Another look at the equity-premium puzzle (29/1/07)

1) The quality of fit of the various models I've been investigating to the M-P data set is not good. In particular, the riskless rate always turns out to be almost constant, which is seriously at odds with the observed data. Maybe a more realistic tale to be told is that $\log(y_t / y_{t-1}) = \xi_t$ is indeed an IID sequence, but period-t consumption

$$c_t = z_t \, y_t$$

where (z_t) is an IID-sequence of $(0,1)$-valued random variables. The interpretation is that $(1-z_t)$ is the proportion of national output taken in tax by the government in year t. If this is the model, then the valuation of the stock takes the form

$$S_t = \frac{1}{U'(c_t)} E_t\left[\sum_{j\geq 0} \beta^j U'(c_{t+j})\, c_{t+j}\right]$$

$$= c_t + \frac{1}{U'(c_t)} E_t\left[\sum_{j\geq 1} \beta^j U'(c_{t+j})\, c_{t+j}\right]$$

$$= c_t + \frac{E z^{1-R}}{U'(c_t)} E_t\left[\sum_{j\geq 1} \beta^j y_{t+j}^{1-R}\right]$$

$$= c_t + \frac{y_t^{1-R}\, E z^{1-R}}{U'(c_t)} E_t\left[\frac{\beta e^{-\nu\mu + \frac{1}{2}\nu^2\sigma^2}}{1-\beta e^{-\nu\mu+\frac{1}{2}\nu^2\sigma^2}}\right] \qquad (\nu \equiv R-1)$$

$$= y_t\left[z_t + z_t^R\, E(z^{1-R})\, E_t \frac{\beta e^{-\nu\mu+\frac{1}{2}\nu^2\sigma^2}}{1-\beta e^{-\nu\mu+\frac{1}{2}\nu^2\sigma^2}}\right],$$

rather as before, and the story for the riskless rate is

$$B_t = \frac{\beta}{U'(c_t)} E_t\left[U'(c_{t+1})\right]$$

$$= \frac{\beta\, E z^{-R}}{U'(c_t)} E_t\left[y_{t+1}^{-R}\right]$$

$$= \frac{\beta\, y_t^{-R}\, E z^{-R}}{U'(c_t)} E_t\left[e^{-R\mu + \frac{1}{2}\sigma^2 R^2}\right]$$

$$= \beta\, z_t^R \left(E z^{-R}\right) E_t\left[e^{-\mu R + \frac{1}{2}\sigma^2 R^2}\right].$$

2) When we try fitting to data, let's assume that $z \sim B(a,b)$, so that

$$E z^\lambda = \frac{\Gamma(a+b)}{\Gamma(a)} \frac{\Gamma(a+\lambda)}{\Gamma(a+b+\lambda)}$$

As a first attempt, let's not do too much that's Bayesian, just suppose that the agent knows all the parameters. Actually, if we plan to fit the z_t to the bond data, there is the danger of values of z in excess of 1, so I think it probably makes more sense to propose $z \sim \Gamma(a,b)$, so $E z^\lambda = b^{-\lambda} \Gamma(a+\lambda)/\Gamma(a)$. For convergence of the stock price we must have $\beta \exp(-\nu\mu + \frac{1}{2}\nu^2\sigma^2) < 1$. The parameters of the problem are

$$(\beta, R, \mu, \sigma^2, a, b, v) \equiv \theta$$ (v is the variance of noise in observing S_t)

If these are given, we go to the bond data to deduce the z_t, then to the consumption data to work out the y_t. The likelihood is constructed from that. Note however that for the expectation $E[z^{-R}]$ to be finite, we need $a > R$, so let's work with $\tilde{a} \equiv a - R > 0$ as the fifth parameter.

3) Let's take a Bayesian point of view, as on p 33 of WN XXVI. With H_t as defined there, we shall have

$$S_t = y_t \left[z_t + z_t^{\ R} E(z^{1-R}) \frac{\beta e^{\nu^2/2\tau} H_t(-\nu)}{H_t(0) - \beta e^{\nu^2/2\tau} H_t(-\nu)} \right]$$

$$B_t = \beta\ z_t^R (E z^{-R})\ e^{R^2/2\tau} \frac{\{ H_t(-R) - \beta e^{\nu^2/2\tau} H_t(-R-\nu) \}}{H_t(0) - \beta e^{\nu^2/2\tau} H_t(-\nu)}$$

As a first attempt, we could suppose that B was observed with no error (whence we deduce the z_t), S_t is observed with no error (whence we deduce the y_t), and finally put all the observational noise into the consumption growth.

But perhaps it will be better to allow observational error on all three series; this will for example allow us to take a $B(a,b)$ dist^n for z without any faking. Suppose noise is additive Gaussian in the logs. Parameters of the problem are:

$$(\beta, R, \kappa_0, \tau, \mu, a, b, v_1, v_2, v_3 ; \xi ; z)$$

(assume $m_0 = 0$ with no great loss of generality). Notice that while we can define $y_0 = 1$ wlog, we must be ready to find the value of z_0, so z is 1 longer than ξ.

Question of Alexander Cherny (25/2/07).

Can we have a cts martingale $(X_t)_{t\geq 0}$ s.t. $X_t \sim N(0,t)$ for all $t \geq 0$, other than Brownian motion?

Perhaps we might have a diffusion example

$$dX_t = \sigma(t, X_t)\, dW_t \quad ?$$

(1) The transition density has to satisfy the KFE

$$\frac{\partial p_t}{\partial t}(x,y) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(v(t,y)p_t(x,y)\right) = 0$$

where $v(t,y) \equiv \sigma(t,y)^2$. The KFE must be satisfied by $p_t(0,y) = g_t(y) \equiv \exp\{-y^2/2t\}/\sqrt{2\pi t}$, so we must have

$$g_t + \frac{1}{2}\left\{ v\, g_{yy} + 2 v_y g_y + v_{yy} g \right\} = 0$$

or equivalently

$$\frac{1}{2}\left(\frac{y^2}{t^2} - \frac{1}{t}\right)(v-1) - \frac{y}{t} v_y + \frac{1}{2} v_{yy} = 0.$$

This can be solved t by t, and one solution is $v \equiv 1$. A solution to the homogeneous equation is $v(t,y) = v^0(t,y) \equiv \exp\{y^2/2t\}$. The other one is $y\, v^0(t,y)$.

This would suggest we use (for some $\varepsilon > 0$)

$$\sigma(t,y) = \left(1 + \varepsilon \exp(y^2/2t)\right)^{\frac{1}{2}}$$

as the volatility function, but do we then have that the diffusion is a martingale?
What if we turn on the extra piece after $t = 1$?

A variant of habit formation (27/2/07)

1) Suppose we have the traditional dynamics for wealth and EWMA consumption

$$\begin{cases} dw_t = r w_t + \theta_t \left(\sigma \, dW_t + (\mu - r)\, dt \right) - c_t \, dt \\ d\bar{c}_t = \lambda (c_t - \bar{c}_t)\, dt \end{cases}$$

with objective

$$E \int_0^\infty e^{-\rho t} U(c_t / \bar{c}_t)\, dt$$

[Constantinides uses the less realistic objective $E\int_0^\infty e^{-\rho t} U(c_t - \bar{c}_t)\, dt$, which reduces to the standard problem when we work with the modified variable $\tilde{w} = w - r^{-1}\bar{c}$ - in effect, you have to guarantee your future consumption stream at at least $\bar{c}$ by putting wealth into the bank, then you play with the rest].

2) If we try to get the value function $V(w, \bar{c})$, we notice that for any $\alpha > 0$

$$V(\alpha w, \alpha \bar{c}) = V(w, \bar{c})$$

and so there is a simpler prescription $V(w, \bar{c}) = v(w/\bar{c})$. Thus the HJB equation

$$\sup_{\theta, c} \left[U\left(\tfrac{c}{\bar{c}}\right) - \rho V + (rw + \theta(\mu - r) - c) V_w + \tfrac{1}{2}\theta^2\sigma^2 V_{ww} + \lambda (c - \bar{c}) V_{\bar{c}} \right] = 0$$

becomes

$$\sup_{\theta, c} \left[U\left(\tfrac{c}{\bar{c}}\right) - \rho v + (rw + \theta(\mu - r) - c)\frac{1}{\bar{c}} v' + \frac{1}{2}\left(\frac{\theta}{\bar{c}}\right)^2 \sigma^2 v'' - \frac{\lambda w}{\bar{c}^2}(c - \bar{c}) v' \right] = 0$$

Writing $w/\bar{c} \equiv t$, $c/\bar{c} \equiv \xi$, $\theta = \bar{c} y$, we get

$$\sup_{\xi, y} \left[U(\xi) - \rho v + (rt + (\mu - r) y - \xi) v' + \tfrac{1}{2} y^2 \sigma^2 v'' - \lambda t (\xi - 1) v' \right] = 0$$

We get $y^* = -(\mu - r) v' / \sigma^2 v''$, ξ^* from $U'(\xi) = (1 + \lambda t) v'$, and then

$$\boxed{\tilde{U}\left((1+\lambda t) v'\right) - \rho v + r t v' - \tfrac{1}{2}\kappa^2 v'^2 / v'' + \lambda t v' = 0}$$

3) The usual dual variables trick gets us to $[J(z) = v(t) - tz, \; z = v'(t)]$

$$\boxed{0 = \tilde{U}\left(z(1 - \lambda J')\right) - \rho J + (\rho - r - \lambda) z J' + \tfrac{1}{2}\kappa^2 z^2 J''}$$

Non-linear this time. For $\lambda = 0$, $J(z) = \gamma^{-1}\tilde{U}(z)$, with $\gamma = R^{-1}(\rho + (R-1)(r + \kappa^2/2R))$.

It doesn't look like this one can be done in closed form, but we could write

$$J(z) = g(\log z)$$

to tidy up the linear part; with $\log z = s$, we have

$$0 = \tilde{u}(e^s - \lambda g'(s)) - \rho g(s) + (\rho - r - \lambda - \tfrac{1}{2}\kappa^2) g'(s) + \tfrac{1}{2}\kappa^2 g''(s)$$

4) Solving the nonlinear ODE for v by numerical means should be OK, but we'll need some boundary conditions. For very small t, there will be negligible investment in the stock, and the equation will be approximately (since $\lambda t \ll 1$)

$$0 = \tilde{u}(v') - \rho v + (r+\lambda) t v',$$

solved by $v(t) = A\, u(t)$, where $A = \left\{ \dfrac{\rho + (R-1)(r+\lambda)}{R} \right\}^{-R}$ ...?

No, this is not correct! The story should really be that $\bar{c}_t \sim \bar{c}_0 e^{-\lambda t}$, since consumption will be extremely small, and this is like the Merton problem with $\rho' \equiv \rho - \lambda(1-R)$, so solution is the usual Merton solution

$$v(t) = \gamma_M^{-R}\, u(t), \qquad \gamma_M = R^{-1}\{ \rho' + (R-1)(r + \kappa^2/2R) \}$$
$$= R^{-1}\{ \rho + (R-1)(r + \lambda + \kappa^2/2R) \}$$

How about large values of t? One approximation would simply be to take $v' = 0$. A little more subtle is to study the dual equation near $z=0$, where $-J'(z)$ is very big so the equation is approximately

$$\tilde{u}(-\lambda z J') - \rho J + (\rho - r - \lambda) z J' + \tfrac{1}{2}\kappa^2 z^2 J'' = 0$$

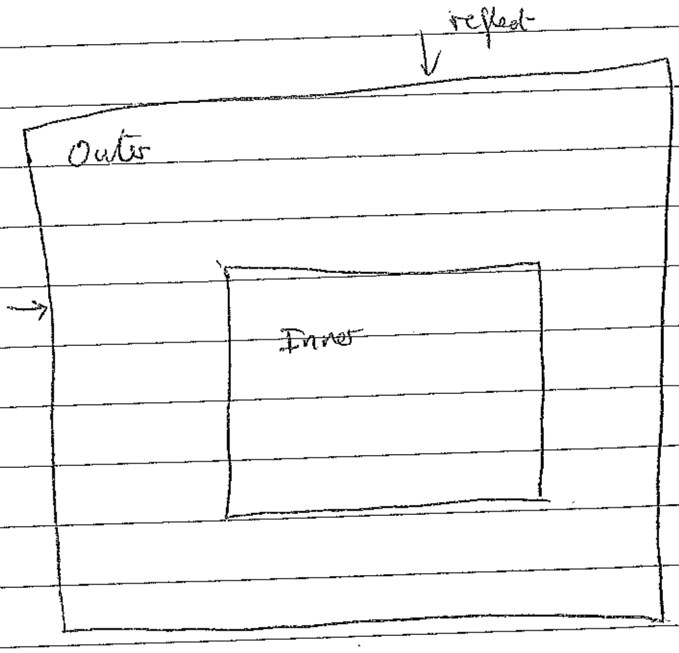
This would have a power solution approximately, $J(z) = A z^{-\beta}$, where β solves

$$-\rho - \beta(\rho - r - \lambda) + \tfrac{1}{2}\kappa^2 \beta(\beta+1) = 0, \quad \beta > -1,$$

the idea being that the term $\tilde{u}(-\lambda z J')$ will be much smaller order. Writing $\beta + 1 \equiv \tilde{R}$, this would suggest that for large t we have $v(t) \sim \text{const}\, t^{1-1/\tilde{R}}$ and hence at the highest grid point t_n, we would have

$$t v'(t) - (1 - 1/\tilde{R}) v(t) = 0,$$

which gives the boundary condition (very similar to $v' = 0$).


reflect
Outer
Inner

Numerical solutions to HJB (1/3/07)

If we have some diffusion-based stochastic control example, we typically will have the HJB equation as

$$\sup_a \mathcal{L} V(x,a) + F(x,a) = 0$$

where a is the control. To deal with this numerically, we'd make some grid for the x-values and try to approximate the diffusion operator $\mathcal{L}$ onto that grid. The problem though will happen at the edges of the region, and how we define the operator there. In order to exploit the monotonicity properties of policy improvement, we really need a genuinely Markovian approximation, yet absorbing or reflecting bcs at the edges of the region of interest can be way off. Using natural bcs for the PDE loses the Markovian interpretation, and 'policy improvement' goes unstable in such cases.

What I think needs to be done is to work out what is going on in some outer region around the inner region (= region of interest) as well. In this outer region, we will suppose that the policy takes some suitably-chosen fixed form, and at the outside edges of the outer region, we'll do reflecting bcs (because we need there to be a green f$^{\text{c}}$ for the process in the outer region. The grid in the outer region may be quite a lot coarser than in the inner region - we are only computing it in order to get good bcs at the edge of the inner region. Suppose we express the HJB equations (discretised) as

$$\sup \left\{ F(x,a) - (\rho - Q) V(x,a) \right\} = 0$$

and we partition $(\rho - Q)$ as

$$\rho - Q = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \begin{array}{l} \text{Inner} \\ \text{Outer} \end{array}$$

Then we have for policy improvement

$$(\rho - Q) V_n(\cdot, \alpha_n) = F(\cdot, \alpha_n) \qquad (n \geq 0)$$

and α_{n+1} is the action maximising

$$F(\cdot, a) - (\rho - Q) V_{n+1}(\cdot, a).$$

But notice that $F(\cdot, a)$ partitions as

$$F = \begin{pmatrix} f(\cdot, a) \\ g \end{pmatrix}$$

where g never changes, because we do not allow the policy to be varied in the outer region.

In implementing this, it is necessary to be careful when κ is small, but also when κ is large, for then the exponentials can explode. The ratios remain sensible, but we need to scale the numerator + denominator to prevent nastiness. Writing $x_p' = x_p - x_0$, $x_m' = x_m - x_0 < 0 < x_p'$, we get

$$\Delta' = x_p' \left\{ \exp(-\kappa^+ (x_p' - x_m')) - \exp(-\kappa^+ x_p' - \kappa^- |x_m'|) \right\}$$
$$+ |x_m'| \left\{ \exp(-\kappa^- (x_p' - x_m')) - \exp(-\kappa^+ x_p' - \kappa^- |x_m'|) \right\}$$

$$\cancel{q(x_0, x_p) = (\Delta')^{-1} \mu \left(\exp(-\kappa^+ (x_p' - x_m')) - \exp(-\kappa^+ x_p - \kappa^- |x_m|) \right)}$$

$$q(x_0, x_m) = \frac{\mu}{\Delta'} \left(\exp(-\kappa^+ x_p - \kappa^- |x_m|) - \exp(-\kappa^- (x_p' - x_m')) \right) \qquad (\mu = -\sigma^2 \kappa / 2)$$

To compute the value function on the inner region, we therefore have to solve

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} V \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

or again

$$(A - BD^{-1}C)\,V = f - BD^{-1}g \qquad (*)$$

How this gets used is that at the beginning we calculate and store $D^{-1}C$ and $D^{-1}g$; these never get changed. To calculate the value of a new policy (which means that we are using new A, B), we use $(*)$. This will be a true implementation of policy improvement, so should work very well...

But it's probably too complicated; why not just solve over the entire grid, and then just throw away the solution on the outer grid!

One point where care is needed is in constructing a Markovian approximation to the diffusion. The nice way to do this is to think that if we have three consecutive grid points $x_m < x_0 < x_p$, then we approximate the diffusion by a constant drift constant vol BM, with drift $\mu = \mu(x_0)$, vol $\sigma = \sigma(x_0)$; we then make the mean time to hit $\{x_m, x_p\}$ match for both the chain and the drifting BM. Some calculations give us

$$q(x_0, x_p) = \frac{-\mu\left(e^{\kappa x_0} - e^{\kappa x_m}\right)}{\Delta}$$

$$q(x_0, x_m) = \frac{-\mu\left(e^{\kappa x_p} - e^{\kappa x_0}\right)}{\Delta} \qquad \left(\kappa = -2\mu/\sigma^2\right)$$

$$-q(x_0, x_0) = \frac{-\mu\left(e^{\kappa x_p} - e^{\kappa x_m}\right)}{\Delta}$$

where $\Delta \equiv (x_p - x_0)\left(e^{\kappa x_m} - e^{\kappa x_0}\right) + (x_0 - x_m)\left(e^{\kappa x_p} - e^{\kappa x_0}\right)$. To be honest, we ought to replace the diffusion by the chain, and then optimise the equations with the chain generator. The snag is that the dependence of the chain generator on the control parameters is not so pretty. I think the way we could get round this would be to pretend we are optimising the differential operator to come up with controls in the various states which we think should do better.

<u>Notes</u> (i) if the utility is unbounded below, (eg, CRRA, $R>1$), then the threat of loss K doesn't really have much impact. So for this example, need to use a utility that's bounded below (and quite possibly above too)

(ii) If we use the two-R utility (reverse of p9) then we see that for large w, the value looks like the Merton solution for the larger R, plus a constant. Thus

$$\int_0^\infty \eta e^{-\eta x}\big(V(w-x)-V(w)\big)\,dx \doteq \frac{\gamma_M^{-R}\, w^{1-R}}{1-R}\int \eta e^{-\eta x}\Big(\big(1-\tfrac{x}{w}\big)^{1-R}-1\Big)\,dx$$

$$\approx \gamma_M^{-R}\, w^{1-R}\int \eta e^{-\eta x}\big(-x/w\big)\,dx$$

$$= -\gamma_M^{-R}\, w^{-R}/\eta$$

Thus for large w, the value of q chosen will be near to satisfying

$$\big(q\,p(q)\big)' = 1/\eta$$

When $p(q) = \varepsilon^{-1}(q^{-1/\beta}-1)$ as on p9, we obtain

$$q = \left(\frac{\beta-1}{\beta}\;\frac{\eta}{\varepsilon+\eta}\right)^{1/\beta}$$

If it turns out that in some states we <u>don't</u> do better, then don't change the control in those states this time.

<u>Insurance examples with choice of volume of business</u>

If the dynamics of wealth are

$$dw_t = rw_t\,dt + \theta_t(\sigma\,dW_t + (\mu-r)dt) - \delta_t dt + q_t p(q_t)\,dt - dC_t$$

where C is a compound poisson process of $\exp(\eta)$ claims arriving at rate η, and if we get a penalty K for the firm going broke, the HJB equation is

$$\sup_{\delta,q,\theta}\Big[U(\delta) - \rho V + (rw + \theta(\mu-r) - \delta + q\,p(q))V' + \tfrac{1}{2}\sigma^2\theta^2 V'' + q\int_0^\infty \eta e^{-\eta x}\big(V(w-x) - V(w)\big)\,dx\Big] = 0$$

with $V(x) = -K \;\; \forall x<0$. For $x \geq 0$, we might actually have value $> -K$, since we could take on some insurance business + use it to finance growth.

To do some policy improvement, the computation of the value from a new policy is likely to be a bit unpleasant, because the presence of the integral term makes the linear system non-sparse. However, what may work is to start from some policy (eg, no insurance business at all, optimal policy for just the investment problem) Call this π_0, with value V_0. Then recursively obtain π_{n+1}, V_{n+1} by

$$\pi_{n+1} = \underset{\pi=(\delta,q,\theta)}{\operatorname{argmax}}\Big[U(\delta) - \rho V_n + (rw + \theta(\mu-r) - \delta + q\,p(q))V_n' + \tfrac{1}{2}\sigma^2\theta^2 V_n'' + q\int_0^\infty \eta e^{-\eta x}\big(V_n(w-x) - V_n(w)\big)\,dx\Big]$$

and then V_{n+1} to solve

$$U(\delta_{n+1}) - \rho V + \{rw + \theta_{n+1}(\mu-r) - \delta_{n+1} + q_{n+1}p(q_{n+1})\}V' + \tfrac{1}{2}\sigma^2\theta_n^2 V'' + q\int_0^\infty \eta e^{-\eta x}\{V_n(w-x) - V_n(w)\}\,dx = 0$$

This is still a sparse linear system, and V <u>should</u> be improved, as we have the interpretation that at first claim you go back to the previous V_n, but have been doing better up til then.

