Stochastic volatility models for stock evolution (1/4/2010)

(1) The starting point is an attempt to use a diverse beliefs story to explain various effects such as liquidity etc. Suppose there is an asset in total supply A, and at time 1 it will deliver X. Agent j thinks $X \sim N(\mu_j, v_j)$ and is CARA with coeff of absolute risk aversion equal to γ_j. Thus if the market price is S_0 at time 0, his demand for the asset will be

$$\theta_j = \frac{\mu_j - S_0}{\gamma_j v_j}$$

and to clear the market, $\sum \theta_j = A$, we deduce the time-0 price will be

$$S_0 = \sum_j p_j (\mu_j - \bar{a}) \equiv \bar{\mu} - \bar{a}$$

where $p_j \propto 1/\gamma_j v_j$, $\sum p_j = 1$, $\bar{a} = A / \sum(\gamma_j v_j)^{-1}$. The total amount of trading is

$$\sum |\theta_j| = \sum \left| \frac{\mu_j - \bar{\mu} + a}{\gamma_j v_j} \right| .$$

If we suppose zero net supply we see

(i) Bigger spread of the values μ_j leads to more liquidity

(ii) Bigger values of v_j leads to less liquidity.

Now we could in a discrete-time model attempt to embody these effects by supposing that the agents observe the log returns Y_t and suppose that

$$\begin{cases} Y_{t+1} = Y_t + \mu_t + \varepsilon_{t+1} \\ \mu_{t+1} = \mu_t + \varepsilon'_{t+1} \end{cases}$$

where the $\varepsilon, \varepsilon'$ may be correlated, and the variances of these need not be the same for each agent. This would give rise to some conventional KF story where different agents estimate μ by some different EWMA of the ΔY_t. This would allow us to study what happens if estimates of μ vary across time and across agents, but we would not be able to study effects of time-varying variance in this simple story.

(2) One direction we could explore would be to try some sort of feedback from prices into beliefs. Thus for the filtering story, we would see ΔY_{t+1} for the updating; but perhaps some (or even all) the agents only observe ΔS_{t+1} and

think this is ΔY_{t+1} – after all, ΔY_{t+1} is the true mean μ_t plus some noise, and the market price is some weighted average of the individual agents' estimates of μ.

[NB: I've tried to do a full REE for this model and it is hopeless. So what I would propose is that we should do just a single-period optimisation, with each agent at each stage just looking ahead one step. This is rather faked, but it seems to me to be no more faked than an REE story where agents calculate present value of stock on the basis of some model which they assume will continue without change of structure for ever. If we take this view, the observed stock price needs to be interpreted with care; it is not of course the market clearing price for the return ΔY_{t+1} about to be received, since buying the stock gives us all future returns...]

(3) Another hopefully useful direction is to suppose some stochastic evolution of the volatility. So let's suppose that ΔY_{t+1} has a density

$$\frac{1}{\sigma} f\left(\frac{y-\mu}{\sigma}\right)$$

conditional on the value μ of the mean and σ of the vol, and let's suppose there is a transition density $p(m,s;\mu,\sigma)$ for the pair (μ,σ). Then the posterior $\pi_t(\cdot,\cdot)$ updates as

$$\pi_{t+1}(\mu,\sigma) \propto \frac{1}{\sigma} f\left(\frac{Y_{t+1}-\mu}{\sigma}\right) \int \pi_t(m,s)\, p(m,s;\mu,\sigma)\, dm\, ds$$

We could include GARCH into such a model; or we could have a stochastic vol model where

$$p(m,s;\mu,\sigma) = g(m,\mu)\, h(s,\sigma)$$

separating the moves of μ and of σ into two separate processes independent of each other. Of course, we don't expect such independence to survive after filtering.

(4) Possible evolutions of the volatility? One simple one would be to do an n-state Markov chain. This would produce some expanding population of KFs, but we could chuck away all but the 25 most likely ones and that would probably do fine.

