A building-block calculation (12/4/08)

1) Suppose we have a Markov process in $[a,b]$ which evolves as $\sigma W_t + \mu dt$ in the interior, reflects off b with local time L, and when it reaches a it jumps immediately to $x_0 \in [a,b]$. If we define

$$f_\lambda(x) \equiv E^x\Big[\int_0^\infty e^{-\lambda t}(k\,dt + dL_t) - \sum_t \varepsilon e^{-\lambda t} I_{\{X_{t-}=a\}} \Big]$$

can we calculate the mean under the invariant law of f_p?

2) As usual, we have

$$\tfrac{1}{2}\sigma^2 f_\lambda'' + \mu f_\lambda' - \lambda f_\lambda + k = 0, \quad f_\lambda'(b) = 1, \quad f_\lambda(a) = f_\lambda(x) - \varepsilon$$

which is solved by

$$f_\lambda(x) = \lambda^{-1}k + A e^{-\alpha x} + B e^{\beta x}$$

with $\alpha \equiv (\mu + \sqrt{\mu^2 + 2\sigma^2\lambda})/\sigma^2$, $\beta = (\sqrt{\mu^2 + 2\sigma^2\lambda} - \mu)/\sigma^2$. We can deduce after a few calculations that

$$A = \frac{e^{\beta x_0} - e^{\beta a} - \varepsilon\beta e^{\beta b}}{\beta e^{\beta b}(e^{-\alpha a} - e^{-\alpha x_0}) - \alpha e^{-\alpha b}(e^{\beta x_0} - e^{\beta a})},$$

$$B = \frac{e^{-\alpha a} - e^{-\alpha x_0} - \alpha\varepsilon e^{-\alpha b}}{\beta e^{\beta b}(e^{-\alpha a} - e^{-\alpha x_0}) - \alpha e^{-\alpha b}(e^{\beta x_0} - e^{\beta a})}$$

3) If $\mu > 0$, then as $\lambda \downarrow 0$ $\beta \sim \lambda/\mu$, $\alpha \to 2\mu/\sigma^2 \equiv c > 0$, and

$$\lambda B \sim \mu \frac{e^{-ca} - e^{-cx_0} - \varepsilon c e^{-cb}}{e^{-ca} - e^{-cx_0} + (a - x_0)c e^{-cb}},$$

so $\lim_{\lambda\downarrow 0} \lambda R_\lambda f_p = \rho^{-1} \lim_{\lambda \downarrow 0} \lambda f_\lambda$

$$= \rho^{-1}\Big\{ k + \mu \frac{e^{-ca} - e^{-cx_0} - \varepsilon c e^{-cb}}{e^{-ca} - e^{-cx_0} + (a-x)c e^{-cb}} \Big\}. \qquad (1)$$

4) If $\mu < 0$, then as $\lambda \downarrow 0$, $\alpha \sim \lambda/|\mu|$, $\beta \to 2|\mu|/\sigma^2 \equiv c$, and

$$\lim_{\lambda\downarrow 0} \lambda R_\lambda f_p = \rho^{-1}\Big\{ k + |\mu| \frac{e^{cx_0} - e^{ca} - \varepsilon c e^{cb}}{e^{ca} - e^{cx_0} + C e^{cb}(x_0 - a)} \Big\} \qquad (2)$$

5) If we think of maximising this over $b \geq x_0$ for $\mu > 0$ the best choice is

$$\begin{cases} b = x_0 & \text{if } \varepsilon < x_0 - a \\ b = +\infty & \text{if } \varepsilon > x_0 - a \end{cases}$$

and for $\mu < 0$ the best choice is actually the same.

So for a realisable solution, it seems we shall have to insist that

$$\boxed{\varepsilon < x_0 - a}$$

and that being the case we shall always use $\boxed{b = x_0}$

6) Suppose we do that, and now with x_0, ε fixed try to find the best a. Calculus gives for $\mu > 0$ that we should solve

$$0 = (c\varepsilon + 1)e^{ca} - e^{cx_0}(c\varepsilon + 1 + ca - cx_0)$$

For $b = x_0$, we always have the function at (1) is decreasing when $a = x_0 - \varepsilon$. It may be decreasing for all $a \in [0, x_0 - \varepsilon]$, but often we shall see a maximum.

7) For $\mu < 0$, we find that in order to obtain a maximum, we should be solving

$$(1 - c\varepsilon)e^{-ca} - e^{-cx_0}(c(x_0 - a - \varepsilon) + 1) = 0$$

At $a = x_0 - \varepsilon$, this is always negative. However, it can be that the maximising a is in the interior $(0, x_0 - \varepsilon)$, especially when $x_0 - \varepsilon$ is large.

8) Perhaps more to the point is to suppose that $\boxed{\varepsilon = \theta a}$ for some $\theta \in (0,1)$, that is, loss on default is proportional to a. We can find interior maxima whether $\mu > 0$ or $\mu < 0$. It appears that when $\mu > 0$ we do have an interior maximum, however large θ may be. This is true; the equation for the derivative at $a = 0$ of the function at (1) to vanish is

$$\theta(1 - e^{cx} + cx) = -(1 - e^{cx} + cxe^{cx})$$

or
$$\theta = -\frac{e^{cx} - 1 - cxe^{cx}}{e^{cx} - 1 - cx}$$

which is always ≥ 1. So however small the wealth after bankruptcy, it is always worth stopping short of $a = 0$, when $\mu > 0$. This is not true always if $\mu < 0$, though it is sometimes.

9) A useful bit of asymptotics?

$$\lim_{\mu \to 0} (1) = \left\{ \frac{\sigma^2(x_0 - a) - \sigma^2\varepsilon}{(2b - x_0 - a)(x_0 - a)} + k \right\} \rho^{-1}$$

10) In the context of the story we're trying to tell, the condition $\varepsilon < x_0 - a$ makes no sense; we suffer some losses on default, & we have to make those up as well as the drop $x_0 - a$ so as to be ready to restart the firm with wealth x_0, so this won't happen here!

A better tale probably would be that on default you pay the losses, and go back into a randomly-chosen project. In this case then, we have to solve

$$\tfrac{1}{2}\sigma^2 f_\lambda'' + \mu f_\lambda' - \lambda f_\lambda + k = 0, \qquad f(a) = -\varepsilon + q \equiv K, \quad f'(b) = 1,$$

where q is value of a randomly selected project, and $\varepsilon = (x_0 - a) + \theta a$ if there are proportional losses θa on default. This preserves some dependence on the discounting, and will result in some implicit equation for q. The ODE is solved by

$$f_\lambda(x) = \frac{k}{\lambda} + \beta e^{-\alpha x} \frac{\left(Ke^{\beta b} - e^{\beta b} k/\lambda - \beta^{-1} e^{\beta a}\right)}{\alpha e^{-\alpha b + \beta a} + \beta e^{-\alpha a + \beta b}} + \alpha e^{\beta x} \frac{\left(Ke^{-\alpha b} - e^{-\alpha b} k/\lambda + \alpha^{-1} e^{-\alpha a}\right)}{\alpha e^{-\alpha b + \beta a} + \beta e^{-\alpha a + \beta b}}$$

If we try to optimize this over b, we end up (Maple) with the equation

$$0 = (\alpha + \beta)\left\{k - \lambda K\right\} - \tfrac{1}{2}\sigma^2 \left\{\beta^2 e^{\alpha(b-a)} - \alpha^2 e^{\beta(a-b)}\right\} \qquad (1)$$

which clearly can have at most one root.

If we try to optimize over a, we get the equation

$$0 = k - \lambda q + \lambda \xi + \frac{\sigma^2(\alpha+\beta)}{2(e^{\beta t} - e^{-\alpha t})} - \frac{\sigma^2}{2}\left\{\alpha\beta(b-t) + \beta + \frac{\alpha+\beta}{e^{\alpha t + \beta t} - 1}\right\}$$

where $t \equiv b - a$

Looks like the best we can hope for is something numerical.

Stop-loss calculation (16/4/08)

(i) This question arose at CCP, so let's work through as far as we can the story in Phil Dybvig's million dollar paper. Assume $r=0$, $\mu>0$, $dS=(\sigma dW+\mu dt)S$ and so

$$S_t = \exp(\sigma W_t + (\mu-\tfrac{1}{2}\sigma^2)t) \equiv \exp(\sigma X_t)$$

so initial wealth is 1. Thus X is a BM with drift $c \equiv (\mu-\tfrac{1}{2}\sigma^2)/\sigma$.

The simple stop-loss rule says "hold S til T, or until it first hits $e^{\sigma b}<1$". This gives a terminal wealth with a distribution which we shall say more on shortly, but that distribution of terminal wealth could be made most cheaply if it were of the form $w_T = \varphi_0(\zeta_T)$ for some decreasing φ_0. Equivalently here, we get $w_T = \varphi(X_T)$ for some increasing φ.

(ii) Let $\tau \equiv \inf\{t: X_t = b\}$ be the time we stop. Then the usual reflection + Girsanov story gives us

$$P[X_T\in dy,\ \tau>T]/dy = e^{cy-\frac{1}{2}c^2T}\left[p_T(0,y) - p_T(2b,y)\right]$$

and hence

$$P[X_T>y,\ \tau>T] = \int_y^\infty e^{cz-\frac{1}{2}c^2T}\{p_T(0,z)-p_T(2b,z)\}\,dz$$

$$= \bar\Phi\left(\frac{y-cT}{\sqrt{T}}\right) - e^{2cb}\,\bar\Phi\left(\frac{y-2b-cT}{\sqrt{T}}\right).$$

Thus for $z>e^{\sigma b}$,

$$\boxed{P[w_T>z] = \bar\Phi\left(\frac{\sigma^{-1}\log z - cT}{\sqrt{T}}\right) - e^{2cb}\,\bar\Phi\left(\frac{\sigma^{-1}\log z - 2b - cT}{\sqrt{T}}\right)}$$

(iii) We also have $P(X_T>x) = \bar\Phi\left(\frac{x-cT}{\sqrt{T}}\right)$ so this enables us to find the φ s.t. $w_T \overset{\mathcal{D}}{=} \varphi(X_T)$. We then solve numerically to find

$$\psi(t,x) = E\left[\varphi(X_T)\zeta_T \mid X_t = x\right]$$

which will deliver the wealth at earlier times. Then easy to get the deltas.

State-price density is

$$\zeta_t = e^{-\mu W_t/\sigma - \frac{1}{2}\mu^2/\sigma^2}$$

$$= \exp\left[-\frac{\mu}{\sigma^2}(\sigma X_t - (\mu-\tfrac{1}{2}\sigma^2)t) - \frac{1}{2}\frac{\mu^2}{\sigma^2}t\right]$$

$$= \exp\left[-\frac{\mu}{\sigma}X_t + \frac{1}{2}\left(\frac{\mu^2}{\sigma^2}-\mu\right)t\right]$$

[Numerical values come out small, $\sim 1\%$ ...]

Bayesian agents with log utilities (16/4/08)

1) Suppose that we have (for now) a single productive asset generating dividend δ_t according to

$$d\delta_t = \sigma \delta_t \, dX_t$$

where under $\mathbb{P}_0$ the process X is a standard Brownian motion. Let's suppose that agent i believes that the law is really $\mathbb{P}_{a_i}$, where

$$\Lambda^i_t \equiv \frac{d\mathbb{P}_{a_i}}{d\mathbb{P}_0}\bigg|_{\mathcal{F}_t} = \exp\left[a_i X_t - \tfrac{1}{2} a_i^2 t \right]$$

so agent i thinks the drift in X is equal to a_i, not 0. The equilibrium/market clearing story as before is that

$$e^{-\rho_i t} \, U_i'(c^i_t) \, \Lambda^i_t = \nu_i \zeta_t \qquad , \quad \sum c^i_t = \delta_t$$

for constants ν_i. If we assume all agents are log investors, with possibly differing ρ_i, we shall have

$$\boxed{\sum_i e^{-\rho_i t} \Lambda^i_t / \nu_i = \delta_t \zeta_t \, .}$$

One very important consequence of this is that

$$\zeta_t S_t = E_t\left[\int_t^\infty \zeta_u \delta_u \, du \right] = \sum_i e^{-\rho_i t} \Lambda^i_t / \nu_i \rho_i$$

and hence

$$\boxed{S_t = \delta_t \, \frac{\sum_i e^{-\rho_i t} \Lambda^i_t / \nu_i \rho_i}{\sum_i e^{-\rho_i t} \Lambda^i_t / \nu_i}} \, .$$

This is simple enough to work with!

2) The model has various parameters; the number N of agents, volatility σ, a_i, ρ_i, ν_i, together with the actual drift α in X, and the irrelevant initial value δ_0. Stacking these into a vector θ, we shall have

$$S_t = f(t, X_t; \theta)$$

$$\zeta_t = g(t, X_t; \theta)$$

for certain quite explicit functions f, g. Now we could compare theoretical averages over some interval with historical values, as Kurz does.

— better to calculate first two moments? $\longrightarrow$

Notice for equity

$$S_t = N_2(t, X_t)/D_1(t, X_t) = \frac{\sum_i \exp\left(-\rho_i t + (a_i + \sigma)X_t - \frac{1}{2}(a_i^2 + \sigma^2)t\right)/\nu_i \rho_i}{\sum_i \exp\left(-\rho_i t + a_i X_t - \frac{1}{2}a_i^2 t\right)/\nu_i}$$

3) This needs some care. To calculate average values, we are computing

$$E\ \frac{1}{T}\int_0^T \varphi(t, X_t)\,dt = \int_0^T \frac{dt}{T}\int \frac{e^{-x^2/2}}{\sqrt{2\pi}}\varphi(t, \alpha t + x\sqrt{t})\,dx$$

$$= \int_0^1 ds \int_0^1 dy\ \varphi(sT,\ \alpha sT + \sqrt{sT}\, q(y))$$

where $q = \Phi^{-1}$ is the Gaussian quantile function. At a first attempt, I propose to put points at the centres of subsquares of the unit square, and average the values of $\varphi(sT, \alpha sT + \sqrt{sT}\, q(y))$ over these points.

The mean value of price/dividend ratio is got by using

$$\varphi_1(t,x) = \frac{\sum e^{\gamma_i t} \exp(a_i x - \frac{1}{2}a_i^2 t)/\gamma_i \rho_i}{\sum e^{\gamma_i t} \exp(a_i x - \frac{1}{2}a_i^2 t)/\gamma_i} \equiv \frac{N_1(t,x)}{D_1(t,x)}$$

For the SD of this, we do Itô on $\varphi(t, X_t)$, to get

$$d\varphi_1 = \frac{D_1 N_1' - D_1' N_1}{D_1^2}\,dX + \frac{D_1 \dot{N}_1 - \dot{D}_1 N_1}{D_1^2}\,dt + \frac{1}{2}\,\frac{D_1^2 N_1'' + 2N_1 D_1'^2 - 2D_1 N_1' D_1' - N_1 D_1 D_1''}{D_1^3}$$

so the next statistic will involve $(D_1 N_1' - D_1' N_1)/D_1^2$ so it looks like it will be helpful to calculate $\xi_i = \exp(a_i x - \frac{1}{2}a_i^2 t - \rho_i t)$ at each grid point, and probably rescale all these to mass ≤ 1.

Crunching out the return on equity, we get

$$-\frac{\dot{D}_1}{D_1} + \frac{N_1''}{2N_1} - \frac{D_1''}{2D_1} + \frac{\dot{N}_1}{N_1} - \frac{\sigma D_1'}{D_1} + \frac{D_1'^2}{D_1^2} + \sigma\frac{N_1'}{N_1} - \frac{N_1' D_1'}{N_1 D_1}$$

$$+ \alpha\left(\sigma + \frac{N_1'}{N_1} - \frac{D_1'}{D_1}\right).$$

The riskless rate is found to be

$$-\frac{1}{2}\left(\frac{D_1''}{D_1} - \left(\frac{D_1'}{D_1}\right)^2\right) - \frac{\dot{D}_1}{D_1} - \alpha\frac{D_1'}{D_1} + \sigma\alpha - \frac{1}{2}\sigma^2 = r(t,X) + \frac{1}{2}\left(\frac{D_1'}{D_1} - \sigma\right)^2$$

and to find its instantaneous vol we get

$$\frac{\partial r}{\partial X} = -\frac{D_1 \dot{D}_1' + \alpha D_1 D_1'' - D_1'(\dot{D}_1 + \alpha D_1')}{D_1^2}$$

Equity premium is just growth rate of stock - riskless rate. Instantaneous correlation between stock + dividend is of course 1.

Bayesian agents with log utilities again (29/4/08)

(1) Let's consider this again in the context of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}_0)$ being a standard Wiener prob'y space, and suppose we are given

$$\left.\begin{aligned} d\delta_t &= \delta_t \,(a_t\,dX_t + b_t\,dt) \\ d\Lambda^j_t &= \Lambda^j_t \;\alpha^j_t\,dX_t \end{aligned}\right\}$$

where the canonical process X is a $\mathbb{P}_0$ - BM. Using the fundamental relationship

$$\Lambda^j_t e^{-\rho_j t} U_j'(c^j_t) = \nu_j \zeta_t$$

we get

$$\zeta_t \delta_t = \sum_j e^{-\rho_j t} \Lambda^j_t / \nu_j$$

$$c^j_t = e^{-\rho_j t} \Lambda^j_t / \nu_j \zeta_t$$

and hence

$$w^j_t = \rho_j^{-1} c^j_t ,$$

$$S_t = \sum_j w^j_t = \Big\{ \sum_j e^{-\rho_j t} \Lambda^j_t / \nu_j \rho_j \Big\} / \zeta_t .$$

2) What is agent j's portfolio process? We have $\zeta_t = \sum e^{-\rho_j t} \Lambda^j_t / \nu_j \delta_t$, so

$$d\zeta_t = \sum_j \frac{e^{-\rho_j t}\Lambda^j_t}{\nu_j \delta_t} \Big[(\alpha^j_t - a_t)\,dX_t + (a_t^2 - \alpha^j_t a_t - b_t - \rho_j)\,dt \Big]$$

which allows us to express $d\zeta_t = \zeta_t(-\kappa_t\,dX_t - r_t\,dt)$ in terms of fundamentals.
Now

$$dw^j_t = w^j_t \big\{ -\rho_j\,dt + (\alpha^j_t + \kappa_t)\,dW_t + (r_t + \kappa_t^2 + \alpha^j_t \kappa_t)\,dt \big\}$$

and $S_t = \sum_j w^j_t$, so we have

$$dw^j_t = (r_t w^j_t - c^j_t)\,dt + \pi^j_t (dS_t - r_t S_t\,dt),$$

where

$$\boxed{\pi^j_t = \frac{w^j_t(\alpha^j_t + \kappa_t)}{\sum_i w^i_t(\alpha^i_t + \kappa_t)}}$$

Notice that if all agents agree, then $\alpha^j_t = \alpha_t$, $W^j_t = \frac{e^{-\rho_j t}}{\nu_j \rho_j} \Lambda_t / \zeta_t$

and
$$\pi^j_t = \frac{e^{-\rho_j t}/\nu_j \rho_j}{\sum_i e^{-\rho_i t}/\nu_i \rho_i}$$

which is C^∞, so no volatility! This in some sense explains volume of trade.

Building block calculation again (30/4/08)

(i) On p3 we were solving

$$\begin{cases} \frac{1}{2}\sigma^2 f_\lambda'' + \mu f_\lambda' - \lambda f_\lambda + k = 0 \quad ; \quad f'(b) = 1 \\ f(a) = q - x_0 + (1-\theta)a \end{cases}$$

in the interval $[a,b] \ni x_0$ fixed. If we have any solution, then if we hold b fixed and add a multiple of

$$g_0(x) \equiv \frac{e^{\beta(x-b)}}{\beta} + \frac{e^{-\alpha(x-b)}}{\alpha}$$

to f, we keep $f'(b)=1$ and raise f everywhere: eventually, we just have contact at some value of a, and at that point, $f(a) = q - x_0 + (1-\theta)a$, and $f'(a) = 1-\theta$.

(ii) What's the second derivative of f at the point a where smooth contact occurs?

$$\tfrac{1}{2}\sigma^2 f''(a) = -\mu(1-\theta) + \lambda(q - x_0 + (1-\theta)a) - k$$

$$> 0 \quad \text{iff}$$

$$\boxed{a > \underline{a} \equiv \frac{k + \mu(1-\theta) - \lambda(q-x_0)}{\lambda(1-\theta)}}$$

If we had $f''(a) < 0$, then the slope f' must decrease over $(1-\theta)$. If we represent $f'(x) = Ae^{-\alpha x} + Be^{\beta x}$, a little thought shows this can only happen if $B>0$, $A>0$, or if $A > 0 > B$. In the second case, f is concave, so never rises above $q - x_0 + (1-a)x$; in the second, it does, but either way, perturbing by adding a multiple of $g_0(\cdot)$ will improve. So we only need to consider optimising over $a > \underline{a}$.