An insurance example with feedback (2/3/07)

1) This goes back to an idea sketched out in the Munich notes, that the volume $v(p)$ of insurance business you get is a decreasing function $v(p)$ of the premium rate p you charge. The wealth dynamics are therefore

$$dw_t = rw_t\,dt + \theta_t\left(\sigma dW_t + (\mu-r)dt\right) - \delta_t\,dt + p v(p)\,dt - \underbrace{\left(a\sqrt{v(p)}\,dW'_t + b\,v(p)\,dt\right)}_{=dC}$$

where we'll assume $b>0$ as seems reasonable from the interpretation. If the objective is

$$\sup E\left[\int_0^\tau e^{-\rho t}U(\delta_t)\,dt - Ke^{-\rho\tau}\right]$$

then the HJB is

$$0 = \sup_{\theta,\delta,p}\left[U(\delta) - \rho V + (rw + \theta(\mu-r) - \delta + pv - bv)V' + \tfrac{1}{2}(\sigma^2\theta^2 + a^2 v)V''\right]$$

and we find as usual $U'(\delta) = V'$, $\theta = -(\mu-r)V'/\sigma^2 V''$, and for the choice of p,

$$(v + pv')V' = v'\left(bV' - \tfrac{1}{2}a^2 V''\right).$$

so that

$$\boxed{p = \left(b - \frac{a^2 V''}{2V'}\right) - \frac{v}{v'}}$$

As an example, suppose that for some $\beta > 1$ we have $v(p) = (1+\varepsilon p)^{-\beta}$. Then $-v/v' = (1+\varepsilon p)/\beta\varepsilon$ and therefore the best value of p will be

$$\boxed{p = \frac{\beta\varepsilon A + 1}{(\beta-1)\varepsilon}}, \qquad A \equiv b - a^2V''/2V'$$

This should go down to the same numerical approach – policy improvement – as the previous habit formation example.

2) This example is a bit silly, because the agent could discontinue the business if w fell too low, and then he's just doing ordinary Merton; so the K is irrelevant, and probably it makes more sense to let the volume of business be the control variable, with p a function of that. Write q for $v(p)$, the volume of business (since v is being over-used!) We have that $p = \varepsilon^{-1}(q^{-1/\beta} - 1)$. The optimisation over q gives us

$$\boxed{q = \left(\frac{\beta}{\beta-1}(1+\varepsilon A)\right)^{-\beta}}$$

Alternative choice of U would be to take

$$I(y) = \left(y^{1/R_1} + k y^{1/R_2}\right)^{-1} \qquad (R_1 > 1 > R_2)$$

Then

$$a \equiv 1/R_1 < b \equiv 1/R_2$$

$$\tilde{U}(y) = \int_y^\infty \frac{dx}{x^a + k x^b}$$

$$= \frac{k^{(b-1)/(b-a)}}{k(b-a)} \int_{\frac{ky^{b-a}}{1+ky^{b-a}}}^1 t^{(1-b)/(b-a)} (1-t)^{(a-1)/(b-a)} \, dt$$

an incomplete $B(\alpha, \beta)$ integral $\alpha = \frac{1-a}{b-a}$, $\beta = \frac{b-1}{b-a}$

Notice that if we start at zero wealth, we will certainly not do any risky asset, but we will do some insurance business, so at 0 we see

$$\sup_{q,\delta} \left[U(\delta) - \rho V(0) + (q p(q) - \delta) V'(0) - q (V(0) + K) \right] = 0$$

This is only correct when we have jumps, as in the example on p 8 (for this diffusion example, at zero wealth we cannot do anything)

Note: $U \circ I(y) = \tilde{U}(y) + y I(y)$, which can be useful.

but we have to observe the bounds $0 \le q \le 1$, so that we will take $q = 1$ if

$$\beta \epsilon A \geq 1$$

3) We might alternatively let q be unbounded, and suppose that $p(q) = Bq^{-\beta}$, for some $\beta \in (0,1)$. This then leads to optimal q

$$q = \left(B(1-\beta)/A \right)^{1/\beta}$$

which looks quite similar. However, these diffusion approximations to the claims process are very stupid near to $w=0$; the qualitative behaviour is completely wrong.

4) Another idea would be ~~to~~ suppose that the claims dist^n is a compound Poisson process with exponential jumps, coming at rate λ, where λ (the volume of business) is a control variable, with $p(\lambda) = B\lambda^{-\beta}$. This time, bankruptcy cannot be avoided, so the value of K matters. The equations become

$$\sup \Big[U(\delta) - \rho V + \{ rw + \theta(\mu - r) - \delta + \lambda p(\lambda) \} V' + \tfrac{1}{2}\sigma^2\theta^2 V'' - \lambda \int_0^\infty \eta e^{-\eta x} \{ V(w) - V(w-x) \} dx \Big]$$

where $V(w) = -K$ for $w < 0$, though we don't expect that $V(0) = -K$. (Notice that the assumption of exponential jump sizes isn't important either for theory or numerics, but it does mean that we can calculate $\int \eta e^{-\eta x} V(w-x)\,dx$ more easily).

We would have optimal λ from

$$\lambda = \left(B(1-\beta) V'/A \right)^{1/\beta}, \qquad A = \int_0^\infty \eta e^{-\eta x} \{ V(w) - V(w-x) \} dx.$$

5) But this is silly: if $R > 1$ then we prefer the firm to go bust to keeping going at small wealth levels. We really need a U which doesn't go to $-\infty$ at zero, and we'd ideally like something with $R > 1$ at large wealth levels. Best I can do just now is to take

$$I(y) = \begin{cases} y^{-1/R_1} & 0 < y \le 1 \\ \left(\dfrac{a+1}{a+y} \right)^{+1/R_2} & y \geq 1 \end{cases}$$

for $R_1 > 1 > R_2$, $a \equiv (R_1/R_2) - 1$ to make $I(y)$ C^1. Then

$$\tilde{U}(y) = \begin{cases} \dfrac{(a+y)^{1-1/R_2}(a+1)^{1/R_2}}{1/R_2 - 1} & \text{for } y \geq 1 \\ \dfrac{y^{1-1/R_1} - 1}{1/R_1 - 1} + \dfrac{R_1}{1-R_2} & \text{for } 0 \le y \le 1 \end{cases}$$

Recursive utility examples (7/3/07)

Suppose we have some conventional wealth dynamics

$$dw_t = r w_t \, dt + \theta_t (\sigma dW_t + (\mu - r) dt) - c_t \, dt$$

with now the objective

$$\max U_0$$

where $(U_t)_{0 \le t \le T}$ is a recursive utility process with the properties

$$U_t + \int_0^t F(s, c_s, U_s) ds = M_t = E_t \left[\int_0^T F(s, c_s, U_s) ds + G(w_T) \right]$$

where we'll suppose $F(s, \cdot, \cdot)$ is concave increasing in its last two arguments, G is concave increasing. We need to know that for a given (c, w) the BSDE has a unique solution U, but I reckon this can be done (under restrictive conditions if necessary), so let's just press on and see what we get. I reckon that if $V(t, w)$ is the value function, then what we should have is

$$\sup_{\theta, c} \left[\dot{V} + (rw + \theta(\mu - r) - c) V_w + \tfrac{1}{2} \theta^2 \sigma^2 V_{ww} + F(t, c, V) \right] = 0$$

If we tried an infinite horizon example where $F(t, c, U) = e^{-\rho t} f(c, U)$, then we would get analogously

$$\sup_{\theta, c} \left[f(c, V) - \rho V + (rw + \theta(\mu - r) - c) V_w + \tfrac{1}{2} \sigma^2 \theta^2 V_{ww} \right] = 0$$

Is there some recursive recipe to solve this? Could try $f(c, V) = c^\alpha V^\beta$ where $0 < \alpha, \beta < 1$, $\alpha + \beta < 1$ for concave increasing, a Cobb-Douglas sort of thing. Or we might do something additive, $f(c, V) = +\dfrac{c^{1-R_1}}{1-R_1} + \dfrac{V^{1-R_2}}{1-R_2}$.

If we were going to try policy improvement on the infinite horizon problem, maybe the first issue is where to start. One natural idea is to take $\theta \equiv 0$, $c = \lambda w$, for some $\lambda > 0$. In that case, since everything is deterministic, the martingale is constant, and we would have

$$U_t + \int_0^t e^{-\rho s} f(c_s, U_s) ds = \text{const}$$

With the first example, $f(c, V) = c^\alpha V^\beta$, we should find $w_t = w_0 e^{(r-\lambda)t}$, $c_t = \lambda w_t$ and

$$\dot{U}_t + e^{-\rho t} (\lambda w_0)^\alpha e^{\alpha(r-\lambda)t} U_t^\beta = 0.$$

Solving gives

$$\frac{U_t^{1-\beta}}{1-\beta} = \frac{(\lambda w_0)^\alpha \exp\{-(\rho - \alpha(r-\lambda))t\}}{\rho - \alpha(r-\lambda)} + K$$

where we require $\rho > \alpha(r-\lambda)$, and $K=0$ to ensure the natural condition that $\lim_{t\to\infty} U_t = 0$. Thus

$$U_t = \left\{ \frac{(1-\beta)\lambda^\alpha}{\rho - \alpha(r-\lambda)} \left(w_0 e^{(r-\lambda)t}\right)^\alpha e^{-\rho t} \right\}^{1/(1-\beta)}$$

$$\propto \quad w_t^{\alpha/(1-\beta)} \, e^{-\rho t/(1-\beta)}$$

This rate of time decay is not what I'd expect. However, if we were to postulate that the value function would be of the form $V(t,w) = \exp(\gamma t)\varphi(w)$, then the HJB for this would read

$$0 = \sup_{\theta, c} \left[\gamma e^{\gamma t}\varphi + (rw + \theta(\mu - r) - c)\, e^{\gamma t}\varphi' + \tfrac{1}{2}\theta^2\sigma^2 e^{\gamma t}\varphi'' + e^{-\rho t} c^\alpha e^{\gamma\beta t}\varphi^\beta \right]$$

and optimising over c will give

$$\alpha c^{\alpha-1} = e^{\gamma(1-\beta)t + \rho t}\, \varphi' / \varphi^\beta$$

~~so for this not to depend on t we must have $\gamma = -\rho/(1-\beta)$, and then~~

$$0 = \sup_\theta \left[\gamma e^{\gamma t}\varphi + (rw + \theta(\mu - r))e^{\gamma t}\varphi' + \tfrac{1}{2}\theta^2\sigma^2 e^{\gamma t}\varphi'' + e^{\gamma t}(\alpha\varphi')^{\alpha/(\alpha-1)}(1-\alpha)\varphi^{\beta/(1-\alpha)} \right]$$

whence

$$\boxed{0 = \gamma\varphi + rw\varphi' - \tfrac{1}{2}\kappa^2 \, \varphi'^2/\varphi'' + (1-\alpha)(\alpha\varphi')^{\alpha/(\alpha-1)} \varphi^{\beta/(1-\alpha)}}$$

Doing the dual variables trick leads to

$$\boxed{\begin{aligned} 0 = &-\frac{\rho}{1-\beta} J + \left(\frac{\rho}{1-\beta} - r\right) z J' + \tfrac{1}{2}\kappa^2 z^2 J'' \\ &+ (1-\alpha)(\alpha z)^{\alpha/(\alpha-1)} (J - zJ')^{\beta/(1-\alpha)} \end{aligned}}$$

Optimal investment with randomised rate of return (18/3/07)

1) Suppose the dynamics is

$$\begin{cases} dw_t = r w_t dt + \theta_t (\sigma dW_t + (\mu_t - r) dt) - \delta_t dt \\ d\mu_t = \sigma_m dW'_t + \beta(m - \mu_t) dt \end{cases}$$

where $dW dW' = \eta dt$, usual objective

$$V(w, \mu) = \sup E\left[\int_0^\infty e^{-\rho t} U(\delta_t) dt \,\Big|\, w_0 = w, \mu_0 = \mu\right]$$

By scaling, $V(w,\mu) = f(\mu) U(w)$, and various familiar calculations lead us to

$$\boxed{\begin{aligned} 0 = (1-R)\tilde{U}(f) - (\rho + r(R-1))f + \beta(m-\mu) f' + \tfrac{1}{2}\sigma_m^2 f'' \\ + \tfrac{1}{2}(1-R) f \left(\mu - r + \eta \sigma \sigma_m f'/f\right)^2 / \sigma^2 R \end{aligned}}$$

The equation here is exactly the same as we get in the Vasicek interest-rate example, though the variable r there becomes constant r here, the variable μ here becomes a constant there.

2) How does the story change if we do not observe μ_t, but only see Y_t, where

$$dY_t = \sigma dW_t + (\mu_t - m) dt \quad ?$$

If we write $z_t = \mu_t - m$, the innovations martingale $dV_t \equiv dY_t - \hat{z}_t dt$, we shall have (in steady state) that

$$\begin{cases} d\hat{z}_t = 2\rho\tilde{\rho}\sigma_m \sigma^{-1} dV_t - \beta \hat{z}_t dt \\ dY_t = dV_t + \hat{z}_t dt \end{cases}$$

after some calculations (see p 29-30 of WN) Here, $\tilde{\rho} \equiv \sqrt{1-\rho^2}$. Thus the point is that the equations are the <u>same</u> as for the perfectly observed case, only the volatility of the OU process $\hat{z}$ is actually smaller than the volatility σ_m of z.

3) Writing $\delta = wt$, $\theta = wx$, we have to pick t, x to achieve an extremum (min if $R>1$, else max) of

$$\tfrac{1}{2}\sigma_m^2 f'' + \{\beta(m-\mu) + \sigma\sigma_m \eta (1-R)x\} f'$$
$$- \{\rho + (R-1)(r + x(\mu - r) - t) + \tfrac{1}{2}\sigma^2 x^2 R(1-R)\} f + t^{1-R}$$

to be equated to zero. Policy improvement once again. But the BCs are problematic ... so just try recursive method?

Optimal investment: business cycle example (22/3/07)

Suppose that the rate of growth μ_t of the stocks in the world story is a function of time, non-random and known. We have as usual

$$dw_t = r w_t dt + \theta_t(\sigma dW_t + (\mu_t - r)dt) - \delta_t dt$$

and the objective $V(w,t) \equiv \sup E\left[\int_t^\infty e^{-\rho(s-t)} U(\delta_s)\,ds \mid w_t = w\right]$ scales for CRRA utility, giving $V(w,t) = h(t)U(w)$. The HJB story becomes

$$\cancel{0 = \sup\left[U - \rho V + \dot V + (rw + \theta(\mu - r) - \delta)V_w + \tfrac12 \theta^2\sigma^2 V_{ww}\right]}$$

and setting $\theta = wy$, $\delta = ws$, we find

$$\cancel{0 = \sup U(w)\left[s^{1-R} - \rho h + \dot h + (1-R)h(r + y(\mu - r) - s) - \tfrac12 y^2\sigma^2 R(1-R)h\right]}$$

$$= U(w)\left[(1-R)\tilde U(h) - \rho h + \dot h + r(1-R)h + h(1-R)\frac{1}{2R}(\mu - r)\cdot a^{-1}(\mu - r)\right]$$

where $a \equiv \sigma\sigma^T$. If we recall that $\pi^* = R^{-1}a^{-1}(\mu - r)$, we shall have the ODE

$$0 = (1-R)\tilde U(h) + \dot h - h\{\rho + (R-1)(r + \tfrac{R}{2}\pi^*\cdot a\pi^*)\}$$

$$= R h^{1-1/R} + \dot h - h\{\rho + (R-1)(r + \tfrac12 R\,\pi^*\cdot a\pi^*)\}.$$

If h is periodic, we solve this with periodic boundary conditions $h(0) = h(T)$. Note that if μ is constant, this comes out to be what it should be. If we abbreviate $g(t) \equiv \rho + (R-1)\{r + \tfrac12 R\,\pi^*(t)\cdot a\pi^*(t)\}$, we have the ODE

$$R h^{1-1/R} + \dot h - hg = 0$$

If we set $h(t) = f(t)^R$, we learn that

$$R f^{R-1} + R f^{R-1}\dot f - f^R g = 0$$

so that

$$R + R\dot f - gf = 0$$

Using integrating factor $G(t) = R^{-1}\int_0^t g(s)\,ds$, we have

$$\boxed{f(t) = e^{G(t)}\left\{f(0) - \int_0^t e^{-G(s)}\,ds\right\}}$$

The condition $f(T) = f(0)$ gives us the result

$$\boxed{f(0) = \frac{\int_0^T e^{-G_s}\,ds}{1 - e^{-G_T}}}$$

No search required.., and this checks out also for the case of constant μ.

Some liquidity models? (28/3/07)

1) Suppose we imagine a world with a continuum of agents, and n bonds which deliver coupons at rates $\varepsilon_1, \dots, \varepsilon_n$. We seek a solution where bond i trades for fixed cost S^i, but the holding H^i_t can only be altered at speed at most k_i : $|h^i_t| \le k_i$, where $h^i_t = \dot{H}^i_t$.
Agent i is subject to income shocks, so each agent has wealth dynamics

$$dw_t = r w_t\,dt + w_t(\sigma dW_t + (\mu - r)\,dt) - c_t\,dt + H_t\cdot\varepsilon\,dt - h_t \cdot S\,dt.$$

We intend to solve the problem of maximising $E\int_0^\infty e^{-\rho t} U(c_t)\,dt$, and use this to understand what equilibrium values of k, S should hold.

2) The HJB for this is

$$\sup_{c,\,|h^i|\le k_i} \Big[U(c) - \rho V + (\mu w - c + H\cdot\varepsilon - h\cdot S) V_w + \tfrac12 \sigma^2 w^2 V_{ww} + h\cdot D_H V \Big] = 0$$

The constraint on h^i destroys scaling in the CRRA problem, and nothing interesting happens in the CARA case, since policy is independent of wealth level, so no bonds get traded once initial set up is done. Thus we are going to be stuck with a truly $(n+1)$-dimensional problem.

3) So what about $n=1$, which compares an illiquid bond with an infinitely liquid bond (= bank account)? May get somewhere with this. Perhaps it will be better to do the wealth dynamics

$$dw_t = r w_t\,dt + \theta_t w_t(\sigma dW_t + (\mu - r)\,dt) - c_t\,dt + (H_t\cdot\varepsilon - h_t\cdot S)\,dt$$

which we could interpret as the agent has a riskless source of income (the bank account) and a risky source of income (entrepreneurship), and he has to choose to distribute his wealth between them. This gets us to

$$\sup_{c,\,|h|\le k} \Big[U(c) - \rho V + (rw + \theta w(\mu - r) - c) V_w + \tfrac12 \sigma^2 \theta^2 V_{ww} + (H\cdot\varepsilon - h\cdot S) V_w + h\cdot D_H V \Big] = 0$$

Now we can change variables from (w, H) to (z, H), where $z = V_w(w, H)$ and we get in the usual way

$$\sup_{|h|\le k} \Big[\tilde{U}(z) - \rho J + (\rho - r) z J_z + \tfrac12 \kappa^2 z^2 J_{zz} + z\,H\cdot\varepsilon + h\cdot(D_H J - zS) \Big] = 0.$$

The quadratic $Q(t) \equiv \frac12 \kappa^2 t(t-1) + (\rho - r)t - \rho$ has roots $-\alpha < 0 < 1 < \beta$. The equation to be solved, as it depends on z, can be solved in almost closed form, up to two constants, in this case, functions of H.

$$\mathcal{L} = \tfrac{1}{2}\kappa^2 z^2 \frac{\partial^2}{\partial z^2} + \rho - \lambda) z \frac{\partial}{\partial z} - \rho$$

Notice that the PDE we are solving takes one of the three forms

(i) $\tilde{u}(z) + \mathcal{L}J + z\, H\cdot\varepsilon = 0$ (when $J_H = zS$)

(ii) $\tilde{u}(z) + \mathcal{L}J + z\, H\cdot\varepsilon \pm k(J_H - zS) = 0$

Holding H fixed and solving (i), we have solution

$$\frac{H\cdot\varepsilon}{r}\, z - \frac{\tilde{u}(z)}{\rho(1-1/R)} + A(H)z^{-\alpha} + B(H)z^{\beta}$$

Bearing in mind that we must also have $J_H = zS$, the only way this can hold throughout an open set is if

$$A' = B' = 0, \qquad S = \varepsilon/r$$

So either there's no interesting story here or else the form (i) of the equation doesn't happen in an open set.

However, this is a complete market (at least from the mathematical standpoint of a single agent) so it shouldn't be surprising if we find prices $S = \varepsilon/r$.

But maybe we do just get two regions, with (i) just the boundary between them?

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4) This looks like it's getting too complicated. Let's try to simplify, by supposing that there are just two types of bonds delivering coupons at constant rates ε_1, ε_2, costing constant prices S_1, S_2, and an agent's wealth evolves as

$$dw_t = w_t(\sigma\, dW_t + \mu dt) - q\, dt + H_t\cdot\varepsilon\, dt - S\cdot\Delta H_t.$$

The agent chooses an intensity of buying, $-1, 0$ or 1, for each of the bonds. This represents the decision whether or not to try to buy or sell the bonds. A maximum of n_j bonds of type j may be held. When the agent ~~tries~~ to buy/sell a j-bond, he meets a counterparty at rate λ_j to trade with. The parameters $\lambda_1, \lambda_2, S_1, S_2$ will be fixed by market clearing in equilibrium.

Of course, no buying is possible if wealth is too low. Likewise, in the implementation, we will suppose that selling is not allowed if the sale of the bond would carry wealth over the upper threshold for wealth which we consider.

A few words about boundary conditions. At the upper boundary for w, we'll suppose we have pure reflection. When w falls to the bottom of the grid, we'll suppose that we consume proportional to wealth, until we manage to sell a bond (if we have one). When we get to sell the bond, we'll suppose that we jump as if from the lowest value w_1 of wealth.