Another story would be to try a (continuous-time) shot-noise model for the variance v_t:

$$dv_t = -\beta v_t\, dt + dZ_t$$

where Z is some suitable subordinator. (We might even wish to shift v up by some floor level $\underline{v}$, but the changes needed for this are trivial.) We then get

$$v_t = e^{-\beta t} v_0 + \int_0^t e^{\beta(u-t)} dZ_u$$

so

$$E \exp(s v_t) = \exp(s v_0 e^{-\beta t}) \exp\left\{ \int_0^t \psi(s e^{\beta(u-t)}) du \right\}$$

where ψ is the characteristic exponent of Z. Developing this further, we would have

$$\int_0^t \psi(s e^{-\beta u}) du = \int_0^t \int_0^\infty \{\exp(s e^{-\beta u} x) - 1\} q(x) dx \, du$$

assuming Z is driftless, Lévy measure has density q;

$$= \int_0^t \int_0^\infty (\exp(sy) - 1) \, q(e^{\beta u} y) e^{\beta u} dy \, du$$

$$= \int_0^\infty (\exp(sy) - 1) \left\{ \int_0^t q(y e^{\beta u}) e^{\beta u} du \right\} dy$$

$$= \int_0^\infty (e^{sy} - 1) \left(\int_1^{e^{\beta t}} q(yv) dv \right) dy$$

which is clearly infinitely divisible.

One interesting case would be if we had $q(x) = k e^{-\alpha x}$, for then

$$\int_1^{e^{\beta t}} q(yv) dv = k \int_1^{e^{\beta t}} \exp(-\alpha y v) dv$$

$$= \frac{k}{y\alpha} \left\{ \exp(-\alpha y) - \exp(-\alpha y e^{\beta t}) \right\}$$

so in the limit as $t \to \infty$ we get the Lévy density is that of a Gamma process:

$$E e^{-\lambda v_\infty} = \left(\frac{\alpha}{\alpha + \lambda} \right)^{k/\alpha}.$$

For finite t, we see

$$E e^{-\lambda v_t} = \exp(-\lambda v_0 e^{-\beta t}) \left\{ \frac{\alpha (\lambda + \alpha e^{\beta t})}{\alpha e^{\beta t} (\lambda + \alpha)} \right\}^{k/\alpha}.$$

Some thoughts on a paper of Carr + Lee (25/4/10)

(1) I was asked to discuss a paper of Carr + Lee for the Warwick FORC meeting where they consider the pricing of variance swaps in a log Lévy situation (actually a time-change of log-Lévy, but that's not so important) The idea is that the futures price (more generally, discounted asset price) at time t, F, is represented as $F_t = \exp(Y_t)$ for some Lévy process Y_t, and F is a martingale. Let ψ denote the Lévy exponent

$$E \exp(-\lambda Y_t) = \exp(-t\psi(\lambda)) \qquad (\lambda > 0).$$

The variance swap pays at time T the amount $[Y]_T$. Now we notice that $[Y]_t$ is itself a subordinator, so for a suitable constant b,

$$E[Y]_t = b \, E \log F_T$$

Then the argument goes that we can approximate $\log(F)$ by linear combinations of call options / put options on F with suitable strikes, therefore we know the price of $\log F$, hence (from the exact knowledge of the Lévy dynamics) we can calculate $E[Y]_t$ and the constant b. So we may price if not hedge the variance swap.

(2) But if we're allowed to assume that puts/calls of all strikes are available for static hedging, why can't we suppose they are available for dynamic hedging?!

For any $\lambda > 0$, we have, taking $g_\lambda(K) \equiv \lambda(\lambda+1) K^{-\lambda-2}$, that

$$\int_0^\infty g_\lambda(K)(K-F)^+ \, dK = F^{-\lambda} = e^{-\lambda Y}$$

So if $P_t(F_T, K)$ is the time-t price of a put with strike K when spot is F_t, we see that

$$\int_0^\infty g_\lambda(K) \; P_T(F_T, K) \, dK = F_T^{-\lambda} = e^{-\lambda Y_T}$$

and

$$\int_0^\infty g_\lambda(K) \; P_t(F_t, K) \, dK = e^{-\lambda Y_t - (T-t)\psi(\lambda)}$$

(where for simplicity let's assume $r=0$, $Y_0 = 0$.) For all $\lambda > 0$, this is a traded asset, so we also have that

$$\lim_{\lambda \downarrow 0} \frac{e^{-\lambda Y_t + t\psi(\lambda)} - 1}{\lambda} = M_t \equiv -Y_t + t\psi'(\lambda)$$

is a traded asset. Notice that

$$\frac{1}{\lambda}\left\{e^{-\lambda Y_t + t\psi(\lambda)} - 1\right\} = \frac{e^{T\psi(\lambda)}}{\lambda}\int_0^\infty g_\lambda(K)\left\{P_t(F_t,K) - P_0(F_0,K)\right\} dK$$

$$\rightarrow \int_0^\infty \left(P_t(F_t,K) - P_0(1,K)\right)\frac{dK}{K^2} \equiv M_t$$

as $\lambda \downarrow 0$.