(iii) We have a solution $f(x;a)$ indexed by the point $a > \underline{a}$ of smooth contact to $q - x_0 + (1-\theta)x$. The coefficient of $e^{\beta x}$ in this is $\propto$ $\lambda(q-x_0) + \lambda a(1-\theta) - k + \frac{1}{2}\sigma^2\beta(1-\theta) > (\mu + \frac{1}{2}\sigma^2\beta)(1-\theta)$ if $a > \underline{a}$, so we conclude that $f'(x;a)$ will for large enough x exceed 1.

Using Maple, I also find that

$$\frac{\partial}{\partial a} f(x_0;a) < 0 \iff f''(a;a) > 0$$

so we have to move a to the left to improve $f(x_0;a)$; what eventually stops this is that the place where $f' = 1$ comes down to x_0. [Again Maple: $\frac{\partial}{\partial a} f'(x_0;a) < 0$ if $a > \underline{a}$] So the story we get is that $\boxed{b = x_0}$

Equity dilution (6/5/08)

(i) This story arises in discussion with Jean-Paul Décamps + Stéphane Villeneuve. Suppose N_t denotes number of issued shares at time t, and that the dynamics of the firm's cash account is

$$dx_t = \sigma dW_t + \mu dt - d\Delta_t + p_t dN_t$$

where Δ_t is the dividend process (to be chosen by the firm); aim is to maximise equity value. Suppose $S(x,n)$ is value of one share when $x_t = x$, $N_t = n$. Then $Y_t = e^{-\rho t} N_t S(x_t, N_t) + \int_0^t e^{-\rho s} d\Delta_s$ is a supermartingale, and a martingale under optimal control. Assume N increases only when $x=0$, and that the price p_t received is $(1-\varepsilon)$ times the current market price of the stock. By Itô,

$$dY_t \doteq e^{-\rho t} N_t \left(-\rho S + \mu S_x' + \tfrac{1}{2}\sigma^2 S_{xx}'' \right) dt + \left(1 - N S_x \right) e^{-\rho t} d\Delta_t$$

$$+ e^{-\rho t} \left\{ S + p N S_x + N S_N \right\} dN$$

From this we learn that

$$\begin{cases} n S_x(x,n) \geq 1, & \text{equality where dividends are paid} \\ \mathcal{L} S \equiv \left(\tfrac{1}{2}\sigma^2 D^2 + \mu D - \rho\right) S = 0 \\ S(0,n) + (1-\varepsilon) S(0,n)\, n S_x(0,n) + n S_n(0,n) = 0 & (*) \end{cases}$$

(ii) Suppose that $\psi_\pm(\cdot)$ are the in/decreasing positive solutions of $\mathcal{L}\psi = 0$, with $\psi(0) = 1$ (in fact, for this example, $\psi_+(x) = e^{\beta x}$, $\psi_-(x) = e^{-\alpha x}$, where $\beta > 0 > -\alpha$ solve the quadratic $\tfrac{1}{2}\sigma^2 z^2 + \mu z - \rho = 0$). The solution must be of the form

$$S(x,n) = A(n)\,\psi_-(x) + B(n)\,\psi_+(x) \qquad 0 \leq x \leq x^*(n)$$

where we see (by considering adding multiples of $\psi_+(x) - \psi_-(x)$ to $S(\cdot,n)$) that at $x^*(n)$ there must be $S_x = 1/n$, $S_{xx} = 0$. Solving for S in terms of $x^*(n)$ gives us

$$A(n) = \frac{-\psi_+''}{n(\psi_-''\psi_+' - \psi_-'\psi_+'')}, \qquad B(n) = \frac{\psi_-''}{n(\psi_-''\psi_+' - \psi_-'\psi_+'')}$$

where derivatives are evaluated at $x^*(n)$. Combined with $(*)$ we get ODE for $x^*(\cdot)$

(iii) Let's take the solution a bit further. Set $v(x,n) \equiv n\,S(x,n)$ so that we have

$$\begin{cases} v_x \geq 1 \\ \mathcal{L} v = 0 \\ (1-\varepsilon)\dfrac{v(0,n)}{n}\, v_x(0,n) + v_n(0,n) = 0 \end{cases}$$

and also for each n, $\inf_x \{ v_x(x,n) \} = 1$. Let's observe that we have

$$v(x,n) = v(0,n)\,\psi_-(x) + k(n)\,(\psi_+(x) - \psi_-(x))$$

for some $k(n)$, and to get $\inf v_x = 0$ we look at minimising $(v(0,n) - k(n))\,\psi_-'(x) + k(n)\,\psi_+'(x)$. For any sensible diffusion, we shall have $\psi_+'(x) \to \infty$ $(x \to \infty)$, so we'll have to insist that $k(n) > 0$, and that at the minimising x we get

$$(v(0,n) - k(n))\,\psi_-''(x) = -k(n)\,\psi_+''(x)$$

$$\Rightarrow \quad q(x) \equiv \frac{\psi_+''(x)}{\psi_-''(x)} = \frac{k(n) - v(0,n)}{k(n)}.$$

Let's take the minimising $x \equiv \xi$ as the independent variable, & try to find $n^*(\xi)$. We have

$$v(0,n) = k(n)\,(1 - q(\xi))$$

so only viable ξ are those for which $q(\xi) < 1$. For such ξ, we now have the minimised derivative condition:

$$1 = k\,\psi_+'(\xi) + (v_0 - k)\,\psi_-'(\xi)$$

$$= \frac{v(0,n)}{1 - q(\xi)} \left\{ \psi_+'(\xi) - q(\xi)\,\psi_-'(\xi) \right\}$$

$$\Rightarrow \quad v(0, n^*(\xi)) = \frac{1 - q(\xi)}{\psi_+'(\xi) - q(\xi)\,\psi_-'(\xi)}$$

This may be easier to work with, especially if we were to take the more realistic dynamics

$$dx = \sigma\, dW + \mu\, dt + r x\, dt - d\Delta + p\, dN, \qquad \rho = r$$

for which the fundamental solutions $\psi_\pm$ are not so simple.

Equilibria with non-negative cash constraint (6/5/08)

(i) Suppose that asset i delivers a dividend stream δ^i_t ($i=1,\dots,n$) measured in consumption goods. There is a price index process p_t which is the time-t price of one unit of consumption good in cash. The cash price S^i_t of asset i evolves as

$$dS^i_t = S^i_t\left\{\sigma_{ij}(t)\,dW^j_t + \mu^i_t\,dt\right\}. \qquad \text{(ex-div)}$$

Thus if an agent holds portfolio θ_t (θ^i_t is the number of units of asset i) then the wealth (denominated in cash) evolves as

$$dw_t = \theta_t\cdot\left(dS_t + p_t\delta_t\,dt\right) - c_t\,dt$$

where c_t is the cash value of consumption (so get c_t/p_t in terms of goods). Let's assume the agent j has objective

$$E^j\left[\int_0^\infty e^{-\rho_j t}\log(c_t/p_t)\,dt\right] = E^0\left[\int_0^\infty e^{-\rho_j t}\Lambda^j_t\,\log(c_t/p_t)\,dt\right]$$

and must work subject to the constraint that cash holdings stay non-negative:

$$\boxed{w_t \geq \theta_t\cdot S_t}$$

(ii) If we introduce a Lagrangian* ζ_t, $d\zeta_t = \zeta_t(a_t\,dW_t + b_t\,dt)$ to account for the wealth dynamics in the usual way, then the Lagrangian form is

$$\max E\int_0^\infty\Big\{\Lambda_t U(t, c_t/p_t) + \zeta_t\big(\theta_t\cdot(S_t\mu_t + p_t\delta_t) - c_t\big) + w_t\zeta_t b_t + \zeta_t\theta_t\cdot S_t\sigma_t a_t$$
$$+\zeta_t\eta_t\left(w_t - \theta_t\cdot S_t - z_t\right)\Big\}\,dt + \zeta_0 w_0$$

where η_t is a multiplier for cash constraint, $z_t \geq 0$

$$= \max E\int_0^\infty\Big\{\Lambda_t\tilde{U}(t, \zeta_t p_t/\Lambda_t) + \zeta_t\theta_t\cdot S_t\left(\tilde{\mu}_t + \sigma_t a_t - \eta_t 1\right) \qquad \left[\tilde{\mu}_t \equiv \mu_t + p_t\delta_t/S_t\right]$$
$$+ w_t\zeta_t(\eta_t + b_t)\Big\}\,dt + \zeta_0 w_0$$

[$\eta_t \geq 0$ for dual feasibility, and $\eta_t + b_t \leq 0$, and
$\tilde{\mu}_t + \sigma_t a_t - \eta_t 1 = 0$ for dual-feasibility]

We have $\tilde{U}(t,y) = e^{-\rho t}\left[-1 - \rho t - \log y\right]$, so the dual problem is

---

* We should write ζ^j to indicate dependence on agent, but we'll drop that for this section

$(\zeta'_t \equiv \zeta_t/\zeta_0)$

$$\min E^0 \int_0^\infty \Lambda_t e^{\rho t}\Big[-1-\rho t - \log \rho_0 - \log(\zeta'_t/\Lambda_t) - \log \zeta_0\Big]\,dt \quad + w_0\zeta_0$$

$$\text{subject to } \eta_t \geq 0, \qquad \tilde{\mu}_t + \sigma_t a_t - \eta_t 1 = 0, \quad \eta_t + b_t \leq 0$$

$$= \min\Big\{ -\frac{1}{\rho}\log\zeta_0 + w_0\zeta_0 - \frac{2}{\rho} - E\int_0^\infty e^{\rho t}\Lambda_t \log(\rho_t/\Lambda_t)\,dt - E\int_0^\infty \Lambda_t e^{-\rho t}\log\zeta'_t\,dt\Big]$$

The optimisation over ζ_0 gives $\zeta_0 = 1/\rho w_0$, but otherwise all the action is in the last term, where we have

$$\log \zeta_t = \int_0^t a_s\cdot dW_s - \int_0^t (\eta_s + \tfrac12|a_s|^2)\,ds \quad + \log\zeta_0 .$$

Let's suppose that $d\Lambda_t = \Lambda_t\,\alpha_t\cdot dW_t$. Then

$$E\Big[\int_0^\infty \Lambda_t e^{-\rho t}\log\zeta'_t\,dt\Big]$$

$$= E\int_0^\infty \Lambda_t e^{-\rho t}\Big\{\int_0^t (\alpha_s\cdot a_s - \eta_s - \tfrac12|a_s|^2)\,ds\Big\}\,dt$$

$$= E\Big[\int_0^\infty (\alpha_s\cdot a_s - \eta_s - \tfrac12|a_s|^2)\,\rho^{-1}e^{-\rho s}\Lambda_s\,ds\Big]$$

Maximising over $\eta \geq 0$ leads easily to

$$\boxed{\eta_t = \big(\alpha_t\cdot\sigma_t^{-1}1 - 1 + 1\cdot(\sigma_t\sigma_t^T)^{-1}\tilde{\mu}_t\big)^+ / 1\cdot(\sigma\sigma^T)^{-1}1}$$

(iii) (9/5/08). Let's now understand better the case of a single risky asset, and multiple agents. Specialising the above, the agents all agree on the stock and the price level, so all have the same σ and $\tilde{\mu}$, but don't agree on α, η, so agent j takes

$$\boxed{\begin{aligned}\eta^j_t &= (\sigma_t\alpha^j_t - \sigma_t^2 + \tilde{\mu}_t)^+\\ a^j_t &= \max\Big\{-\frac{\tilde{\mu}_t}{\sigma_t},\ \alpha^j_t - \sigma_t\Big\}\end{aligned}} \quad = (\eta^j_t - \tilde{\mu}_t)/\sigma_t,$$

and uses his own SPD $d\zeta^j_t = \zeta^j_t\{a^j_t\,dW_t - \eta^j_t\,dt\}$. His consumption then is

$$c^j_t = e^{-\rho_j t}\Lambda_t / \zeta^j_t$$

We can do some calculations;

$$d(\zeta^j_t w^j_t) \doteq \zeta^j_t\Big[-c^j_t - \eta^j_t(w^j_t - \theta^j_t S_t)\Big]dt \quad = -c^j_t\zeta^j_t\,dt,$$

using complementary slackness. Therefore

$$\zeta_t^j w_t^j + \int_0^t \zeta_u^j c_u^j \, du \quad \text{is a martingale}$$

Thus* $\zeta_t^j w_t^j = E_t\left[\int_t^\infty \zeta_u^j c_u^j \, du\right] = E_t\left[\int_t^\infty e^{-\rho_j u} \Lambda_u^j \, du\right] = \rho_j^{-1} e^{-\rho_j t} \Lambda_t^j$, whence

$$\boxed{\rho_j w_t^j = c_t^j = e^{-\rho_j t} \Lambda_t^j / \zeta_t^j}$$

From this we quickly obtain

$$d w_t^j = w_t^j \left[(\alpha_t^j - a_t^j) \, dW_t - \rho_j \, dt + \left(\eta_t^j + (a_t^j)^2 - a_t^j \alpha_t^j\right) dt \right]$$

and so

$$\boxed{\theta_t^j = w_t^j \left(\frac{\alpha_t^j - a_t^j}{\sigma_t \, S_t} \right)}$$

The drifts in the wealth dynamics $dw^j = \theta^j (dS + p\delta \, dt) - c^j dt$ also match up.
Notice that

$$\frac{\alpha_t^j - a_t^j}{\sigma_t} = \min\left\{ 1, \frac{\sigma_t \alpha_t^j + \tilde{\mu}_t}{\sigma_t^2} \right\}$$

so that

$$w_t^j - \theta_t^j S_t = w_t^j \left\{ 1 - \frac{\alpha_t^j - a_t^j}{\sigma_t} \right\}$$

and the cash holding of agent j becomes

$$w_t^j \left(\frac{\sigma_t^2 - \alpha_t^j \sigma_t - \tilde{\mu}_t}{\sigma_t^2} \right)^+ \geq 0$$

This allows us to clear the cash market, given the total money supply.

(iv) If we let ζ_t^0 be the 'positive-cash' SPD, defined by $d\zeta^0 = \zeta^0 \left(\frac{-\tilde{\mu}}{\sigma}\right) dW$ then a little calculation gives us

$$\zeta_t^j S_t + \int_0^t \zeta_u^j p_u \delta_u \, du \quad \text{is a martingale for all } j = 0, 1, \ldots, J.$$

---

* If we didn't have $\lim_{t\to\infty} \zeta_t^j w_t^j = 0$, then agent j could get the consumption for $< w_0^j$, and thus contradicts optimality

Building block calculation again (12/5/08)

(i) I think I got things a bit scrambled before. The dynamics of the value of the firm will be

$$dV_t = \sigma dW_t + \mu dt - \delta_t dt$$

where the aim will be to max

$$f(x) = E^{x_0}\left[\int_0^{\tau} e^{-\lambda s}\, \delta_s\, ds + e^{-\lambda\tau}\left(q - (x_0 - X_\tau) - \theta X_\tau \right) \right]$$

So we get $\sup_{\delta} \left[\frac{1}{2}\sigma^2 f'' + (\mu-\delta) f' - \lambda f + \delta \right] = 0 \Rightarrow f' \geq 1$ in the continuation region, and is equal to 1 where dividends get paid out. If we have found optimal $[a,b]$, where default at a, pay dividends at b, we have

$$f(x) = A e^{-\alpha(x-a)} + B e^{\beta(x-a)}$$

with $f'(x) \geq 1$ in $[a,b]$, $f'(b) = 1$, $x_0 \in [a,b]$. Now if we added a multiple of $g(x) = e^{\beta(x-a)} - e^{-\alpha(x-a)}$ to f, we would improve f everywhere, and we would shift the point where $f' = 1$ to the right. The only way that this improvement ever can stop is if $f'(b) = 1$, $f''(b) = 0$, so we must have

$$f(x) = \frac{\alpha^2 e^{\beta(x-b)} - \beta^2 e^{-\alpha(x-b)}}{\alpha\beta(\alpha+\beta)}.$$

But note that $f(x_0)$ decreases with b, for x_0 fixed. Thus optimality requires $\boxed{b = x_0}$ and

$$f(x) = \left\{ \alpha^2 e^{\beta(x-x_0)} - \beta^2 e^{-\alpha(x-x_0)} \right\} / \alpha\beta(\alpha+\beta).$$

(ii) Does this work at $x = a$? We certainly have f is concave in $[0, x_0]$ so we require

$$\boxed{\begin{aligned} f(x_0) &= (\alpha-\beta)/\alpha\beta > q - \theta x_0 \\ f(0) &= (\alpha^2 e^{-\beta x_0} - \beta^2 e^{+\alpha x_0})/\alpha\beta(\alpha+\beta) \leqslant q - x_0 \end{aligned}}$$

The first implies that $\mu > 0$. We have some base function

$$g(x) = \begin{cases} \dfrac{(\alpha^2 e^{\beta x} - \beta^2 e^{-\alpha x})}{\alpha\beta(\alpha+\beta)} & x \leq 0 \\[2ex] \dfrac{\alpha-\beta}{\alpha\beta} + x & x \geq 0 \end{cases}$$

which is plainly monotone increasing, slope ≥ 1. We then have

$$f(x) = g(x-b)$$

for some b. The requirement that $f(0) \leq q - x_0$ is the condition $g(-b) \leq q - x_0$, so $\alpha^2 e^{-\beta b} - \beta^2 e^{\alpha b} \leq (q-x_0)(\alpha+\beta)\alpha\beta$. This says that we must have $b \geq b_0$ for some b_0. Given that, the way we maximise $f(x)$ is to take b as small as possible, that is, $b = b_0$.

So we have to have

$$\begin{cases} f(0) = g(-b) \leq q - x_0 = g(-b_0) \\ f(x_0) = g(0) = (\alpha-\beta)/\alpha\beta = \mu/2 \geq q - \theta x_0 \end{cases}$$

The second of these can be interpreted as a requirement that μ must be big enough to merit funding, and in this way the loss parameter θ gets into the story. Otherwise, the influence of θ is problematic; if we try to maximise $f(x)$ over choice of b, clearly the best choice is $b = b_0$, and in that case you just run until $w_t = 0$, so the value of θ is irrelevant. If it turns out that $x_0 > b_0$, then what you do is immediately pay out dividends to lower the firm value to b_0. This seems odd ... we might instead insist that

$$b \geq \max\{b_0, x_0\}$$

which would not be unrealistic - any lender would want to ensure that the borrowers were putting their own money at risk. Once we impose this, there will be defaults when $x_0 > b_0$, and the loss rate θ will matter.

[I tried log-Brownian wealth dynamics, but the snag with this was how to model the interest repayments on borrowing; then I tried $dw_t = w_t(\sigma dW_t + \mu dt) - \varepsilon dt - \delta_t dt$, but the snag this time was the Kummer functions ...]