If we're getting sales at rate η ($\eta = 0, \lambda_1, \lambda_2$ or $\lambda_1+\lambda_2$), the best rate at which to consume from wealth is γw_0,

$$\gamma = R^{-1}\{\rho + \eta + (R-1)(\mu - \tfrac{1}{2}\sigma^2 R)\}$$

and we acquire utility

$$U(\gamma w_0)/\gamma = \gamma^{-R} U(w_0)$$

on the way here. The subsequent utility gets discounted by a factor $\eta/(\eta+\rho)$. Thus if we have positive amounts of each bond, the solution at the lowest point w_1 of the w_1 grid is

$$\gamma^{-R} U(w_1) + \frac{\tilde{\lambda}_1}{\tilde{\lambda}_1+\tilde{\lambda}_2}\,\frac{\eta}{\eta+\rho}\, V(w_1+S_1;\, n_1-1,\, n_2) + \frac{\tilde{\lambda}_2}{\tilde{\lambda}_1+\tilde{\lambda}_2}\,\frac{\eta}{\eta+\rho}\, V(w_1+S_2;\, n_1,\, n_2-1)$$

where $\tilde{\lambda}_j = \lambda_j I_{\{n_j>0\}}$.

$$\boxed{\text{Set } F(y,q) = \int_0^y v_p(1,z)\, z^q \, dz}$$

needs $a+1-k' > 0$, the well-posed condition $\longrightarrow$

$k_1 = 1+k$

Guarantees problem with Phil Dybvig (19/4/07)

1) In this story, we find an expression for the value f^n

$$V = \rho^{-1}(1-p)\,U(L_0) + \int_0^\infty r_\rho(1,y)\,\Phi\left(\frac{\lambda y}{p}, L_0\right) dy$$

$$+\, a_0 \int_{L_0}^\infty U'(x)\left\{1 \wedge \left(\frac{p z^* U'(x)}{\lambda}\right)^a\right\} dx$$

where

$$\Phi(\lambda, L) = \begin{cases} p\,\tilde{U}(\lambda) & \text{if } \lambda < U'(L) \\ p(1+K)\,\tilde{U}\left(\frac{\lambda}{1+K}\right) - pK\,U(L) & \text{if } \lambda > (1+K)\,U'(L) \\ p\left(U(L) - L\lambda\right) & \text{else} \end{cases}$$

and

$$r_\rho(x,y) = c_0 \left(\frac{x}{y} \wedge 1\right)^b \left(\frac{y}{x} \wedge 1\right)^a y^{-1} \qquad \left(c_0 = 2/k^2(a+b)\right)$$

is resolvent density.

How can we compute these numerically for CRRA example?

2) It seems best to split the first integral at $y_0 = pU'(L_0)/\lambda$, $y_1 = p(1+K)U'(L_0)/\lambda$ and then calculate

$$I_1 = \int_0^{y_0} r_\rho(1,y)\,p\,\tilde{U}(\lambda y/p)\,dy = \int_0^{y_0} r_\rho(1,y)\,y^{1-1/R}\,dy \;\; p\,\tilde{U}(\lambda/p) \qquad (R' = 1/R)$$

$$= p\,\tilde{U}(\lambda/p)\;F(y_0, 1-R')$$

$$I_2 = \int_{y_0}^{y_1} r_\rho(1,y)\,p\left(U(L_0) - L_0\frac{\lambda y}{p}\right) dy$$

$$= p\,U(L_0)\left\{F(y_1,0) - F(y_0,0)\right\} - \lambda L_0\left\{F(y_1,1) - F(y_0,1)\right\}$$

$$I_3 = \int_{y_1}^\infty p\,r_\rho(1,y)\left\{K_1\,\tilde{U}\left(\frac{\lambda y}{pK_1}\right) - K\,U(L_0)\right\} dy$$

$$= K_1 p\;\tilde{U}\left(\frac{\lambda}{pK_1}\right)\left\{F(\infty, 1-R') - F(y_1, 1-R')\right\} - pK\,U(L_0)\left\{F(\infty,0) - F(y_1,0)\right\}.$$

For the final integral, we split at $x_1 = I(\lambda/pz^*)$, to give the result

$$a_0 \cdot \frac{(x_1 \vee L_0)^{1-R} - L_0^{1-R}}{1-R} + a_0 \left(\frac{p\zeta^*}{\lambda}\right)^{a} \frac{(x_1 \vee L_0)^{1-(1+a)R}}{-1+(1+a)R}$$

3) The final element of the story is how w_0 is related to λ, and this is via

$$\tfrac{1}{2} K^2 w_0 = \int_0^1 dt \int_1^\infty dy \; t^a y^{-b} \, I\left(\frac{\lambda t y/p}{g(\max\{y, ty/x_0\})}\right)$$

where $g(x) = (1+K) \wedge (1 \vee (\zeta^* x))$, $\quad x_0 \equiv p\zeta^* U'(L_0)/\lambda$. We can take this in parts, firstly

$$\int_0^{x_0 \wedge 1} dt \int_1^\infty dy \; t^a y^{-b} \, I\left(\frac{\lambda t y/p}{g(y)}\right)$$

$$= \int_0^{x_0 \wedge 1} t^{a - 1/R} \, dt \cdot \left(\frac{\lambda}{p}\right)^{-1/R} \int_1^\infty y^{-b-1/R} \, g(y)^{1/R} \, dy$$

$$= \left(\frac{\lambda}{p}\right)^{-1/R} \frac{(x_0 \wedge 1)^{1+a-R'}}{1+a-R'} \int_1^\infty y^{-b-1/R} \, g(y)^{1/R} \, dy$$

$$= \left(\frac{\lambda}{p}\right)^{-R'} \frac{(x_0 \wedge 1)^{1+a-R'}}{1+a-R'} \left[\frac{\left(1 \vee \frac{1}{\zeta^*}\right)^{1-b-R'} - 1}{1-b-R'} + (\zeta^*)^{R'} \frac{\left(1 \vee \frac{1+K}{\zeta^*}\right)^{1-b} - \left(1 \vee \frac{1}{\zeta^*}\right)^{1-b}}{1-b} + (1+K)^{R'} \frac{\left(1 \vee \frac{1+K}{\zeta^*}\right)^{1-b-R'}}{-1+b+R'} \right].$$

NO

When $t > x_0$, we have $g(\max\{y, ty/x_0\}) = g(ty/x_0) = K_1 \wedge (\zeta^* t y / x_0)$ for $y > 1$ $\quad (K_1 = 1+K)$

so $$\int_{x_0 \wedge 1}^1 dt \int_1^\infty dy \; t^a y^{-b} \, I\left(\frac{\lambda t y/p}{g(ty/x_0)}\right)$$

$$= \int_{x_0 \wedge 1}^1 dt \int_1^{x_0 K_1 / t\zeta^*} t^a y^{-b} \, I\left(\frac{\lambda x_0}{p\zeta^*}\right) dy + \int_{x_0 \wedge 1}^1 dt \int_{x_0 K_1/t\zeta^*}^\infty t^a y^{-b} (ty)^{-1/R} \, I\left(\frac{\lambda}{pK_1}\right)$$

$$= I\left(\frac{\lambda x_0}{p\zeta^*}\right) \int_{x_0 \wedge 1}^1 t^a \frac{1}{1-b} \left(\left(\frac{x_0 K_1}{t\zeta^*}\right)^{1-b} - 1\right) dt + I\left(\frac{\lambda}{pK_1}\right) \int_{x_0 \wedge 1}^1 t^{a-1/R} \frac{1}{b+1/R-1} \left(\frac{x_0 K_1}{t \zeta^*}\right)^{1-b-1/R} dt$$

$$= - I\left(\frac{\lambda x_0}{p\zeta^*}\right) \frac{1}{1-b} \frac{1}{a+1} \left(1 - (x_0 \wedge 1)^{a+1}\right) + \left\{ I\left(\frac{\lambda x_0}{p\zeta^*}\right) \left(\frac{x_0 K_1}{\zeta^*}\right)^{1-b} \frac{1}{1-b} + I\left(\frac{\lambda}{pK_1}\right) \frac{1}{b+1/R-1} \left(\frac{x_0 K_1}{\zeta^*}\right)^{1-b-1/R} \right\} \cdot \frac{1}{a+b} \left(1 - (x_0 \wedge 1)^{a+b}\right)$$

$$= - I\left(\frac{\lambda x_0}{p\zeta^*}\right) \frac{1}{1-b} \frac{1}{a+1} \left(1 - (x_0 \wedge 1)^{a+1}\right) + I\left(\frac{\lambda}{pK_1}\right) \left(\frac{x_0 K_1}{\zeta^*}\right)^{1-b-R'} \frac{R'}{(1-b)(b+R'-1)} \frac{1}{a+b} \left(1 - (x_0 \wedge 1)^{a+b}\right)$$

For $t > x_0$, we have $\int_{x_0\wedge 1}^{1} dt \int_1^{\infty} t^a y^{-b} \left(\frac{\lambda t y}{p}\right)^{-R'} g\left(\frac{ty}{x_0}\right)^{R'} dy$

so taking the y-integral for fixed t we get $\left(\frac{\lambda}{p}\right)^{-R'} t^{a-R'}$ times

$$J = \int_1^{\infty} y^{-b-R'} g\left(ty/x_0\right)^{R'} dy = \int_1^{y_0\vee 1} y^{-b-R'} dy + \int_{y_0\vee 1}^{y_1\vee 1} y^{-b-R'} \left(\frac{z^* t y}{x_0}\right)^{R'}$$

$$+ \int_{y_1\vee 1}^{\infty} y^{-b-R} K_1^{R'} dy \qquad \left[\begin{array}{l} y_0 = x_0/tz^* \\ y_1 = K_1 y_0 \end{array}\right]$$

$$= \frac{1}{b+R'-1}\left\{1-(y_0\vee 1)^{-(b+R'-1)}\right\} + \left(\frac{z^* t}{x_0}\right)^{R'} \frac{1}{b-1}\left\{(y_0\vee 1)^{-(b-1)} - (y_1\vee 1)^{-(b-1)}\right\}$$

$$+ K_1^{R'} \frac{1}{b+R'-1} (y_1 \vee 1)^{-(b+R'-1)}.$$

We now integrate $J = J(t)$, multiplied by $t^{a-R'}$, from 0 to q. The integral breaks at $t_0 = x_0/z^*$ and at $t_1 = K_1 x_0/z^* > t_0$, so we get

$$\int_0^{t_0\wedge q} t^{a-R'} J(t)\,dt + \int_{t_0\wedge q}^{t_1\wedge q} t^{a-R'} J(t)\,dt + \int_{t_1\wedge q}^{q} t^{a-R'} J(t)\,dt$$

$$= I_1 + I_2 + I_3$$

where

$$I_1 = \int_0^{t_0\wedge q} t^{a-R'}\left[\frac{1}{b+R'-1}\left\{1-\left(\frac{tz^*}{x_0}\right)^{b+R'-1}\right\} + \left(\frac{z^* t}{x_0}\right)^{R'}\frac{1}{b-1}\left\{\left(\frac{tz^*}{x_0}\right)^{b-1} - \left(\frac{tz^*}{K_1 x_0}\right)^{b-1}\right\}\right.$$

$$\left. + \frac{K_1^{R'}}{b+R'-1}\left(\frac{tz^*}{K_1 x_0}\right)^{b+R'-1}\right] dt$$

$$= \frac{(t_0\wedge q)^{a+1-R'}}{(a+1-R')(b+R'-1)} + \frac{(t_0\wedge q)^{a+b}}{a+b}\left(\frac{z^*}{x_0}\right)^{b+R'-1} \frac{R'}{(b-1)(b+R'-1)}\left(1-K_1^{1-b}\right),$$

$$I_2 = \int_{t_0\wedge q}^{t_1\wedge q} t^{a-R'}\left[\left(\frac{z^* t}{x_0}\right)^{R'}\frac{1}{b-1}\left\{1-\left(\frac{tz^*}{K_1 x_0}\right)^{b-1}\right\} + \frac{K_1^{R'}}{b+R'-1}\left(\frac{tz^*}{K_1 x_0}\right)^{b+R'-1}\right] dt$$

$$= \left(\frac{z^*}{x_0}\right)^{R'} \frac{(t_1\wedge q)^{a+1} - (t_0\wedge q)^{a+1}}{(a+1)(b-1)} + \frac{(t_1\wedge q)^{a+b} - (t_0\wedge q)^{a+b}}{a+b} \frac{-R'}{(b-1)(b+R'-1)} K_1^{1-b}\left(\frac{z^*}{x_0}\right)^{b+R'-1}$$

$$I_3 = \int_{t_1\wedge q}^{q} t^{a-R'} \frac{K_1^{R'}}{b+R'-1}\,dt$$

$$= \frac{q^{a+1-R'} - (t_1\wedge q)^{a+1-R'}}{a+1-R'} \cdot \frac{K_1^{R'}}{b+R'-1}$$

Finally, don't forget the factor $(\lambda/p)^{-R'}$!!

Arbitrage + equilibrium (26/4/07)

(i) It seems that a lot of APT is addressing the wrong question! We like APT because, although we'd prefer to do an equilibrium analysis, there are relatively few examples where we can carry out the equilibrium calculations; APT on the other hand facilitates calculation. Since any equilibrium pricing system must be arbitrage free, we have enlarged the set of possible pricing systems; the problem is that we've made it way too large – equilibrium prices actually satisfy further properties beyond no-arbitrage which make their behaviour much more regular.

(ii) To fix ideas, we will begin with $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ satisfying the usual conditions. In APT, we start with an $\mathbb{R}^d$-valued semimartingale $(S_t)_{0\le t\le T}$ satisfying some form of the NA condition, + look for a measure which is (near to) an EMM. In an equilibrium story, we begin from the measure (marginal utility of aggregate consumption) + try to find the equilibrium prices $(S_t)_{0\le t\le T}$ – the opposite way round!

Let's set up some ideas for the equilibrium story. Let's define

$$\mathcal{E}_n = \Big\{ \sum_{i=1}^n Z_i \, (T_{i-1}, T_i] \; : \; 0 \le T_0 \le T_1 \le \dots \le T_n \le T \text{ stopping times}, \; Z_i \in b\mathcal{F}_{T_i} \Big\},$$

and set $\mathcal{E} = \bigcup_n \mathcal{E}_n$. Then evidently $\mathcal{E}$ is a vector space (and even an algebra). Suppose next that we have d assets, asset j delivering a dividend δ^j at time T, and suppose $\Delta = \sum q_j \delta^j$, where q_j is net supply of asset j. The objective of the agent is to

$$\max \; E \; U(w_T)$$

where w_T is the consumption good available at time T. We assume that U may depend on ω, but that $U(\cdot, \omega)$ is strictly increasing, concave $\forall \omega$. If we are to get an equilibrium solution, must be that $w_T = \Delta$, so we shall insist that

$$U(\Delta) \in L^1.$$

An important rôle is played by

$$V = \{ \eta \in L^0(\mathcal{F}_T) \; : \; \text{for some } \varepsilon > 0, \; U(\Delta + t\eta) \in L^1 \;\; \forall\, |t| \le \varepsilon \},$$

the vector space (U is concave!) of feasible directions at Δ. Notice that if $\eta \in V$ then

$$E\, t^{-1}\big(U(\Delta + t\eta) - U(\Delta) \big) \uparrow E\, \eta\, U'(\Delta) \quad \text{as } t \downarrow 0, \quad \text{and} \quad -\infty < E\, \eta U'(\Delta) < \infty$$

since the function $t \mapsto E\{U(\Delta+t\eta) - U(\Delta)\}$ is concave and finite-valued. Thus $\eta U'(\Delta) \in L^1$ for all $\eta \in V$. (V is a vector space of contingent claims which are priced by marginal pricing).

Notice that we don't necessarily have $U'(\Delta) \in L^1$. Let's assume though that

$$\delta^j \in V \quad \text{for each } j.$$

If this were to fail, the asset j for which it's not true would never be traded, so the agent would just have to hold it to the end. The effect of this could be absorbed into $U(\cdot)$ so we don't lose much generality doing this assumption.

(iii) Let's now suppose that there is some $\nu \in V$ such that $P(\nu = 0) = 0$. We have for any $a > 0$ that

$$U(\Delta) \geq \tfrac{1}{2} U(\Delta - a) + \tfrac{1}{2} U(\Delta + a)$$

so that $U(\Delta + a) \leq 2U(\Delta) - U(\Delta - a)$. Thus if we were to take $a = t\nu^-$, we'd get $U(\Delta + t\nu^-) \in L^1$, and hence $|\nu| \in V$. Thus it's as good to assume there's a $\nu \in V$ st. $P(\nu \leq 0) = 0$.

Any contingent claims in the space V can have relative prices ascribed to them at times $t \in [0,T)$ by the recipe

$$\tilde{Y}_t = \frac{Y_t}{N_t} = \frac{E_t[U'(\Delta)\eta]}{E_t[U'(\Delta)\nu]}$$

expressing the time-t price of η in terms of units of the numeraire. We shall fix the numeraire ν, with associated $N_t \equiv E_t[U'(\Delta)\nu] > 0$; the choice is essentially arbitrary, but has to be expressed in terms of a tradable - thus if $1 \notin V$, we can't use 1 as the numeraire. (Ideally, we could use one of the positive-net-supply assets for the numeraire, but I don't believe this is strictly needed.)

The optimisation problem to be solved is to

$$\max_{\theta \in \mathcal{E}} E\left[U(w_T \nu)\right]$$

where

$$w_t = \tilde{\theta}_t \cdot [1; \tilde{S}_t] \qquad \tilde{S}_t = E_t[U'(\Delta)\delta]/N_t$$

solves

$$dw_t = \theta_t \cdot d\tilde{S}_t$$

with initial condition $w_0 = (0\ q)\cdot(1, \tilde{S}_0) = q\cdot\tilde{S}_0$. The confirmation that $\theta \equiv q \in \mathcal{E}$ is optimal comes by noticing that if $\theta = q + H \in \mathcal{E}$, we have

$$E[U(w_T\nu)] \leq E\left[U(\Delta) + U'(\Delta)(H\cdot\tilde{S})_T\,\nu\right] \quad \text{by concavity of } U$$

$$= E\,U(\Delta) + E\left[N_T (H\cdot\tilde{S})_T\right]$$

$$= E\,U(\Delta)$$

since it's an easy exercise to show $N_t (H\cdot\tilde{S})_t$ is a martingale for every $H \in \mathcal{E}$.

To sum up: if the prices $(\tilde{S}_t)_{0\le t\le T}$ of the positive net-supply assets, expressed in units of some tradable numeraire, arise from equilibrium, then they are martingales in the prob defined by their numeraire.

(iv) Now with dividends and consumption. Assets deliver dividend stream $\delta_t \, dA_t$ with $\Delta_t \equiv 1\cdot\delta_t$ the aggregate dividend. The agent's objective is

$$\sup E\Big[\int_{(0,T]} U(s, c_s)\, dA_s \Big],$$

so as before we define the space V of feasible consumption perturbations to be

$$V = \{ \eta : \text{for some } \varepsilon > 0, \ \int_{(0,T]} U(s, \Delta_s + a\eta_s)\, dA_s \in L^1 \quad \forall |a| \le \varepsilon \}.$$

Assuming there is some $\nu > 0$ in V, and all δ^i are in V, we can take the numeraire to be

$$N_t \equiv E_t\Big[\int_{(0,T]} U'(s,\Delta_s)\, \nu_s \, dA_s \Big],$$

a UI martingale, and price the (ex-div) stocks by marginal pricing

$$\tilde{S}_t = E_t\Big[\int_{(t,T]} U'(s,\Delta_s)\, \delta_s \, dA_s \Big] / N_t \equiv S_t / N_t$$

Set $\lambda_t \equiv U'(t,\Delta_t)/N_t$, which converts time-$t$ consumption goods into units of N. We will consider as admissible only $\theta \in \mathcal{E}$, and $c \ge 0$ such that $E\big[\int_{(0,T]} U'(s,\Delta_s)\, c_s \, dA_s\big] < \infty$. Assume $\delta \ge 0$, and Δ is admissible consumption. The wealth equation for wealth in units of N is

$$w_t = \theta^0_t + \theta_t\cdot\tilde{S}_t = w_0 + \int_{(0,t]} \theta_u \cdot d\tilde{S}_u + \int_{(0,t]} (\theta_u\cdot\delta_u - c_u)\, \lambda_u \, dA_u$$

To stop unbounded consumption, assume $w_T = 0$. If we take simple $\theta = Z\, I_{(\tau,T]}$, we can calculate

$$N_t \Big\{ \int_{(0,t]} \theta_u \cdot d\tilde{S}_u + \int_{(0,t]} \theta_u\cdot\delta_u \lambda_u \, dA_u \Big\} = Z\, N_t \Big(\tilde{S}_t - \tilde{S}_{t\wedge\tau} + \int_{t\wedge\tau}^{t} \delta_u \lambda_u \, dA_u \Big)$$

$$\doteq Z \Big\{ S_t - S_{t\wedge\tau} + N_t \int_{(t\wedge\tau,\, t]} \delta_u \lambda_u \, dA_u \Big\}$$

($\doteq$ means differs by martingale)

$$\doteq Z \Big\{ -\int_{(t\wedge\tau,\, t]} N_s \lambda_s \delta_s \, dA_s + N_t \int_{(t\wedge\tau,\, t]} \delta_u \lambda_u \, dA_u \Big\}$$

$$= -\int_{(0,t]} \theta_u\cdot\delta_u \, \lambda_u N_u \, dA_u + N_t \int_{(0,t]} \theta_u\cdot\delta_u \, \lambda_u \, dA_u$$

Using non-negativity δ, N and boundedness of θ, this is a UI martingale (optional projections!) Now we use this to prove that $(1, \Delta)$ is the optimal (θ, c) subject to the constraint $w_T = 0$.