Differentiating once more, we learn that

$$M_t^2 + t\psi''(0) \quad \text{is tradable.}$$

But notice that $[M] = [Y]$, and that

$$M_t^2 - [M]_t \equiv M_t^2 - [Y]_t = 2\int_0^t M_{s-}\, dM_s$$

is tradable, hence $[Y]_t$ is tradable.

We can similarly deduce for $M_t^2 + t\psi''(0)$ that the representation in terms of puts is

$$\int_0^\infty \left(1 + T\psi'(0) - \log K\right)\left(P_t(F_t,K) - P_0(1,K)\right)\frac{dK}{K^2}$$

$$= \left(1 + T\psi'(0)\right)M_t - \int_0^\infty \frac{\log K}{K^2}\left\{P_t(F_t,K) - P_0(1,K)\right\} dK$$

Notice that the hedging position for M_t and $M_t^2 + t\,\psi''(0)$ is a <u>static</u> position in puts.

Shot noise story again (3/5/10)

(1) Can we see any of the nice infinitely divisible laws as invariant laws of a shot-noise process? If we did a shot-noise process driven by a subordinator with density q to its Lévy measure as on p.3, then the limiting distribution of the variance will be expressed as

$$E\, e^{-\lambda \xi_0} = \exp\left[- \int_0^\infty (1-e^{-\lambda y}) \left(\int_1^\infty q(yx)\, dx \right) dy \right]$$

Now for a generalised IG with density proportional to

$$y^{\alpha-1} \exp\left\{ - \tfrac{a^2}{2y} - \tfrac{1}{2} c^2 y \right\}$$

we find that the LT is

$$E \exp(-\lambda Z) = \left(\frac{c^2}{c^2+2\lambda} \right)^{\alpha/2} \frac{K_\alpha(a\sqrt{c^2+2\lambda})}{K_\alpha(ac)}.$$

Now according to formula (6) on p 79 of Watson

$$\frac{d}{dz}\left(\frac{K_\nu(z)}{z^\nu} \right) = - \frac{K_{\nu+1}(z)}{z^\nu}$$

so that

$$\frac{d}{d\lambda} \log E \exp(-\lambda Z) = - \frac{a}{\sqrt{c^2+2\lambda}} \cdot \frac{K_{\alpha+1}(z)}{z^\alpha} \frac{z^\alpha}{K_\alpha(z)} \qquad (z \equiv a\sqrt{c^2+2\lambda})$$

$$= \frac{-a \, K_{\alpha+1}(z)}{(c^2+2\lambda)^{1/2} K_\alpha(z)}$$

According to Pitman-Yor, this is (the negative of) a CM function of z^2.

(2) Special case: $\alpha = -\tfrac{1}{2}$. Here the generalised IG is a drifting BM first passage density, and

$$E \exp(-\lambda Z) = \exp\left\{ -a\left(\sqrt{c^2+2\lambda} - c \right) \right\} \qquad (\text{for } c>0)$$

$$= \exp\left[-a \int_0^\infty (1-e^{-\lambda t})\, e^{-c^2 t/2} \frac{dt}{\sqrt{2\pi t^3}} \right]$$

As for the shot-noise interpretation we demand

$$\int_1^\infty q(yx)\, dx = a e^{-c^2 y/2} / \sqrt{2\pi y^3} \iff \int_y^\infty q(s)\, ds = a e^{-c^2 y/2} / \sqrt{2\pi y}$$

$$\iff \boxed{ q(y) = \frac{a}{2\sqrt{2\pi y^3}} e^{-c^2 y/2} \left\{ c^2 y + 1 \right\} }$$

Investment timing and corporate structure (20/5/10)

1) Takashi is looking at a model where you want to invest in a production process which will produce a cashflow $(X_t)_{t\geq 0}$, and at the moment of investment you have to decide about the corporate structure.

