution.

A question of Alexander Cherny (11/1/07)

Alexander asks: if X_t is a continuous martingale, $X_t \sim N(0,t)$ $\forall t \geq 0$, is it the case that X is a Brownian motion?

We have some cts adapted process A_t such that $X_t^2 - A_t$ is a local martingale, and the question now is whether $A_t = t$.

Notice that $|X_t|^p$ is a submartingale for all $p \geq 1$, so for any t_0, $\{ |X_{t \wedge \tau}|^p : t \leq t_0, \tau \text{ a stopping time} \}$ is uniformly integrable. Now consider the Hermite polynomials

$$\sum_{n \geq 0} \frac{\theta^n}{n!} H_n(t,x) \equiv \exp(\theta x - \theta^2 t/2)$$

$$= \sum_{n\geq 0} \frac{\theta^n}{n!} \left(x - \frac{\theta t}{2} \right)^n$$

We have $H_0 = 1$, $H_1 = x$, $H_2 = x^2 - t$, $H_{m+1} = x H_m - mt H_{m-1}$. Always $H_n(A_t, X_t)$ is a local martingale, so if

$$H_{2n}(t,x) = \sum_{r=0}^{n} b_{2n,r} \, t^r x^{2(n-r)},$$

by stopping at a stopping time $\tau \leq T$ which reduces $H_{2n}(A_t, X_t)$ we deduce from OST that

$$b_{2n,n} \, E A_\tau^n = - \sum_{r=0}^{n-1} b_{2n,r} \, E\left[A_\tau^r X_\tau^{2n-2r} \right]$$

whence

$$|b_{2n,n}| . E A_\tau^n \leq \sum_{r=0}^{n-1} |b_{2n,r}| \left(E A_\tau^n \right)^{r/n} \left(E X_\tau^{2n} \right)^{(n-r)/n}$$

Letting $\tau \uparrow T < \infty$, and using the UI property of $|X|^p$, we can conclude that

$$\boxed{|b_{2n,n}| \; E A_T^n \leq \sum_{r=0}^{n-1} |b_{2n,r}| \left(E A_T^n \right)^{r/n} \left(E X_T^{2n} \right)^{(n-r)/n}}$$

We have that $E X_T^{2n} = T^n (2n)! / 2^n n!$ and hence we deduce that all moments of A_T are finite. Writing $\xi_n^n \equiv E A_T^n / E X_T^{2n}$ for $T = 1$ (say) we have

$$\boxed{|b_{2n,n}| \; \xi_n^n \leq \sum_{r=0}^{n-1} |b_{2n,r}| \; \xi_n^r}$$

so this even provides an upper bound for ξ_n. Can also show

$$\boxed{b_{2n,r} = \left(-\frac{1}{2}\right)^r \frac{2n!}{r! \, (2(n-r))!}}$$

so $|b_{2n,n}| = E[X_1^{2n}]$.

Steering your portfolio to the end-of-day position (16/1/07)

1) Suppose we have a stock modelled for simplicity as

$$dS_t = \sigma\, dW_t + \mu\, dt$$

and we aim to have the number θ_T of units of stock held at the end of the day to be close to some linear function $a + b S_T$. Suppose we incur (liquidity) losses $-\frac{1}{2} k_0 |\dot\theta_t|^2$ for changing holding at rate $\dot\theta_t$. Then the objective we propose is

$$V(t, \theta, S) = \sup E_t\left[\int_t^T \left(\theta_u \mu - \tfrac{1}{2} k_0 |\dot\theta_u|^2\right) du - \tfrac{1}{2} k_1 \left(\theta_T - a - b S_T\right)^2\right]$$

since the wealth $dw_t = \theta_t\, dS_t - \frac{1}{2} k_0 |\dot\theta_t|^2\, dt$. Now the HJB equation we get is

$$0 = \sup_{\dot\theta}\left[\mu\theta - \tfrac{1}{2} k_0 |\dot\theta|^2 + \dot V + \dot\theta V_\theta + \mu V_S + \tfrac{1}{2}\sigma^2 V_{SS}\right]$$

with condition $V(T, \theta, S) = -\frac{1}{2} k_1 (\theta - a - bS)^2$ as the terminal condition.