[OVER]

We have for any feasible (θ, c)

$$E\left[\int_{[0,T]} U(s,c_s)\,dA_s\right] = E\left[\int_{[0,T]} U(s,c_s)\,dA_s + N_T W_T\right]$$

$$\le E\Big[\int_{[0,T]} U(s,\Delta_s)\,dA_s + \int_{[0,T]} U'(s,\Delta_s)\,(c_s-\Delta_s)\,dA_s$$

$$+ N_T w_0 + N_T\Big\{\int_{[0,T]} \theta_u\,d\tilde{S}_u + \int_{[0,T]} \theta_u\cdot\delta_u\,\lambda_u\,dA_u\Big\}$$

$$- N_T\int_{[0,T]} c_u\,\lambda_u\,dA_u\Big]$$

$$= E\Big[\int_{[0,T]} U(s,\Delta_s)\,dA_s + \int_{[0,T]} \lambda_s N_s c_s\,dA_s - N_T\int_{[0,T]} \lambda_s c_s\,dA_s$$

$$- \int_{[0,T]} U'(s,\Delta_s)\,\Delta_s\,dA_s + N_T w_0\Big] \qquad \left(\begin{array}{c}\text{using result on}\\ \text{previous page}\end{array}\right)$$

$$= E\left[\int_{[0,T]} U(s,\Delta_s)\,dA_s + N_T w_0 - \int_{[0,T]} U'(s,\Delta_s)\,\Delta_s\,dA_s\right] \qquad (\;\;''\;\;)$$

To finish, suppose $\theta \equiv 1$, $c \equiv \Delta$ and use

$$0 = N_T W_T$$

$$= N_T w_0 + N_T\, 1\cdot(\tilde{S}_T - \tilde{S}_0)$$

$$= N_T w_0 - N_T(1\cdot\tilde{S}_0) \qquad (\tilde{S}_T = 0)$$

$$\Rightarrow \quad E(N_T w_0) = E\left[N_T(1\cdot\tilde{S}_0)\right] = E\, N_0(1\cdot\tilde{S}_0) = E\, 1\cdot S_0 = E\left[\int_{[0,T]} U'(s,\Delta_s)\,\Delta_s\,dA_s\right].$$

Thus $(1,\Delta)$ is optimal.

Remarks on a question of Thomas Breuer (15/5/07)

(i) Thomas is interested in questions concerning portfolio optimisation under some dynamic riskmeasure constraints. One example could be where we have the objective

$$\max E\, U(w_T),$$

where $dw_t = rw_t\,dt + \theta_t\left(\sigma\, dW_t + (\mu - r)\,dt\right)$ and

subject to the constraint

$$P\left[\inf_{0\le s\le T} w_s < c\right] \le \varepsilon$$

for some $c < w_0$, and $\varepsilon > 0$. In Lagrangian form, if $\underline{w}_t \equiv \inf\{w_s : s \le t\}$, we get

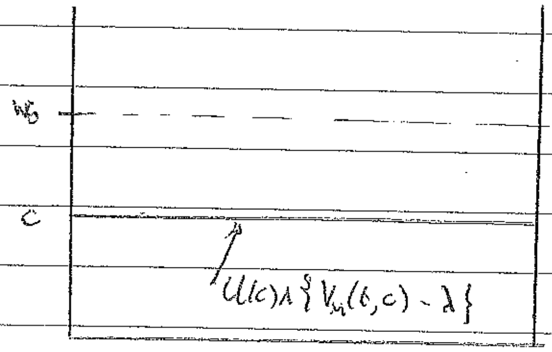
$$\max E\left[U(w_T) - \lambda I_{\{\underline{w}_T < c\}}\right].$$

What is the form of the solution to this? If the wealth never hits c, then we do not pay the penalty λ. If the wealth falls to c, then we must either stop and get terminal utility $U(c)$, else we continue, having paid the penalty λ. If we do this, we expect to end up on average with $V_M(t,c) - \lambda$, where t is the time we hit c, and V_M is the Merton value function. Thus the optimal control problem is to $\max E\, F(\tau, w_\tau)$ where $\tau = \inf\{t : w_t = c\}$ and $F(t,c) = U(c) \vee \{V_M(t,c) - \lambda\}$, $F(T,x) = U(x)$.

We have

$$V_M(t,w) = U(w)\exp\{-\gamma(T-t)\}$$

where $\gamma = (R-1)(r + \kappa^2/2R)$.



(ii) How would it look if we were to impose the condition

$$P\left[\inf_{0\le s\le T} w_s \le c\right] \le \varepsilon\ ?$$

My guess is that you now do a randomised policy; at time 0, you pick whether or not to observe the constraint $\underline{w}_T > c$. If you do not observe it, you just do Merton. If you do observe it, you just reserve a cash amount c, and invest the remainder.

Modelling futures prices (15/5/07)

In the conventional story, we have for $t < T$ that

$$F_{tT} = E_t^* F_T$$

so we need to model the spot F_T, or perhaps more usefully $x_T \equiv \log F_T$. A simple notion for this would be to take

$$dX_t = \sigma \, dW_t + B(m_t - X_t)\, dt$$

where σ is $d \times d$ non-singular, B is $d \times d$, m_t is a $d \times 1$ deterministic function of time (to allow for business cycle, or annual seasonality, ...) and then to let $x_t = v \cdot X_t$ for some fixed vector v. We have of course that $(\tau \equiv T-t > 0)$

$$X_T = \exp(-\tau B) X_t + \int_t^T \exp(-(T-s)B)\, \sigma \, dW_s + \int_t^T \exp(-(T-s)B) B m_s \, ds$$

and hence

$$F_{tT} = \exp\left[v \cdot e^{-\tau B} X_t + v \cdot \int_t^T e^{-(T-s)B} B m_s \, ds + \tfrac{1}{2} V(T-t) \right]$$

where $V(\tau) \equiv v \cdot \int_0^\tau e^{-tB} \sigma \sigma^T e^{-tB^T} v \, dt$.

(i) The univariate case of this is the most simple, but would imply that futures prices of different maturities were perfectly correlated ... credible?

(ii) Michael Dempster has done pretty much the same thing, and finds quite good fits.

Care!! H_n^{γ} isn't necessarily bounded...

Constructing an EMM (21/5/07)

Suppose we are given some semimartingale $(S_t)_{0\le t\le T}$ price process. Let's see how we might be able to construct an EMM by utility maximisation. First, let

$$\mathcal{E}_n \equiv \left\{ H = \sum_{i=1}^{2^n T} X_i \,\big((i-1)2^{-n}, i2^{-n}\big] \;:\; X_i \in b\mathcal{F}_{(i-1)2^{-n}} \right\}$$

be the set of bounded elementary integrands changing only at times in $\mathbb{D}_n = 2^{-n}\mathbb{Z}$.

(i) Suppose we set $\eta \equiv \sup_{0\le t\le T}|S_t|$ and make an initial measure change using density $\propto \exp(-\eta^a)$ for some $a>1$ which we'll have more to say about.

If there's to be no arbitrage, there certainly is no arbitrage if we restrict to $H\in\mathcal{E}_n$, and this is a discrete-time result, where we know everything. So if we consider the problem

$$\sup_{H\in\mathcal{E}_n} E\, U(H\cdot S)$$

where $U(x) = -\exp(-x)$, $H\cdot S \equiv \int_{(0,T]} H_t\, dS_t$, then there exists optimising H_n^*, and $Z_n = \lambda_n \exp(-H_n^*\cdot S) = \lambda_n U'(H_n^*\cdot S)$ is an EMM (for $(S_t)_{t\in\mathbb{D}_n}$) where

$$\lambda_n^{-1} = E\exp(-H_n^*\cdot S);$$

we have

$$E\, Z_n (S_\tau - S_0) = 0$$

for any stopping time τ with values in $\mathbb{D}_n$. Of course, we'd also have $E Z_m(S_\tau - S_0)=0$ $\forall m\ge n$. Now we have

$$\sup_{H\in\mathcal{E}_n} E\,U(H\cdot S) = \sup_{H\in\mathcal{E}_n} E\left[U(H\cdot S) - U'(H_n^*\cdot S)\, H\cdot S\right]$$

$$\le E\left[\tilde{U}(\lambda_n^{-1} Z_n)\right]$$

with equality when $H=H_n^*$. We have $\tilde U(y) = y\log y - y$ here, and clearly these increase with n; we have

$$-\lambda_n^{-1} = E\,U(H_n^*\cdot S) = E\,\tilde U(\lambda_n^{-1}Z_n)$$

$$= \lambda_n^{-1}\left\{E\,Z_n\log Z_n + \log\lambda_n^{-1} - 1\right\}$$

$\boxed{\text{Assuming that } \lambda_n \not\to 0,}$ we get that $\sup_n E(Z_n\log Z_n)<\infty$. There's a lemma of Delbaen-Schachermayer (a variant of a result of Komlos) that we can take convex (finite) combinations $\tilde Z_n = \sum_{j\ge n} w_{nj} Z_j$ which converge a.s. to limit Z_∞, which has finite relative entropy $E[Z_\infty \log Z_\infty]<\infty$ by convexity and Fatou. The question however is whether or not this gives us an EMM; do we have $E\, Z_\infty(S_\tau - S_0)=0$ $\forall\tau$ with values in $\mathbb{D}_n$? It seems that the way to handle this is via Orlicz spaces + Luxemburg norms. The idea here is to study convex $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ which are increasing, $x^{-1}\varphi(x) \downarrow 0$ $(x \downarrow 0)$, and $x^{-1}\varphi(x) \uparrow \infty$ $(x \uparrow \infty)$. We let

$$\tilde{\varphi}(y) \equiv \sup_{x \geq 0} \{ xy - \varphi(x) \}$$

so that $\varphi(x) + \tilde{\varphi}(y) \geq xy$ (Young's inequality). The Luxemburg norm

$$\|X\|_\varphi \equiv \inf \{ \lambda > 0 : E\, \varphi(|X|/\lambda) \leq 1 \}$$

makes the Orlicz space a Banach space, and we have the Hölder-type inequality

$$E\,|XY| \leq \|X\|_\varphi \, \|Y\|_{\tilde{\varphi}}$$

We want to apply this to

$$|E\,(Z_\infty - Z_n)(S_2 - S_0)| \leq E\,|Z_\infty - Z_n|\,|S_2 - S_0|$$

$$\leq \|Z_\infty - Z_n\|_\varphi \, \|S_2 - S_0\|_{\tilde{\varphi}}$$

Now we want (a) $\|Z_\infty - Z_n\|_\varphi \to 0$ (b) $\|S_2 - S_0\|_{\tilde{\varphi}}$ finite. We know that $Z_\infty - Z_n \to 0$ a.s., but we need more than that. If we could find Orlicz f^{ns} φ_1, φ_2 such that

$$\varphi_1(\varphi_2(x)) = (1+x)\log(1+x) - (1+x) \equiv \varphi_0(x)$$

then $\varphi_1(E\,\varphi_2(|Z_n|/\lambda)) \leq E\,\varphi_1 \circ \varphi_2(|Z_n|/\lambda) = E\,\varphi_0(|Z_n|/\lambda) \leq 1$ if $\lambda \geq \lambda_0 \equiv \sup_n \|Z_n\|_{\varphi_0}$; so for $\lambda \geq \lambda_0$ we get $\{ \varphi_2(|Z_n|/\lambda) : n \geq 1 \}$ is a UI family and $(Z_n) \in L_{\varphi_2}$. Since φ_0 is of moderate growth, we shall even have that $\{ \varphi_2(k|Z_n|) : n \geq 1 \}$ is UI for any $k > 0$. So let's define φ_2 via

$$\varphi_2'(x) = (\log(1+x))^\beta$$

for some fixed $\beta \in (0,1)$ (this is good, because $\varphi_2'(\cdot)$ increases from 0 to infinity) and then define φ_1 by

$$\varphi_1'(\varphi_2(x)) \cdot \varphi_2'(x) = \varphi'(x)$$

$$\therefore \varphi_1'(\varphi_2(x)) = (\log(1+x))^{1-\beta}$$

which also increases from 0 to infinity. We have

$$\|X\|_\varphi \equiv \inf \{ \lambda > 0 : \int \varphi(x/\lambda)\, P(|X| \in dx) \leq 1 \}$$

$$= \inf\left\{\lambda>0 : \int_0^\infty \left(\int_0^{x/\lambda} \varphi'(t)\,dt\right) P(|X|\in dx) \le 1\right\}$$

$$= \inf\left\{\lambda>0 : \int_0^\infty \varphi'(t)\, P\big(|X|>\lambda t\big)\,dt \le 1\right\}$$

so in order to prove $\|X_n\|_\varphi \to 0$ it is equivalent to prove that for any $\lambda>0$ we have

$$\int_0^\infty \varphi'(t)\, P\big(|X_n|>\lambda t\big)\,dt \to 0.$$

We want to prove $\|Z_n - Z_\infty\|_{\varphi_2} \to 0$, so we consider

$$\int_0^\infty \varphi_2'(t)\, P\big(|Z_n - Z_\infty| > \lambda t\big)\,dt$$

$$\le \int_0^M \varphi_2'(t)\, P\big(|Z_n - Z_\infty|>\lambda t\big)\,dt + \int_M^\infty \varphi_2'(t)\left\{P\big(|Z_n|>\tfrac{\lambda t}{2}\big) + P\big(|Z_\infty|>\lambda t/2\big)\right\}dt,$$

where we will soon decide how to choose M. Suppose given $\varepsilon>0$. We note that

$$\int_M^\infty \varphi_2'(t)\, P(|Z_n|>\lambda t/2)\,dt = \int_M^\infty \varphi_2'(t)\int_{\lambda t/2}^\infty P(|Z_n|\in dz)\,dt$$

$$= E\left\{\varphi_2\left(\frac{2|Z_n|}{\lambda}\right) - \varphi_2(M)\right\}^+$$

Now since φ_1, φ_2 are of moderate growth, $\{\varphi_2(2|Z_n|/\lambda)\}$ is UI, so pick M so big that this is uniformly less than $\varepsilon/3$, finally select n_0 so large that $\forall n \ge n_0$ we get $\int_0^M \varphi_2'(t)\, P(|Z_n - Z_\infty|>\lambda t)\,dt < \varepsilon/3$. This nails $\|Z_n - Z\|_{\varphi_2} \to 0$, and we just need to ensure that $\|S_\tau - S_0\|_{\tilde\varphi_2} < \infty$. For this it's enough that

$$\|\eta\|_{\tilde\varphi_2} < \infty.$$

We have

$$\tilde\varphi_2(y) = \int_0^y \left(e^{v^{1/\beta}} - 1\right) dv \sim \text{const}\; e^{y^{1/\beta}} \quad \text{for large } y$$

We also have $E\,\tilde\varphi_2(\eta/\lambda) \le \text{const.}\; E\exp\left(\frac{\eta}{\lambda}\right)^{1/\beta}$, so if we were to have chosen $a > 1/\beta$, then we get for all $\lambda>0$ that $E\,\tilde\varphi_2(\eta/\lambda) < \infty$, and thus $S_\tau \in L_{\tilde\varphi_2}$, which we need. But we can of course select a, β so that $a\beta>1$, $\beta\in(0,1)$, so it's all OK.

Shot-noise dividend processes (23/5/07)

(i) It's an easy exercise to show that if Z is an increasing Lévy process (in $\mathbb{R}^d$) with exponent $\psi(\lambda) = -\log E \exp(-\lambda Z_1)$ then

$$E \exp\left(-\int_0^t h(s)\, dZ_s\right) = \exp\left\{-\int_0^t \psi(h_s)\, ds\right\}$$

for any $h: \mathbb{R}^+ \to (\mathbb{R}^+)^d$ measurable. Thus if we attempt to model a MV dividend process as

$$\delta_t = \int_{-\infty}^t e^{a(s-t)}\, dZ_s$$

we shall have $E \exp(-\lambda\delta_t) = \exp\left\{-\int_0^\infty \psi(\lambda e^{-as})\, ds\right\}$. Let's study this through the univariate projections

$$\psi_v(t) \equiv \psi(tv) \qquad t \geq 0,\ v \in (\mathbb{R}^+)^d.$$

(ii) Let's observe that (now writing ψ as an abbreviation for ψ_v, with Lévy measure μ as abbreviation for μ_v, etc ...)

$$\psi(\lambda) = \int_0^\infty (1-e^{-\lambda x})\, \mu(dx) = \int_0^\infty \lambda e^{-\lambda x}\, \bar{\mu}(x)\, dx$$

so

$$\int_0^\infty \psi(\lambda e^{-as})\, ds = \int_0^\infty \int_0^\infty \lambda e^{-as} e^{-\lambda e^{-as} x}\, \bar{\mu}(x)\, dx\, ds$$

$e^{-as}x = y$

$$= \int_0^\infty \lambda e^{-\lambda y} \left(\int_0^\infty \bar{\mu}(y e^{as})\, ds\right) dy$$

This exhibits $\mathcal{L}(\delta_t)$ as infinitely divisible. The process δ_t is Markovian, with

$$\delta_{t+h} = e^{-ah}\delta_t + \int_t^{t+h} e^{a(s-t-h)}\, dZ_s$$

so we are also going to be interested in the law of $\int_0^h e^{-as}\, dZ_s$.

(iii) If we just integrate over a finite interval we get

$$E \exp\left(-\lambda \int_0^T e^{-as}\, dZ_s\right) = \exp\left\{-\int_0^T \psi(\lambda e^{-as})\, ds\right\}$$

and

$$\int_0^T \psi(\lambda e^{-as})\, ds = \int_0^\infty \lambda e^{-\lambda y} \left(\int_0^T \bar{\mu}(y e^{as})\, ds\right) dy$$

$$\equiv \int_0^\infty \lambda e^{-\lambda y}\, \bar{\nu}_T(y)\, dy.$$

Say. The extent to which such a model may be tractable depends on how nice $\bar{\nu}_T$ is.

(iv) When we want to do equilibrium pricing in a CRRA setting, if $\Delta_t \equiv 1\cdot\delta_t$, we shall need to compute things like

$$E\left[\int_0^\infty e^{-\rho t}\, U'(\Delta_t)\,\delta_t\,dt \;\Big|\; \delta_0=\delta\right]$$

$$= E\int_0^\infty e^{-\rho t}\left(\Delta_0+(\Delta_t-\Delta_0)\right)^{-R}\left(\delta_0+(\delta_t-\delta_0)\right)dt$$

and the law of the increment $\delta_t-\delta_0$ is something we have some knowledge of. Going on,

$$= E\int_0^\infty e^{-\rho t}\left(e^{-at}\Delta_0+1\cdot\xi_t\right)^{-R}\left(e^{-at}\delta_0+\xi_t\right)dt$$

where $\xi_t \equiv \int_0^t e^{-cs}\,dZ_s$;

$$= E\int_0^\infty e^{-\rho t}\left\{\int_0^\infty x^{R-1}\exp\left(-\left(e^{-at}\Delta_0+1\cdot\xi_t\right)x\right)\frac{dx}{\Gamma(R)}\right\}\left(e^{-at}\delta_0+\xi_t\right)dt$$

$$\boxed{= \int_0^\infty \frac{x^{R-1}\,dx}{\Gamma(R)}\int_0^\infty \exp\left(-\rho t-e^{-at}\Delta_0 x\right)dt\; E\left\{\exp\left(-x\,1\cdot\xi_t\right)\left(e^{-at}\delta_0+\xi_t\right)\right\}}$$

The next step therefore is to understand $E\exp(-v\cdot\xi_t)$.

(v) Notice that in the univariate story

$$\bar V_T(y) = \int_0^T \bar\mu(ye^{as})\,ds$$

$$\Rightarrow\quad \bar V_T'(y) = \int_0^T e^{as}\,\bar\mu'(ye^{as})\,ds$$

$$\Rightarrow\quad ay\,\bar V_T'(y) = \int_y^{ye^{aT}}\bar\mu'(v)\,dv = \bar\mu(ye^{aT})-\bar\mu(y)$$

(vi) Let's try an example where $\boxed{\psi(v)=\sum_{i=1}^d c_i v_i/(v_i+b_i)}$ for positive constants c_i, b_i; the components are thus independent. The Laplace exponent of $\int_0^T e^{-as}\,dZ_s^i$ will therefore be

$$\lambda\mapsto \int_0^T \frac{c_i\,\lambda e^{-as}}{b_i+\lambda e^{-as}}\,ds = \frac{c_i}{a}\log\left\{\frac{b_i+\lambda}{b_i+\lambda e^{-aT}}\right\}$$

and hence

$$E\exp\{-y\cdot\xi_t\} = \prod_{i=1}^d\left(\frac{b_i+y_i e^{-at}}{b_i+y_i}\right)^{c_i/a}$$

so as $t\to\infty$ we get that the components of ξ are <u>independent gammas</u>.

Keeping up with the Jones' (9/6/07)

1) We could consider a situation where two agents each derive utility from consumption, as well as some effect of consumption relative to each other:

$$\max E \int_0^\infty e^{-\rho_i t} U_i(c_i(t), c_{3-i}(t))\,dt \qquad (i=1,2)$$

where each has the conventional dynamics $dw_i = rw_i dt + \theta_i(\sigma dW + (\mu - r)dt) - c_i dt$.

The solution in terms of the state-price density will be

$$U_i'(c_i(t), c_{3-i}(t))\, e^{-\rho_i t} = \lambda_i \zeta_t \qquad (i=1,2)$$

and now the question is whether we can find any examples where this can be done in closed form.

2) Could try

$$U_i(c_i, c_{3-i}) = \frac{c_i^{1-R_i}}{1-R} \left(c_i/c_{3-i}\right)^{\alpha_i}$$

where $(1-R_i)\alpha_i > 0$ as a natural restriction. The equations for optimality now read

$$\begin{cases} \dfrac{1-R_1+\alpha_1}{1-R_1}\, c_1^{\alpha_1 - R_1}\, c_2^{-\alpha_1} = \lambda_1 e^{\rho_1 t}\zeta_t \\[2ex] \dfrac{1-R_2+\alpha_2}{1-R_2}\, c_2^{\alpha_2 - R_2}\, c_1^{-\alpha_2} = \lambda_2 e^{\rho_2 t}\zeta_t \end{cases}$$

Taking a ratio gives

$$\frac{(1-R_1+\alpha_1)(1-R_2)}{(1-R_2+\alpha_2)(1-R_1)}\, c_1^{\alpha_1+\alpha_2-R_1}\, c_2^{R_2-\alpha_1-\alpha_2} = \frac{\lambda_1}{\lambda_2} e^{(\rho_1-\rho_2)t}$$

$$\Rightarrow \quad c_1 = b\, c_2^{\beta} e^{\gamma t} \qquad b = \left(\frac{\lambda_1}{\lambda_2}\frac{(1-R_2+\alpha_2)(1-R_1)}{(1-R_1+\alpha_1)(1-R_2)}\right)^{1/\alpha_1+\alpha_2-R_1}$$

$$\beta = (-R_2+\alpha_1+\alpha_2)/(\alpha_1+\alpha_2-R_1), \quad \gamma = \frac{\rho_1-\rho_2}{\alpha_1+\alpha_2-R_1}$$

so returning this to the simultaneous equations lead after a while to

$$\boxed{\begin{aligned} c_2^{-\tilde R_2} &= k_1^{p_2} k_2^{1-p_2} e^{\tilde\rho_2 t}\zeta_t \\ c_1^{-\tilde R_1} &= k_1^{1-p_1} k_2^{p_1} e^{\tilde\rho_1 t}\zeta_t \end{aligned}}$$

where

$$k_i = \lambda_i(1-R_i)/(1-R_i+\alpha_i), \quad p_i = \alpha_i/(\alpha_1+\alpha_2-R_{3-i}), \quad \tilde R_i = (\alpha_1 R_2 + \alpha_2 R_1 - R_1R_2)/(\alpha_1+\alpha_2-R_{3-i})$$

$$= R_i(1-p_i) + p_i R_{3-i}$$

and $\tilde\rho_i = p_i\rho_{3-i} + (1-p_i)\rho_i$.