Equilibria with non-negative cash again (16/5/08)

(i) We saw earlier that for a single risky asset

$$\zeta^j_t w^j_t = \zeta^j_t c^j_t/\rho_j = e^{\gamma_j t}\Lambda^j_t/\rho_j$$

and we also obtained

$$\begin{cases} \zeta^0_t w^j_t + \int_0^t \zeta^0_u c^j_u \, du & \text{is a martingale} \\ \zeta^0_t S_t + \int_0^t \zeta^0_u p_u \delta_u \, du & \text{———} \end{cases}$$

Summing on j, $\sum_j w^j_t = S_t + m_t$, where m_t is the money supply at time t, which implies $\zeta^0_t m_t$ is a martingale ... <u>but how could this be if $m_t = e^t$, say??</u>
We can't allow completely arbitrary m_t, it seems ... but another way of saying this is that <u>we can't insist on a zero nominal interest rate</u>

(ii) Let's see how things look if we allow an interest rate r_t. Agent j now solves

$$\max E\int_0^\infty \Lambda^j_t U_j(t, \tilde{c}^j_t/\tilde{p}_t)\, dt$$

subject to wealth dynamics

$$d\tilde{w}^j_t = \theta^j_t(d\tilde{S}_t + \tilde{p}_t\delta_t\, dt) - \tilde{c}^j_t\, dt$$

$$= \theta^j_t \tilde{S}_t(\sigma dW_t + \hat{\mu}_t dt) - \tilde{c}^j_t\, dt,$$

where $\sim$ denotes discounted quantities $\tilde{c}^j_t \equiv c^j_t e^{-\int_0^t r_s ds}$, $\hat{\mu}_t \equiv \mu_t - r_t + \tilde{p}_t\delta_t/\tilde{S}_t$
and the non-negative cash constraint

$$\tilde{w}^j_t \geq \theta^j_t \tilde{S}_t .$$

The optimisation proceeds as before: we get

$$\boxed{\tilde{c}^j_t = \Lambda^j_t e^{\gamma_j t}/\zeta^j_t}$$

where $$\boxed{d\zeta^j_t = \zeta^j_t\left[\frac{\eta^j - \hat{\mu}}{\sigma}\, dW - \eta^j\, dt\right]}$$

and

$$\boxed{\begin{aligned} \eta^j_t &= \sigma_t(\alpha^j_t - \beta_t)^+ \\ \beta_t &\equiv \sigma_t - \hat{\mu}_t/\sigma_t \end{aligned}}$$

We shall have

$$\zeta^j_t \, \tilde{w}^j_t = E_t\left[\int_t^\infty \zeta^j_u \, \tilde{c}^j_u \, du\right] = \rho_j^{-1} \zeta^j_t \, \tilde{c}^j_t$$

so that $$\boxed{c^j_t = \rho_j \, w^j_t}$$

(iii). Straightforward calculations lead to the information:

(1) $$\zeta^j_t \, \tilde{w}^j_t + \int_0^t \zeta^j_u \, \tilde{c}^j_u \, du \quad \text{is a martingale}$$

(2) $$\zeta^0_t \, \tilde{w}^j_t + \int_0^t \zeta^0_u \, \tilde{c}^j_u \, du \quad \text{is a martingale}$$

(3) $$d\left(\frac{\Lambda^j_t \zeta^0_t}{\zeta^j_t}\right) = \frac{\Lambda^j_t \zeta^0_t}{\zeta^j_t}\left(\alpha^j_t - \frac{\eta_t}{\sigma_t}\right) dW_t = \frac{\Lambda^j_t \zeta^0_t}{\zeta^j_t}\left(\alpha^j_t \wedge \beta_t\right) dW_t$$

(4) $$\zeta^j_t \, \tilde{S}_t + \int_0^t \zeta^j_u \, \tilde{p}_u \delta_u \, du \quad \text{is a martingale} \quad \forall j = 0, 1, \dots, J$$

(5) $$\theta^j_t \, \tilde{S}_t = \tilde{w}^j_t \min\left\{1, \frac{\hat{\mu}_t + \sigma_t \alpha^j_t}{\sigma_t^2}\right\}$$

$$= \tilde{w}^j_t \left[1 + \min\left\{0, \frac{\alpha^j_t - \beta_t}{\sigma_t}\right\}\right] = \tilde{w}^j_t \left\{1 - \left(\frac{\beta_t - \alpha^j_t}{\sigma_t}\right)^+\right\}.$$

Summing (2) over j, and using m_t to denote time-t money supply, $\tilde{m}_t = m_t e^{-\int_0^t r_s ds}$, we see that

$$\zeta^0_t \left(\tilde{S}_t + \tilde{m}_t\right) + \int_0^t \zeta^0_u \, \tilde{p}_u \delta_u \, du \quad \text{is a martingale},$$

by market clearing, so in particular

$$\zeta^0_t \, \tilde{m}_t \quad \text{is a martingale}.$$

(iv) If we set $L^j_t = \Lambda^j_t \zeta^0_t / \zeta^j_t$, then we shall find

$$\tilde{c}^j_t = \frac{e^{-\rho_j t} L^j_t}{\zeta^0_t}$$

so this is like an unconstrained Bayesian agent's story, where agent j replaces his belief α^j on the drift with $\alpha^j \wedge \beta$.

Returning to the notationally simpler case $r \equiv 0$, we have $\zeta_t^0 m_t$ is a martingale. If m is to be continuous and FV, this implies m is constant.

After some calculations we can prove

$$d\left(\frac{w^j}{S}\right) = \frac{w^j}{S}\left[-(\beta-\alpha^j)^+ (dW-\beta dt) + \left(\frac{\rho\delta}{S} - \rho_j\right) dt \right]$$

Moreover,

$$\sum_j w_t^j = S_t + m_t = \sum_j \theta^j S_t + w_t^j \left(\frac{\beta_t - \alpha_t^j}{\sigma_t}\right)^+$$

(6) $$\Rightarrow \boxed{m_t = \sum_j w_t^j \left(\frac{\beta_t - \alpha_t^j}{\sigma_t}\right)^+}$$

(v) For this section, <u>assume</u> $m_t = m_0$ <u>is constant</u>. Summing (5) over j, after dividing by S_t gives

$$1 = \sum \frac{w_t^j}{S_t} - \frac{m_0}{S_t}$$

so differentiating tells us

$$0 = \sum_j \frac{w^j}{S}\left[-(\beta-\alpha^j)^+ (dW-\beta dt) + \left(\frac{\rho\delta}{S} - \rho_j\right) dt \right] + \frac{m_0}{S}\left\{ \sigma(dW-\beta dt) - \frac{\rho\delta}{S} dt \right\}$$

Comparing coefficients of $dW - \beta dt$ tells us (6); what remains is market clearing for consumption, so nothing new.

(vi) Notice the following consequence of (5): using $S_t + m_t = \sum w_t^j$, looking at the local martingale terms only, we get

$$dm = \sum dw^j - dS = \sum \theta^j (dS + \rho\delta dt) - dS + \cdots$$

$\Rightarrow$ <u>no local martingale term in m</u>. So m <u>will be constant</u> (assuming its continuous)

(vii) Note another <u>possibly useful fact</u>,

(7) $$\boxed{d(\zeta_t^0 S_t) = \zeta_t^0 S_t \left[\beta_t dW_t - (\rho_t \delta_t / S_t) dt \right]}$$

(viii) <u>How are we to reconcile all these relationships?</u> Suppose we give ourselves (β_t, σ_t), continuous semimartingales. Then we deduce L_t^j (using (3)), ζ^0 (from $d\zeta^0 = \zeta^0 \left(-\frac{\mu}{\sigma}\right) dW = \zeta^0 (\beta - \sigma) dW$), and hence we deduce

$\zeta^0_t w^j_t = e^{-\beta t} L^j_t$. Now we can define S via

(8) $$\boxed{S_t = \sum_j w^j_t - m}$$

and the price level p via

(9) $$\boxed{\sum c^j = \sum \rho^j w^j = p\delta.}$$

We can confirm that S defined by (8) actually satisfies the dynamics (7), as follows

$$\sum_j d(\zeta^0 w^j) = \sum_j \zeta^0 w^j \left[(\alpha^j \wedge \beta) dW - \rho_j \, dt \right]$$

$$= \sum_j \zeta^0 w^j \left(\beta - (\beta - \alpha^j)^+ \right) dW - \zeta^0 p \delta \, dt \qquad \text{(from (9))}$$

$$= \sum \zeta^0 w^j \left(\beta - \sigma + \sigma - (\beta - \alpha^j)^+ \right) dW - \zeta^0 p\delta \, dt$$

$$= (\beta - \sigma) \zeta^0 (S+m) dW + \sum_j \zeta^0 w^j \left(1 - \left(\tfrac{\beta - \alpha^j}{\sigma}\right)^+ \right) \sigma \, dW - \zeta^0 p\delta \, dt$$

$$= (\beta - \sigma) \zeta^0 (S+m) dW + S \zeta^0 \sigma \, dW - \zeta^0 p \delta \, dt \qquad \text{(from (5))}$$

$$= \left(-\tfrac{\tilde{\mu}}{\sigma} \right) \zeta^0 (S+m) dW + \sigma \zeta^0 S \, dW - \zeta^0 p\delta \, dt$$

$$= \zeta^0 S \beta \, dW + m \, d\zeta^0 - \zeta^0 p \delta \, dt$$

provided we have the consistency condition

(10) $$\boxed{\sum_j w^j \left(\frac{\beta - \alpha^j}{\sigma} \right)^+ = m.}$$

So the key thing is to choose (β, σ) so that (10) holds, and $\sum w^j \geq m$ always. We then deduce S from (8) and p from (9) – that is, the equilibrium stock price and price level. We then deduce the θ^j from (5), and we're sure to get $\sum \theta^j = 1$.

(ix) Here's another course through the obstacles. Start from β. Now we have

$$d(\zeta^0 w^j) = \zeta^0 w^j \{ (\alpha^j \wedge \beta) dW - \rho_j dt \}$$

so we may (assuming we're given w^j_0, and $\zeta^0_0 = 1$) then deduce $\zeta^0 w^j$. From this

$$\sum_j \rho_j \zeta^0 w^j = \zeta^0 \sum_j c^j = \zeta^0 p\delta$$

so we know $d(\zeta^0 S) = (\zeta^0 S) \beta \, dW - \zeta^0 p\delta \, dt$ which allows us to find $\zeta^0 S$.

(The value of S_0 will be determined by m_0). We then know

$$
\begin{cases}
(11) \quad m\zeta^0 = \sum \zeta^0 w^j - \zeta^0 S \\
(12) \quad m\sigma\zeta^0 = \sum \zeta^0 w^j (\beta - \alpha^j)^+
\end{cases}
$$

from which σ can be deduced, whence $\tilde\mu$ $(\beta = \sigma - \tilde\mu/\sigma)$, and also ζ^0 $(d\zeta^0 = \zeta^0(-\tilde\mu/\sigma)\,dW)$, and then S. We would then have to confirm that S solves $dS = S(\sigma dW + \mu dt)$, and that ζ^0 as it was calculated from (11), (12) satisfies $d\zeta^0 = \zeta^0(-\tilde\mu/\sigma)dW$.

But both of these consistency stories check out! Here's how:

$$
\begin{aligned}
d(m\zeta^0) &= \sum \zeta^0 w^j \{(\alpha^j \wedge \beta) dW - \rho_j\, dt\} - \zeta^0 S \beta dW + \zeta^0 \rho\delta\, dt \\
&= \sum \zeta^0 w^j (\beta - (\beta - \alpha^j)^+) dW - \zeta^0 S \beta dW \\
&= [\beta(\zeta^0 S + m\zeta^0) - m\sigma\zeta^0 - \zeta^0 S\beta] dW \\
&= m\zeta^0 (-\tilde\mu/\sigma) dW,
\end{aligned}
$$

and

$$
\begin{aligned}
dS &= d(\zeta^0 S \cdot \tfrac{1}{\zeta^0}) \\
&= \zeta^0 S \tfrac{1}{\zeta^0}\Big[\beta dW - \frac{\rho\delta}{S} dt + \frac{\tilde\mu}{\sigma} dW + \Big(\frac{\tilde\mu}{\sigma}\Big)^2 dt + \beta\frac{\tilde\mu}{\sigma} dt\Big] \\
&= S\Big[\sigma dW - \frac{\rho\delta}{S} dt + \frac{\tilde\mu}{\sigma}\cdot\sigma dt\Big] \\
&= S[\sigma dW + \mu dt]
\end{aligned}
$$

when we define μ by $\mu = \tilde\mu - \rho\delta/S$.

So what is going on?

(X) <u>My conjecture:</u>

$$\zeta^j_t w^j_t + \int_0^t \zeta^j_u c^j_u\, du \quad \text{is a UI martingale} \quad \forall j$$

$$\zeta^0_t S_t + \int_0^t \zeta^0_u \rho_u \delta_u\, du \quad \text{is a UI martingale}$$

<u>BUT</u> $\quad \zeta^0 w^j_t + \int_0^t \zeta^0_u c^j_u\, du$ is a martingale, <u>which is not</u> UI

This imposes a significant <u>consistency</u> condition. If we choose β, we can derive $\zeta^0 w^j$ and hence $\zeta^0 \rho\delta$, and we must have that the martingale

$$E_t\left[\int_0^\infty \zeta^0_u \rho_u \delta_u \, du\right] = \zeta^0_0 S_0 + \int_0^t \beta_u \zeta^0_u S_u \, dW_u .$$

Another way to express this would be in terms of ξ, defined via

$$d\xi_t = \xi_t \, \beta_t \, dW_t \qquad , \; \xi_0 = 1$$

[which we can use to define a new measure $\tilde{P}$, $d\tilde{P}/dP|_{\mathcal{F}_t} = \xi_t$?] We can easily check that

$$d\left(\frac{\zeta^0 S}{\xi}\right) = -\frac{\zeta^0 \rho \delta}{\xi} dt .$$

To go further, define

$$A_t = \int_0^t \frac{\zeta^0_u \rho_u \delta_u}{\xi_u} \, du$$

and suppose we could find β so that

$$\boxed{A_\infty = S_0}$$

We have

$$\frac{\zeta^0_t S_t}{\xi_t} + A_t = \text{const} = S_0 \qquad (\text{by considering } t=0)$$

and so

$$\xi_t S_0 - \zeta^0_t S_t = \xi_t A_t$$

$$= \int_0^t \zeta^0_u \rho_u \delta_u \, du + \int_0^t A_u \beta_u \xi_u \, dW_u$$

$$\Rightarrow \quad \zeta^0_t S_t + \int_0^t \zeta^0_u \rho_u \delta_u \, du = \xi_t S_0 - \int_0^t A_u \beta_u \xi_u \, dW_u$$

$$= S_0 + \int_0^t \beta_u \xi_u (S_0 - A_u) \, dW_u$$

$$= S_0 + \int_0^t \beta_u \zeta^0_u S_u \, dW_u .$$

Of course, this is implicit in the SDE for S, but what we do gain is the knowledge that $\zeta^0_t S_t = \xi_t(A_\infty - A_t) \to 0$. This isn't sufficient for the martingale to be UI, but it's necessary, so trying to choose β to ensure $A_\infty = S_0$ looks an interesting direction to search.

(xi) Developing this a bit further, with the notation $L^j_t = \Lambda^j_t \zeta^0_t / \zeta^j_t$ as before, we have that $c^j_t = \rho_j w^j_t = e^{-\rho_j t} L^j_t / \zeta^0_t$ so the condition we require is

$$S_0 = \int_0^\infty \sum_j \zeta^0_t \, \rho_j w^j_t / \xi_t \, dt$$

and if we Itô the summand $\zeta^0 w^j/\zeta$ we get

$$d\left(\frac{\zeta^0 w^j}{\zeta}\right) = \frac{\zeta^0 w^j}{\zeta}\left[-(\rho-\alpha^j)^+ (dW - \rho dt) - \rho_j dt\right]$$

Thus if we set $z_t^j = e^{\rho_j t}\, \zeta_t^0 w_t^j/\zeta_t$, we see our condition as

$$\begin{aligned}
S_0 &= \sum_j \int_0^\infty \rho_j e^{-\rho_j t} z_t^j \, dt \\
&= \sum_j z_0^j + \sum_j \int_0^\infty e^{-\rho_j s} \, dz_s^j \\
&= S_0 + m_0 + \int_0^\infty \sum_j e^{-\rho_j s} \, dz_s^j
\end{aligned}$$

So the condition is equivalently

$$\boxed{0 = m_0 + \sum_j \int_0^\infty e^{-\rho_j s} \, dz_s^j}$$

Notice that $dz_t^j = z_t^j \{-(\rho_t - \alpha_t^j)^+ (dW_t - \rho_t dt)\}$, $z_0^j = w_0^j$. If we could find some choice of ρ which would satisfy this for some given m_0, then we could certainly satisfy this for any $m_0' \in (0, m_0)$ just by setting $\rho = \min_j \alpha_j$ once $\sum_j \int_0^t e^{-\rho_j s} dz_s^j$ hits $-m_0'$. This would mean that after that stopping time, all agents (except the most pessimistic) hold zero cash.

Discrete-time strict local martingales (28/6/08)

Bob Jarrow asks whether such can exist. Someone must know this, but the answers are

(i) Yes. Take $X_0 = 0$, $X_1 \sim N(0,1)$ and given $X_1 = x$ we choose $X_2 = x + e^{x^4} Z$, where Z is $N(0,1)$, with $X_k = X_2 \ \forall k \geq 2$.

(ii) No if M is non-negative. Indeed, if $T_n \uparrow \infty$ is a sequence of reducing times, we have $\Delta M_k^n = \Delta M_k \, I_{\{T_n > k\}} \in L^1$. We also have $M_0 \in L^1$. By induction, every M_k is in L^1:

$$E[M_{k+1} : T_n > k] = E[M_k : T_n > k] \uparrow E M_k < \infty$$

so by monotone convergence, $M_{k+1} \in L^1$. Thus $\Delta M_k \in L^1 \ \forall k$, and for any $Z \in b\mathcal{F}_k$

$$0 = E_k[Z \Delta M_k^n] \to E[Z \Delta M_k] \qquad \text{(Dominated)}$$

Calculating various moments for Bayesian log-investors (7/7/08)

(1) Let's return to pp 5–6 where various moments of interest are calculated for multiple Bayesian log investors, and be a bit more explicit about the various things calculated here. Recall $\delta_t = \delta_0 \exp(\sigma X_t - \frac{1}{2}\sigma^2 t)$, and

$$\frac{S_t}{\delta_t} = \frac{\sum e^{-\rho_i t} \Lambda_t^i / \nu_i \rho_i}{\sum e^{-\rho_i t} \Lambda_t^i / \nu_i} \equiv \frac{N_1(t, X_t)}{D_1(t, X_t)}$$

where $N_1(t,x) = \sum \exp(-\rho_i t + a_i x - \frac{1}{2} a_i^2 t) / \nu_i \rho_i$, $D_1(t,x) = \sum \exp(-\rho_i t + a_i x - \frac{1}{2} a_i^2 t)/\nu_i$

We also have

$$S_t = \frac{\delta_t N_1(t, X_t)}{D_1(t, X_t)} = \frac{N_2(t, X_t)}{D_1(t, X_t)},$$

where $N_2(t, X_t) = \sum \exp(-\rho_i t + (a_i + \sigma) X_t - \frac{1}{2}(a_i^2 + \sigma^2)t) / \nu_i \rho_i$, and

$$\zeta_t = D_1(t, X_t) \big/ \delta_t .$$

(a) For the long term PD ratio, we calculate the expectation as at the top of p6.

(b) For the SD of the PD ratio, let's understand that as the vol; notice that

$$d\left(\frac{S}{\delta}\right) = d\left(\frac{N_1}{D_1}\right) = \frac{N_1}{D_1}\left(\frac{N_1'}{N_1} - \frac{D_1'}{D_1}\right) dX + FV$$

so we have to calculate the mean of $\left| N_1'/D_1 - N_1 D_1'/D_1^2 \right|$, just as before

(c) For the risky return on equity, we see that

$$d(\log S) = \frac{N_2' \, dX}{N_2} + \frac{\dot N_2}{N_2} dt + \frac{1}{2}\left(\frac{N_2''}{N_2} - \left(\frac{N_2'}{N_2}\right)^2\right) dt$$

$$- \frac{D_1'}{D_1} dX - \frac{\dot D_1}{D_1} dt - \frac{1}{2}\left(\frac{D_1''}{D_1} - \left(\frac{D_1'}{D_1}\right)^2\right) dt \qquad \left[q = \frac{N_2'}{N_2} - \frac{D_1'}{D_1} \right]$$

$$= q(dW + \alpha dt) + \left(\frac{\dot N_2}{N_2} - \frac{\dot D_1}{D_1}\right) dt + \frac{1}{2}\left\{ \frac{N_2''}{N_2} - \frac{D_1''}{D_1} - \left(\frac{N_2'}{N_2}\right)^2 + \left(\frac{D_1'}{D_1}\right)^2 \right\} dt$$

Hence risky return will be

$$\left(\frac{\dot N_2}{N_2} - \frac{\dot D_1}{D_1}\right) + \alpha q + \frac{1}{2} q^2 + \frac{1}{2}\left\{ \frac{N_2''}{N_2} - \frac{D_1''}{D_1} - \left(\frac{N_2'}{N_2}\right)^2 + \left(\frac{D_1'}{D_1}\right)^2 \right\}$$

(d) The SD of that could be calculated by the averaging story again.

(e) riskless rate comes out as on p6.

Simulating from a CIR SDE (19/8/08)

(i) Suppose we have an SDE
$$dX = \sigma(X)\, dW + \mu(X)\, dt$$
which may be a bit singular in places (as in the CIR SDE:
$$\sigma(X) = v\sqrt{X}, \quad \mu(X) = \alpha - \beta X \quad)$$
If we use the scale function s, which is increasing, defined up to irrelevant affine transformations by
$$s'(x) = \exp\left[- \int^x \frac{2\mu}{\sigma^2}(y)\, dy \right] \qquad \left(= x^{-2\alpha/v^2} \exp\left(\tfrac{2\beta}{v^2} x\right) \text{ for CIR} \right)$$
then $Y_t \equiv s(X_t)$ is a local martingale
$$dY_t = g(Y_t)\, dW_t, \qquad g(y) \equiv (\sigma s') \circ s^{-1}(y).$$
The idea is to simulate from Y, and transform back.