2) We have the notion that

$$V(t, \theta, S) = \tfrac{1}{2}(\theta, S)\, Q(\tau) \begin{pmatrix}\theta \\ S\end{pmatrix} + q(\tau)\cdot\begin{pmatrix}\theta \\ S\end{pmatrix} + \gamma(\tau)$$

$(\tau \equiv T - t)$ for some quadratic form. The HJB equation says

$$0 = \mu\theta + V_\theta^2/2k_0 + \dot V + \mu V_S + \tfrac{1}{2}\sigma^2 V_{SS}$$

$$= \mu\theta + \frac{1}{2k_0}\left(Q_{\theta\theta}\theta + Q_{\theta S} S + q_\theta\right)^2 - \tfrac{1}{2}(\theta, S)\, \dot Q \begin{pmatrix}\theta \\ S\end{pmatrix} - \dot q \begin{pmatrix}\theta \\ S\end{pmatrix}$$

$$+ \mu\left(Q_{SS} S + Q_{\theta S}\theta + q_S\right) + \tfrac{1}{2}\sigma^2 Q_{SS}$$

Picking out the c/es gives us various equations:

θ^2] $\quad \frac{1}{2k_0} Q_{\theta\theta}^2 = \frac{1}{2}\dot Q_{\theta\theta}$

θS] $\quad \frac{1}{k_0} Q_{\theta S} Q_{\theta\theta} = \dot Q_{\theta S}$

S^2] $\quad \frac{1}{k_0} Q_{\theta S}^2 = \dot Q_{SS}$

There are others, but let's just do these for a start.

(Calculations to be checked by Maple...)

The first one is solved by

$$\boxed{Q_{\theta\theta}(\tau) = \frac{-k_0 k_1}{k_0 + k_1 \tau}}$$

the second by

$$\boxed{Q_{\theta S}(\tau) = \frac{b k_0 k_1}{k_0 + k_1 \tau}}$$

and lastly

$$\boxed{Q_{SS}(\tau) = -\frac{b^2 k_0 k_1}{k_0 + k_1 \tau}}$$

Further calculations lead to

$$\boxed{q_\theta(\tau) = \frac{a k_0 k_1 + \mu\left\{(k_0 - b k_0 k_1)\tau + \frac{1}{2} k_1 \tau^2\right\}}{k_0 + k_1 \tau}}$$

This can already be assembled to give the optimal investment rate of change:

$$\dot{\theta}_t = \frac{1}{k_0} V_\theta = \frac{1}{k_0}\left\{Q_{\theta\theta} \cdot \theta + Q_{\theta S} \cdot S + q_\theta\right\}$$

$$= k_0^{-1}(k_0 + k_1\tau)^{-1}\left\{-k_0 k_1 \theta + b k_0 k_1 S + a k_0 k_1 + \mu\left\{(k_0 - b k_0 k_1)\tau + \tfrac{1}{2} k_1 \tau^2\right\}\right\}.$$

(25/2/07)

A special case arises when $b = 0$, and we let $k_1 \to \infty$. Then

$$\boxed{\dot{\theta}_t = \left\{k_0(a - \theta) + \tfrac{1}{2}\mu\tau^2\right\} / k_0 \tau}$$

Some interesting questions / remarks

1) Suppose we see at one moment ($t=0$, say) the entire implied vol surface for $dS_t = \sigma_t S_t \, dW_t$ – can we work out the law of σ?

2) Suppose we see $C(t, S_t; T)$ (i.e. ATM calls for a fixed expiry, as a process) – can we deduce what σ is? (i.e., rcd for $(\sigma_t)_{0 \le t \le T}$ given $\mathcal{F}_s$, $s<T$)

3) If X is a subordinator, $E \exp z X_t = \exp t\psi(z)$, then

$$E \exp z \int_0^t X_u \, du = E \exp\Big(z \int_0^t (t-u) \, dX_u \Big) = \exp\Big\{ \int_0^t \psi(uz) \, du \Big\}$$

4) Costis Skiadas says a lot of this risk measurement stuff is well known in recursive utility literature. "quasi-linear utility" is a phrase used to describe things. Dynamic risk measures appear as "cash invariant recursive utilities". Walter Schachermayer reckons the cash-invariant feature is different from what is already known. "Gorman aggregation" also a phrase used

5) Can we get some mileage out of expressing a BSDE in Stratonovich form? Probably not:

$X_t = X_0 + \int_0^t \sigma(X_s) dW_s + \int_0^t b(X_s) \, ds$, $Y_t = g(X_1) + \int_t^1 f(X_s, Y_s, Z_s) ds - \int_t^1 Z_s \, dW_s$ requires that Z be adapted (without this, there are many solutions), and in the context of an ODE version, this is not a condition that makes a lot of sense

6) In the BS model, is the map $(k, T) \longrightarrow E(S_T - K)^+$ log-concave? Plots of the surface suggest that it is, but numerical examples demonstrate conclusively that it's not.

7) Suppose we have an N-state Markov chain with Q-matrix Q. What is its likelihood ratio martingale relative to the law under which all jumps occur at rate 1?

It's

$$\Lambda_t = \prod_{s \le t} q(\xi_{s-}, \xi_s) \, . \exp\Big\{ \int_0^t \{ N-1 - q(\xi_s) \} \, ds \Big\}$$

where the terms in the product are 1 for non-jump times, and $q(\xi) \equiv \sum_{j \ne \xi} q_{\xi j}$ as usual.

8) If $Y \sim B(\frac{1}{2}, \frac{n}{2})$, then

$$n(1-Y)/Y \sim Z^2, \quad \text{where } Z \sim t_n$$