~~This looks like a standard CRRA solution for each agent~~, but do we have $\tilde R_1, \tilde R_2 > 0$? Not necessarily, because we don't know $p_1, p_2 \in (0,1)$.

Liquidity again (13/6/07)

(i) Suppose the density of quotes at relative price $x > 0$ is $q(x)\,\Delta t$. Then if we aim to acquire $h\Delta t$ units of the asset, we will purchase up to relative price s, where

$$h = \int_1^s q(x)\,dx$$

The total price we pay to acquire this will be

$$S_t \int_1^s x\,q(x)\,dx \quad \Delta t$$

so the loss we book doing this trade will be

$$S_t \left(\int_1^s x\,q(x)\,dx - h \right) \Delta t = S_t \int_1^s (x-1)\,q(x)\,dx \quad \Delta t$$

$$= S_t \; \ell(h) \; \Delta t .$$

Notice that $\ell(h) \geq 0$, and

$$\frac{d\ell}{dh} = \frac{(s-1)\,q(s)}{q(s)} = s-1$$

is increasing with h, and ≥ -1. Thus we can and shall suppose that

ℓ is convex, non-negative, $\quad \ell(x)+x$ is increasing, $\quad \ell'(-\infty) = -1$

. A natural example would be

$$\ell(x) = \frac{e^{\varepsilon x} - 1}{\varepsilon} - x .$$

(ii) Assume $r = 0 = \mu$, so that the wealth dynamics are simply

$$dw_t = H_t\,dS_t - S_t\,\ell(h_t)\,dt$$

$$= d(H_t S_t) - S_t \{ h_t + \ell(h_t) \}\,dt$$

Therefore

$$\boxed{w_t = w_0 + H_t S_t - H_0 S_0 - \int_0^t S_u\, f(h_u)\,du}$$

where $f(x) \equiv \ell(x) + x$. If we are trying to hedge some contingent claim η, we face the problem

$$\sup E\, U\Big(H_T S_T - \int_0^T S_u\, f(h_u)\,du + \eta - a \Big)$$

$$= \sup E\, U\Big(H_0 S_T + \int_0^T \big(h_u S_T - S_u f(h_u) \big)\,du + \eta - a \Big).$$

Here, a is some constant. If there exists an optimal hedging process h^*, then if we set

$$y^* \equiv H_0 S_T + \int_0^T (h_u^* S_T - S_u f(h_u^*))\,du + \eta - a$$

we get by the usual FOC argument that

$$E\left[U'(y^*) \int_0^T (\delta_u S_T - S_u f'(h_u^*)\delta_u)\,du\right] = 0$$

for any perturbation δ, whence

$$\boxed{E_t[S_T U'(y^*)] = S_t f'(h_t^*) E_t U'(y^*).}$$

Letting

$$Z_t \equiv c.E_t[U'(y^*)]$$

be the change of measure associated to the optimal y^*, we conclude that

$$\boxed{Z_t S_t f'(h_t^*) = E_t[Z_T S_T] \text{ is a martingale.}}$$

(iii) Defining a new measure Q by $dQ/dP|_{\mathcal{F}_T} \propto U'(y^*)$, we see that $S_t f'(h_t^*)$ is a Q-martingale. This looks hard to do very much with, as so often is the case ...

... but maybe there is something a little simpler that could be done...

(iv) Suppose we start with H_0 of stock, x_0 of cash, and follow the investment strategy h. Then at time T the replication portfolio stands at

$$H_0 S_0 + x_0 + \int_0^T H_u\,dS_u \quad = \xi$$

while the liquidity losses amount to $\int_0^T S_u\, \ell(h_u)\,du$. We want to be close to some contingent claim

$$\eta = \eta_0 + \int_0^T \theta_u\,dS_u$$

So why not try the optimisation problem

$$\min_h \left[\tfrac{1}{2} E(\eta - \xi)^2 + E\int_0^T S_u\, \ell(h_u)\,du\right]$$

$$= \min_h \left\{\tfrac{1}{2}(\eta_0 - x_0 - H_0 S_0)^2 + E\int_0^T \tfrac{1}{2}(H_u - \theta_u)^2 \sigma^2 S_u^2\,du + E\int_0^T S_u\, \ell(h_u)\,du\right]?$$

If we set $V(t,H,S) \equiv \inf E\left[\int_t^T \tfrac{1}{2}(H_u-\theta_u)^2\sigma^2 S_u^2\,du + \int_t^T S_u\,\ell(h_u)\,du \,\middle|\, H_t = H, S_t = S\right]$

then the usual story gives

$$\boxed{0 = \inf_h \left[V_t + h V_H + \tfrac{1}{2}\sigma^2 S^2 V_{SS} + \tfrac{1}{2}\sigma^2 S^2 (H-\theta)^2 + S\,\ell(h)\right]}$$

We could solve this by policy improvement, for example. So if we start with $h^{(0)}=0$ and solve

$$\mathcal{L} V^{(0)} + h^{(0)} V^{(0)}_H + S\ell(h^{(0)}) = 0 \qquad \mathcal{L} = \partial_t + \tfrac{1}{2}\sigma^2 S^2 \partial_{SS} + \tfrac{1}{2}\sigma^2 S^2 (H-\theta)^2$$

we get

$$V^{(0)}(t,S,H) = E_t\left[\int_t^T \tfrac{1}{2}\sigma^2 S_u^2 (H-\theta_u)^2\,du \,\Big|\, S_t = S\right]$$

$$= \tfrac{1}{2}\sigma^2 S^2 \left\{ H^2 a(t) - 2H\, b(t,S) + c(t,S)\right\}$$

and this then gives by policy improvement

$$S\ell'(h^{(1)}) = -\sigma^2 S^2 \left(H a(t) - b(t,S)\right)$$

If we really do have $\ell' \geq -1$, then for large H this will be violated, so the behaviour will be to sell at top speed, in effect, throw H away. However, the structure of $V^{(0)}$ won't be repeated for later $V^{(n)}$.

(v) How about some approximations, or bounds? To handle this, I'd suggest we take $\ell(h) = \tfrac{1}{2}\varepsilon h^2$ (so $\ell' \not\geq -1$, meaning for $h<0$ we have costly disposal of the asset... not totally absurd, and for small ε we probably won't get near to this situation). The HJB for this problem is now

$$V_t + \tfrac{1}{2}\sigma^2 S^2 V_{SS} + \tfrac{1}{2}\sigma^2 S^2 (H-\theta)^2 - \tfrac{1}{2}\, V_H^2/\varepsilon S = 0$$

For ε small, we expect V, V_{SS}, V_t all to be small, but the term $\tfrac{1}{2}\sigma^2 S^2 (H-\theta)^2$ is typically not small. Thus for HJB to hold, we need to have approximately

$$\frac{V_H}{\sqrt{\varepsilon S}} \doteqdot \sigma S(H-\theta) \quad \Rightarrow \quad h = -V_H/\varepsilon S \approx -\sigma S(H-\theta)/\sqrt{\varepsilon S}$$

This suggests that using the control $\boxed{h = -\sigma S(H-\theta)/\sqrt{\varepsilon S}}$ should be good. For the value under this control, we solve

$$\boxed{\mathcal{L} V \equiv V_t + \tfrac{1}{2}\sigma^2 S^2 V_{SS} - \frac{\sigma S(H-\theta)}{\sqrt{\varepsilon S}} V_H + \sigma^2 S^2 (H-\theta)^2 = 0, \qquad V(T,S,H) = 0}$$

with probabilistic solution

$$V(t,H,S) = E\left[\int_t^T \sigma^2 S_u^2 (H_u - \theta_u)^2\,du \,\Big|\, S_t = S,\ H_t = H\right] \geq 0$$

(vi) Let us seek an upper bound for V of the form $F(t,H,S) = \tfrac{1}{2}\varphi(t,S)(H-\theta)^2 = \tfrac{1}{2}\varphi(t,S)(H - q_S(t,S))^2$, where $q_t + \tfrac{1}{2}\sigma^2 S^2 q_{SS} = 0$, and therefore $q_{tS} + \tfrac{1}{2}\sigma^2 S^2 q_{SSS} = -\sigma^2 S q_{SS}$. Take $\varphi(T,\cdot) = 0$ as the boundary condition.

$$\mathcal{L}(\Theta f) = \Theta \mathcal{L} f + \sigma^2 S^2 \Theta_S f_S - f \sigma^2 S \Theta_S$$

$$= \Theta \mathcal{L} f + \sigma^2 S \Theta_S (S f_S - f)$$

After some calculations, we obtain

$$\mathcal{L}F = \tfrac{1}{2}(H-\theta)^2\left[\varphi_t + \tfrac{1}{2}\sigma^2S^2\varphi_{SS} - \frac{2\sigma S}{\sqrt{\varepsilon S}}\varphi + 2\sigma^2S^2\right] + \tfrac{1}{2}\sigma^2S^2\varphi\, q_{SS}^2$$

$$+\sigma^2S^2 q_{SS}(H-\theta)(\varphi - S\varphi_S)$$

but the last term here spoils things...

(vii) It's natural to conjecture that for adapted φ, previsible H,

$$E\left[\varphi_t^2\left(\int_0^t H_u\,dW_u\right)^2\right] \le E\left[\int_0^t \varphi_t^2 H_u^2\,du\right]$$

but with $H\equiv 1$, $\varphi_t = W_t$, we see this is false in general.

(viii) (27/6/07). Let's write $V(t,H,S) \equiv v(t,Y,S)$ where $Y\equiv H-\theta \equiv H - q_S$. Then $V_t = v_t - \theta_t v_Y$, $V_S = v_S - \theta_S v_Y$, $V_{SS} = v_{SS} - 2\theta_S v_{SY} + \theta_S^2 v_{YY} - \theta_{SS}v_Y$, so the PDE for V becomes (exploiting $\theta_t + \tfrac{1}{2}\sigma^2S^2\theta_{SS} + \sigma^2S\theta_S = 0$)

$$\boxed{\tfrac{1}{2}\sigma^2S^2\{v_{SS} - 2\theta_S v_{SY} + \theta_S^2 v_{YY}\} + \sigma^2 S\theta_S v_Y + v_t - \sigma\sqrt{\tfrac{S}{\varepsilon}}\,Y v_Y + \sigma^2S^2Y^2 = 0}$$

We now can look for a solution of the form

$$v(t,Y,S) = a(t,S)Y^2 + b(t,S)Y + c(t,S)$$

which leads to the coupled system

$$\boxed{\begin{aligned}
&\tfrac{1}{2}\sigma^2S^2 a_{SS} + a_t - 2\sigma\sqrt{\tfrac{S}{\varepsilon}}\,a + \sigma^2S^2 = 0\\
&\tfrac{1}{2}\sigma^2S^2(b_{SS} - 4\theta_S a_S) + 2a\sigma^2S\theta_S + b_t - \sigma\sqrt{\tfrac{S}{\varepsilon}}\,b = 0\\
&\tfrac{1}{2}\sigma^2S^2(c_{SS} - 2\theta_S b_S + 2a\theta_S^2) + \sigma^2S\theta_S b + c_t = 0
\end{aligned}}$$

(checked via Maple).

One solution to the first equation is

$$\boxed{a^{(0)}(t,S) = \frac{3\sigma^2}{32}\varepsilon S + \frac{\sigma\sqrt{\varepsilon}}{2}S^{3/2}} \qquad !!$$

Next, if we set $b = \tilde{b} + 2\theta a$, we find $\left[\mathcal{L}\equiv \tfrac{1}{2}\sigma^2S^2\frac{\partial^2}{\partial S^2} + \frac{\partial}{\partial t}\right]$ the second eq^n is

$$\boxed{\mathcal{L}\tilde{b} - \sigma\sqrt{\tfrac{S}{\varepsilon}}\,\tilde{b} + 2\theta\left\{\sigma a\sqrt{\tfrac{S}{\varepsilon}} - \sigma^2S^2\right\} = 0.}$$

The third equation gives an equation for $\tilde{c} \equiv c - \theta b$:

If $b = -2a\theta + \tilde{b}$, we get

$$0 = \mathcal{L}\tilde{b} - \sigma\sqrt{\frac{S}{\varepsilon}}\,\tilde{b} + 2\sigma^2 S\theta_s\{a - S a_s\}$$

and if $c = a\theta^2 + \tilde{c}$, we have

$$\mathcal{L}\tilde{c} + \theta\mathcal{L}\tilde{b} + a\sigma^2 S\theta_s = 0$$

For small x, $f(x) \sim 8x/\sigma\sqrt{\varepsilon}$

for large x, $f(x) \sim x^{3/8} \exp\left(\frac{8}{\sqrt{\sigma}}\left(\frac{x}{\varepsilon}\right)^{1/4}\right)\left(\frac{\varepsilon^{1/4}\sqrt{\sigma}}{16\pi}\right)^{1/2}$

$$\boxed{\mathcal{L}\tilde{c} + \theta \mathcal{L} b + \sigma^2 S^2 a \theta_s^2 = 0}$$

(ix) It's probably more direct to argue that

$$V(t,H,S) = a(t,S)H^2 + b(t,S)H + c$$

from which

$$(*)\quad \boxed{\begin{array}{l} \mathcal{L}a - 2\sigma\sqrt{\frac{S}{\varepsilon}}\,a + \sigma^2 S^2 = 0 \\ \mathcal{L}b - \sigma\sqrt{\frac{S}{\varepsilon}}\,(b - 2a\theta) - 2\sigma^2 S^2\theta = 0 \\ \mathcal{L}c + \sigma\sqrt{\frac{S}{\varepsilon}}\,\theta b + \sigma^2\theta^2 S^2 = 0 \end{array}}$$

(same PDE as for $\tilde{a}$)

(same PDE as for $\tilde{b}$)

(x) Some related stuff. The magic solution $a^{(0)}$ to the first equation leads us to consider

$$E\left[\exp\left(-\int_0^t 2\sigma\sqrt{S_u/\varepsilon}\,du\right) S_t^\lambda\right] \qquad \text{for } \lambda = 1, 3/2$$

We can try to see the exponential term in the expectation as part of a change-of-measure martingale. If we can solve

$$\tfrac{1}{2}\sigma^2 S^2 f''(S) = 2\sigma\sqrt{\tfrac{S}{\varepsilon}}\, f(S)$$

then $Z_t \equiv f(S_t)\exp\left(-\int_0^t 2\sigma\sqrt{S_u/\varepsilon}\,du\right)$ is a (positive) local martingale. The solution we get is

$$\boxed{f(S) = \sqrt{S}\, I_1\left(\frac{8}{\sqrt{\sigma}}\left(\frac{S}{\varepsilon}\right)^{1/4}\right).}$$

We also have the change of drift from

$$Z_t^{-1}\,dZ_t = \frac{f'(S_t)}{f(S_t)}\,dS_t = \sigma\,\frac{S_t f'(S_t)}{f(S_t)}\,dW_t$$

and Maple gives us

$$g(S) \equiv \frac{Sf'(S)}{f(S)} = \frac{\xi^2\, I_1(\xi)}{4\xi I_0(\xi) - 8I_1(\xi)} \qquad \xi \equiv \frac{8}{\sqrt{\sigma}}\left(\frac{S}{\varepsilon}\right)^{1/4}$$

The graph of this is non-negative, increasing, going up v. slowly ($S^{1/4}$ according to the asymptotic). Gradient at 0 is infinite. $S^{-1/4} g(S)$ decreases to positive finite limit. The process S under the changed drift will reach $+\infty$ in finite time, therefore the "change of measure" local martingale Z is a local martingale, but not a martingale! However, we have $S^\lambda / f(S)$ is bounded uniformly on $[0,\infty)$ for $\lambda = 1, 3/2$

Back to the story for a. We shall write $a = a^{(0)} - \bar{a}$, where $\bar{a}$ solves

$$\mathcal{L}\bar{a} - 2\sigma\sqrt{\tfrac{S}{\varepsilon}}\,\bar{a} = 0, \qquad \bar{a}(T,S) = a^{(0)}(T,S) = \frac{3\sigma^2}{32}\varepsilon S + \frac{\sigma\sqrt{\varepsilon}}{2} S^{3/2}$$

Thus

$$\bar{a}(t,S) = E_t\left[\exp\left(-\int_t^T 2\sigma\sqrt{S_u/\varepsilon}\,du\right) a^{(0)}(T,S_T)\right] \quad > 0$$

$$= E_t\left[Z_T \quad a^{(0)}(T,S_T)/f(S_T)\right] e^{\int_0^t 2\sigma\sqrt{S_u/\varepsilon}\,du}$$

$$\leq \sup_x\left\{\frac{a^{(0)}(T,x)}{f(x)}\right\}.\ f(S_t)$$

For large S_t, this is a completely useless bound! For the record, a few lines of calculation give

$$\sup_x \frac{a^{(0)}(T,x)}{f(x)} = \text{const } \varepsilon^{3/2}.$$

Probably the most useful conclusion is going to be

$$\boxed{a(t,S) \leq a^{(0)}(t,S) = \frac{3\sigma^2}{32}\varepsilon S + \frac{\sigma\sqrt{\varepsilon}}{2} S^{3/2}}$$

(xi) Maybe more to the point is to try the representation

$$V(t,H,S) = a(t,S)(H-\theta)^2 + b(t,S)(H-\theta) + c(t,S)$$

leading to the equations

$$\begin{cases} \mathcal{L}a - 2\sigma\sqrt{\frac{S}{\varepsilon}}\,a + \sigma^2 S^2 = 0 \\ \mathcal{L}b - \sigma\sqrt{\frac{S}{\varepsilon}}\,b + 2\sigma^2 S(a - Sa_S)\,\theta_S = 0 \\ \mathcal{L}c + \sigma^2 S\theta_S(b - Sb_S) + \sigma^2 S^2 a\,\theta_S^2 = 0 \end{cases} \quad \checkmark$$

Notice $\mathcal{L}(Sa_S) = S\frac{\partial}{\partial S}(\mathcal{L}a) = \sigma\sqrt{\frac{S}{\varepsilon}}\,a + 2\sigma\frac{S^{3/2}a_S}{\sqrt{\varepsilon}} \quad 2\sigma^2 S^2$

(xii) The strategy now is to show that a, Sa_S are both small in some sense; deduce that b is small; then deduce that c is small. Firstly,

$$a^{(0)} - Sa^{(0)}_S = -\sigma\sqrt{\varepsilon}\,S^{3/2}/4$$

Next, writing $A_{t,T} \equiv \int_t^T 2\sigma\sqrt{S_u/\varepsilon}\,du$, we shall have

$$\bar{a}(t,S) = E\left[\exp(-A_{t,T})\,a^{(0)}(S_T)\right]$$

$$\therefore\ \bar{a}_S = E\left[e^{-A_{t,T}}\left\{\frac{S_T}{S}\,a^{(0)}_S(S_T) - \int_t^T 2\sigma\sqrt{\frac{S_u}{\varepsilon}}\frac{du}{2S}\,.\,a^{(0)}(S_T)\right\}\right]$$

$$\Rightarrow \bar{a} - S\bar{a}_S = E\left[e^{-A_{t,T}} \left\{ -\frac{\sigma\sqrt{\varepsilon}}{4} S_T^{3/2} + \frac{1}{2} A_{t,T}\, a^{(0)}(S_T) \right\} \right]$$

This gives the bound

$$|\bar{a} - S\bar{a}_S| \le \frac{\sigma\sqrt{\varepsilon}}{4} S_t^{3/2} e^{3\sigma^2\tau/8} + \frac{1}{2e} \left\{ \frac{3\sigma^2}{32} \varepsilon S_t + \frac{\sigma\sqrt{\varepsilon}}{2} S_t^{3/2} e^{3\sigma^2\tau/8} \right\} \qquad (\tau \equiv T-t)$$

$$= \frac{3\sigma^2}{64e} \varepsilon S_t + (1+e^{-1}) \frac{\sigma\sqrt{\varepsilon}}{4} S_t^{3/2} e^{3\sigma^2\tau/8},$$

and $|a^{(0)} - Sa_S^{(0)}| = \frac{\sigma\sqrt{\varepsilon}}{4} S^{3/2}$.