(ii) For the CIR example, there are two cases to consider

Case 1: $\alpha \geq \frac{1}{2}v^2$. In this case, 0 is inaccessible and the scale function $s : (0,\infty) \to \mathbb{R}$, $s^{-1} : \mathbb{R} \to (0,\infty)$, and there is no problem about 0 (unless we were to start there).

Case 2: $\alpha < \frac{1}{2}v^2$. This time, s' is integrable at 0, so we may as well suppose $s(0) = 0$. Indeed, $s(x) \sim x^{1-\varepsilon}$ near $x = 0$, where $\varepsilon = 2\alpha/v^2 < 1$.

What we do here is to extend s antisymmetrically into $(-\infty, 0)$ so $s(-y) = -s(y)$ $\forall y > 0$, and simulate the local martingale
$$dY_t = g(|Y_t|)\, dW_t.$$
The simulated path of the SDE X is obtained as $|s^{-1}(Y_t)|$.

(iii) How would we simulate from the SDE
$$dY = g(Y)\, dW \; ?$$
We could try to do some sort of Milstein scheme, but if g has singularities this can be problematic. Better is to realise Y as a time change of BM:
$$A_t = \int_0^t \frac{ds}{g(B_s)^2}$$
for some BM B, and then set $\tau_t = \inf\{s : A_s > t\}$, $Y_t = B(\tau_t)$.

What's the good way to do this?

$$E\left[L(T,x)\,;\,B(T)=\xi\right] = \int_0^\infty \lambda e^{-\lambda T}\left(\int_0^T p_t(0,x)\, p_{T-t}(x,\xi)\,dt\right)dT$$

$$= \frac{1}{\lambda}\ \frac{\theta}{2} e^{-\theta|x|}\ \frac{\theta}{2} e^{-\theta|x-\xi|}$$

$$P\left(B(T)=\xi\right) = \frac{\theta}{2} e^{-\theta|\xi|},\ \infty$$

$$E\left[L(T,x)\,\middle|\,B(T)=\xi\right] = \frac{1}{\theta}\exp\left\{-\theta|x| - \theta|x-\xi| + \theta|\xi|\right\}.$$

Worth having: $E^0\left[L(\tau,a)\,\middle|\,X_\tau=\delta\right] = \begin{cases} \dfrac{(a+\delta)^2}{\delta} & a\in(-\delta,0) \\ \dfrac{\delta^2-a^2}{\delta} & a\in(0,\delta) \end{cases}$

Could be useful to get conditional ~~mean~~ exit time given whether you went up or down.

(iv) The obvious thing would be to sample the BM at the times h, $2h, 3h, \dots$ giving values $Z_1, Z_2, Z_3, \dots$ and then to record corresponding clock times $a_1, a_2, \dots$, where

$$a_{i+1} - a_i = h / g(Z_i)^2$$

However, this is too crude, and will come unstuck if g has zeros, as in the case of the CIR SDE.

The better thing to do is to sample the BM at times $T_1 < T_2 < \dots$, where $T_{i+1} - T_i$ are IID $\exp(\frac{1}{2}\theta^2)$. The point of this is that it allows us a much cleaner way to handle the increments of A.

Notice that

$$Z_{i+1} - Z_i \equiv B(T_{i+1}) - B(T_i) \text{ are IID}$$

with density $\frac{1}{2}\theta e^{-\theta|x|}dx$, so these are easily simulated. Now consider the local time $L(T_1, x)$ of the first increment (all are similar, we just do the first one for notational simplicity)

Suppose that $Z_1 \equiv B(T_1) = \xi > 0$.

Then
$$E\left[L(T_1, x) \mid Z_0, Z_1\right] = \begin{cases} \frac{1}{\theta} e^{2\theta x} & (x \leq 0) \\ \frac{1}{\theta} & (0 \leq x \leq \xi) \\ \frac{1}{\theta} e^{-2\theta(x-\xi)} & (x \geq \xi) \end{cases}$$

Thus conditional on $Z_1 = \xi$, the mean of $A(T_1)$ is

$$\int_{-\infty}^{0} \frac{1}{\theta} e^{2\theta x} \frac{dx}{g(x)^2} + \frac{1}{\theta}\int_0^{\xi} \frac{dx}{g(x)^2} + \frac{1}{\theta}\int_{\xi}^{\infty} e^{-2\theta(x-\xi)} \frac{dx}{g(x)^2} \qquad (*)$$

and we use this as the (proxy for) the increment of A. Notice that this requires us to have the indefinite integral of $g(x)^{-2}dx$, and the convolution of $e^{-\theta x} I_{\{x>0\}}$ with $g(x)^{-2}dx$, which can be calculated and stored in a lookup table before the simulation starts.

(v) But maybe the best thing is to sample BM at times of crossing a grid of spacing δ
$E^0[L(\tau, x)] = (\delta - x) \wedge (\delta + x)$ for $|x| \leq \delta$, $\tau = \inf\{t : |B_t| = \delta\}$. Use this to compute the mean of $A(\tau)$.

Risk measure constraints on optimal investment + contracts (21/8/08)

Let's pick up the point Phil Dybvig made on my paper on contracting for optimal investment under risk constraints.

(i) The principal is going to offer the agent $\varphi(X)$, where X is the wealth generated, and $X = \psi(\zeta)$ for some decreasing ψ. Let's suppose that the risk measure constraints are on the net wealth

$$h(\zeta) = \psi(\zeta) - \varphi(\psi(\zeta))$$

and now the Lagrangian form of the problem is

$$\max E\Big[U_p(h(\zeta)) + \beta U_A(\psi(\zeta) - h(\zeta)) + \lambda\zeta(w_0 - \psi(\zeta)) + \sum_i \alpha_i \{ g_i(\zeta)\, h(\zeta) - b_i - z_i \} \Big]$$

just as we found before. Now optimising over $\psi(\zeta)$ would give

$$\beta U_A'(\psi(\zeta) - h(\zeta)) = \lambda\zeta \quad = \beta U_A'(\varphi(\psi(\zeta)))$$

if we could optimise freely, but we do need ψ to be decreasing.

(ii) If we write $G(x) \equiv \sum_i \alpha_i g_i(F_\zeta^{-1}(x))$ $(x \in (0,1))$ and $q(x) \equiv \psi(F_\zeta^{-1}(x))$ then the aim is to

$$\boxed{\max \int_0^1 \{ U_p(q(x) - \varphi(q(x))) + q(x)G(x) \}\, dx}$$

over <u>decreasing</u> functions q, where

$$\boxed{U_A'(\varphi(q(x)))\, \varphi'(q(x)) = \lambda F_\zeta^{-1}(x).}$$

Extending Bolton-Freixas: the story for the firm (24/8/08)

1) Let's take a conventional classical growth model for the firm, where there is depreciation of capital at rate $\delta K_t\, dt$, output is $F(K_t, L_t)$, where F is concave increasing, homogeneous of degree 1, wage rate is w, and there is some reinvestment of a fraction $1-\theta$ of the output. The rest of the output is used to pay the coupons ρdt due on the initial bond issue (of face value 1) as well as some dividends (perhaps negative). The evolution of the capital of the firm is modelled as

$$dK_t = -\delta K_t\, dt + \sigma K_t\, dW_t + K_t a(1-\theta)\, dt$$

where $a \equiv \sup\{F(1,L) - wL\}$ is the best net output rate for unit capital. [This also determines an optimal workforce required]. Thus the dynamics of K are

$$dK_t = \sigma K_t\, dW_t + \mu K_t\, dt \qquad [\mu = a(1-\theta) - \delta]$$

$$= \sigma K_t\, d\tilde{W}_t + (r - a\theta) K_t\, dt$$

where $\tilde{W}_t$ is a BM in the risk-neutral measure (under which the growth rate of the firm without payout of dividends must be r).

2) Assuming for now that the fraction θ of output passed to bondholders and shareholders is given, the only decision of the firm's management is the choice of bankruptcy level. Let us assume that a fraction $p \in (0,1)$ of the firm's value survives default, and that the shareholders are left with $(pK-1)^+$, the bondholders with $pK \wedge 1$ if default happens at K. There is the constraint that the time-0 value of the bonds must be equal to 1, and this will be achieved by setting the coupon rate ρ to the correct value.

The shareholders get dividends at rate $Kb - \rho$, where $b \equiv a\theta$ for short so the stock price as a function of K will solve

$$-rS + \tfrac{1}{2}\sigma^2 K^2 S'' + (r-b)KS' + bK - \rho = 0$$

with smooth pasting to $(pK-1)^+$ at the bankruptcy point ξ. The homogeneous equation is solved by functions of the form

$$K \mapsto AK^{-\alpha} + BK^{\beta}$$

where $-\alpha < 0 < 1 < \beta$ are roots of $\frac{1}{2}\sigma^2 t(t-1) + (r-b)t - r = 0$. In view of the fact that $S(K) \sim K$ for large K, it has to be that $B=0$, and if we have a smooth-pasted solution at $K = \xi < 1/p$ then

$$\left.\begin{aligned} A\xi^{-\alpha} + \xi - \tfrac{\rho}{r} &= 0 \\ -\alpha A \xi^{-\alpha-1} + 1 &= 0 \end{aligned}\right\}$$

using the particular solution $K - \rho/r$ to the ODE. From this we get quickly that
$$\boxed{\xi = \frac{\rho\,\alpha}{r(1+\alpha)}, \quad A = \frac{1}{\alpha}\xi^{1+\alpha}.}$$

Thus the coupon rate has to be proportional to the default barrier ξ. The value of the bond we calculate as

$$B(K_0) = E^{K_0}\left[\int_0^{\tau} \rho\, e^{-rs}\, ds + e^{-r\tau} p\xi\right]$$

$$= \frac{\rho}{r} + \left(p\xi - \frac{\rho}{r}\right)\left(\frac{K_0}{\xi}\right)^{-\alpha}$$

in this instance, and this has to be issued at par, that is, $B(K_0) = 1$. Hence

$$\boxed{\left(\frac{K_0}{\xi}\right)^{\alpha} = \frac{\left(\frac{1+\alpha}{\alpha} - p\right)\xi}{\left(\frac{1+\alpha}{\alpha}\xi - 1\right)}}$$

by substituting ρ in terms of ξ. Thus we have learnt how the initial level of capital must depend on the bankruptcy level ξ. Notice this can only work for ξ in the range

$$\xi \in \left(\frac{\alpha}{1+\alpha}, \frac{1}{p}\right).$$

3) Could it be that the optimal decision of the firm would involve stopping at $\xi \geq 1/p$? In principle it could, but it turns out that this is not compatible with par issuance of the bond. Indeed, if $\xi \geq 1/p$, the bondholders get their initial investment back, so it must be that $\rho = r$. Thus the smooth pasting conditions at ξ for S read in this case

$$\left.\begin{aligned} A\xi^{-\alpha} + \xi - 1 &= p\xi - 1 \\ -\alpha A\xi^{-\alpha-1} + 1 &= p \end{aligned}\right\} \Rightarrow \left.\begin{aligned} A\xi^{-\alpha-1} &= p-1 \\ A\xi^{-\alpha-1} &= (1-p)/\alpha \end{aligned}\right\}$$

which cannot be satisfied simultaneously.

4) We now envisage some large economy of many firms in some sort of steady state. The firms are like those analysed already, but the various parameters $(\sigma, a, \delta, \theta, p)$ may vary from one firm to another. The large collection of households will hold shares + bonds issued by the firms, and will be represented by a single representative agent whose preferences are described by

$$E \int_0^\infty e^{-\lambda t} U(C_t)\, dt$$

where λ is a parameter greater than any of the individual firm's $a-\delta$, and C_t is the aggregate consumption rate. Now because we envisage a large number (continuum) of independent firms of each of the types, the aggregate consumption C_t will be deterministic, so as far as the agent representing the households is concerned, the marginal pricing of a random cash flow (x_t) would be by $E\left[\int_0^\infty e^{-\lambda t} x_t\, dt\right]$.

If the agent chooses to invest in a firm of the given type, how would the split between equity/bonds be done? (the initial K_0 can be varied by varying ξ – or equivalently ρ) We would compute the value of the stock and of the bond and maximise the total value per unit of initial capital. Let's look at two possible ways we might do this.

(a) The firm runs til default and then stops. Share value solves

$$-\lambda f + \tfrac{1}{2}\sigma^2 K^2 f'' + \mu K f' + (bK - \rho) = 0, \quad f(\xi) = 0$$

leading as before to a solution

$$f(K) = A\, K^{-\alpha'} - \frac{\rho}{\lambda} + \frac{b}{\lambda - \mu} K$$

where $-\alpha' < 0 < 1 < \beta'$ are roots of $\tfrac{1}{2}\sigma^2 t(t-1) + \mu t - \lambda = 0$, and A is chosen to make $f(\xi)$ vanish.

The bond valuation is simply

$$g(K) = \frac{\rho}{\lambda} + \left(p\xi - \rho/\lambda\right)\left(K_0/\xi\right)^{-\alpha'}$$

where the bond starts with initial capital 1, initial value of K is K_0.

(b) When the firm fails, the bondholders + shareholders restart the firm, shareholders contributing $K_0 - 1$, bondholders contributing $1 - p\xi$, and then off it goes again. If we abbreviate $(K_0/\xi)^{-\alpha'} = E^{K_0}(e^{-\lambda\tau}) \equiv q$ we get the valuation of equity V_S satisfies

$$V_S = f(K_0) + q\{-(K_0-1)+V_S\}$$

and the valuation of the bond satisfies

$$V_{Bond} = \frac{\rho}{\lambda}(1-q) + q\{-(1-p\xi)+V_B\}$$

Hence

$$V_S = \frac{f(K_0)-q(K_0-1)}{1-q}, \quad V_{Bond} = \frac{\frac{\rho}{\lambda}(1-q) - q(1-p\xi)}{1-q}.$$

5) <u>Bank funding</u>. The major difference I propose between funding by issuing corporate bonds, and funding by taking a bank loan, is that the coupon paid on a bank loan is $\rho_B = r + \varepsilon'$, where ε' is the spread required by the bank. Let's also assume that there's a cost per unit time c to cover the bank's administration of the loan, and we end up with a net spread of $\varepsilon \equiv \varepsilon' - c$.

This gives us the bankruptcy level

$$\xi_B = \frac{\rho_B\ \alpha}{r(1+\alpha)}$$

exactly as before, where α is exactly as before, because the firm's management just maximises share value.

How does it look for the bank? The bank has subjective discount rate λ_B, and if $-\alpha_B$ is the negative root of

$$\tfrac{1}{2}\sigma^2 t(t-1) + \mu t - \lambda_B$$

and we set

$$q(K_0,\lambda_B) = E^{K_0}\left[e^{-\lambda_B\tau}\right] = (K_0/\xi_B)^{-\alpha_B}$$

then the bank values the cash flow as

$$V_B = \frac{\varepsilon(1-q(K_0,\lambda_B))/\lambda_B - q(K_0,\lambda_B)\{1-p_B\xi_B\}}{1-q(K_0,\lambda_B)}.$$

The households would value the same cash flow slightly differently, using $\lambda = \lambda_H$ in place of λ_B. If the bank had its own initial capital W_B, and the households had placed equity capital Q_B with the bank, then the households receive a fraction $Q_B/(w_B+Q_B)$ of that profit stream. The households also receive the value of the equity in the firm, which they value in the same manner as before. Thus the households should be in a position to choose K_0 to maximise the value per unit of input capital (16-1).

6) But this is not how it works for a central planner equilibrium. Let's lay out the bits. For notational simplicity, let's suppose the different types of firm are indexed by a, the maximal rate of output (of course, we should include distribution proportion θ etc, but this keeps it simple). There will only be finitely many types of firms considered in the analysis.

Let firms of type a receive $k_H(a)$ in capital which is used to fund the firm by issuance of equity + perpetual corporate debt, a proportion $\pi^*(a)$ of the capital being put to bonds. Let the value to the household representative agent H of unit bond-funded investment in a type-a firm be denote $\varphi(a)$.

Suppose firms of type a receive a total of $k_B(a)$ units of capital which will be split among a bank loan and equity in the ration $\pi_B(a) : 1-\pi_B(a)$. Let the value to H of the equity part of unit capital so invested be $\varphi^B(a,\pi)$. Let the value to H of the loan part of unit capital so invested be $\psi_E^B(a,\pi)$, and let the value to the bank of the loan part so invested be $\psi_B^B(a,\pi)$ — this may differ from $\psi_E^B(a,\pi)$ because the bank may use a different subjective discount factor, but aside from that it would be the same. Thus if we are looking at a central planner who mixes the objective of H together with ω times the objective of the bank, unit investment in a type-a firm would be worth

$$\varphi^B(a,\pi) + \frac{Q}{w^B+Q}\psi_E^B(a,\pi) + \omega\frac{w^B}{w^B+Q}\psi_B^B(a,\pi).$$

Here, Q is the equity invested in banks by H.

Suppose that H assigns capital D_B to riskless bank deposits, and D_G to riskless government deposits. Then the objective which the central planner wishes to maximise is

$$\frac{(D_G+D_B)r}{\lambda_H} + \sum_a\Big[k_H(a)\varphi(a) + k_B(a)\big\{\varphi^B(a,\pi) + q\,\psi_E^B(a,\pi)\big\}\Big] + \omega\sum_a k_B(a)(1-q)\psi_B^B(a,\pi)$$

$[q \equiv Q/(w^B+Q)]$ subject to various constraints:

$$Q + \sum_a\{k_H(a) + k_B(a)(1-\pi_B(a))\} + D_B + D_G = \bar{K}, \qquad [\text{multiplier } \eta]$$

where $\bar{K}$ is total available capital of households;

$$k_B(a) + k_H(a) \leq f(a),$$ where $f(a)$ is total capacity of firms of type a;

$$\sum k_B(a)\pi_B(a) = w^B + Q + D_B \leq (w^B + Q)/\kappa \qquad [\text{multipliers } y_1, y_2]$$

(total amount of banks' loans cannot exceed their assets, nor some multiple of their capital assets. It never helps the banks to accept deposits which they don't lend out, as they simply redeposit this money with the government. So we'll assume an equality constraint here). We'll ignore the capacity constraint in setting up the Lagrangian, since the problem will be linear and the solution subject to capacity constraint will be obvious. So the Lagrangian will be

$$\frac{(D_G + D_B)^\nu}{\lambda_H} + \sum_a \Big[k_H(a)\varphi(a) + k_B(a)\big\{ \varphi^B(a,\pi) + q\,\psi_E^B(a,\pi) + \omega(1-q)\,\psi_B^B(a,\pi) \big\} \Big]$$

$$+\eta \Big[\bar{K} - D_G - D_B - \frac{q w^B}{1-q} - \sum_a \{ k_H(a) + k_B(a)(1-\pi_B(a)) \} \Big]$$

$Q = \frac{q w^B}{1-q}$

$$+ y_1 \Big\{ w^B + Q + D_B - \sum k_B(a)\pi_B(a) \Big\} + y_2 \Big\{ \frac{w^B + Q}{\kappa} - \sum k_B(a)\pi_B(a) \Big]$$

The maximisation over $\pi_B(a)$ can be done numerically, then the max over $k_H(a)$, $k_B(a)$ is maxing a linear functional ... not entirely clear how to deal nicely with max over q ..

Maybe best is to just restrict to one bank to begin with, and see how this looks

7) Ergodic averaging.