Seems to me that the decision process is about maximising the value of equity. Suppose that at time 0 the shareholders have cash Q_0 which they invest at constant riskless rate r until the moment T when they decide to start the factory. We'll suppose that the cashflow per $\underbrace{\text{unit of investment}}_{\equiv\text{ factory}}$ is $(X_t)_{t\geq 0}$ satisfying

$$dX_t = X_t(\sigma\, dW_t + \mu\, dt)$$

where W is a BM in the pricing measure, and $\mu < r$ for a well posed problem.

2) In effect, we combine two familiar technologies here; valuation of a given corporate structure, and an infinite-horizon optimal stopping problem. The second is really so conventional that we don't need to spend time on it here, the first uses standard techniques, but the solutions depend on the assumed forms of corporate structure, of which we discuss just two.

(i) Debt-equity financing

Here we issue debt with face value D_T, attracting a constant coupon $c\,dt$ until default (chosen by the firm). Assuming a tax rate τ and a recovery rate ρ on default (both in $[0,1]$) the value to the shareholders is

$$(Q_T + D_T)\, E^{x_0}\left[\int_0^S e^{-ru}(X_u - c)(1-\tau)\,du\right](r-\mu)/x_0$$

where $x_0 = X_T$ is the value we go in at, and the value to the debtholders (per unit of investment) is

$$E^{x_0}\left[\int_0^S e^{-ru}\,c\,du + e^{-rS}\frac{\rho X_S}{r-\mu}\right]\cdot\frac{r-\mu}{x_0}$$

where $S \equiv \inf\{t : X_t = b\}$ is the default time.

How is the default level b chosen? The value of equity (per unit of initial investment) solves

$$\tfrac{1}{2}\sigma^2 x^2 f'' + \mu x f' - rf + (1-\tau)(x-c) = 0$$

in the continuation region, with $f = f' = 0$ at b when optimally chosen.

Setting $-\alpha < 0$, $\beta > 1$ as the roots of

$$\tfrac{1}{2}\sigma^2 z(z-1) + \mu z - r = 0$$

we see that the general solution to the ODE which remains $O(x)$ as $x \to \infty$ will be

$$f(x) = A\left(x/b\right)^{-\alpha} + (1-\kappa)\left\{\frac{x}{r-\mu} - \frac{c}{r}\right\}$$

If we now require that $f(b) = f'(b) = 0$, we see that

$$\left.\begin{aligned} \alpha A &= b\,\frac{1-\kappa}{r-\mu} \\ A &= -(1-\kappa)\left(\frac{b}{r-\mu} - \frac{c}{r}\right) \end{aligned}\right\}$$

$$= \frac{b}{\alpha}\,\frac{1-\kappa}{r-\mu}$$

whence

$$\boxed{\;b = \frac{c\,\alpha(r-\mu)}{r(1+\alpha)}\;}$$

So for a given coupon c, this tells us where to put the default barrier b. The value of debt now needs to be understood. The debtholders put $D_T \equiv \lambda Q_T$ into firm at start up, and receive coupons $\frac{(r-\mu)}{x_0} Q_T(1+\lambda)\,c$ until bankruptcy, at which point they get ρ times the value of the factory, which is $Q_T(1+\lambda)\,X_\tau/(r-\mu) \cdot \left(\frac{r-\mu}{x_0}\right)$. So what we have to have is

$$\lambda Q_T = (1+\lambda)Q_T\left[\frac{r-\mu}{x_0}\frac{c}{r}\left(1-\left(\frac{x}{b}\right)^{-\alpha}\right) + \left(\frac{x}{b}\right)^{-\alpha}\frac{\rho\, b}{r-\mu}\,\frac{r-\mu}{x_0}\right]$$