From the PDE for b we get

$$b(t,S) = E_t\left[\int_t^T e^{-\frac{1}{2}A_{t,u}}\, 2\sigma^2 S_u (a - Sa_S)\, \theta_S \, du \right]$$

so if we assume that θ_S is uniformly bounded by K_1 then we have

$$|b(t,S)| \le 2K_1\sigma^2 \int_t^T E_t\left[e^{-\frac{1}{2}A_{t,u}} S_u \, |a - Sa_S|(u, S_u) \right] du$$

$$\le 2K_1\sigma^2 \int_t^T E_t\left[e^{-\frac{1}{2}A_{t,u}} S_u \left\{ \frac{3\sigma^2}{64e} \varepsilon S_u + (2+e^{-1}) \frac{\sigma\sqrt{\varepsilon}}{4} S_u^{3/2} e^{3\sigma^2(T-u)/8} \right\} \right] du$$

$$\le 2K_1\sigma^2 \left\{ \frac{3\sigma^2}{64e} \varepsilon S_t^2 \frac{e^{\sigma^2\tau} - 1}{\sigma^2} + (2+e^{-1}) \frac{\sigma\sqrt{\varepsilon}}{4} S_t^{5/2} e^{3\sigma^2 T/8 - 15\sigma^2 t/8} \left(e^{3\sigma^2\tau/2} - 1\right) \cdot \frac{2}{3\sigma^2} \right\}$$

$$\le c_{11}\, \varepsilon S_t^2 + c_{12} \sqrt{\varepsilon}\, S_t^{5/2}$$

Finally, to bound c, we will be OK if we can bound $b - Sb_S$. For this we shall also need to assume θ_{SS} is uniformly bounded. We also require a bound on $S^2 a_{SS}$ of the form $S^2 a_{SS} \le c_{21}\, \varepsilon S^{q_1} + c_{22} \sqrt{\varepsilon}\, S^{q_2}$ for some positive powers q_1, q_2; this can be established rather as we bounded $S\bar{a}_S$. Then we have

$$\mathcal{L}(b - Sb_S) = \mathcal{L}b - S\frac{\partial}{\partial S}\mathcal{L}b = S^2 \frac{\partial}{\partial S}\left(-\frac{1}{S}\mathcal{L}b \right)$$

$$= S^2 \frac{\partial}{\partial S} \left\{ -\frac{\sigma}{\sqrt{\varepsilon S}} b + 2\sigma^2 (a - Sa_S)\theta_S \right\}$$

$$= S^2 \left[\frac{\sigma}{2\sqrt{\varepsilon} S^{3/2}} b - \frac{\sigma}{\sqrt{\varepsilon S}} b_S + 2\sigma^2 (a - Sa_S)\theta_{SS} - 2\sigma^2 S a_{SS}\, \theta_S \right]$$

$$= \sigma\sqrt{\frac{S}{\varepsilon}} (b - Sb_S) - \frac{\sigma}{2}\sqrt{\frac{S}{\varepsilon}}\, b + 2\sigma^2 S^2 (a - Sa_S)\theta_{SS} - 2\sigma^2 S^3 a_{SS}\, \theta_S$$

and hence

Solving HJB in the form

$$V = a(t,S)\,(H-\theta(t,S))^2 + b(t,S)\,(H-\theta(t,S)) + c(t,S)$$

leads to

$$\begin{cases} \mathcal{L}a + \frac{1}{2}\sigma^2 S^2 - \frac{2a^2}{\varepsilon S} = 0 \\ \mathcal{L}b - \frac{2ab}{\varepsilon S} + 2S\sigma^2\theta_S(a - Sa_S) = 0 \\ \mathcal{L}c - \frac{1}{2}\frac{b^2}{\varepsilon S} + S^2\sigma^2\theta_S^2 a + \theta_S\left(+S\sigma^2(b - Sb_S) - 2a\sigma S\theta(\sigma-1)\right) = 0 \end{cases}$$

$$\mathcal{L}(b - Sb_S) - \sigma\sqrt{\tfrac{S}{\varepsilon}}\,(b - Sb_S) = -\frac{\sigma}{2}\sqrt{\frac{S}{\varepsilon}}\, b + 2\sigma^2 S^2 (a - Sa_S)\,\theta_{SS} - 2\sigma^2 S^3 a_{SS}\,\theta_S$$

Since the RHS is bounded by terms of the form $\sqrt{\varepsilon}\, S^{q_1} + \varepsilon S^{q_2}$ we deduce the same for $b - Sb_S$, and then a similar bound for c. [A general result to this effect would be a clean way to do this]

(xiii) But might it not be easier to solve the HJB equation

$$\mathcal{L}V + \tfrac{1}{2}\sigma^2 S^2 (H-\theta)^2 - \frac{V_H^2}{2\varepsilon S} = 0$$

by seeking a solution of the form $V = aH^2 + bH + c$?? This leads to

$$\begin{cases} \mathcal{L}a + \frac{1}{2}\sigma^2 S^2 - \dfrac{2a^2}{\varepsilon S} = 0 & (1) \\[2ex] \mathcal{L}b - \sigma^2 S^2 \theta - \dfrac{2ab}{\varepsilon S} = 0 \\[2ex] \mathcal{L}c + \frac{1}{2}\sigma^2 S^2 \theta^2 - \dfrac{b^2}{2\varepsilon S} = 0 \end{cases}$$

with zero boundary conditions at $t = T$. Everything depends on solving the first (non-linear) PDE: if

$$\Psi(t, S; \alpha) \equiv E\left[\int_t^T \exp\left(-\int_t^u 2\alpha(v, S_v)\,\frac{dv}{\varepsilon S_v}\right)\cdot \tfrac{1}{2}\sigma^2 S_u^2\, du \,\Big|\, S_t = S\right]$$

then a solution to (1) is a function such that $\Psi(t,S;\alpha) = \alpha(t,S)$. We have the obvious inequality $\alpha \le \beta \Rightarrow \Psi(\cdot,\cdot;\alpha) \ge \Psi(\cdot,\cdot;\beta)$, and we can set about solving recursively, defining $a^{(0)} \equiv 0$, $a^{(n+1)} \equiv \Psi(\cdot,\cdot;a^{(n)})$. We see that

$$0 \equiv a^{(0)} \le a^{(1)}, \quad a^{(0)} \le a^{(2)} \le a^{(1)}$$

and inductively

$$a^{(2n-2)} \le a^{(2n)} \le a^{(2n-1)} \le a^{(2n-3)}$$

So the even terms converge monotonically to a limit, as do the odd terms; the limit of $a^{(2n+1)} \ge \lim a^{(2n)}$, but are they equal? Notice that if $\bar{a}$ is the limit of the odd terms, $\underline{a}$ the limit of the even terms, then

$$\begin{cases} \mathcal{L}\bar{a} + \frac{1}{2}\sigma^2 S^2 - \dfrac{2\,\bar{a}\,\underline{a}}{\varepsilon S} = 0 \\[2ex] \mathcal{L}\underline{a} + \frac{1}{2}\sigma^2 S^2 - \dfrac{2\,\underline{a}\,\bar{a}}{\varepsilon S} = 0 \end{cases}$$

and hence $\bar{a} = a$. As for uniqueness, we have that if $\tilde{a}$ is another non-negative solution then

$$\tilde{a} = \Psi(\tilde{a}) \leq \Psi(0) = \Psi(a^{(0)}) = a^{(1)}$$

whence

$$\tilde{a} = \Psi(\tilde{a}) \geq \Psi(a^{(1)}) = a^{(2)}$$

and continuing, $\tilde{a} = a = \bar{a}$.

(xiv) Numerical solution of PDE. Writing $a(t,S) \equiv f(t, \log S) \equiv f(t,y)$, we convert PDE(1) into

$$f_t + \tfrac{1}{2}\sigma^2 (f_{yy} - f_y) + \tfrac{1}{2}\sigma^2 e^{2y} - \tfrac{2}{\varepsilon} e^{-y} f^2 = 0$$

Suppose we put down a grid of y-points $(y^i)_{i=1}^{K}$ and a grid of t-points $(t_n)_{n=1}^{N}$, and build a matrix M which approximates $\frac{1}{2}\sigma^2(D^2 - D)$. If f_n denotes the solution at $t = t_n$, then the C-N equation we need to solve would be

$$\tfrac{1}{2}\sigma^2 e^{2y} + \tfrac{1}{2}M(f_{n+1} + f_n) - \tfrac{1}{\varepsilon} e^{-y}(f_{n+1}^2 + f_n^2) = \frac{f_n - f_{n+1}}{t_{n+1} - t_n} \equiv \frac{f_n - f_{n+1}}{\Delta t_{n+1}}$$

This isn't linear in f_n, so if we try to express $f_n = g_n + \Delta f_n$, where g_n is a known vector, then we get to leading order

$$\tfrac{1}{2}\sigma^2 e^{2y} + \tfrac{1}{2}M(f_{n+1} + g_n + \Delta f_n) - \tfrac{1}{\varepsilon} e^{-y}\left(f_{n+1}^2 + g_n^2 + 2 g_n \Delta f_n\right) = \frac{g_n - f_{n+1} + \Delta f_n}{\Delta t_{n+1}}$$

The obvious choice is to use $g_n = f_{n+1}$, but it may be insufficiently accurate, so we may have to iterate this step a few times. Rearranging this to make Δf_n the subject

$$\left(I - \frac{\Delta t_{n+1}}{2}\left\{M - \frac{4}{\varepsilon} e^{-y} g_n\right\}\right)\Delta f_n = f_{n+1} - g_n + \Delta t_{n+1}\left\{\tfrac{1}{2}\sigma^2 e^{2y} + \tfrac{1}{2}M(f_{n+1} + g_n) - \tfrac{1}{\varepsilon} e^{-y}(f_{n+1}^2 + g_n^2)\right\}$$

or in terms of f_n,

$$\boxed{\left(I - \frac{\Delta t_{n+1}}{2}\left\{M - \frac{4}{\varepsilon} e^{-y} g_n\right\}\right) f_n = f_{n+1} + \Delta t_{n+1}\left\{\tfrac{1}{2}\sigma^2 e^{2y} + \tfrac{1}{2} M f_{n+1} - \tfrac{1}{\varepsilon} e^{-y}(f_{n+1}^2 - g_n^2)\right\}}$$

For solving a linear equation

$$\mathcal{L} f - \rho(t,S) f + \varphi(t,S) = 0$$

becomes

$$\boxed{\left(I - \frac{\Delta t_{n+1}}{2}(M - \rho_n)\right) f_n = \left(I + \frac{\Delta t_{n+1}}{2}(M - \rho_{n+1})\right) f_{n+1} + \tfrac{1}{2}\Delta t_{n+1}(\varphi_n + \varphi_{n+1})}$$

(XV) Taking the bounds further. I want to have a bound that proves that as $\varepsilon \downarrow 0$ the loss goes to zero. Introduce change of variables

$$s = \varepsilon z$$

with $a(t,s) = \alpha(t,z) \equiv \alpha(t, s/\varepsilon)$, $b(t,s) = \beta(t,z)$, $c(t,s) = \gamma(t,z)$, so that the PDEs from the approximate optimal solution are now

$$\left[\tilde{\mathcal{L}} = \tfrac{1}{2}\sigma^2 z^2 \frac{\partial^2}{\partial z^2} + \frac{\partial}{\partial t}\right]$$

$$\begin{cases} \tilde{\mathcal{L}}\alpha - 2\sigma\sqrt{z}\,\alpha + \sigma^2\varepsilon^2 z^2 = 0 & (1.1) \\ \tilde{\mathcal{L}}\beta - \sigma\sqrt{z}\,\beta + 2\sigma^2 \varepsilon z(\alpha - z\alpha_z)\theta_s = 0 & (1.2) \\ \tilde{\mathcal{L}}c + \sigma^2 \varepsilon z\,\theta_s(\beta - z\beta_z) + \sigma^2\varepsilon^2 z^2 \alpha\,\theta_s^2 = 0 & (1.3) \end{cases}$$

with exact solution $\alpha^{(0)} = \varepsilon^2\left(\frac{3\sigma^2}{32} z + \frac{\sigma}{2} z^{3/2}\right)$ for the first equation. The transformed PDEs from the HJB become (see reverse of p 39)

$$\begin{cases} \tilde{\mathcal{L}}(\varepsilon^{-2}a) + \tfrac{1}{2}\sigma^2 z^2 - \frac{2}{z}(\varepsilon^{-2}a)^2 = 0 & (2.1) \\ \tilde{\mathcal{L}}(\varepsilon^{-2}b) - \frac{2}{z}(\varepsilon^{-2}a)(\varepsilon^{-2}b) + 2\varepsilon z\sigma^2\theta_s(a - za_z)\varepsilon^{-2} = 0 & (2.2) \\ \tilde{\mathcal{L}}(\varepsilon^{-2}c) - \tfrac{1}{2}(\varepsilon^{-2}b)^2 + z^2\sigma^2\theta_s^2(\varepsilon^{-2}a) + \theta_s\{\varepsilon z\sigma^2\,\varepsilon^{-2}(b - zb_z) - 2\varepsilon^{-2}a\,\sigma\,\varepsilon z\,\theta(\sigma-1)\} = 0 & (2.3) \end{cases}$$

First equation has a unique solution $a^*(t,z)$ for $\varepsilon^{-2}a$, and we also know by comparison with the approximate optimal solution that

$$0 < a^*(t,z) \leq \frac{3\sigma^2}{32} z + \frac{\sigma}{2} z^{3/2}.$$

[We obtain by routine calculations

$$\mathcal{L}(a^* - z a^*_z) = \frac{4a^*}{z}(a^* - z a^*_z) + \tfrac{1}{2}\sigma^2 z^2$$

so that $q \equiv -a^* + z a^*_z \geq 0$,

$$q(t,z) = E\left[\int_t^T \exp\left(-\int_t^s 4 z_u^{-1} a^*(u, z_u)\,du\right) \tfrac{1}{2}\sigma^2 z_s^2\,ds\right],$$

if this helps]

Better? We already know that $|\alpha - z\alpha_z| \leq \varepsilon^2(k_0 z^{3/2} + k_1 z)$, and so going to (1.2), if we assume θ_s is bounded we can majorise β by solving the PDE

$$\tilde{\mathcal{L}}g - \sigma\sqrt{z}\,g + k\varepsilon^3 z(z^{3/2} + z) = 0$$

and this we can solve explicitly with $g^{(0)}(t,z) = \varepsilon^3\left\{\frac{3\sigma}{8}k(1+\sigma^{-1})z + k(1+\sigma^{-1})z^{3/2} + \frac{k}{\sigma}z^2\right\}$.

Thus we have a bound of the form

$$|\beta(t,z)| \leq k\,\varepsilon^3 (z + z^2)$$

Next, very like an earlier analysis we obtain

$$\tilde{L}(\beta - z\beta_z) = L\beta - z\frac{\partial}{\partial z} L\beta$$

$$= \sigma\sqrt{z}(\beta - z\beta_z) - \frac{\sigma}{2}\sqrt{z}\,\beta - 2\sigma^2\varepsilon z^3 \alpha_{zz}\,\theta_s + 2\sigma^2\varepsilon^2 z^2(\alpha - z\alpha_z)\,\theta_{ss}$$

and we shall have similarly to the first time that

$$|z^2 \alpha_{zz}| \leq k\,\varepsilon^2 (z + z^{3/2})$$

so if we assume θ_{ss} is bounded, we get a bound

$$|\beta - z\beta_z| \leq k\left\{\varepsilon^3 (z^{3/2} + z^{5/2}) + \varepsilon^4 (z^{7/2} + z^3)\right\}$$

Doing a similar story for c, we get the bound

$$|c| \leq k\,\varepsilon^4 z^{5/2}\left(1 + z + \varepsilon(z^{3/2} + z^2)\right) + k\,\varepsilon^4 z^3 (1 + \sqrt{z})$$

Thus $(z \equiv S/\varepsilon)$ for fixed S we get that the objective $\to 0$ as $\varepsilon \downarrow 0$.

$$(z-a_1)\cdot\tau_1(z-a_1) + (Cz-a_2)\cdot\tau_2(Cz-a_2)$$

$$= (z-b)(\tau_1 + C^{T}\tau_2 C)(z-b)$$

$$+ (x_2 - \tilde{C}^T x_1)(I + \tilde{C}^T\tilde{C})^{-1}(x_2 - \tilde{C}^T x_1),$$

$$\left.\begin{aligned} \tilde{C} &\equiv \tau_1^{-\frac{1}{2}} C^T \tau_2^{\frac{1}{2}} \\ x_i &= \tau_i^{\frac{1}{2}} a_i \\ (I + \tilde{C}\tilde{C}^T)\tau_1^{\frac{1}{2}} b &= x_1 + \tilde{C} x_2 \end{aligned}\right\}$$

[Check with Scilab simulation - this calculation is OK]

Kalman filtering with occasional jumps (21/7/07)

1) Suppose we have basic dynamics

$$X_{t+1} = AX_t' + \varepsilon_t \qquad \varepsilon_t \sim N(0, V_\varepsilon)$$

where $X_t' = X_t$ with prob^y $1-p$; and $X_t' = X_t + \eta_t$ with prob^y p $(\eta_t \sim N(0, V_\eta))$. Then we observe

$$Y_t = CX_t + \delta_t \qquad \delta_t \sim N(0, V_\delta)$$

How do we filter X_t from $\mathcal{Y}_t = \sigma(\{Y_r : r \leq t\})$?

2) Try to do this by modifying KF. If we have posterior density $\pi_t(x)$ for X_t, and if we set $\gamma(x, V) = \exp(-\frac{1}{2}x \cdot V^{-1}x) / \sqrt{\det V}\,(2\pi)^{-d/2}$ for the Gaussian density, then we obtain

$$P(X_{t+1} = x, Y_t = y) = (1-p) \int \pi_t(x')\ \gamma(x - Ax', V_\varepsilon)\, \gamma(y - Cx, V_\delta)\, dx'$$

$$+ p \iint \pi_t(z)\ \gamma(x' - z, V_\eta)\, \gamma(x - Ax', V_\varepsilon)\, \gamma(y - Cx, V_\delta)\, dz\, dx'$$

$$(*) \qquad = \int \gamma(x - Ax', V_\varepsilon)\, \gamma(y - Cx, V_\delta) \Big\{ (1-p)\pi_t(x') + p \int \pi_t(z)\, \gamma(x' - z, V_\eta)\, dz \Big\}\, dx'.$$

Thus if $\pi_t(\cdot)$ is expressed as a mixture of Gaussian densities, we will find π_{t+1} expressed as a mixture of Gaussian densities. In general, there will be 2^t terms in π_t, but if p is small most of these are likely to be negligible.

3) If τ_1, τ_2 are $d \times d$ invertible symmetric, then a calculation we need repeatedly is

$$(z - a_1) \cdot \tau_1 (z - a_1) + (z - a_2) \cdot \tau_2 (z - a_2)$$

$$= \big(z - (\tau_1 + \tau_2)^{-1}(\tau_1 a_1 + \tau_2 a_2)\big) \cdot (\tau_1 + \tau_2)\big(z - (\tau_1 + \tau_2)^{-1}(\tau_1 a_1 + \tau_2 a_2)\big)$$

$$+ (a_1 - a_2) \cdot \tau_1 (\tau_1 + \tau_2)^{-1} \tau_2\ (a_1 - a_2)$$

4) If we were to take $\pi_t(z) = \gamma(z - m_t, V_t)$, then with $\tau_t = V_t^{-1}$ etc

$$\int \pi_t(z)\, \gamma(x' - z, V_\eta)\, dz = \gamma(-x' + m_t, V_t + V_\eta)$$

and integrating in (*) w.r.t. x' will give two terms like

$$\gamma(x - Am_t, V_\varepsilon + A^T V A)\ \gamma(y - Cx, V_\delta),$$

where $V = V_t$ with weight $(1-p)$, $V = V_t + V_\eta$ with weight p. Each of these terms contributes to the posterior density, giving

$$\mathcal{N}(x-b,\ \tilde{V})\left\{\frac{\det(V_\xi)\det(V_\xi+A^TVA)}{\det(\tilde{V})}\right\}^{\frac{1}{2}}\exp\left\{-\tfrac{1}{2}(x_2-\tilde{C}x_1)^T(I+\tilde{C}^T\tilde{C})^{-1}(x_2-\tilde{C}x_1)\right\}$$

where
$$\begin{cases}\tilde{V}^{-1} = \tau_1 + C^T\tau_2 C \\ \tilde{C} \equiv \tau_1^{-\frac{1}{2}}C^T\tau_2^{\frac{1}{2}} \\ x_1 = \tau_1^{\frac{1}{2}}Am_t \\ x_2 = \tau_2^{\frac{1}{2}}y\end{cases}$$
$\tau_1 = (V_\xi + A^TVA)^{-1}$, $\tau_2 = V_\xi^{-1}$

This is interesting because if we didn't have the possibility of two choices for V, all the terms with $\sqrt{\det\ldots}$ and the exponential of the quadratic stuff would cancel out, and we'd just get a Gaussian posterior, but now the weighting of the two normals depends in an explicit but quite complicated way on y, and other things.

5) An example of interest in FX could be to have a vector (z_t) of country effects satisfying
$$(z_{t+1} - m_{t+1}) = B(z_t - m_{t+1}) + \omega_{t+1}$$
where $m_{t+1} = m_t$ with prob^y $(1-p)$, and $m_{t+1} = m_t$ + noise w.p. p. Thus if $X_t = \begin{pmatrix} z_t \\ m_t\end{pmatrix}$ we get
$$\begin{pmatrix} z_{t+1} \\ m_{t+1}\end{pmatrix} = \begin{pmatrix} B & I-B \\ 0 & I\end{pmatrix}\begin{pmatrix} z_t \\ m_t'\end{pmatrix} + \begin{pmatrix}\omega_{t+1} \\ 0\end{pmatrix}$$

and we see $Y_t = Cz_t$, where $C = \left[\begin{array}{c|c} -1 & \\ -1 & I \\ -1 & \end{array}\right]$.

Two-bonds problem again (21/8/07)

1) In the two-bonds study with Jose (+ Wei), we have finitely-many bonds of each type for each agent, and you can only buy/sell these one at a time. Maybe it will be better to allow bonds to be bought/sold simultaneously?

So let's do the following. Let x_t denote cash held at time t, and $y_i(t)$ the number of units of bond i held at time t, $i=1,2$. Bonds of type i trade at price b_i, and deliver a constant dividend stream $\varepsilon_i\, dt$. Then the dynamics will be

$$\boxed{\begin{aligned} dx_t &= x_t(\sigma\, dW_t + \mu\, dt) + (\varepsilon_1 y_1(t) + \varepsilon_2 y_2(t))\, dt - b_1\, dy_1(t) - b_2\, dy_2(t) - c_t\, dt \\ dy_i(t) &= \theta_i(t)\, y_i(t)\, dt \qquad (i=1,2) \end{aligned}}$$

where we impose $\boxed{\theta_i(t) \geq -\lambda_i}$ $(i=1,2)$. The reason for this is that we should not be able to sell bonds we don't have, so reducing our holding of bond i can only be done by offering bonds on the market, and the rate of finding a buyer for each bond is constant, different for each bond. When it comes to buying, we may buy at any rate we wish (subject to the availability of bonds for sale - but this will be OK in equilibrium). So what we expect will happen is some mix of local-time-style purchase of bonds, no trading, or sell-off of bonds.