We need to understand better how the banks + households value the cashflows they receive in this story. The general shape is common to all; the diffusion K evolves as $dK = K(\sigma dW + \mu dt)$ generating cashflow $g(K_t)dt$ until the time when K first hits ξ, at which point there is a cost C and the process gets reset at distinguished value K_0. The value starting from initial capital K is thus

$$E^K\Big[\int_0^\tau e^{-\gamma s} g(K_s)\,ds + e^{-\gamma\tau}\{-C + V_0\}\Big] \equiv V(k).$$

Now the function $V(\cdot)$ is of the form $R_\gamma f$ for some suitable f, where (R_λ) is the resolvent of the ergodic Markov process, and we require to find the ergodic average, obtained by averaging over the starting point K with the invariant distribution. We get this as

$$\lim_{\varepsilon\downarrow 0} \varepsilon R_\varepsilon V = \lim_{\varepsilon\downarrow 0} \varepsilon R_\varepsilon R_\gamma f$$

$$= \lim_{\varepsilon\downarrow 0} \frac{\varepsilon}{\gamma-\varepsilon}(R_\varepsilon - R_\gamma) f$$

$$= \frac{1}{\gamma} \lim_{\varepsilon\downarrow 0} \varepsilon R_\varepsilon f$$

and to calculate this limit, we could use any value of K for the argument of $R_\varepsilon f$, since the limit will not depend on which argument is chosen. So the natural choice is to use $K = K_0$, for then we have

$$R_\varepsilon f(K_0) = \frac{E^{K_0}\left[\int_0^\tau e^{-\varepsilon t} g(K_t)\,dt\right] - C E^{K} e^{-\varepsilon\tau}}{1 - E^{K_0}\left[e^{-\varepsilon\tau}\right]}$$

whence

$$\boxed{\lim_{\varepsilon\downarrow 0} \varepsilon R_\varepsilon f(K_0) = \frac{E^{K_0}\left[\int_0^\tau g(K_t)\,dt\right] - C}{E^{K_0}[\tau]}}$$

For our applications, we only ever need to deal with the case where g is linear. After some calculations, we arrive at the conclusion

$$\boxed{\begin{aligned} E^{K_0}[\tau] &= -\log(K_0/\xi)/(\mu - \tfrac{1}{2}\sigma^2) \\ E^{K_0}\left[\int_0^\tau K_s\,ds\right] &= -(K_0 - \xi)/\mu \end{aligned}}$$

and require $\boxed{\mu < 0}$ for the second expectation to be finite.

Notice that the agent's discount factor γ only enters as the prefactor γ^{-1}. The households therefore regard the corporate bond paying coupon ρ as worth γ^{-1} times

$$\rho + \frac{(1-b\xi)(\mu - \tfrac{1}{2}\sigma^2)}{\log(K_0/\xi)}$$

and the equity as worth

$$\frac{b(K_0 - \xi)(\mu - \tfrac{1}{2}\sigma^2)}{\mu \log(K_0/\xi)} - \rho + \frac{(K_0 - 1)(\mu - \tfrac{1}{2}\sigma^2)}{\log(K_0/\xi)}$$

Bank runs (5/9/08)

Angus is interested to tell a continuous time story for bank runs, and in discussion we come up with this. Let K_t denote the level of bank capital at time t, D_t the level of deposits, and L_t the face value of all (perpetual) loans that are made at time t. The riskless rate r is constant, and loans pay interest at rate $r+s$. The bank must observe the inequalities

$$\left.\begin{aligned} L_t &\leq K_t + D_t \\ L_t &\leq \nu K_t \end{aligned}\right\}$$

for some constant ν. If one of these inequalities is about to be violated, the bank has to call in loans, but they only raise a fraction p of the face value by doing this. Thus we get dynamics

$$dK_t = (r+s)L_t\,dt - rD_t\,dt - dL_t^+ + p\,dL_t^- - cL_t\,dt$$

$$dD_t = D_t(\sigma dW_t + \mu dt) - \alpha\,dL_t^-$$

where the term $cL_t dt$ denotes costs and dividend payments made by the bank, and the term $-\alpha dL_t^-$ reflects the tendency for bad news to frighten away the depositors.

If we define the value function

$$V(K,D,L) \equiv \sup E\left[\int_0^\tau e^{-rs} L_s\,ds \,\Big|\, K_0=K,\ D_0=D,\ L_0=L\right]$$

then we have the HJB equations

$$\begin{cases} V_K \geq V_L, \quad V_L + \alpha V_D \geq p V_K \\[2ex] \tfrac{1}{2}\sigma^2 D^2 V_{DD} + \mu D V_D + \{(r+s)L - rD - cL\}V_K - rV + L = 0 \end{cases}$$

Notice also the scaling relation

$$V(\lambda K, \lambda D, \lambda L) = \lambda V(K,D,L) \qquad \forall \lambda > 0$$

Writing $x = K/L$, $y = D/L$, $V(K,D,L) = Lv(x,y)$, we have

$$\boxed{\begin{aligned} &\tfrac{1}{2}\sigma^2 y^2 v_{yy} + \mu y v_y + (r+s+c-ry)v_x - rv + 1 = 0, \\[2ex] &v_x \geq v - xv_x - yv_y \geq pv_x - \alpha v_y, \qquad \nu x \geq 1, \quad x+y \geq 1 \end{aligned}}$$

Continuous-time particle filtering? (11/9/08)

Let's look at a very simple example of this. Suppose we have underlying diffusion X_t - which we'll suppose for now is BM $(\mathbb{R}^d)$) and we observe

$$dY_t = dW_t + h(X_t)\,dt \qquad h: \mathbb{R}^d \to \mathbb{R}^m$$

What about filtering X?

The likelihood (relative to Wiener measure for (X,Y)) of the processes is just

$$\exp\Big[\int_0^t h(X_s)\cdot dY_s - \tfrac{1}{2}\int_0^t |h(X_s)|^2\,ds \Big] \qquad (*)$$

So this gives the posterior density (wrto Wiener) of $X_{[0,t]}$ given $\mathcal{Y}_t$.

(0) You might simulate X from BMs and then reweight the paths by $(*)$

(1) Maybe a better idea is to try to identify a ML path; minimise

$$\int_0^t \tfrac{1}{2}|\dot{x}_s|^2\,ds - \int_0^t h(x_s)\cdot dY_s + \tfrac{1}{2}\int_0^t |h(x_s)|^2\,ds$$

If we do calculus of variations, and perturb x to $x+\eta$, assuming we have $x_0=0$ fixed, the FOC is

$$0 = \int_0^t \{ (\dot{x}_s, \dot{\eta}_s) - (\dot{Y}_s, Dh(x_s)\eta_s) + (h(x_s), Dh(x_s)\eta_s) \}\,ds$$

$$= \big[\dot{x}_s \eta_s\big]_0^t - \int_0^t \ddot{x}_s \eta_s\,ds + \int_0^t (h(x_s) - \dot{Y}_s, Dh(x_s)\eta_s)\,ds$$

so this would imply that

$$\begin{cases} 0 = -\ddot{x}_s + (h(x_s) - \dot{Y}_s)\circ Dh(x_s) \\ 0 = \dot{x}_t \end{cases}$$

This is a shooting-type ODE - solving wouldn't be an issue, but getting $\dot{x}_t$ would, and moreover this condition would keep changing as you advance t.

Once you've identified the optimal x, you could simulate a bunch of BMs and at the optimal x_t^* to them. This gives you various paths $x_t^* + W_t^{(i)}$ whose likelihoods relative to Wiener measure can be computed, as can the posterior likelihoods, and the ratios of this give the posterior weights!

PDCB again (24/9/08)

(i) Let's return to the no-calling problem (see WN , p 15) taking some independent variable x. Then the equations which govern the solution are ($t \equiv 1/\theta$)

$$(1) \quad r(\alpha+\beta)(n-m)\frac{ds}{dx} = \frac{1}{\xi}\frac{d\eta}{dx}\Big[m\rho'\psi_0(t) - r\xi\big(\psi_1(t) + t\psi_1'(t)\big)\Big]$$

$$(2) \quad r(\alpha+\beta)(n-m)\frac{p\xi}{m} = r\xi\psi_1(\theta) - \rho(n-m\tau)\psi_0(\theta)$$

$$(3) \quad r(\alpha+\beta)(n-m)A = m\rho'\psi_0(t) - r\eta\psi_1(t).$$

There are three equations, and it seems important to make good choices of three dependent variables and one independent variable.

(ii). On the RHS of (1) we see ($t \to \infty$)

$$m\rho'\psi_0'(t) - r\xi(\psi_1(t) + t\psi_1'(t))$$

$$= \beta(\alpha m\rho' - r(\alpha+1)\xi)t^{\beta-1} + (\alpha+\beta)r\xi - \alpha(m\rho'\beta - (\beta-1)r\xi)t^{-\alpha-1}$$

$$= \beta m q (mt)^{\beta-1} + (\alpha+\beta)\xi - \alpha(m\rho'\beta - (\beta-1)r\xi)t^{-\alpha-1}$$

where we define

$$(4) \quad \boxed{q \equiv \frac{\alpha m\rho' - r(\alpha+1)\xi}{m^\beta}} \quad \to 0 \quad \text{as } m \to 0$$

(iii) If we look at (2) we see on the RHS

$$r\eta\theta\psi_1(\theta) - \rho(n-m\tau)\psi_0(\theta)$$

$$= [r\eta(\beta-1) - \rho(n-m\tau)\beta]\theta^{-\alpha} + \rho(n-m\tau)(\alpha+\beta) - r(\alpha+\beta)\eta\theta$$
$$+ [r\eta(\alpha+1) - \alpha\rho(n-m\tau)]\theta^\beta$$

$$\equiv -z\left(\frac{m}{\theta}\right)^\alpha + \rho(n-m\tau)(\alpha+\beta) - r(\alpha+\beta)\eta\theta + [r(\alpha+1)\eta - \alpha\rho(n-m\tau)]\theta^\beta$$

where we define

$$(5) \quad \boxed{z \equiv \frac{-r\eta(\beta-1) + \rho(n-m\tau)\beta}{m^\alpha}}$$

Using the fact that $m/\theta = \eta m/\xi \to n\beta(1+\alpha)/(\beta-1)(1-rc)$ as $m \to 0$, letting $m \to 0$ in (2) leads to the conclusion that z has a limit

(6) $$\boxed{z_0 = n\rho(\alpha+\beta)\left[1 - \frac{\alpha p(1-rc)}{1+\alpha}\right]\left(\frac{(\beta-1)(1-rc)}{n\beta(\alpha+1)}\right)^{\alpha}}$$

(iv) Now we shall transform everything to the independent variable m, and the dependent variables s, q, z. We get from (4), (5) respectively that

(7) $$\xi = \frac{\alpha m \rho' - q m^{\beta}}{r(\alpha+1)}$$

(8) $$\eta = \frac{\rho(n-mrc)\beta - z m^{\alpha}}{r(\beta-1)}$$

and hence an expression for $\theta = \xi/\eta$ in terms of m, q, z. We therefore write the three equations as

$$r(\alpha+\beta)(n-m)\frac{ds}{dm} = \frac{d\eta}{dm}\left[\beta\frac{m}{\xi}q\left(\frac{m}{\theta}\right)^{\beta-1} + r(\alpha+\beta) - \alpha\theta^{1+\alpha}\left\{\beta\rho'\frac{m}{\xi} - r(\beta-1)\right\}\right]$$

$$r(\alpha+\beta)(n-m)p\frac{\xi}{m} = -z\left(\frac{m}{\theta}\right)^{\alpha} + \rho(\alpha+\beta)(n-mrc) - r(\alpha+\beta)\xi$$
$$+ \theta^{\beta}\left[r(\alpha+1)\eta - \alpha\rho(n-mrc)\right]$$

$$r(\alpha+\beta)(n-m)s = \left(\frac{m}{\theta}\right)^{\beta}q + \theta^{\alpha}\left[\beta m\rho' - r(\beta-1)\xi\right] + (\alpha+\beta)(r\eta - m\rho')$$

This is all fine if $\alpha \geq 1$, but if $\alpha < 1$ we see a term in $m^{\alpha-1}$ in $d\eta/dm$. To deal with this, just switch to independent variable $x \equiv m^{\alpha}$!!

Simulating spread out samples (25/9/08)

Suppose we want to simulate points from a $N(0, I_d)$ distribution in such a way as to approximate $E[\varphi(X)]$ for some test function φ, and we want to perhaps spread points out a bit more than just IID draws; we want to do something to spread the points, perhaps with some reweighting

(i) One thing you could try is to simulate X_{k+1} conditional on $X_1, \ldots, X_k$ in a non-independent fashion. We could for example simulate a point according to the Gaussian density f, and then reject it with probability

$$h(x) = \Big(\prod_{i=1}^{k} f(x - X_i) \Big)^{\alpha} \Big/ b$$

where b is chosen to make $\sup_x h(x) \leq 1$. Thus

$$h(x) = b^{-1} \exp\Big\{ -\frac{\alpha}{2} k |x - \bar{X}|^2 - \frac{\alpha}{2} \sum |X_i - \bar{X}|^2 \Big\} (2\pi)^{-d\alpha k/2}$$

We would then have to attach weight $\tilde{h}(X_k)$ to point X_k, where

$$\tilde{h}(x) = \frac{1}{1 - h(x)} \cdot \int f(y)(1 - h(y))\, dy$$

The rejection prob^y is

$$\int f(x)\, h(x)\, dx = (2\pi)^{-\alpha dk/2}\, b^{-1} \exp\Big[-\frac{\alpha k}{1+\alpha k} \frac{|\bar{X}|^2}{2} - \frac{\alpha}{2} S_{xx} \Big] (1 + \alpha k)^{-d/2}$$

where $S_{xx} = \sum |X_i - \bar{X}|^2$. If we write $\bar{h} \equiv \sup h(x) = b^{-1} \exp\{-\frac{\alpha}{2} S_{xx}\} (2\pi)^{-\alpha dk/2}$ we shall see that the rejection prob^y will be

$$\bar{h}\, (1 + \alpha k)^{-d/2} \exp\Big[-\frac{\alpha k}{1 + \alpha k} \frac{|\bar{X}|^2}{2} \Big]$$

A natural choice would be $\alpha = \varepsilon / k$, whereupon the rejection prob^y becomes

$$\bar{h}\, (1 + \varepsilon)^{-d/2} \exp\Big\{ -\frac{1}{2} \frac{\varepsilon}{1 + \varepsilon} |\bar{X}|^2 \Big\}$$

but this will be close to $\bar{h}(1+\varepsilon)^{-d/2}$ for large sample size

Some more building block calculations (1/10/08)

(i) Suppose we have

$$dK = \sigma K\, dW + (r-b) K\, dt$$

as underlying dynamics, and we wish to find

$$f(K) = E^K \Big[\int_0^{\tau\wedge T} e^{-\lambda s} (a_0 + b_0 K_s)\, ds + e^{-\lambda\tau} I_{\{\tau<T\}}\, q_0 + e^{-\lambda T} I_{\{T<\tau\}} (c_0 + d_0 K_T) \Big]$$

where $\tau = \inf\{t : K_t = \xi\}$, $T \sim \exp(\nu)$ independent. Then we have

$$f(K) = E^K \Big[\int_0^{\tau} e^{-(\lambda+\nu)s} (a_0 + b_0 K_s)\, ds + e^{-(\lambda+\nu)\tau} q_0 + \int_0^{\tau} \nu e^{-(\nu+\lambda)s} (c_0 + d_0 K_s)\, ds \Big]$$

$$= E^K \Big[\int_0^{\tau} e^{-(\lambda+\nu)s} (\tilde a + \tilde b K_s)\, ds + q_0 e^{-(\lambda+\nu)\tau} \Big]$$

where $\tilde a \equiv a_0 + \nu c_0$, $\tilde b \equiv b_0 + \nu d_0$. Then clearly f solves

$$\tfrac12 \sigma^2 K^2 f'' + (r-b) K f' - (\lambda+\nu) f + (\tilde a + \tilde b K) = 0, \qquad f(\xi) = q_0$$

and the solution takes the form

$$\boxed{\begin{aligned} f(K) &= A_0 \left(\frac{K}{\xi}\right)^{-\alpha} + \frac{\tilde a}{\lambda+\nu} + \frac{\tilde b K}{\nu+b+\lambda-r} \\ &= q_0 \left(\frac{K}{\xi}\right)^{-\alpha} + \frac{\tilde a}{\lambda+\nu}\left(1 - \left(\frac{K}{\xi}\right)^{-\alpha}\right) + \frac{\tilde b}{\nu+b+\lambda-r}\left(K - \xi\left(\frac{K}{\xi}\right)^{-\alpha}\right) \end{aligned}}$$

where $-\alpha$ is a root of $\tfrac12\sigma^2 t(t-1) + (r-b)t - \lambda - \nu = 0$, where we assume $\boxed{\nu + b + \lambda - r > 0}$

(ii) Suppose next that we want to record the same payments up to time $\tau\wedge T$, but then we have to restart at $K = K_0$ and keep repeating. Then we get the value is

$$F(K) = f(K) + E^K\big(e^{-\lambda(\tau\wedge T)}\big) F(K_0)$$

$$= f(K) + E^K\Big[e^{-\lambda\tau} I_{\{\tau<T\}} + e^{-\lambda T} I_{\{T<\tau\}} \Big] F(K_0)$$

$$= f(K) + \Big\{ \frac{\nu}{\lambda+\nu} + \frac{\lambda}{\lambda+\nu} E e^{-(\lambda+\nu)\tau} \Big\} F(K_0)$$

$$\Rightarrow \quad \boxed{\lambda\big(1 - E^{K_0} e^{-(\lambda+\nu)\tau}\big) F(K_0) = (\lambda+\nu) f(K_0).}$$

'Expectations Theory' for term structure (3/10/08)

Clive Bowsher has been looking at a model for term structure which (in continuous time) would have the structural property

$$-\log P(t,T) = g(T-t) + E_t\left(\int_t^T r_s\,ds\right).$$

Using the notation from §5 of "Which model for term structure of interest rates should one use?" this would imply that (in one-dimensional setting)

$$+\int_0^t r_s\,ds - \log P(t,T) = +\int_0^t \Sigma(s,T)\,dW_s + \tfrac{1}{2}\int_0^t \Sigma(s,T)^2\,ds$$

$$= g(T-t) + E_t\left(\int_0^T r_s\,ds\right)$$

and hence

$$E_t\left(\int_0^T r_s\,ds\right) - \int_0^t \Sigma(s,T)\,dW_s = g(T-t) - \tfrac{1}{2}\int_0^t \Sigma(s,T)^2\,ds$$

is a continuous FV martingale, therefore constant. Thus

$$g'(T-t) + \tfrac{1}{2}\Sigma(t,T)^2 = 0$$

which implies that $\Sigma(t,T)$ is a function only of $T-t$;

$$\Sigma(t,T) = \int_t^T \sigma(t,s)\,ds = h(T-t)$$

$$\Rightarrow \quad \sigma(t,T) = h'(T-t)$$

and this forces the conclusion that the vol process σ of the HJM characterisation of the model is deterministic, and a function of $(T-t)$ only.

Thus all of these models are Gaussian, and time-invariant; quite a restricted class, in fact.

The conjecture that

$$Y^{(k)}_t = \sup_{t \le t_k < t_{k-1} < \dots < t_1 \le T} \sum_{i=1}^{k} \left(Z_{t_i} - M^{(i)}_{t_i} + M^{(i)}_t \right) \quad \mathbb{P}\text{-a.s.}$$

is rejected numerically!

An example of multiple optimal stopping (21/10/08)

Discussions with Christian Bender + Nizar Touzi on the dual characterization of the solution of a multiple optimal leave me a bit confused. Let's work through a simple reasonably explicit example and see if this aids understanding.

(1) At times $t = 0, 1, \dots, T$, you are offered random variable Z_t which are IID $\exp(1)$. Suppose you get k choices (so $k=1$ is the standard optimal stopping problem), with unused choices thrown away. What is the optimal policy?

Let $V(t, z; k)$ be the value if at time t you still have k choices, and the current Z-value is z. Then

$$V(t, z; k) = \max\{ z + E[V(t+1, Z; k-1)], \; E[V(t+1, Z; k)] \}$$

Thus if we set $E\, V(t, Z; k) \equiv a_t(k)$, we have from the Bellman equation that

$$a_t(k) = E[\max\{ Z + a_{t+1}(k-1), \; a_{t+1}(k) \}$$

Also, it is clear that $a_t(k)$ increases with k, and decreases with t, so

$$\boxed{\begin{aligned} a_t(k) &= a_{t+1}(k) + E\left(Z - (a_{t+1}(k) - a_{t+1}(k-1))\right)^+ \\ &= a_{t+1}(k) + \exp\{-(a_{t+1}(k) - a_{t+1}(k-1))\} \end{aligned}}$$

We have $a_t(0) \equiv 0$, and $a_t(k) = T - t + 1$ if $T - t \le k - 1$.