This therefore determines $\lambda = \lambda(c)$; the value to the shareholders is

$$Q_T(1+\lambda)\left[(1-\kappa)\left(\frac{x}{r-\mu} - \frac{c}{r}\right) + \left(\frac{x}{b}\right)^{-\alpha}\frac{b}{\alpha}\,\frac{1-\kappa}{r-\mu}\right]\frac{r-\mu}{x_0}$$

which must be maximised over c for each x_0

(ii) Soft refinancing The idea here is that if the firm value falls too low, you renegotiate the debt, using Nash bargaining ideas. The story Takashi tells looks to me to be more complicated than it needs be; at the moment that X falls to the renegotiation trigger level k, what in effect happens is that the value of the firm if default occurs will be

$$\frac{r-\mu}{x_0} \cdot \frac{(Q_T + D_T)\, k\rho}{(r-\mu)}$$

and if default does not occur then the value is

$$\frac{r-\mu}{x_0} \cdot \frac{(Q_T + D_T)\, k}{r-\mu}$$

As the surplus $\frac{r-\mu}{x_0}(1-\rho)\, k\, (Q_T + D_T)/(r-\mu)$ is split between the shareholders and debtholders in the ratio $\eta : 1-\eta$.

Thus the time-T value of equity will be

$$(Q_T + D_T)\, E^{x_0}\left[\int_0^S e^{-rt}(X_t - c)(1-\tau)\, dt + e^{-rS}(1-\rho)\eta X_S/(r-\mu)\right]\frac{r-\mu}{x_0}$$

and the time-T value of debt will be

$$(Q_T + D_T)\, E^{x_0}\left[\int_0^S e^{-rt} c\, dt + e^{-rS}\frac{X_S}{r-\mu}\{\rho + (1-\eta)(1-\rho)\}\right]\frac{r-\mu}{x_0}$$

(so with $\eta = 0$, it's like the previous situation with full recovery)

So there's a common story:

Shareholders get $(Q+D)(1-\tau)(X_t - c)\frac{r-\mu}{x_0}$ } until X drops to some critical

Bondholders get $(Q+D)\, c\, \frac{r-\mu}{x_0}$ }

level x^*, and at that time shareholders get payment of $A_Q\, x^*(Q+D)\left(\frac{r-\mu}{x_0}\right)$

bondholders get payment of $A_D\, x^*(Q+D)\left(\frac{r-\mu}{x_0}\right)$

That's all. This time, the shareholders' best choice of b for a given c is

$$\boxed{b = \frac{\alpha(1-\tau)(r-\mu)\, c}{r(1+\alpha)\left[1-\tau - A_Q(r-\mu)\right]}} \equiv Kc, \text{ say}$$

Consistent with earlier expression when $A_Q = 0$

Now we see that the value of the debt is $(Q+D)$ times $(r-\mu)/x_0$ times

$$\frac{c}{r} + \left(\frac{x}{b}\right)^{-\alpha}\left[A_D\, b - \frac{c}{r}\right]$$

so if we expect the bondholders to invest $D = \lambda Q$, we have to have

$$\lambda = (1+\lambda)\left[\frac{c}{r} + \left(\frac{x}{b}\right)^{-\alpha}\left(A_D b - \frac{c}{r}\right)\right]\frac{r-\mu}{x_0}$$

So the story is quite clear; if we choose some coupon rate c, we deduce the barrier b, then we get λ, and we see that the value to the shareholders is

$$Q\,(1+\lambda(c))\left\{(1-\varepsilon)\left(\frac{x}{r-\mu} - \frac{c}{r}\right) + A\left(\frac{x}{b}\right)^{-\alpha}\right\}\frac{r-\mu}{x_0}$$

where $A = b\left\{A_Q - \frac{1-\varepsilon}{r-\mu}\right\} + (1-\varepsilon)\frac{c}{r} = \frac{(1-\varepsilon)c}{r(1+\alpha)}$

For a given x_0 we choose c to maximise this.