Suppose the objective is to get

$$V(x, y_1, y_2) = \sup E\left[\int_0^\infty e^{-\rho t} U(c_t)\, dt \,\Big|\, x_0 = x,\ y_i(0) = y_i\right]$$

where U is CRRA. Then the HJB equation would say

$$0 = \sup_{\theta_i,\, c}\Big[U(c) - \rho V + \tfrac{1}{2}\sigma^2 x^2 V_{xx} + (\varepsilon_1 y_1 + \varepsilon_2 y_2 - \theta_1 b_1 y_1 - \theta_2 b_2 y_2 - c) V_x + \mu x V_x + \theta_1 y_1 V_1 + \theta_2 y_2 V_2\Big]$$

$$= \tilde{U}(V_x) - \rho V + \tfrac{1}{2}\sigma^2 x^2 V_{xx} + (\varepsilon_1 y_1 + \varepsilon_2 y_2) V_x + \lambda_1 y_1 (b_1 V_x - V_1)^+ + \lambda_2 y_2 (b_2 V_x - V_2)^+ + \mu x V_x$$

with conditions

$$V_1 \leq b_1 V_x, \qquad V_2 \leq b_2 V_x.$$

The scaling property $V(ax, ay_1, ay_2) = a^{1-R} V(x, y_1, y_2)$ reduces the dimension of the problem by 1.

2) An approximation to the dynamics would be to suppose $-\lambda_i \leq \theta_i(t) \leq K$ for some

large K, for then we would get HJB equation

$$0 = \tilde{U}(V_x) - \rho V + \tfrac{1}{2}\sigma^2 x^2 V_{xx} + (\varepsilon_1 y_1 + \varepsilon_2 y_2 + \mu x)V_x + \lambda_1 y_1 (b_1 V_x - V_1)^+$$
$$+ \lambda_2 y_2 (b_2 V_x - V_2)^+ + K y_1 (V_1 - b_1 V_x)^+ + K y_2 (V_2 - b_2 V_x)^+ .$$

For exploiting the change of variables in the scaling, maybe the simplest is to use $V(x, y_1, y_2) = y_1^{1-R} v(\frac{x}{y_1}, \frac{y_2}{y_1})$. The HJB equation then comes out as

$$0 = \tilde{U}(v_\xi) - \rho v + \tfrac{1}{2}\sigma^2 \xi^2 v_{\xi\xi} + (\varepsilon_1 + \varepsilon_2 \eta + \mu\xi) v_\xi + \lambda_1 ((\xi + b_1) v_\xi + \eta v_\eta - (1-R)v)^+$$
$$+ \lambda_2 \eta (b_2 v_\xi - v_\eta)^+ + K((1-R)v - \eta v_\eta - (\xi + b_1) v_\xi)^+ + K(v_\eta - b_2 v_\xi)^+ \eta .$$

Here, of course, $\xi \equiv x/y_1$, $\eta \equiv y_2/y_1$. This is of course the same as

$$0 = \sup_{c,\, \theta_i} \Big[U(c) - \rho v + \tfrac{1}{2}\sigma^2 \xi^2 v_{\xi\xi} + (\varepsilon_1 + \varepsilon_2 \eta + \mu \xi - c) v_\xi + \theta_1 ((1-R) v - \eta v_\eta - (\xi + b_1) v_\xi)$$
$$+ \theta_2 (v_\eta - b_2 v_\xi) \Big],$$

which is identifiable as a modified control problem.

3) An alternative use of scaling is to write

$$V(x, y_1, y_2) = x^{1-R} v(\tfrac{y_1}{x}, \tfrac{y_2}{x}) \equiv x^{1-R} v(\eta_1, \eta_2),$$

and doing this gives us

$$0 = \tilde{U}((1-R)v - \eta_1 v_1 - \eta_2 v_2) - \tilde{\rho} v + \tfrac{1}{2}\sigma^2 (\eta_1^2 v_{11} + 2\eta_1\eta_2 v_{12} + \eta_2^2 v_{22})$$
$$- (\varepsilon_1 \eta_1 + \varepsilon_2 \eta_2 + \mu - \sigma^2 R)\eta_1 v_1 - (\varepsilon_1 \eta_1 + \varepsilon_2 \eta_2 + \mu - \sigma^2 R)\eta_2 v_2$$
$$+ \lambda_1 \eta_1 (b_1 (1-R) v - b_1 \eta_2 v_2 - (1 + \eta_1 b_1) v_1)^+$$
$$+ \lambda_2 \eta_2 (b_2 (1-R) v - b_2 \eta_1 v_1 - (1 + \eta_2 b_2) v_2)^+$$

with $\tilde{\rho} \equiv \rho + \mu(R-1) - \tfrac{1}{2}\sigma^2 R(R-1) + (\varepsilon_1\eta_1 + \varepsilon_2\eta_2)(R-1)$

4) Again, we could try to do the scaling in the form

$$V(x, y_1, y_2) = x^{1-R} v(\log(\tfrac{y_1}{x}), \log(\tfrac{y_2}{x})) \equiv x^{1-R} v(\eta_1, \eta_2).$$

This time, the form we find (Maple once more!) is

$$0 = \tilde{U}\big((1-R)v - v_1 - v_2\big) - \tilde\rho v + \tfrac12\sigma^2(v_{11}+2v_{12}+v_{22}) - (\mu-\sigma^2R+\tfrac12\sigma^2+\varepsilon_1e^{\eta_1}+\varepsilon_2e^{\eta_2})v_1$$
$$-(\mu-\sigma^2R+\tfrac12\sigma^2+\varepsilon_1e^{\eta_1}+\varepsilon_2e^{\eta_2})v_2$$
$$+\lambda_1\big(b_1e^{\eta_1}(1-R)v - b_1e^{\eta_1}v_2 - (1+b_1e^{\eta_1})v_1\big)^+$$
$$+\lambda_2\big(b_2e^{\eta_2}(1-R)v - b_2e^{\eta_2}v_1 - (1+b_2e^{\eta_2})v_2\big)^+$$
$$\Big[+K\big((1+b_1e^{\eta_1})v_1 + b_1e^{\eta_1}v_2 - b_1e^{\eta_1}(1-R)v\big)^+ + K\big((1+b_2e^{\eta_2})v_2 + b_2e^{\eta_2}v_1 - b_2e^{\eta_2}(1-R)v\big)^+\Big]$$

(the latter just for the approximation.) Again,

$$\tilde\rho = \rho + (\mu-\tfrac12\sigma^2R)(R-1) + (\varepsilon_1e^{\eta_1}+\varepsilon_2e^{\eta_2})(R-1).$$

Re-expressing this, we have (with $\boxed{v_x \equiv (1-R)v - v_1 - v_2}$)

$$\boxed{\begin{aligned}0 = \tilde U(v_x) - \tilde\rho v + \tfrac12\sigma^2(v_{11}+2v_{12}+v_{22}) - (\mu-\sigma^2R+\tfrac12\sigma^2+\varepsilon_1e^{\eta_1}+\varepsilon_2e^{\eta_2})(v_1+v_2)\\ +\lambda_1(b_1e^{\eta_1}v_x - v_1)^+ + \lambda_2(b_2e^{\eta_2}v_x - v_2)^+\\ \big[+K(v_1 - b_1e^{\eta_1}v_x)^+ + K(v_2 - b_2e^{\eta_2}v_x)^+\big]\end{aligned}}$$

This can be expressed as the solution to an optimization:

$$\boxed{\begin{aligned}0 = \sup_{c,\theta_i}\Big[U(c) - \tilde\rho v + \tfrac12\sigma^2(v_{11}+2v_{12}+v_{22}) - (\mu-\sigma^2R+\tfrac12\sigma^2+\varepsilon_1e^{\eta_1}+\varepsilon_2e^{\eta_2})(v_1+v_2)\\ - c v_x + \theta_1(v_1 - b_1e^{\eta_1}v_x) + \theta_2(v_2 - b_2e^{\eta_2}v_x)\Big]\end{aligned}}$$

where $-\lambda_i \le \theta_i \le K$. Notice how the non-controlled parts of the diffusion operator correspond to a story where

$$d\eta_1(t) = d\eta_2(t) = \sigma dW_t - (\mu-\sigma^2R+\tfrac12\sigma^2+\varepsilon_1e^{\eta_1}+\varepsilon_2e^{\eta_2})\,dt.$$

5) We may also do a transformation of the dynamics and objective, dividing by z_t which solves $dz_t = z_t(\sigma dW_t + \mu dt)$. If $\tilde x_t = x_t/z_t$, and similarly, and if we make the initial switch $(\varepsilon_i, y_i) \to (\varepsilon_i/b_i, b_i y_i)$ we may and shall wlog assume $\boxed{b_1 = b_2 = 1}$ so that the original dynamics

$$\begin{cases} dx_t = x_t(\sigma dW_t + \mu dt) - c\,dt + \varepsilon\cdot y_t\,dt - \theta_t\cdot y_t\,dt \\ y_i(t)^{-1}dy_i(t) = \theta_i(t)\,dt \end{cases}$$

leads to

$$\boxed{\begin{cases} d\tilde x_t = -\tilde c_t\,dt + \varepsilon\cdot\tilde y_t\,dt - \theta_t\cdot\tilde y_t\,dt \\ \tilde y_i^{-1}d\tilde y_i = \theta_i\,dt - \sigma d\tilde W + (\sigma^2R-\mu)\,dt \end{cases}}$$

where the objective is transformed:

$$E\int_0^\infty e^{-\rho t}U(c_t)\,dt = E\int_0^\infty e^{-\rho t} z_t^{1-R}\, U(\tilde{c}_t)\,dt$$

$$= E\int_0^\infty \exp\left(-\rho t + \sigma(1-R)W_t + (1-R)(\mu-\sigma^2/2)t\right) U(\tilde{c}_t)\,dt$$

$$= \tilde{E}\int_0^\infty \exp\left\{-\rho t + (1-R)\left(\mu-\frac{\sigma^2}{2}\right)t + \frac{\sigma^2}{2}(1-R)^2 t\right\} U(\tilde{c}_t)\,dt$$

$$\left[W_t = \tilde{W}_t + \sigma(1-R)t\right]$$

$$= \tilde{E}\int_0^\infty e^{-\tilde{\rho} t}\, U(\tilde{c}_t)\,dt$$

where $\tilde{\rho} \equiv \rho + (R-1)(\mu - \frac{\sigma^2}{2}R)$. The HJB equation in this form looks like

$$0 = \tilde{U}(V_x) - \tilde{\rho}\, V + \varepsilon\cdot y\, V_x + a\, y\cdot V_y + \tfrac{1}{2}\sigma^2\left(y_1^2 V_{11} + 2y_1y_2V_{12} + y_2^2 V_{22}\right)$$

$$+ \sum_{i=1}^{2} \lambda_i y_i (V_x - V_i)^+ \qquad \left[V_i \le V_x \text{ is needed}\right]$$

As before, writing $V(x, y_1, y_2) = x^{1-R}\, v(\log(y_1/x), \log(y_2/x))$ leads to $\left[V_x \equiv (1-R)v - v_1 - v_2,\right]$

$$\boxed{\begin{aligned}0 = \tilde{U}(v_x) - (\tilde{\rho} + (R-1)\varepsilon\cdot y)v + (a - \tfrac{1}{2}\sigma^2 - \varepsilon\cdot y)(v_1+v_2) + \tfrac{1}{2}\sigma^2(v_{11} + 2v_{12} + v_{22}) \\ + \sum_{i=1}^{2} \lambda_i\,(y_i v_x - v_i)^+,\end{aligned}}$$

$$(y_i v_x \ge v_i)$$

This is the same as

$$0 = \sup_{\substack{\theta_i \ge -\lambda_i,\\ c}} \left[U(c) - c\left((1-R)v - v_1 - v_2\right) + \sum_{i=1}^{2} \theta_i(v_i - y_i v_x)\right]$$

$$-(\tilde{\rho} + (R-1)\varepsilon\cdot y)v + \tfrac{1}{2}\sigma^2(v_{11} + 2v_{12} + v_{22}) + (a - \tfrac{1}{2}\sigma^2 - \varepsilon\cdot y)(v_1 + v_2)$$

where the variables are $\eta_1, \eta_2 \equiv \log y_1, \log y_2$.

(The bonds cost 1 and deliver coupon εdt.) $\longrightarrow$

Optimal investment/consumption with illiquid bond (24/8/07)

1) Let's do the following. You hold two accounts, one is your general account (x_t) and the other is your illiquid bond account:

$$\begin{cases} dx_t = x_t(\sigma dW_t + \mu dt) + \varepsilon y_t\,dt - p\theta_t y_t\,dt - c_t\,dt \\ dy_t = \theta_t y_t\,dt \end{cases}$$

where θ is constrained: $-\lambda \le \theta \le \kappa$. The value function

$$V(x,y) = \sup E\left[\int_0^\infty e^{-\rho t} U(c_t)\,dt \,\Big|\, x(0)=x,\ y(0)=y\right]$$

solves the HJB equation

$$0 = \sup\left[U(c) - \rho V + ((\varepsilon-\theta)y - c + \mu x)V_x + \tfrac{1}{2}\sigma^2x^2V_{xx} + \theta y V_y\right]$$

$$= \tilde{U}(V_x) - \rho V + \tfrac{1}{2}\sigma^2x^2V_{xx} + (\varepsilon y + \mu x)V_x + \kappa y(V_y - pV_x)^+ + \lambda y(pV_x - V_y)^+.$$

There is also the scaling property $V(x,y) = y^{1-R}V(x/y, 1) \equiv y^{1-R}v(x/y)$. Writing $\xi \equiv x/y$, we can transform this to

$$\boxed{\begin{aligned} 0 = \tilde{U}(v') - \rho v + \tfrac{1}{2}\sigma^2\xi^2 v'' + (\varepsilon+\mu\xi)v' + ((1-R)v - (\xi+p)v')^+\kappa \\ + ((p+\xi)v' - (1-R)v)^+\lambda \end{aligned}} \qquad (1)$$

As an optimal control problem, we could consider this as

$$\boxed{0 = \sup_{\substack{c\ge0 \\ -\lambda\le\theta\le\kappa}}\left[U(c) - \rho v + \tfrac{1}{2}\sigma^2\xi^2v'' + (\varepsilon + \mu\xi - c)v' + \theta((1-R)v - (p+\xi)v')\right]} \qquad (2)$$

This should be no problem numerically; probably there's nothing we can do theoretically.

2) Transforming the problem? We could introduce the benchmark process $dz_t = z_t(\sigma dW_t + \mu dt)$ with $\tilde{x}_t = x_t/z_t$ etc, to obtain dynamics (with bond price p possibly $\neq 1$ reintroduced)

$$d\tilde{x}_t = (\varepsilon\tilde{y}_t - \tilde{c}_t - \theta_t p\,\tilde{y}_t)\,dt$$

$$d\tilde{y}_t = \tilde{y}_t(\theta_t\,dt - \sigma d\tilde{W} + (\sigma^2R - \mu)\,dt), \qquad d\tilde{W} = dW_t - \sigma(1-R)dt$$

with objective

$$\max \tilde{E}\left[\int_0^\infty e^{-\tilde{\rho}t}U(\tilde{c}_t)\,dt\right]$$

where $\tilde{\rho} = \rho + (R-1)(\mu - \tfrac{\sigma^2}{2}R)$.

The HJB for this becomes ($\eta \equiv \tilde{y}/\tilde{x}$)

$$0 = \sup_{c,\theta}\Big[U(c) - \tilde{\rho} v - c v_x + \tfrac{1}{2}\sigma^2\eta^2 v'' + a\eta v' + \varepsilon\eta v_x + \theta\eta\{ v' - p\, v_x \} \Big] \qquad (3)$$

where $v_x \equiv (1-R)v - \eta v'$.

A further change of variables would come if we write

$$V(\tilde{x},\tilde{y}) = \tilde{x}^{1-R}\, v(\log(\tilde{y}/\tilde{x})) \equiv \tilde{x}^{1-R}\, v(z)$$

which leads to

$$\boxed{0 = \sup\Big[U(c) - c v_x - \tilde{\rho} v + \tfrac{1}{2}\sigma^2(v'' - v') + \varepsilon e^{z} v_x + \theta\{ v' - p e^{z} v_x \} + a v' \Big]} \qquad (4)$$

Where $a \equiv \sigma^2 R - \mu$, $v_x \equiv (1-R)v - v'$. For a given choice of c, θ, the value solves the 2nd order ODE

$$0 = U(c) - (\tilde{\rho} + (1-R)(c - \varepsilon e^{z} + \theta p e^{z}))v + \{ a - \tfrac{1}{2}\sigma^2 + (c - \varepsilon e^{z} + \theta p e^{z}) + \theta \} v' + \tfrac{1}{2}\sigma^2 v'' \qquad (5)$$

3) Boundary conditions? It may be better to work with the form (2) of the problem, where there's an easier interpretation of what is sensible for the BC. At the lower ξ-value, I propose that we throw away the cash, and just insist that the agent consumes the coupons for ever (i.e., the holding of the bond may not be altered). At the highest ξ-value, the story would be that we reflect (since we expect that buying bonds will be easy, this is not a bad story).

Given the choice of c, θ, we are solving

$$0 = U(c) - (\rho + \theta(R-1))v + (\varepsilon + \mu\xi - \theta(p+\xi) - c)v' + \tfrac{1}{2}\sigma^2\xi^2 v''.$$

PDCCB study again (9/9/07)

1) Since it looks inevitable that we have to do things numerically, let's work with $m = 0, h, 2h, \dots$, $Mh \equiv n$ (=1 by standard transformation) so we discretise m, but let's stick with continuous V. The point is that whatever $m > 0$, there has to be some low enough $\xi(m)$ where the firm defaults. For each m, we try initially to determine a single interval $(\xi(m), \eta(m))$ over which nothing happens, with conversion at $\eta(m)$; if we can find that, and the corresponding S stays above $\underline{S}(m,V) = \left(\frac{V}{n} \wedge \frac{V-mK}{n-m}\right)^+$, that's OK, otherwise we get an interval $(\xi(m), \underline{\eta}(m))$ controlled by firm at both ends, and then try to find a similar interval above $V = nK$.

At any place where bondholders want to convert, we have to have

$$B(kh,V) = \frac{1}{k}\left\{ S((k-1)h,V) + (k-1)\,B((k-1)h,V) \right\}$$

$$S(kh,V) = S((k-1)h,V) \geq B(kh,V) \qquad (\Rightarrow\ S((k-1)h,V) \geq B((k-1)h,V))$$

2) Suppose we can't make the one-interval solution without clipping $\underline{S}(m,\cdot)$. Then below $V = nK$ we know how S is going to behave. We can't have $(\xi, \underline{\eta})$ with $\underline{\eta} > nK$, so $S(m, nK) = K$.

What happens to the right of nK? Certainly $S(m,\cdot)$ must be continuous at nK, so we either get lift-off from $\underline{S}$ somewhere to the right of nK — this must happen smoothly, and therefore the upper end of the interval will be controlled by B — or else we get non-smooth liftoff of S from K at $V = nK$. In the second of these situations, we have already ruled out the possibility that the firm controls the upper end, so whatever happens <u>the upper end is controlled by bondholders.</u> Of course, there may be no such interval.

3) We have from the ODEs that

$$\begin{cases} S(m,V) = \dfrac{rV - n\rho'}{r(n-m)} + \dfrac{\rho'}{r} + a_S(m)\,V^{-\alpha} + b_S(m)\,V^{\beta} \\[2ex] B(m,V) = \dfrac{\rho}{r} + a_B(m)\,V^{-\alpha} + b_B(m)\,V^{\beta} \end{cases}$$

so this suggests we consider ($z = V^{\alpha+\beta}$)

$$V^{\alpha} S(m,V) = a_S(m) + b_S(m)\,z + (n-m)^{-1} z^{(\alpha+1)/(\alpha+\beta)} - \frac{m\rho'}{r(n-m)} z^{\alpha/(\alpha+\beta)}$$

$$\equiv s(m,z)$$

with the similar transformation

$$V^{\alpha} B(m,V) = a_B(m) + b_B(m)\,z + r^{-1}\rho\, z^{\alpha/(\alpha+\beta)} \equiv b(m,z).$$

4) How to do numerical solution? Perhaps the way to do this would be to assign to a pair (z_-, z) $\in (0, \eta^{\alpha+\beta})^2$ the penalty computed by calculating the (a_s, b_s) that would take s from $\underline{s}(z_-)$ at z_- to old-$s(z)$ at z, similarly the (a_B, b_B) that would take s from old-$b(z)$ at z to $\underline{b}(z)$ at z, and then returning the sum of squares of gradient mismatches.

Some thoughts on a paper of Barbers + Xiong (18/9/07)

The basic idea is that agents do not like to sell an asset at a loss, so the idea is to propose an element of the objective which rewards selling high relative to the basis. At the moment, for simplicity we'll assume that we're fully invested in the single risky asset at all times, except when we (instantaneously) go in and out of the single risky asset. Take CRRA U, and let $\tau_1 < \tau_2 < \dots$ be the times when we reset the wealth. Wealth evolves as

$$dx_t = x_t(\sigma dW_t + \mu dt) - c_t dt$$

with objective

$$V(x,p) = \sup E\left\{ \int_0^\infty e^{-\delta t} U(c_t)\,dt + \alpha \sum_{i\geq 1} e^{-\delta\tau_i} U(x_{\tau_i})\, \varphi\left(\frac{x_{\tau_i}}{x_{\tau_{i-1}}}\right) \;\middle|\; x_0 = x,\ x_{\tau_0} = p \right]$$

where p is the initial value of the basis (that is, the price we bought in at). When we reset, we lose a proportion ε of the value in transaction costs. We get the HJB equation

$$0 = \sup\left[U(c) - \delta V + (\mu x - c) V_x + \tfrac{1}{2}\sigma^2 x^2 V_{xx} ,\ - V(x,p) + \alpha U(x)\, \varphi(x/p) + V((1-\varepsilon)x, (1-\varepsilon)x) \right]$$

as well as scaling:

$$V(\lambda x, \lambda p) = \lambda^{1-R} V(x,p) \Rightarrow V(x,p) = p^{1-R} v(x/p) \equiv p^{1-R} V(\tfrac{x}{p}, 1)$$

so if we write $z \equiv x/p$,

$$0 = \max\left[\tilde{U}(v') - \delta v + \mu z v' + \tfrac{1}{2}\sigma^2 z^2 v'' ,\ -v(z) + \alpha U(z)\varphi(z) + (1-\varepsilon)^{1-R} z^{1-R} V(1,1) \right]$$

$$= \max\left[\tilde{U}(v') - \delta v + \mu z v' + \tfrac{1}{2}\sigma^2 z^2 v'' ,\ -v(z) + \alpha U(z)\varphi(z) + (1-\varepsilon)^{1-R} z^{1-R} v(1) \right]$$

It's tempting to guess a solution of the form

$$v(z) = a\,U(z) \quad (z \leq z_*) \ ; \quad = \alpha U(z)\varphi(z) + (1-\varepsilon)^{1-R} z^{1-R} v(1) \quad (z \geq z_*)$$

but I'm not sure this can be what happens.