(2) The Snell envelope for k choices is

$$Y_t^{(k)} = V(t, Z_t; k) = (Z_t + a_{t+1}(k-1)) \vee a_{t+1}(k)$$

$$= a_{t+1}(k) + (Z_t - \Delta a_{t+1}(k))^+$$

where $\Delta a_t(k) \equiv a_t(k) - a_t(k-1)$. The martingale of the Doob decomposition is

$$\boxed{\begin{aligned} \Delta M_t^{(k)} &= (Z_t - \Delta a_{t+1}(k))^+ - E\left((Z_t - \Delta a_{t+1}(k))^+\right) \\ \text{and} \quad \Delta A_{t+1}^{(k)} &= Y_t^{(k)} - E_t\left[Y_{t+1}^{(k)}\right] = (Z_t - \Delta a_t(k))^+ \end{aligned}}$$

Filtering by least action again (25/10/08)

1) Generalising the story on p36, we could suppose $Z = \binom{X}{Y}$ solves some (nice) SDE in $\mathbb{R}^d$

$$dZ_t = \sigma(t, Z_t)\, dW_t + \mu(Z_t)\, dt$$

where we observe Y and are looking for Z. Suppose we have a prior density $\exp(-\varphi(x))$ for X_0; in that case, the least-action problem is to

$$\min \left\{ \varphi(x_0) + \int_0^T \psi(t, x_t, p_t)\, dt \right\}$$

where $p_t \equiv \dot{x}_t$ and $\psi(t, x_t, \dot{x}_t) \equiv \frac{1}{2} \left| \sigma(z)^{-1} \left\{ \binom{\dot{x}_t}{\dot{y}_t} - \mu\binom{x_t}{y_t} \right\} \right|^2$. To deal with this practically, we would probably do a piecewise-linear interpolation of Y between time points at which it was observed.

2) The calculus of variations gives us for small perturbation η from optimal x

$$0 = \eta_0 \cdot D\varphi(x_0) + \int_0^T \{ D_x\psi \cdot \eta_t + D_p\psi \cdot \dot{\eta}_t \}\, dt$$

$$= \eta_0 \cdot D\varphi(x_0) + \int_0^T \eta_t^j \left\{ D_{x_j}\psi - \frac{\partial}{\partial t} D_{p_j}\psi - \dot{x}_k D_{p_j} D_{x_k}\psi - \dot{p}_k D_{p_j} D_{p_k}\psi \right\} dt$$

$$+ \left[\eta_t^j D_{p_j}\psi \right]_0^T$$

$\Rightarrow$ we need for optimality

$$\begin{cases} D_j\varphi(x_0) = D_{p_j}\psi(0, x_0, p_0) \\ D_{x_j}\psi - \frac{\partial}{\partial t} D_{p_j}\psi - \dot{x}_k D_{p_j} D_{x_k}\psi - \dot{p}_k D_{p_j} D_{p_k}\psi = 0 \\ D_{p_j}\psi(T, x_T, p_T) = 0 \end{cases}$$

3) <u>Special case</u>: $\sigma = I$, $\mu = \nabla U$, so $\psi = \frac{1}{2}\left| \binom{p}{\dot{y}} - \nabla U\binom{x}{y} \right|^2$, and

$$D_{x_j}\psi = -\left\{ \binom{p}{\dot{y}} - \nabla U(z) \right\}_k D_{jk}U(z), \quad D_{p_j}\psi = \left\{ \binom{p}{\dot{y}} - \mu(z) \right\}_j$$

$$D_{p_k} D_{z_j}\psi = -D_{jk}U(z), \quad D_{p_j} D_{p_k}\psi = \delta_{jk}$$

4) <u>A simple example</u> Suppose that the underlying signal X is a 1-D OU

$$dX = \sigma_x\, dW - \beta_x X\, dt$$

to which we add an independent OU noise y

$$dy = \sigma_y \, dW' - \beta_y \, y \, dt$$

to get the observation process $Y = X + y$. Then $Z = [X; Y]$ satisfies

$$dZ = \begin{bmatrix} \sigma_x & 0 \\ \sigma_x & \sigma_y \end{bmatrix} \begin{pmatrix} dW \\ dW' \end{pmatrix} + \underbrace{\begin{bmatrix} -\beta_x & 0 \\ -\beta_x + \beta_y & -\beta_y \end{bmatrix}}_{A} \begin{pmatrix} X \\ Y \end{pmatrix} dt$$

Therefore

$$\psi(t,x,p) = \frac{1}{2}\left(\begin{pmatrix} p \\ \mathring{Y} \end{pmatrix} - A\begin{pmatrix} x \\ Y \end{pmatrix}\right)^T M \left(\begin{pmatrix} p \\ \mathring{Y} \end{pmatrix} - A\begin{pmatrix} x \\ Y \end{pmatrix}\right)$$

where $M \equiv (\sigma\sigma^T)^{-1}$,

$$= \frac{1}{2}\left(p * v1 - x * a1 + b\right)^T M \left(\text{——} \right) \qquad v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad a_1 = A(:,1)$$

$$= \frac{1}{2}\Big\{ p^2 \, v1 \cdot M v1 + x^2 \, a1 \cdot M a1 + b \cdot Mb$$
$$- 2px \, a1 \cdot M v1 + 2p \, v1 \cdot Mb - 2x \, a1 \cdot Mb \Big\}$$

where $b = \mathring{Y} * v_2 - a_2 Y$, $\mathring{b} = \mathring{\mathring{Y}} * v_2 - a_2 \mathring{Y}$

Beauty contests (14/11/08)

(1) Let's return to the diverse beliefs work with Angus, in particular, the way it looks for log agents. We shall suppose that all agents have the same ρ for simplicity. Then we shall have

$$c_t^j = e^{-\rho t} \Lambda_t^j / \nu_j \zeta_t \qquad , \quad \delta_t \zeta_t = e^{-\rho t} \sum \Lambda_t^j / \nu_j$$

We have easily $w_0^j = 1/\rho \nu_j \zeta_0$ and by some irrelevant scaling we can assume that the ν_j^{-1} sum to 1. Thus $\boxed{\delta_t \zeta_t = e^{-\rho t} \bar{\Lambda}_t}$ where

$\bar{\Lambda}_t = \sum \nu_j^{-1} \Lambda_t^j$ is the density of the initial-wealth-weighted average of the agents' beliefs. Notice that

$$d\bar{\Lambda}_t = \bar{\Lambda}_t \frac{\sum \nu_j^{-1} \Lambda_t^j \alpha_t^j \, dX_t}{\bar{\Lambda}_t} \equiv \bar{\Lambda}_t \left(\sum p_t^j \alpha_t^j \right) dX_t$$

$$\equiv \bar{\Lambda}_t \, \bar{\alpha}_t \, dX_t$$

where $p_t^j = \nu_j^{-1} \Lambda_t^j / \bar{\Lambda}_t$, a probability distribution.

(2) Would it benefit agents to <u>profess different beliefs</u>? Suppose that agent j actually believes the drift is α_t^j, but pretends to believe that it is $\tilde{\alpha}_t^j$, with associated likelihood-ratio martingale $\tilde{\Lambda}_t^j$. He acts as if he believed what he professes, so the state price density is derived as before

$$\delta_t \tilde{\zeta}_t = e^{-\rho t} \bar{\Lambda}_t = e^{-\rho t} \sum_j \nu_j^{-1} \tilde{\Lambda}_t^j .$$

Agent j has objective

$$\max E^0 \int_0^\infty e^{-\rho t} \Lambda_t^j \log\left(e^{-\rho t} \tilde{\Lambda}_t^j \nu_j^{-1} / \tilde{\zeta}_t \right) dt$$

$$\doteq \max E^0 \int_0^\infty e^{-\rho t} \Lambda_t^j \log \left(\tilde{\Lambda}_t^j / \bar{\Lambda}_t \right) dt \qquad \text{(equality to within irrelevant constants)}$$

The objective to be maximised is equivalently

$$E^0 \int_0^\infty e^{-\rho t} \Lambda_t^j \log\left(\tilde{\Lambda}_t^j / \bar{\Lambda}_t \right) dt$$

$$= \bar{E} \int_0^\infty e^{-\rho t} \frac{\Lambda_t^j}{\bar{\Lambda}_t} \log\left(\tilde{\Lambda}_t^j / \bar{\Lambda}_t \right) dt$$

Under $\bar{P}$, we have $dX = d\bar{X} + \bar{\alpha}_t dt$, where $\bar{\alpha}_t = \sum_j p_t^j \tilde{\alpha}_t^j$ is the weighted average drift and $\bar{X}$ is a $\bar{P}$-Brownian motion. We have that

$$\Lambda^j / \bar{\Lambda} = \mathcal{E}((\alpha^j - \bar{\alpha})\cdot \bar{X})_t$$

the Doléans exponential martingale. Thus the objective we have to work on is

$$\bar{E} \int_0^\infty e^{-\rho t} \, \mathcal{E}((\alpha^j - \bar{\alpha})\cdot\bar{X})_t \left\{ \int_0^t (\tilde{\alpha}^j_s - \bar{\alpha}_s)\, d\bar{X}_s - \tfrac{1}{2}\int_0^t (\tilde{\alpha}^j_s - \bar{\alpha}_s)^2 ds \right\} dt$$

$$= \bar{E} \int_0^\infty e^{-\rho t} \int_0^t \mathcal{E}((\alpha^j - \bar{\alpha})\cdot\bar{X})_s \left\{ (\alpha^j_s - \bar{\alpha}_s)(\tilde{\alpha}^j_s - \bar{\alpha}_s) - \tfrac{1}{2}(\tilde{\alpha}^j_s - \bar{\alpha}_s)^2 \right\} ds \, dt$$

$$= \bar{E} \int_0^\infty \rho^{-1} e^{-\rho s} \, \mathcal{E}((\alpha^j - \bar{\alpha})\cdot\bar{X})_s \, (\tilde{\alpha}^j_s - \bar{\alpha}_s)(\alpha^j_s - \tfrac{1}{2}\bar{\alpha}_s - \tfrac{1}{2}\tilde{\alpha}^j_s) \, ds$$

$$= E^0 \int_0^\infty \rho^{-1} e^{-\rho s} \Lambda^j_s \, (\tilde{\alpha}^j_s - \bar{\alpha}_s)(\alpha^j_s - \tfrac{1}{2}(\bar{\alpha}_s + \tilde{\alpha}^j_s)) \, ds.$$

This is to be maximised over the choice of the professed beliefs $\tilde{\alpha}^j$. We suppose that the other $\tilde{\alpha}^i$ are for the time being held fixed, and we drop the subscript for time, to keep things clean.

Notice that $\bar{\alpha} = p^j \tilde{\alpha}^j + b_j \equiv p^j \tilde{\alpha}^j + \sum_{i \neq j} p^i \tilde{\alpha}^i$, so our job is to maximise the quadratic

$$(\tilde{\alpha}^j (1 - p^j) - b_j)(\alpha^j - \tfrac{1}{2} b_j - \tfrac{1}{2}(1 + p^j)\tilde{\alpha}^j)$$

which is achieved when

$$0 = -(1 - (p^j)^2)\tilde{\alpha}^j + \tfrac{1}{2} b_j (1 + p^j) + (1 - p_j)(\alpha^j - \tfrac{1}{2} b_j)$$

$$= -(1 - (p^j)^2)\tilde{\alpha}^j + p_j b_j + (1 - p_j)\alpha^j$$

$$\Rightarrow \quad \boxed{\tilde{\alpha}^j = \frac{p_j b_j}{1 - (p^j)^2} + \frac{\alpha^j}{1 + p_j}}$$

Then we can proceed to simplify further:

$$\frac{1 - p_j^2}{p_j} \tilde{\alpha}^j = b_j + \frac{\alpha^j (1 - p_j)}{p_j}$$

$$\Rightarrow \left(\frac{1-p_j^2}{p_j} + p_j\right)\tilde{\alpha}^j = B + \frac{\alpha^j(1-p_j)}{p_j} \qquad B \equiv \sum p_i \tilde{\alpha}^i$$

$$\Rightarrow p_j \tilde{\alpha}^j = p_j^2 B + \alpha^j p_j (1-p_j)$$

so if we sum on j we learn that

$$(1 - \sum p_j^2) B = \sum \alpha^j p_j (1-p_j)$$

$$\Rightarrow \boxed{B = \frac{\sum \alpha^j p_j (1-p_j)}{\sum p_j (1-p_j)}}$$

and

$$\boxed{\tilde{\alpha}^j = p_j B + (1-p_j)\alpha^j}$$

Thus agent j will profess a convex combination of the beliefs of the population and himself!

Impact of 'wrong' data? (16/12/08)

1) A question I put to Michael Li was "Suppose people believe a gamma-gaussian world, but the data is actually t_6?" There are two effects of a big observation; the observer should revise his views about the mean, and about the variance. To do this reasonably cleanly, I think we need to let the mean and the precision move a little. We'll suppose that the mean does a zero-mean Gaussian random walk with variance ε/τ. How about the precision?

2) Suppose $X \sim \Gamma(\alpha,\beta)$, and $Q \sim B(a,b)$. Then the distribution of $Y \equiv XQ$ is given by

$$P(Y \le y) = \int_0^1 \frac{q^{a-1}(1-q)^{b-1}}{B(a,b)} \cdot \int_0^{y/q} (\beta x)^{\alpha-1} e^{-\beta x} \frac{\beta\, dx}{\Gamma(\alpha)}\, dq$$

So $$P(Y \in dy)/dy = \int_0^1 \frac{q^{a-1}(1-q)^{b-1}}{B(a,b)} \frac{1}{q} \left(\frac{\beta y}{q}\right)^{\alpha-1} e^{-\beta y/q} \frac{\beta\, dq}{\Gamma(\alpha)}$$

thus

$$e^{\beta y} P(Y \in dy)/dy = \int_0^1 \frac{q^{a-1}(1-q)^{b-1}}{B(a,b)} q^{-\alpha} (\beta y)^{\alpha-1} e^{-\beta y(\frac{1}{q}-1)} \frac{\beta\, dq}{\Gamma(\alpha)}$$

$\frac{1}{q} - 1 \equiv v$

$q = \frac{1}{1+v}$

$dq = \frac{-dv}{(1+v)^2}$

$$= \int_0^\infty (1+v)^{\alpha-a-1} \left(\frac{v}{1+v}\right)^{b-1} (\beta y)^{\alpha-1} e^{-\beta v y} \frac{\beta\, dv}{\Gamma(\alpha)B(a,b)}$$

$$= (\beta y)^{\alpha-1} \int_0^\infty v^{b-1} e^{-\beta v y} \frac{\beta\, dv}{\Gamma(a)\Gamma(b)} \qquad \text{if } \boxed{\alpha = a+b}$$

$$= \beta(\beta y)^{a-1}/\Gamma(a)$$

Thus if we multiply a $\Gamma(\alpha)$ random variable by an independent $B(a, \alpha-a)$ random variable, we get a $\Gamma(a)$ random variable. This is useful here, because it can prevent the shape parameter of the gamma posterior from drifting off to infinity.

3) Suppose now that at time t, given $\mathcal{Y}_t$, we have a gamma-gaussian posterior for (μ, τ): density

$$\propto (\beta\tau)^{\alpha-1} \exp\{-\beta\tau - \frac{K\tau}{2}(\mu-\hat{\mu})^2\} \sqrt{\tau}$$

so the density for the next (y, μ, τ) will be

(??)

$$\propto \int_0^1 \frac{q^{a-1}(1-q)^{b-1}}{B(a,b)}\,dq \int \frac{e^{-x^2\tau/2\varepsilon}}{\sqrt{\varepsilon/\tau}}\,dx \; \frac{1}{q}\left(\frac{\beta\tau}{q}\right)^{\alpha-1} e^{-\beta\tau/q} \exp\left\{-\frac{K\tau}{2}(\mu-\hat{\mu}-x)^2-(y-\mu)^2\tau/2\right\}\tau^{\frac{1}{2}}$$

$$\propto \int_0^1 q^{a-1}(1-q)^{b-1}\,dq \; \frac{1}{q}\left(\frac{\beta\tau}{q}\right)^{\alpha-1} \exp\left\{-\frac{\beta\tau}{q}-\frac{(y-\mu)^2\tau}{2}-\frac{\tau}{2}(\mu-\hat{\mu})^2\frac{K}{1+K\varepsilon}\right\}\sqrt{\tau^2}$$

$$\propto e^{-\beta\tau-\tau(y-\mu)^2/2-\frac{1}{2}K\tau(\mu-\hat{\mu})^2/(1+K\varepsilon)} \int_0^\infty \frac{v^{b-1}}{(1+v)^{a+b}}(1+v)^\alpha(\beta\tau)^{\alpha-1}e^{-\beta\tau v}\,dv \cdot \sqrt{\tau^2}$$

$$\left[\left(\tfrac{1}{q}-1\right)=v\right]$$

$$\propto \exp\left[-\beta\tau-\frac{\tau}{2}(y-\mu)^2-\frac{K\tau}{2(1+K\varepsilon)}(\mu-\hat{\mu})^2\right](\beta\tau)^{a-1}\sqrt{\tau^2}$$

which again a gamma-Gaussian posterior, expressible as

$$(\beta\tau)^{a+\frac{1}{2}-1}e^{-\beta\tau}\exp\left\{-\frac{K'\tau}{2}(\mu-\tilde{\mu})^2-\frac{\theta}{1+\theta}\frac{\tau}{2}(y-\hat{\mu})^2\right\}\sqrt{\tau}$$

where $K' = 1+\frac{K}{1+K\varepsilon}$, $K'\tilde{\mu} = y+\frac{K\hat{\mu}}{1+K\varepsilon}$, $\theta = \frac{K}{1+K\varepsilon}$.

Hence the updating is expressed as

$$\begin{cases} \alpha' = a+\frac{1}{2} \\ \beta' = \beta+\frac{\theta}{1+\theta}(y-\hat{\mu})^2 \\ \hat{\mu}' \equiv \tilde{\mu} = (y+\theta\hat{\mu})/(1+\theta) \\ K' = 1+\theta \\ \theta \equiv K/(1+K\varepsilon) \end{cases}$$

Notice that we could by taking $a = \alpha-\frac{1}{2}$ keep the value of α constant. This has some appeal. Since $E(1/\tau) = \beta/(\alpha-1)$, we would want to fix on a value of α which was greater than 1. There is also a steady-state value of K:

↑ but

$$K^* = \frac{1}{2}\left\{1+\sqrt{1+4\varepsilon^{-1}}\right\}.$$

4) Another idea would be to multiply the gamma variable τ by an independent Beta, and then to multiply by some scalar $\lambda > 1$. This then goes through in a similar fashion; we shall have that the equations of the previous page are unaltered except for

$$\beta' = \beta/\lambda + \frac{\theta}{1+\theta}(y-\hat{\mu})^2$$

Since the net effect of multiplying by $\lambda B(a, \alpha-a)$ is to multiply the mean by $\lambda a/\alpha$, we could determine λ by the requirement

$$\lambda = \frac{\alpha}{a}$$

So if we're keeping α steady at α^* by using $a = \alpha^* - 1/2$, we would get

$$\lambda^* = 2\alpha^*/(2\alpha^*-1).$$

as the choice which imposes no extraneous drift on τ.

5) How do we compute prices from this? We could just pretend that there was no drifting of (μ, τ) - that is, the Bayesian story from the earlier study. We might have a chance with the full distribution, but it looks quite hard.

(6) SNAG: Multiplying τ by a Beta random variable of correct shape will give another Gamma random variable τ' with the required shape parameter, but $K\tau'$ is not the precision of the conditional dist^n of μ given observations!!

Nice updating of an AR(1) model (29/1/09)?

(i) Let's consider an AR(1) model (univariate for now) where we allow the level to which we're mean reverting, and the precision of the noise, vary a bit from time to time. Let $(\mathcal{Y}_t)$ be the observation filtration. We shall suppose that the posterior density $\pi_t(\mu,\tau)$ of (μ_t,τ_t) given $\mathcal{Y}_t$ has gamma-gaussian form

$$\pi_t(\mu,\tau) \propto \exp\left[-\frac{K\tau}{2}(\mu-\hat{\mu}_t)^2 - \beta_t\tau\right] \tau^{\alpha-1}\sqrt{\tau}$$

and we shall show that if we evolve μ_t, τ_t, Y_t appropriately, then this form is preserved.

(ii) Suppose that we carry out the following moves:

(a) $\quad \mu_{t+1} \sim N(\mu_t, (\lambda\tau_t)^{-1})$

(b) $\quad Y_{t+1} - \mu_{t+1} = B(Y_t - \mu_{t+1}) + N(0, \tau_t^{-1})$

(c) $\quad \tau_{t+1} = \gamma Z_{t+1}\tau_t \qquad$ where $Z \sim B(a,b)$, and $\gamma = \frac{a+b}{a}$.

We shall find the 'good' choices of a, b.