For example, what we have as $z \downarrow 0$ is that x/p is so small that there is no possibility we'd reset in meaningful timescales, so we would imagine we get

$$V(x,1) \sim a_0\, U(x)$$

where (after some calculations)

$$a_0^{-1/R} = \left(\delta + \mu(R-1) - \tfrac{1}{2}\sigma^2 R(R-1)\right)/R$$

So for BCs we could set that value at 0 (or ε, better), with reflection at the top. This asymptotic at zero would only permit one choice of a in the form we guessed, which is why it's wrong

More on the callable bonds question (28/9/07)

Maybe we should be focussing more on B, and less on Y. One thing is that our conversion condition really should be that B smooth-pastes to some curve, not that Y smooth-pastes to 0 (though probably these work out the same). To see how this might look, let's return to the no-calling situation. If we have $s(\eta)$ is the curve $S(m, \eta(m))$ parametrised by η, then we must have $B(m, V)$ smooth-pasting to s at $\eta(m)$; matching the values of B and its first derivative give us

$$B(m,V) = s(\eta) + \frac{s(\eta) - \rho/r}{\alpha+\beta}\,\psi_0\!\left(\frac{V}{\eta}\right) + \frac{V s'(\eta)}{\alpha\beta(\alpha+\beta)}\,\psi_0'\!\left(\frac{V}{\eta}\right)$$

which automatically satisfies $\partial B/\partial m = 0$ at η. We still have

$$S(m,V) = \frac{m\rho'\,\psi_0(V/\xi) - rV\psi_1(V/\xi)}{r(\alpha+\beta)(n-m)}$$

and we must match $S(m,\eta) = s(\eta)$, $B(m,\xi) = p\xi/m$, and $\partial S/\partial m = 0$ at η. The last of these is best dealt with via

$$s'(\eta) = \frac{d}{d\eta} S(m(\eta),\eta) = \frac{\partial S}{\partial V}(m,\eta) = \frac{\xi^{-1}\left(m\rho'\psi_0'(1/\theta) - r\xi\psi_1(1/\theta) - r\eta\,\psi_1'(1/\theta)\right)}{r(\alpha+\beta)(n-m)} \tag{1}$$

$$s(\eta) = \frac{m\rho'\,\psi_0(1/\theta) - r\eta\,\psi_1(1/\theta)}{r(\alpha+\beta)(n-m)} \tag{2}$$

$$s + \frac{s - \rho/r}{\alpha+\beta}\,\psi_0(\theta) + \frac{\eta\theta s'}{\alpha\beta(\alpha+\beta)}\,\psi_0'(\theta) = p\theta\eta/m \tag{3}$$

We could use (2) to get m as a function of (η, s, θ), then use (3) to find s' as a function of (η, s, θ), and then finally take (1) and rework it to the form $F(\eta, s, \theta) = 0$ which would determine θ as a function of (η, s), and hence the ODE for s, but it's ugly.

Forward utilities again (26/9/07)

(i) Take a standard (complete) log-Brownian market with state-price density process ζ_t and suppose we have a forward utility $u(t,x)$ which has the properties

(i) $u(t,\cdot)$ is increasing concave satisfying Inada conditions for all t

(ii) $u(\cdot,x)$ is $\mathcal{F}_t$-adapted for all x

(iii) for all $0 \le t < T$ and all x we have

$$u(t,x) = \sup E_t\left[u(T, X^x_{t,T})\right]$$

where $X^x_{t,T}$ denotes a time-t wealth which could be attained from initial wealth x at time t. What is the underlying structure of this set up?

(ii) Write $\tilde{u}(t,\lambda) = \sup\{u(t,x) - \lambda x\}$ for the convex dual. We have

$$u(t,x) - \lambda x \zeta_t = \sup E_t\left[u(T, X^x_{t,T}) - \lambda \zeta_T X^x_{t,T}\right]$$

$$\le E_t\left[\tilde{u}(T, \lambda\zeta_T)\right]$$

from which

$$\tilde{u}(t, \lambda\zeta_t) \le E_t\left[\tilde{u}(T,\lambda\zeta_T)\right]$$

for each $\lambda > 0$.

However, if we fix $t < T$, and $\lambda > 0$, and we consider $x(\lambda)$ defined by

$$\zeta_t\, x(\lambda) = E_t\left[\zeta_T\, I(T, \lambda\zeta_T)\right]$$

then from initial wealth $x(\lambda)$ (an $\mathcal{F}_t$-measurable RV) the optimal time-T wealth to construct will be $I(T,\lambda\zeta_T)$, and

$$u(t, x(\lambda)) - \lambda x(\lambda)\zeta_t = E_t\left[u(T, I(T,\lambda\zeta_T)) - \lambda\zeta_T I(T,\lambda\zeta_T)\right]$$

$$= E_t\left[\tilde{u}(T, \lambda\zeta_T)\right],$$

from which we conclude that

$$\boxed{\tilde{u}(t,\lambda\zeta_t) = E_t\left[\tilde{u}(T,\lambda\zeta_T)\right]}.$$

(iii) We have thus that $M_t(\lambda) \equiv \tilde{u}(t,\lambda\zeta_t)$ is a martingale for each $\lambda > 0$, and $M_t(\cdot)$ is convex decreasing for each t, and this is a characterisation of forward utilities. From this,

$$-\frac{\partial}{\partial\lambda} M_t(\lambda) = -\zeta_t \frac{\partial\tilde{u}}{\partial y}(t,\lambda\zeta_t)$$

(Multivariate version of the representation?)

is a positive martingale, and must be decreasing with λ. One special case of interest would be where $\tilde{u}(\cdot,\cdot)$ is a deterministic function; then

$$0 < -\zeta_t \tilde{u}'(t, \lambda\zeta_t) = -\zeta_t \tilde{u}'(t, \lambda \exp(-\kappa W_t - \tfrac{1}{2}|\kappa|^2 t - rt))$$

$$= h(t, W_t - \kappa^{-1}\log\lambda) \qquad [\kappa = (\mu - r)/\sigma]$$

so that $h(t,x)$ is a positive harmonic function which is increasing in its second argument:

$$h(t,x) = \int_0^\infty \exp(\alpha x - \tfrac{1}{2}\alpha^2 t)\ \mu(d\alpha)$$

for some measure μ. This is the story we did before.

Another interesting case would be where

$$M_t(\lambda) = f(\lambda) X_t$$

where X_t is some positive martingale, f is convex decreasing. Then in this case,

$$\tilde{u}(t, \lambda\zeta_t) = f(\lambda) X_t$$

$$\Rightarrow \tilde{u}(t,y) = X_t f(y/\zeta_t)$$

$$\Rightarrow u(t,x) = \inf\{\tilde{u}(t,y) + yx\}$$

$$= X_t \tilde{f}(x\zeta_t / X_t).$$

Another class of examples is found in M-Z, where they have

$$u(t,x) = Z_t f(A_t, x/Y_t) \qquad \begin{cases} dZ_t = Z_t \varphi_t\, dW_t \\ dY_t = Y_t \delta_t\, (dW_t + \kappa\, dt) \\ dA_t = |\kappa + \varphi - \delta|^2\, dt \end{cases}$$

$$\Rightarrow \tilde{u}(t, \lambda\zeta_t) = Z_t \tilde{f}(A_t, \lambda\zeta_t Y_t / Z_t)$$

with $\tilde{f}$ solving the appropriate PDE: $\boxed{\tfrac{1}{2} y^2 \tilde{f}_{yy} + \tilde{f}_t = 0}$ This reworks after a little calculus to the form given by MZ, namely $f_t f_{xx} = \frac{1}{2} f_x^2$.

(iv) We can make a little progress on the portfolio/wealth process, because $x_t = I(t, \lambda\zeta_t)$ for some fixed $\lambda > 0$, and if the portfolio for x is θ_t, we have

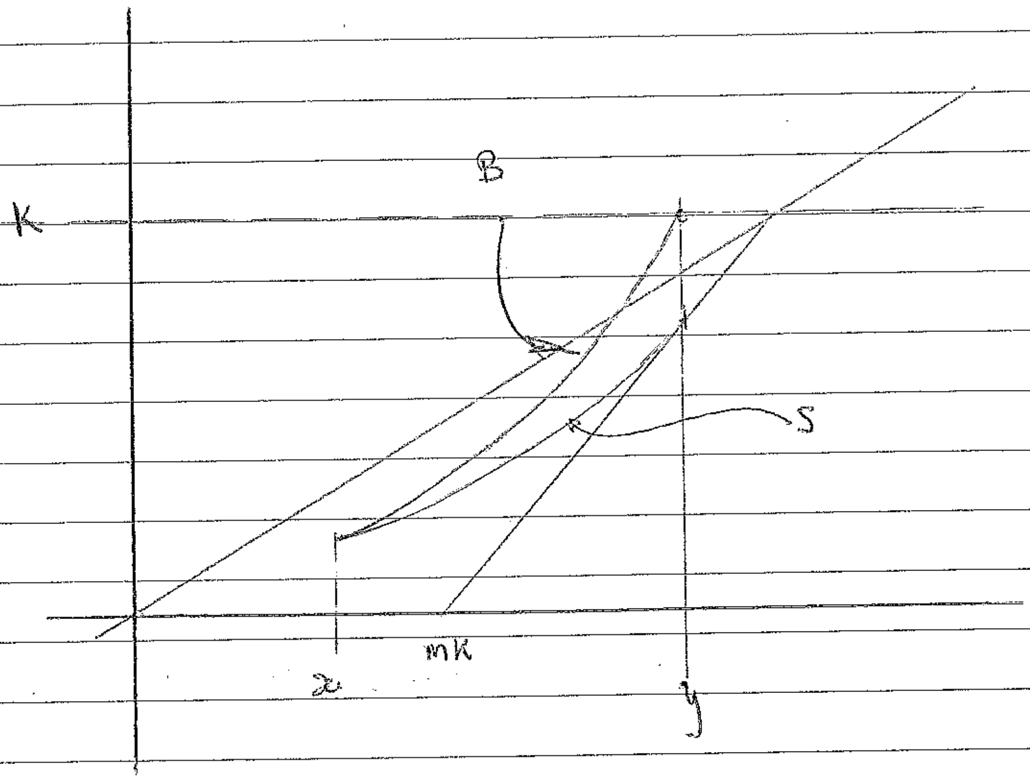
$$d(\zeta_t X_t) = \zeta_t\, dX_t + X_t\, d\zeta_t + \ldots$$

$$= \zeta_t \{\theta_t - \kappa x_t\} dW_t$$

But

$$\zeta_t x_t = \zeta_t I(t, \lambda\zeta_t) = -\frac{\partial}{\partial\lambda} \tilde{u}(t, \lambda\zeta_t)$$

so if we have $\tilde{u}(t, \lambda\zeta_t) \equiv M_t(\lambda)$ sufficiently explicitly, we may find θ_t.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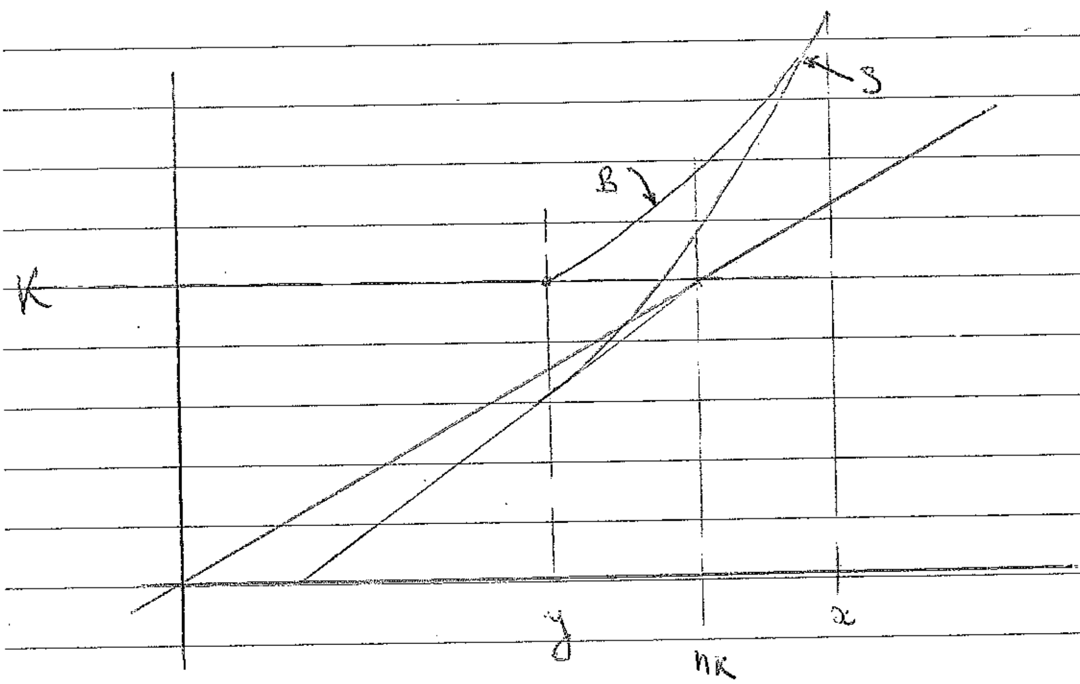
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PDCCB: another possibility (30/9/07)

(i) There appears to be one possibility not considered earlier, where S smooth-pastes to $\frac{V-mK}{n-m}$ in (mK, nK) while bondholders convert elsewhere, at x. Let y be the point of calling, set $\theta = x/y$, $s \equiv S(m, x)$. We could in principle have $x > y$, or $y > x$. We have equations for $Y(m,y) = K - (y-mK)/(n-m) = (nK-y)/(n-m)$, $\frac{\partial S}{\partial m} = 0$ at x, and the equation (4.6) for s, which altogether gives us (using θ as the independent variable)

$$\begin{cases} \cancel{r(\alpha+\beta)(nK-y) = ry\,\psi_1(1/\theta) - \rho(n-mx)\psi_0(1/\theta)} & (1) \\ r(n-m)(\alpha+\beta)\,s = r(\alpha+\beta)(x-mK) + m(\rho'-rK)\,\psi_0(\theta) & (2) \\ (n-m)\,s' = \left\{1 + \frac{m(\rho'-rK)}{r(\alpha+\beta)}\,\psi_0'(\theta)\cdot\frac{1}{y}\right\}\frac{dx}{d\theta} & (3) \end{cases}$$

We can use (1),(2) to give (x, m) as a function of (s,θ), which then allows us to deduce $dx/d\theta$, in terms of s'. Reworking (3) leads to an equation for s'. In more detail, (1)+(2) say

$$A(s,\theta)\begin{pmatrix} x \\ m \end{pmatrix} = \begin{pmatrix} r(\alpha+\beta)nK + \rho n\psi_0(1/\theta) \\ rn(\alpha+\beta)s \end{pmatrix} \equiv \begin{pmatrix} h(\theta) \\ rn(\alpha+\beta)s \end{pmatrix}$$

where

$$A(s,\theta) = \begin{pmatrix} r(\psi_1(1/\theta)+\alpha+\beta)\theta^{-1} & \rho x\psi_0(1/\theta) \\ r(\alpha+\beta) & (\rho'-rK)\psi_0(\theta) + r(\alpha+\beta)(s-K) \end{pmatrix} \equiv \begin{pmatrix} A_{11}(\theta) & A_{12}(\theta) \\ A_{22} & A_{22}(s,\theta) \end{pmatrix}$$

Hence

$$x = \frac{1}{\Delta}\left(A_{22}(s,\theta)\,h(\theta) - A_{12}(\theta)\,rn(\alpha+\beta)s\right) \qquad \left(\Delta \equiv A_{11}A_{22} - A_{21}A_{12}\right).$$

Now $\frac{dx}{d\theta} = \frac{\partial x}{\partial \theta} + \frac{\partial x}{\partial s}\frac{ds}{d\theta}$

So if we set $q(s,\theta) \equiv 1 + \frac{m(\rho'-rK)}{r(\alpha+\beta)y}\psi_0'(1/\theta)$, we shall obtain the ODE for s

$$\boxed{\left(n-m-q\frac{\partial x}{\partial s}\right)s' = q\frac{\partial x}{\partial\theta}}$$

To flesh this out, we'll need to get $\nu \equiv A_{22}(s,\theta)\,h(\theta) - rn(\alpha+\beta)s\,A_{12}(\theta)$, its derivs wrt s,θ, and the same for Δ all coded up.

(ii) This could alter the way we think of the solution in some respects. Firstly, we may ask the question

"Is there a +ve interval containing nK for arbitrarily small m?" If $K<K_c$, the no-calling solution can't do it, so the only possibility would be Case (2,4), an interval $(x(m), y(m))$ where $x(m) \in (mK, nK)$ and $y(m) > nK$. [A block convert at $m_0 > 0$ would likewise be impossible, because S smooth-pasting to $\underline{S}$ in $(mK, nK) \Rightarrow S$ is convex.] Note that in Case (2,4), the only possibility is (call, convert). We couldn't expect $y(m)$ to be increasing, as that would imply block conversion, so if we can assume monotonicity of $y(\cdot)$, we have to have $y(\cdot)$ to be decreasing. The equations that have to hold will be $[\theta(m) \equiv x(m)/y(m)]$

$$Y(m, x(m)) = \frac{r x(m)\, \psi_1(\theta(m)) - \rho(n-m r c)\psi_0(\theta(m))}{r(n-m)(\alpha+\beta)} = \frac{nK - x(m)}{n-m}$$

Letting $m \to 0$ we get in the limit the equations (dropping subscript 0)

$$r x\, \psi_1(\theta) - \rho n \psi_0(\theta) = r(\alpha+\beta)(nK - x)$$

which are the same as

$$r x\left((\alpha+1)\theta^{\beta-1} + (\beta-1)\theta^{-\alpha-1}\right) - n\rho\psi_0(\theta) = r(\alpha+\beta)\, nK. \qquad (*)$$

Conditions that must be satisfied for the existence of such a solution include

$$K_c > \rho/r > K, \quad \text{and} \quad y_0 > (n\rho/\delta) \vee nK.$$

We want y_0 as large as possible, so if we think of y_0 as fixed, and look at $(*)$, the place where this is maxed as a function of θ would solve

$$r y_0 \left\{(\alpha+1)\beta\theta^{\beta-1} - \alpha(\beta-1)\theta^{-\alpha-1}\right\} = n\rho\, \alpha\beta\left(\theta^{\beta-1} - \theta^{-\alpha-1}\right)$$

i.e. $$r y_0\left((\alpha+1)\beta\,\theta^{\alpha+\beta} - \alpha(\beta-1)\right) = n\rho\,\alpha\beta\left(\theta^{\alpha+\beta} - 1\right)$$

$$\Rightarrow \theta^{\alpha+\beta} = \frac{\alpha(r y_0(\beta-1) - n\rho\beta)}{\beta(r y_0(\alpha+1) - n\rho\alpha)} ~~\in (0,1) \text{ if } y_0 > \frac{n\rho\beta}{r(\beta-1)}~~ \equiv \eta_0$$

~~So this looks quite sensible.~~ So the thing to do is to take θ to be this function of y_0, stick into $(*)$, and find the value of y_0 for which this equality holds... but numerics don't seem to support this. Now we also have

$$~~\frac{\alpha(r y_0(\beta-1) - n\rho\beta)}{\beta(r y_0(\alpha+1) - n\rho\alpha)} \in (0,1) \text{ if } y_0 < \frac{n\rho\alpha}{r(\alpha+1)}~~$$

and it appears this may happen, but this remains to be explored.

Equation $(*)$ gives x as a function of θ, whence $y = x/\theta$. Numerics show that $y(\theta)$ is decreasing from $y(0) = \eta_0$, so this says we should be looking for $x(0)=0$, $y(0) = \eta_0$, and hopefully we can find $x(m) > mK$ to keep this solution rolling along.

Interesting questions

1) If we have a martingale X_t and are told the laws of X_{t_1} and X_{t_2}, $t_1 < t_2$, what are the extremal values of $E\,|X_{t_1} - X_{t_2}|$? $E X_{t_1} X_{t_2}$??
[Clearly, $E X_{t_1} X_{t_2} = E X_{t_1}^2$, ...]

2) Can we construct an equilibrium in an incomplete market where the introduction of a zero-net-supply derivative changes the prices of existing traded assets?

3) Sinai asks: if $A_t = \int_0^t W_u \, du$, $\tau = \inf\{t : A_t = -1\}$, what is asymptotics of $P(\tau \geq t)$ as $t \to \infty$?

4) Dan Stroock has been looking at the WH problem for drifting BM in $[0,1]$, reflected at $0,1$, with the AF $\varphi_t = t - a_0 L_t^0 - a_1 L_t^1$.

5) Steve Ross: doubling strategies are telling us there's something missing from the modelling set-up — price impact? Credit?

6) Does an "insider" really know value of process at some future time(s), or rather has better knowledge of the law? (Baudoin?)

7) Yacine asks what should replace implied vol? How should we characterise $C(t, K)$?

8) In discussions at Princeton (Sep 07) on the events of the summer it emerges that what might be happening is this. With some (insurance-type) product such as CDOs, no-one really knows how to price these things. If one bank starts to sell at 95% of true economic cost, others will be compelled to match that or lose out on the lucrative premium income. This race for the bottom can only be halted by a financial crisis in some form... How does the insurance industry handle this?

9) Asked Chris Sims why prices can rise steadily then drop suddenly but not apparently the other way round. He said it is probably because selling short is harder to do, and effectively impossible for small agents. Does this mean that bubbles are driven by naive little guys?