The likelihood of (μ_{t+1}, τ_t) given $\mathcal{Y}_{t+1}$ is thus proportional to

$$\int \exp\left[-\frac{\tau}{2}(Y_{t+1} - BY_t - (1-B)\mu)^2 - \frac{\lambda\tau}{2}(\mu-x)^2 - \frac{K\tau}{2}(x-\hat{\mu}_t)^2 - \beta_t\tau\right] \tau^{\alpha-1}\sqrt{\tau}\,\tau\, dx$$

$$\propto \exp\left[-\frac{\tau}{2}(1-B)^2(m-\mu)^2 - \frac{c\tau}{2}(\mu-\hat{\mu}_t)^2 - \beta_t\tau\right] \tau^{\alpha},$$

where we define

$$(1-B)m = Y_{t+1} - BY_t, \quad c = \lambda K/(\lambda+K).$$

For brevity, we shall write $Q = \frac{1}{2}(1-B)^2(m-\mu)^2 + \frac{c}{2}(\mu-\hat{\mu}_t)^2$, so that the posterior for (μ_{t+1},τ_t) given $\mathcal{Y}_{t+1}$ is proportional to

$$\exp(-Q\tau - \beta_t\tau)\,\tau^{\alpha}$$

What then is the posterior of $(\mu_{t+1}, \tau_{t+1} \mid \mathcal{Y}_{t+1})$?

(iii) To answer this, take some test function φ and consider $E\left[\varphi(\mu_{t+1},\tau_{t+1})\mid\mathcal{Y}_{t+1}\right]$.

This is proportional to $\qquad \tau = t/\gamma z$

$$\int_0^1 dz \iint \varphi(\mu, \gamma z \tau)\ \exp(-Q\tau - \beta_t \tau)\ z^{a-1}(1-z)^{b-1}\ \tau^\alpha$$

$$= \int d\mu \int_0^1 dz \int_0^\infty \frac{dt}{\gamma z}\ \varphi(\mu,t)\ \exp\left(-(Q+\beta)\, t/\gamma z\right) z^{a-1}(1-z)^{b-1}\left(\frac{t}{\gamma z}\right)^\alpha$$

$$\propto \int d\mu \int dt \int_0^1 dz\ \varphi(\mu,t)\ \exp\left(-(Q+\beta)t/\gamma z\right)(1-z)^{b-1}\ t^\alpha\ z^{a-2-\alpha}$$

$$= \int d\mu \int dt \int_1^\infty \frac{dv}{v^2}\ \varphi(\mu,t)\ \exp\left(-\frac{(Q+\beta)t}{\gamma} v\right) t^\alpha\ v^{\alpha+2-a}\ (v-1)^{b-1}\ v^{1-b}$$

$$= \int d\mu \int dt\ \varphi(\mu,t)\, t^\alpha \int_1^\infty \exp\left(-\frac{Q+\beta}{\gamma} tv\right)(v-1)^{b-1}\ dv$$

provided $\boxed{\alpha + 1 - a - b = 0}$

$$= \int d\mu \int dt\ \varphi(\mu,t)\ t^\alpha \int_0^\infty e^{-\frac{Q+\beta}{\gamma} ts}\ s^{b-1} ds\ e^{-(Q+\beta)t/\gamma}$$

$$\propto \int d\mu \int dt\ \varphi(\mu,t)\left(\frac{\gamma}{t(Q+\beta)}\right)^b\ t^\alpha\ e^{-(Q+\beta)t/\gamma}$$

so posterior for $(\mu_{t+1}, \tau_{t+1} \mid y_{t+1})$ has density

$$\propto t^{\alpha-b}\ (Q+\beta)^{-b}\ \exp\{-(Q+\beta)t/\gamma\}$$

and this is no good for updating unless $\underline{b=0}$. This forces $\gamma = 1$, $\alpha = a-1$, and there is no wiggling of τ.

So the conclusion is that the shaking of τ does not happen, and the posterior for (μ_{t+1}, τ) given y_{t+1} has density proportional to

$$\tau^{\alpha-\frac{1}{2}}\ \exp\left[-\frac{\tau}{2}(1-B)^2(m-\mu)^2 - \frac{c\tau}{2}(\mu-\hat\mu_t)^2 - \beta_t \tau\right]\ \sqrt{\tau}$$

$$= \exp\left[-\frac{\tau}{2}\left((1-B)^2 + c\right)\mu^2 + \mu\tau\left(m(1-B)^2 + c\hat\mu_t\right) - \frac{(1-B)^2}{2}\tau m^2 - \frac{c\tau}{2}\hat\mu_t^2 - \beta_t\tau\right]\tau^\alpha$$

Thus if we fix

$$K = (1-B)^2 + \frac{\lambda K}{\lambda + K} \equiv pK + qK$$

we keep the conditional precision the same, and have posterior

$$\propto \exp\left[-\frac{K\tau}{2}\mu^2 + K\tau\mu\,(pm + q\hat{\mu}_t) - \frac{Kp\tau m^2}{2} - \frac{Kq\tau}{2}\hat{\mu}_t^2 - \beta_t\tau \right] \tau^{\alpha}$$

$$= \exp\left[-\frac{K\tau}{2}(\mu - \hat{\mu}_{t+1})^2 + \frac{1}{2}K\tau\,(pm + q\hat{\mu}_t)^2 - \frac{1}{2}K\tau\,(pm^2 + q\hat{\mu}_t^2) - \beta_t\tau \right] \tau^{\alpha}$$

$$\overline{(\hat{\mu}_{t+1} \equiv pm + q\hat{\mu}_t)}$$

$$= \exp\left[-\frac{K\tau}{2}(\mu - \hat{\mu}_{t+1})^2 - \frac{1}{2}K\tau\, pq(m - \hat{\mu}_t)^2 - \beta_t\tau \right] \tau^{\alpha - \frac{1}{2}} \sqrt{\tau}$$

So the precision of $\mu_{t+1} | y_{t+1}$ stays at $K\tau$, and the parameters of the Gamma for τ update to

$$\begin{cases} \alpha_{t+1} = \alpha_t + \frac{1}{2} \\ \beta_{t+1} = \beta_t + \frac{1}{2}K pq\,(m - \hat{\mu}_t)^2 \end{cases}$$

Structural models of default again (1/2/09)

(i) Let's come back to the setting of Leland, Leland & Toft, Décamps + Villeneuve. Here we have the dynamics

$$dV_t = V_{t-}\left(dZ_t + (r-\delta)\,dt\right)$$

for the value of a firm's assets in the pricing measure, where Z is some Lévy process which is also a martingale (typically σdW in many applications). The firm issues debt at rate $f\,dt$, with maturity profile $\mu(\cdot)$. Write $\Phi(x) = \int_x^\infty \mu(dt)$ for the tail of μ. The firm defaults at stopping time τ, and on default a fraction π of the firm's value is recovered. Debt of face value 1 attracts coupons at rate $c\,dt$.

At any given time, amount of debt currently issued which has to be paid back s units of time in the future is

$$-f\int_0^\infty dv\; \Phi'(s+v) = f\,\Phi(s)$$

and total face value of all issued debt remains constant at

$$F = f\int_0^\infty \Phi(s)\,ds.$$

Debt issued at time 0 with maturity t is worth (unit face value)

$$d(V,\tau,t) \equiv E^V\left[\int_0^{t\wedge\tau} c\,e^{-rs}\,ds + e^{-rt} I_{\{\tau > t\}} + \frac{\pi V_\tau}{F}\,e^{-r\tau} I_{\{\tau\le t\}}\right]$$

so that the total value at time 0 of all issued debt is

$$D_0(V,\tau) = \int_0^\infty d(V,\tau,t)\; f\,\Phi(t)\,dt.$$

The total coupon $C = cF$, and there is a tax rebate $\theta C\,dt$ being paid, so the value of the firm is

$$v_0(V,\tau) = E\left[\int_0^\tau e^{-rs}\left(\delta V_s + \theta C\right)ds + \pi\, e^{-r\tau} V_\tau\right]$$

It seems that everybody deduces that the value at time 0 of the firm's equity is

$$q_0(V,\tau) = v_0(V,\tau) - D_0(V,\tau)$$

and this is I believe incorrect, and the source of the confusion.

(ii) The point is this – which Leland identifies in his critique of DV: not all the payments associated with the issuance + retirement of bonds have been identified.

There is the payment of πV_τ at default; and before default there is payment of coupons at rate $C\,dt$, repayment of capital at rate $f\,dt$, and receipts of the current market value of newly issued bonds. The current market value of newly issued bonds is

$$f\,ds \int_0^\infty \mu(dt)\; d(V,\tau,t)$$

$$= f\,ds\;\; E^V\Big[\int_0^\tau \Phi(t)\,c\,e^{-rt}dt + \int_0^\tau e^{-rt}\mu(dt) + \frac{\pi V_\tau}{F}\,e^{-r\tau}\,\Phi(\tau)\Big]$$

$$\equiv g(V)\,ds,$$

say. Thus the value of debt will be not $D(V,\tau)$, but

$$\tilde D(V,\tau) = E\Big[\int_0^\tau (C+f-g(V_s))\,e^{-rs}ds + e^{-r\tau}\pi V_\tau\Big]$$

(iii) Let's now specialize to $\mu(dt) = me^{-mt}dt$, $\Phi(t) = e^{-mt}$, $f = mF$, and

$$f^{-1}g(V) = E^V\Big[\int_0^\tau ce^{-(m+r)t}dt + m\int_0^\tau e^{-(m+r)t}dt + \frac{\pi V_\tau}{F}\,e^{-(m+r)\tau}\Big]$$

$$= E^V\Big[\frac{m+c}{m+r}\{1-e^{-(m+r)\tau}\} + \frac{\pi V_\tau}{F}\,e^{-(m+r)\tau}\Big]$$

If we use a barrier rule $\tau = \inf\{t : V_t = V_B\}$, then we can now simplify this to

$$f^{-1}g(V) = \frac{m+c}{m+r} + \Big(\frac{\pi V_\tau}{F} - \frac{m+c}{m+r}\Big)\Big(\frac{V}{V_B}\Big)^{-\alpha'}$$

where $-\alpha' < 0 < \beta'$ are the roots of (assume $Z_t = \sigma W_t$)

$$Q_0(t) \equiv \tfrac12\sigma^2 t(t-1) + (r-\delta)t - (m+r) = 0.$$

This allows us to get an ODE to decide what $\tilde D$ should be. Notice that

$$C+f - g(V) = C+f - f\,\frac{m+c}{m+r} - f\Big(\frac{\pi V_B}{F} - \frac{m+c}{m+r}\Big)\Big(\frac{V}{V_B}\Big)^{-\alpha'}$$

$$= \frac{(c+m)Fr}{m+r} + f\Big(\frac{m+c}{m+r} - \frac{\pi V_B}{F}\Big)\Big(\frac{V}{V_B}\Big)^{-\alpha'}$$

$$= a_0 + b_0 V^{-\alpha'}$$

say. We therefore have

$$-r\tilde D + \tfrac12\sigma^2V^2\tilde D'' + (r-\delta)V\tilde D' + a_0 + b_0V^{-\alpha'} = 0, \quad \tilde D(V_B) = \pi V_B$$

This ODE is solved by (rejecting the growing solution at infinity)

$$\tilde{D}(V) = A\left(\frac{V}{V_B}\right)^{-\alpha} + \frac{a_0}{r} + b_1 V^{-\alpha'}$$

where $-\alpha < 0 < \beta$ solve $Q(t) \equiv \frac{1}{2}\sigma^2 t(t-1) + (r-\delta)t - r = 0$, where $b_1 Q(\alpha') + b_0 = 0$, so after simplifying,

$$b_1 = -\frac{b_0}{m}.$$

Matching the BC at $V = V_B$ gives

$$\tilde{D}(V) = \pi V_B \left(\frac{V}{V_B}\right)^{-\alpha} + \frac{a_0}{r}\left(1 - \left(\frac{V}{V_B}\right)^{-\alpha}\right) + b_1\left\{V^{-\alpha'} - V_B^{-\alpha'}\left(\frac{V}{V_B}\right)^{-\alpha}\right\}$$

$$= \pi V_B \left(\frac{V}{V_B}\right)^{-\alpha} + \frac{a_0}{r}\left(1 - \left(\frac{V}{V_B}\right)^{-\alpha}\right) + b_1 V_B^{-\alpha'}\left\{\left(\frac{V}{V_B}\right)^{-\alpha'} - \left(\frac{V}{V_B}\right)^{-\alpha}\right\}$$

Similarly, we calculate

$$v(V) = \frac{\theta C}{r}\left\{1 - \left(\frac{V}{V_B}\right)^{-\alpha}\right\} + V - (1-\pi)V_B\left(\frac{V}{V_B}\right)^{-\alpha}$$

(iv) The correct expression for equity value is $q(V) = v(V) - \tilde{D}(V)$, and if the boundary condition for optimality given by Leland is correct, this would say we need

$$\left.\frac{\partial q}{\partial V}\right|_{V=V_B} = 0$$

Now

$$\left.\frac{\partial v}{\partial V}\right|_{V=V_B} = \frac{\alpha\,\theta C}{r V_B} + 1 + \alpha(1-\pi)$$

$$\left.\frac{\partial \tilde{D}}{\partial V}\right|_{V=V_B} = -\alpha\pi + \frac{a_0 \alpha}{r V_B} + \frac{b_0 V_B^{-\alpha'}}{m}\,\frac{\alpha' - \alpha}{V_B}$$

$$= -\alpha\pi + \frac{a_0\alpha}{r V_B} + \frac{f}{m}\left(\frac{m+c}{m+r} - \frac{\pi V_B}{F}\right)\frac{\alpha'-\alpha}{V_B}$$

Working through the algebra, we get

$$\boxed{V_B = \frac{\dfrac{\alpha' F(m+c)}{m+r} - \dfrac{\alpha \theta C}{r}}{1 + \pi\alpha' + (1-\pi)\alpha}}$$

which is exactly the expression obtained by Leland

Constraints on consumption drawdown (25/2/09)

(i) Suppose we have standard wealth dynamics

$$dw_t = r w_t\,dt + \theta_t(\sigma\,dW_t + (\mu - r)dt) - c_t\,dt$$

but we insist that

$$c_t \geq b\,\bar{c}_t \equiv b \sup_{s\leq t} c_s$$

for some $b \in (0,1)$. How would an agent optimise $E\left[\int_0^\infty e^{-\rho t} U(c_t)\,dt\right]$ for a CRRA utility under this constraint?

(ii) If we set $V(w,\bar{c}) \equiv \sup E\left[\int_0^\infty e^{-\rho t} U(c_t)\,dt \,\middle|\, w_0 = w, \bar{c}_0 = \bar{c}\right]$ then we have the familiar scaling $V(\lambda w, \lambda\bar{c}) = \lambda^{1-R} V(w,\bar{c})$ so we have $V(w,\bar{c}) = \bar{c}^{1-R} v(x)$ where $x \equiv w/\bar{c}$.

(iii) Notice that if w falls to $b\bar{c}/r$, then the interest on the capital is only just enough to cover the locked-in consumption, so we shall have

$$V\left(\frac{b\bar{c}}{r}, \bar{c}\right) = \frac{U(b\bar{c})}{\rho} = \bar{c}^{\,1-R}\,\frac{U(b)}{\rho}$$

$$\therefore \quad \boxed{v(b/r) = U(b)/\rho}$$

(iv) The HJB equations

$$0 = \sup\left[U(c) - \rho V + (rw + \theta(\mu - r) - c)V_w + \tfrac{1}{2}\sigma^2\theta^2 V_{ww}\right]$$

$$0 \geq V_{\bar{c}} = \bar{c}^{-R}\left[(1-R)v(x) - x v'(x)\right]$$

give us in terms of v and $\pi = \theta/\bar{c}$, $t = c/\bar{c}$

$$0 = \sup_{b\leq t\leq 1} \bar{c}^{1-R}\left[U(t) - \rho v + (rx + \pi(\mu - r) - t)v' + \tfrac{1}{2}\sigma^2\pi^2 v''\right]$$

$$\therefore \quad c/\bar{c} = 1 \wedge ((v')^{-1/R} \vee b), \qquad \pi = -(\mu - r)v'/\sigma^2 v''$$

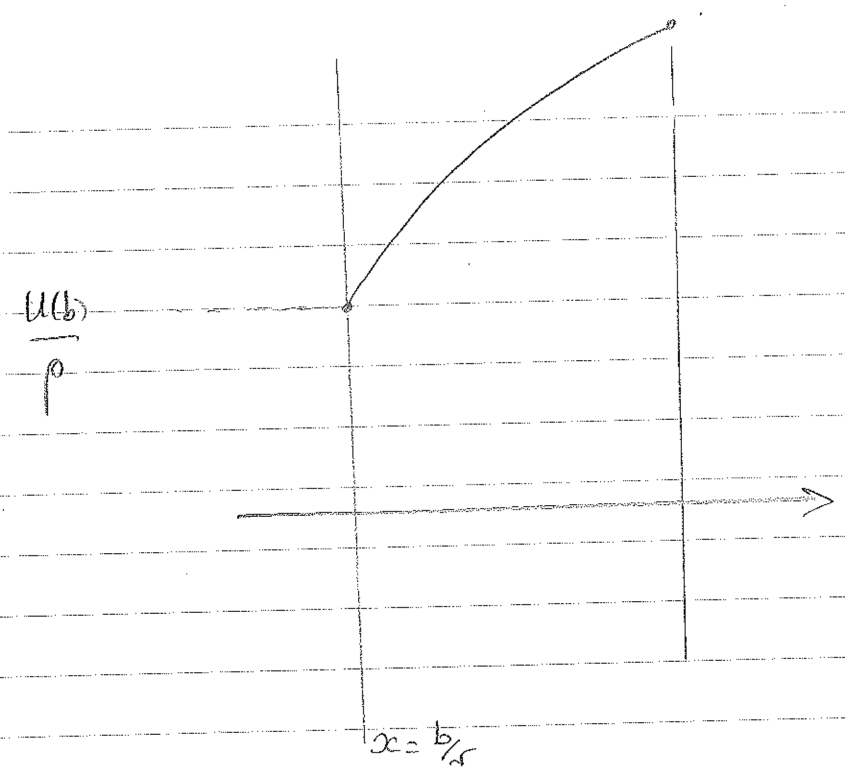
and

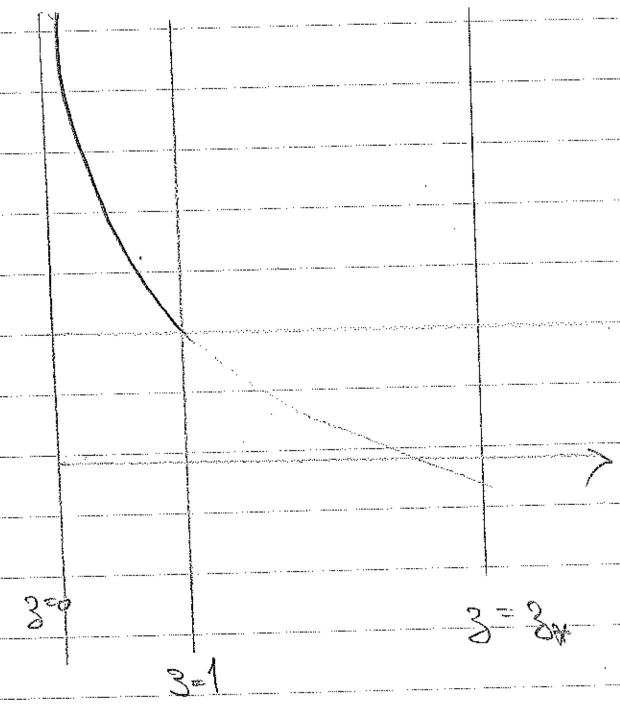
$$\boxed{0 = \tilde{U}_b(v') - \rho v + r x v' - \frac{\kappa^2}{2}\,\frac{v'^2}{v''}}$$

Now observe that we shall have

$$\boxed{b^{-R} \geq (c/\bar{c})^{-R} = v' \geq 1}$$

$$\left[\tilde{U}_b(y) = U(1\wedge(y^{-1/R}\vee b)) - y\{1\wedge(y^{-1/R}\vee b)\}\right]$$





Dual problem

$$0 = \tilde{U}(z) - \rho J + (\rho - r) z J' + \frac{\kappa^2}{2} z^2 J''$$

solved by

$$J = A z^{-\alpha} + B z^{\beta} - \frac{\tilde{U}(z)}{Q(1-1/R)} = A z^{-\alpha} + B z^{\beta} + \frac{\tilde{U}(z)}{\gamma_R}$$

after some calculations, where $-\alpha < 0 < \beta$ are roots of $Q(t) \equiv \frac{1}{2}\kappa^2 t(t-1) + (\rho - r)t - \rho$.

When the wealth gets high enough, we expect that we shall need to raise $\bar{c}$. Therefore when $v' = 1$ we raise $\bar{c}$. So what I think happens is that $J(z) = a\,\tilde{u}(z)$ for some $a > 0$ while $z \leq 1$, with a C^1 join to the solution of the ODE at $z=1$:

$$\begin{cases} a\,\tilde{u}(1) = A + B + \dfrac{\tilde{u}(1)}{\gamma_M} \\ -a = -\alpha A + \beta B - \dfrac{1}{\gamma_M} \end{cases} \qquad \tilde{u}(1) = -\frac{R}{R-1}$$

$$\therefore \begin{pmatrix} 1 & 1 \\ -\alpha & \beta \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \nu \begin{pmatrix} R \\ R-1 \end{pmatrix}, \qquad \nu \equiv \left(\frac{1}{\gamma_M} - a\right)\frac{1}{R-1}$$

$$\text{so } \begin{pmatrix} A \\ B \end{pmatrix} = \frac{\nu}{\alpha+\beta}\begin{pmatrix} \beta & -1 \\ \alpha & 1 \end{pmatrix}\begin{pmatrix} R \\ R-1 \end{pmatrix}$$

$$= \frac{\nu}{\alpha+\beta}\begin{pmatrix} \beta + 1 - R \\ \alpha - 1 + R \end{pmatrix}$$

Alternatively, we could propose a value of z_*, at which we would have to have $J' = -b/r$, $J = u(b)/\rho - z_* b/r$, and then solve

$$0 = \tilde{u}(z) - \rho J + (\rho - r) z J' + \tfrac{1}{2}\kappa^2 z^2 J''$$

back from z_*. Then we would have to check at $z = 1$ whether we get
$J'(1) = (1 - \tfrac{1}{R})\, J(1)$.

Possible stories for model updating (23/3/09).

(i) Suppose we're presented with a sequence of observations $Y_t \sim N(\mu, V)$ where we're trying to infer μ, V. Suppose also that we allow at each stage a jump of μ, with prob p_j, distributed as $N(0, \lambda_j V)$, $j = 1, \dots, J$. How do we do the inference?

(ii) Doing this honestly is rather messy because a Bayesian analysis would carry along some weighted average of Gaussians. Nevertheless, this could be worth knowing. Suppose the posterior at time t is $\sum w_t^i N(\hat{\mu}_t^i, V_t^i)$. Then given new observation Y_{t+1}, and given that the jump was of type j, the posterior likelihood for particle i will become

$$\frac{w_t^i \; p_j \; \exp\left[-\frac{1}{2}(x - \hat{\mu}_t^i)(V_t^i + \lambda_j V)^{-1}(x - \hat{\mu}_t^i) - \frac{1}{2}(Y_{t+1} - x)V^{-1}(Y_{t+1} - x)\right]}{(2\pi)^d \sqrt{\det(V_t^i + \lambda_j V)} \sqrt{\det V}},$$

which is maximised by $x = \hat{\mu}_{t+1}$, where $\hat{\mu}_{t+1}$ satisfies

$$\hat{\mu}_{t+1} - \hat{\mu}_t^i = (V_t^i + \lambda_j V)(V_t^i + (1+\lambda_j)V)^{-1}(Y_{t+1} - \hat{\mu}_t^i).$$

From this, we can work out the new weights of the descendant particles, and we could just keep the big ones. This approach has the virtue that it carries along the possibility of fat tails, but you should ideally know what V is.

(iii) Alternatively, we could substitute the MLE $x = \hat{\mu}_{t+1}$ into the likelihood, and after some calculations we find that the likelihood maximised over x becomes

$$\frac{w_t^i \; p_j \; \exp\left[-\frac{1}{2}(Y_{t+1} - \hat{\mu}_t)\cdot(V_t^i + (1+\lambda_j)V)^{-1}(Y_{t+1} - \hat{\mu}_t)\right]}{(2\pi)^d \sqrt{\det(V_t^i + \lambda_j V)} \sqrt{\det V}}$$

and next maximise over V. This maximisation is achieved by

$$0 = (1+\lambda)\,\Delta\Delta^T - \tilde{V}^{-1}(I + \lambda\tilde{V})^{-1}(I + (1+\lambda)\tilde{V})^3$$

where $\Delta \equiv (V_t^i)^{-\frac{1}{2}}(Y_{t+1} - \hat{\mu}_t)$, $\tilde{V} \equiv (V_t^i)^{-\frac{1}{2}} V (V_t^i)^{-\frac{1}{2}}$. But this doesn't make sense ($\det V = 0$?!)

ML estimation of covariance under constraints (31/3/09)

(i) Suppose we take some MVN sample in $\mathbb{R}^n$, and form the sample covariance $S = m^{-1}\sum_{i=1}^{m}(Y_i-\bar{Y})(Y_i-\bar{Y})^T$. Now we want to find the MLE of V under the restriction that $V = a_0 I + b_0 11^T$. This amounts to minimising

$$\operatorname{tr}(V^{-1}S) - \log\det V^{-1}$$

where V^{-1} is required to be of the form $aI + b11^T$. We might also wish to consider the possibility that $V^{-1} = \Lambda(aI+b11^T)\Lambda$ for some diagonal Λ; in this case, we aim to minimize

$$\operatorname{tr}\left((aI+b11^T)\tilde{S}\right) - \log\det(aI+b11^T) - 2\log\det\Lambda,$$

where $\tilde{S} = \Lambda S \Lambda$.

(ii) Either way, we need to minimise over (a,b) the expression $\operatorname{tr}((aI+b11^T)\tilde{S}) - \log\det(aI+b11^T)$. We have

$$L \equiv \operatorname{tr}\left((aI+b11^T)\tilde{S}\right) - \log\det(aI+b11^T) - 2\log\det\Lambda$$

$$= a\operatorname{tr}\tilde{S} + b\,1^T\tilde{S}1 - n\log a - \log\det\left(I+\frac{b}{a}11^T\right) - 2\log\det\Lambda$$

$$= a\operatorname{tr}\tilde{S} + b\,1^T\tilde{S}1 - n\log a - \log\det\left(I+\frac{nb}{a}e_1e_1^T\right) - 2\log\det\Lambda$$

by rotating 1 to $\sqrt{n}\,e_1 \equiv (\sqrt{n},0,\dots,0)^T$;

$$= a\operatorname{tr}\tilde{S} + b\,1^T\tilde{S}1 - n\log a - \log\left(1+\frac{nb}{a}\right) - 2\log\det\Lambda$$

The first-order conditions are

$$\left.\begin{aligned} 0 &= \frac{\partial L}{\partial a} = \operatorname{tr}\tilde{S} - \frac{n}{a} + \frac{nb}{a(a+nb)} \\ 0 &= \frac{\partial L}{\partial b} = 1^T\tilde{S}1 - \frac{n}{(a+nb)} \end{aligned}\right\}$$

The first of this is equivalently (using the second)

$$0 = \operatorname{tr}\tilde{S} - \frac{n-1}{a} - \frac{1^T\tilde{S}1}{n}$$

whence $$\boxed{\frac{1}{a} = \frac{1}{n-1}\left\{ \mathrm{tr}\,\tilde{S} - \frac{1^T \tilde{S} 1}{n} \right\}}$$

and $$\boxed{b = \frac{n(\mathrm{tr}\,\tilde{S} - 1^T\tilde{S}1)}{1^T\tilde{S}1\,(n\,\mathrm{tr}\,\tilde{S} - 1^T\tilde{S}1)}}$$

So the MLE for V is easily found to be
$$\boxed{\frac{1}{a} I - \frac{\mathrm{tr}\,\tilde{S} - 1^T\tilde{S}1}{n(n-1)} 11^T}$$

which says that on diagonal we get $\frac{1}{n}\mathrm{tr}\,\tilde{S}$, off diagonal $-(\mathrm{tr}\,\tilde{S} - 1^T\tilde{S}1)/n(n-1)$
So we match the trace of $\tilde{S}$, and $1^T\tilde{S}1$.

(iii) How would we now optimise over the diagonal matrix Λ? We have
$$0 = \frac{\partial L}{\partial \lambda_k} = 2\left[a \lambda_k S_{kk} + b(S\lambda)_k - \frac{1}{\lambda_k} \right]$$

which is equivalently
$$a \lambda_k^2 S_{kk} + b \lambda_k (S\lambda)_k - 1 = 0. \qquad (*)$$

We can try to solve this recursively, by using $\lambda^{(n)}$ to define $a = a(\lambda^{(n)})$, $b = b(\lambda^{(n)})$ and $S\lambda^{(n)}$, and then solve the quadratics $(*)$ to give $\lambda^{(n+1)}$. This appears to be quick and stable numerically, though there is of course a scaling indeterminacy.

Better notion for updatings (9/4/09)

(i) Suppose that the idea was that

$$Y_t = \mu_t + \varepsilon_t$$

where ε_t are IID $N(0,V)$ and $\mu_t - \mu_{t-1}$ are IID $N(0,?)$ where variance takes value v_k with prob'y p_k. Suppose we have a posterior $\pi_t(\mu)$ for μ at time t. Then

$$P(Y_{t+1} = y, \text{ variance } k \text{ chosen}, \mu_{t+1} = \mu)$$

$$= p_k \quad (\pi_t * \gamma_{v_k})(\mu) \quad \gamma_V(y-\mu)$$

where $\gamma_V(\cdot)$ is zero-mean Gaussian density with covariance V. If we integrate out the variable μ_{t+1}, we get

$$P(Y_{t+1} = y, \text{ variance } k \text{ chosen}) = \int p_k \, (\pi * \gamma_{v_k})(\mu) \, \gamma_V(y-\mu) \, d\mu$$

$$= p_k \quad f_k(y \mid \mathcal{Y}_t)$$

where $f_k(\cdot \mid \mathcal{Y}_t)$ is conditional density of Y_{t+1} given $\mathcal{Y}_t$ and that variance v_k got picked. This then gives us the different prob'ys of the different possible variances, and we could select the one which is most likely. This is a bit approximate, but uses the notion that one of the K possibilities will be overwhelmingly the most likely

(ii) Chaining the successive observations? As we gather one more observation, we need to update the density of $(Y_1, \ldots, Y_t)$ to the density of $(Y_1, \ldots, Y_{t+1})$. Strictly speaking, this should be done by multiplying by

$$\sum_k p_k \, f_k(Y_{t+1} \mid \mathcal{Y}_t)$$

but it may be OK to multiply just by $\quad p_{k^*} \, f_{k^*}(Y_{t+1} \mid \mathcal{Y}_t)$

Diverse mistaken beliefs (23/4/09)

(i) Let's come back to the diverse beliefs story with log agents, in a discrete-time setting now, where

$$X_t = \log(\delta_t / \delta_{t-1})$$

are IID $N(\mu, \tau^{-1})$ random variables. We still have the optimality FOCs

$$e^{-\rho_j t} \Lambda^j_t / c^j_t = \nu_j \zeta_t \quad \Rightarrow \quad \zeta_t \delta_t = \sum_j e^{-\rho_j t} \Lambda^j_t / \nu_j$$

and deduce the stock price (ex-dividend) to be

$$S_t \zeta_t = \sum_j \frac{e^{-\rho_j t} \Lambda^j_t}{\nu_j (e^{\rho_j} - 1)}$$

similarly to what we get in cts time. <u>What would happen if the agents observing S_t assumed that in fact</u> $S_t = \delta_t / \rho$?

(ii) Suppose you see IID random variables $X_1, X_2, \dots$, all $N(\mu, \tau^{-1})$, and you have a $N(\hat{\mu}_0, (K_0 \tau)^{-1})$ prior over μ? The joint density of $(\mu, X_1, \dots, X_t)$ is

$$\exp\left\{ - \frac{K_0 \tau}{2} (\mu - \hat{\mu}_0)^2 - \frac{\tau}{2} \sum_1^t (X_i - \mu)^2 \right\} \left(\frac{\tau}{2\pi}\right)^{t/2} \sqrt{\frac{K_0 \tau}{2\pi}}$$

$$= \exp\left\{ - \frac{K_t \tau}{2} (\mu - \hat{\mu}_t)^2 - \tfrac{1}{2} \tau K_t (\hat{\mu}_0 - \bar{X}_t)^2 p(1-p) - \tfrac{1}{2} \tau S_{XX} \right\} \left(\frac{\tau}{2\pi}\right)^{t/2} \sqrt{\frac{K_0 \tau}{2\pi}}$$

after some calculation, where

$$K_t = t + K_0$$

$$\bar{X}_t = t^{-1} \sum_1^t X_i \quad , \quad S_{XX} = \sum_1^t (X_i - \bar{X}_t)^2$$

$$K_t \hat{\mu}_t = K_0 \hat{\mu}_0 + t \bar{X}_t \quad , \quad p = \frac{K_0}{K_t}.$$

Integrating out the μ variable, we get a likelihood-ratio martingale:

$$(*) \quad \Lambda_t = \exp\left[-\tfrac{1}{2} \tau S_{XX} - \tfrac{1}{2} \tau K_t (\bar{X}_t - \hat{\mu}_0)^2 p(1-p) \right] \left(\frac{\tau}{2\pi}\right)^{t/2} \sqrt{\frac{K_0}{K_t}} \Big/ L_t$$

$$= \exp\left\{ - \frac{\tau}{2} \sum_1^t X_i^2 + \frac{\tau}{2} \left(K_t \hat{\mu}_t^2 - K_0 \hat{\mu}_0^2 \right) \right\} \left(\frac{\tau}{2\pi}\right)^{t/2} \sqrt{\frac{K_0}{K_t}} \Big/ L_t,$$

where L_t is density of observations in reference measure.

(iii) How does the new observation X_{t+1} cause Λ_t to change to Λ_{t+1}? Suppose we write $X_{t+1} = \hat\mu_t + \varepsilon$. Then we have

$$K_{t+1}\hat\mu_{t+1} = K_t\hat\mu_t + \hat\mu_t + \varepsilon = K_{t+1}\hat\mu_t + \varepsilon \Rightarrow \hat\mu_{t+1} = \hat\mu_t + \frac{\varepsilon}{K_{t+1}}$$

So the change in $K_t\hat\mu_t^2$ will be

$$K_{t+1}\hat\mu_{t+1}^2 - K_t\hat\mu_t^2 = K_{t+1}\left(\hat\mu_t + \frac{\varepsilon}{K_{t+1}}\right)^2 - K_t\hat\mu_t^2$$

$$= \hat\mu_t^2 + 2\varepsilon\hat\mu_t + \varepsilon^2/K_{t+1}$$

so that

$$2\log(\Lambda_{t+1}/\Lambda_t) = -\tau X_{t+1}^2 + \tau\left(\hat\mu_t^2 + 2\varepsilon\hat\mu_t + \frac{\varepsilon^2}{K_{t+1}}\right) + \log(K_t/K_{t+1}) + \log\tau$$

$+2\log(L_{t+1}/L_t)$

$$= \tau\left(-2\varepsilon\hat\mu_t - \varepsilon^2 + 2\varepsilon\hat\mu_t + \varepsilon^2/K_{t+1}\right) + \log(K_t/K_{t+1}) + \log\tau$$

$$= -\tau\varepsilon^2\frac{K_t}{K_{t+1}} + \log\left(K_t/K_{t+1}\right) + \log\tau$$

If we want to evolve the price, we have

$$\frac{S_{t+1}}{\delta_{t+1}} = \frac{S_t e^{\xi}}{\delta_{t+1}} = \frac{\sum_j e^{-\rho_j(t+1)}\Lambda^j_{t+1}/\nu_j(e^{\beta_j}-1)}{\sum_j e^{-\rho_j(t+1)}\Lambda^j_{t+1}/\nu_j}$$

$$= \frac{\sum_j e^{-\rho_j(t+1)}(\Lambda^j_{t+1}/\Lambda^j_t)\Lambda^j_t/\nu_j(e^{\beta_j}-1)}{\sum_j e^{-\rho_j(t+1)}(\Lambda^j_{t+1}/\Lambda^j_t)\Lambda^j_t/\nu_j}$$

and this is a root-finder for $\xi \equiv X_{t+1}$ which appears on the RHS as well as LHS

<u>Note</u>: If we took as the reference measure the law under which all the X_t were IID $N(0,\tau^{-1})$, then $\Lambda_t = \exp[\frac{\tau}{2}(K_t\hat\mu_t^2 - K_0\hat\mu_0^2)]\sqrt{K_0/K_t}$, which ties up with what we know in continuous time.

Numerics (if correct) are indicating very little effect here. Perhaps if $\log S_t$ were an OU process? Actually, things look more interesting with hundreds of agents!

But we also see things settling down as time passes, which is ultimately the learning effect - what about a steady-state AR(1) story?

(iv) Suppose we observe an AR(1) process plus noise:

$$Y_t = X_t + \eta_t \qquad \eta_t \sim N(0, v_\eta)$$

where
$$X_{t+1} = bX_t + \varepsilon_t \qquad \varepsilon_t \sim N(0, v_\varepsilon)$$

and we start off with a $N(\hat{X}_0, v_0)$ prior for X_0. We have the usual table

$$\left(\begin{matrix} X_{t+1} \\ Y_{t+1} \end{matrix} \middle| \mathcal{Y}_t \right) \sim N\left(\begin{pmatrix} b\hat{X}_t \\ b\hat{X}_t \end{pmatrix}, \begin{pmatrix} b^2 v_t + v_\varepsilon & b^2 v_t + v_\varepsilon \\ b^2 v_t + v_\varepsilon & b^2 v_t + v_\varepsilon + v_\eta \end{pmatrix} \right)$$

so
$$\hat{X}_{t+1} - b\hat{X}_t = \frac{b^2 v_t + v_\varepsilon}{b^2 v_t + v_\varepsilon + v_\eta} (Y_{t+1} - b\hat{X}_t)$$

and
$$v_{t+1} = b^2 v_t + v_\varepsilon - \frac{(b^2 v_t + v_\varepsilon)^2}{b^2 v_t + v_\varepsilon + v_\eta} = \frac{(b^2 v_t + v_\varepsilon)\, v_\eta}{b^2 v_t + v_\varepsilon + v_\eta}$$

For the steady-state variance, we solve a quadratic:

$$\boxed{v_\infty = \frac{-(v_\varepsilon + (1-b^2)v_\eta) + \left\{ (v_\varepsilon + (1-b^2)v_\eta)^2 + 4 v_\varepsilon v_\eta \right\}^{1/2}}{2b^2}}$$

(v) Let's suppose for simplicity that the prior variance v_0 is actually v_∞. Then the density of $Y_1, \dots, Y_T$ will be

$$\prod_{j=1}^{T} \exp\left\{ -\frac{1}{2(b^2 v_\infty + v_\varepsilon + v_\eta)} (Y_{t+1} - b\hat{X}_t)^2 \right\} \left(2\pi\,(b^2 v_\infty + v_\varepsilon + v_\eta)\right)^{-1/2}$$

__Notes__ This probably would not produce any very long term price fluctuations away from 'correct' values, because if everyone thinks $\log \delta_t$ is a zero-mean AR(1) process then everyone knows it's going to get back to 0 before long...

Interesting questions:

1) Jean-Paul Décamps + Stephane Villeneuve ask about the Leland-Toft stuff, could we modify the problem so that the optimal default policy is of the form when we first hit zero? Perhaps alter the dividend $\delta(V_t)$?

2) Liquidity vs solvency; suppose there is a firm whose cash account evolves as $\sigma dW_t + \mu dt - dA_t + p_t dN_t$ where N_t is the number of issued shares, A_t cumulative dividends, and p_t is the market value of shares issued at time t (or perhaps $(1-\varepsilon)$ times the market value of shares) - how do we understand this dilution problem? (Stephane V. + J-P. D again)

3) Bas Werker makes the point that for the hi-lo-open-close estimator of correlation it may be worth including terms like H_1^2, $H_1 S_1$ - these won't vary with ρ, of course, but may well reduce variance.

4) Discussions with Sorp + Kalina Acar on what properties one might wish for a model of asset prices. For me, the most important are

(i) Serial correlation of returns allowed for

(ii) Ability to extend to hundreds of assets

(iii) Match the marginals reasonably OK

(iv) 'Volatility clustering' possible

One way you could try to get this would be to model

$$dX^i_t = a^i \cdot dF_t + \sigma_i dW^i_t + \mu_i dt$$

where the factors F may represent a small number of economically significant variables - either these are treated as known (so it's a regression) or treated as unknown, when it's a filtering problem

5) Valuing stock of a company may be non-linear because of the possibility of controlling the company, and thereby making dividend stream more favorable; or there may be a convenience yield to holding the stock if others want to go short.

6) Equilibrium is the cornerstone of microeconomics, but can there be a satisfactory theory of out-of-equilibrium pricing?

7) Nathanaël asks: if you have independent cts semimartingales X, Y, is it the case that $[X,Y] = 0$? Also, if $dX_t = \sigma(X_t)\, dW_t$ in $\mathbb{R}^d$, σ, σ^{-1} bdd cts, is the support of X_t equal to $\mathbb{R}^d$?

8) Marlene Müller + Susanne are doing American option pricing by some hideously complicated method in the Kou-Rogers setting - get Arun Th. to try this in the summer?