[It seems that this can be troublesome numerically - the shareholders generally desire to push λ very high $(1-\varepsilon)$ so that almost all of the huge initial investment of the project is debt funded !? No, this was apparently caused by a missing scaling factor]

1/6/10 Let's suppose that each factory costs I, and once bought delivers cashflow $(X_t)_{t\geq 0}$. Shareholders initially have wealth Q_0. If they decide to invest when X hits x_0, and set action trigger at $b \leq x_0$, then we work out the corresponding c which makes this action trigger optimal ($c = b/K$, in fact), and we then work out the value to bondholders of one factory:

$$d(x_0; b) = E\left[\int_0^S c\, e^{-rs}\, ds + e^{-rS} A_D X_S\right]$$

$$q(x_0; b) = E\left[\int_0^S e^{-rs}(1-\varepsilon)(X_s - c)\, ds + e^{-rS} A_Q X_S\right]$$

The number a of factories bought satisfies

$$Q_0 + a\, d(x_0; b) = aI \Rightarrow a = Q_0 / (I - d(x_0; b))$$

and value to firm is therefore

$$a\, q(x_0; b) - Q_0$$

which for each x_0 should be maximised over b. Might choose instead to compound up the initial cash to $Q_S = e^{rS} Q_0$ .... ?

Least-action filtering: another look (21/5/10)

(1) Let's go back to the setting for least-action filtering, where $Z_t = [X_t; Y_t]$ evolves as

$$dZ_t = \sigma(Z_t)\, dW_t + \mu(Z_t)\, dt$$

and we observe $(Y_t)_{0\le t\le T}$ and need to estimate $(X_t)_{0\le t\le T}$. We then form the log-likelihood

$$-\left\{ \frac{1}{2}\int_0^T \left|\sigma(Z_t)^{-1}(\dot{Z}_t - \mu(Z_t))\right|^2 dt \right\} - \varphi(X_0)$$

$$\equiv -\ \frac{1}{2}\int_0^T \psi(t, x_t, p_t)\, dt \quad - \varphi(x_0) \qquad (p_t \equiv \dot{x}_t)$$

where the prior density for x_0 is $\exp(-\varphi(x))$ and we attempt to maximise this, equivalently, minimise the action

$$\frac{1}{2}\int_0^T \psi(t, x_t, p_t)\,dt + \varphi(x_0).$$

This gets solved by calculus of variations: we get an ODE for x which has to be solved by a shooting method; there's a condition at 0 and at T. As T gets too big, this is going to fail, and we do in any case want some recursive methodology.

(2) If we think of the Euler approximation

$$Z_{t_{i+1}} - Z_{t_i} = \mu(Z_{t_i})(t_{i+1}-t_i) + \sigma(Z_{t_i})\,.\,\Delta W\,\sqrt{t_{i+1}-t_i}$$

to the ODE, then the log-likelihood is $(\Delta t_i \equiv t_i - t_{i-1})$

$$(*)\quad -\frac{1}{2}\sum_{i=1}^{N} \left|\sigma(Z_{t_i})^{-1}\left(\frac{Z_{t_{i+1}}-Z_{t_i}}{\Delta t_{i+1}} - \mu(Z_{t_i})\right)\right|^2 \Delta t_{i+1} \quad - \varphi(X_0)$$

Maximising this over x (Y is given and fixed) is in effect what the least-action method does; we are maximising likelihood over the large vector $(x_0, \dots, x_N)$.

(3) To understand the likelihood surface around the MLE, we need to consider the second derivative. If we can understand how the likelihood changes with x_j, then we have in effect found the <u>posterior covariance</u> for x_j given $(Y_t)_{0\le t\le T}$. We could then start the LA calculation from that intermediate time.

But how does the likelihood vary with x_j? Look at $(*)$: if Δt_i is small, <u>the principal contribution is from the derivative term</u> !!? But this can't be correct; we really need to find the evectors/evalues of the covariance matrix...

(4) There has to be a reference to this somewhere?! But I've not found it.

Suppose σ, b are bounded Lipschitz, and X solves the SDE

$$X_t = x_0 + \int_0^t \sigma(X_s)\,dW_s + \int_0^t b(X_s)\,ds$$

while $X^{(n)}$ solves

$$X^{(n)}_t = x_0 + \int_0^t \sigma\left(X^{(n)}(2^{-n}[2^n s])\right) dW_s + \int_0^t b\left(X^{(n)}(2^{-n}[2^n s])\right) ds.$$

If we set $\Delta_t \equiv X_t - X^{(n)}_t$, then for any $T>0$, any $p \geq 2$ there exists $C = C(p,T,K)$ (where K is a Lipschitz bound, and also a uniform bound on the coefficients) such that for all n, for all $t \in [0,T]$

$$\boxed{E(\Delta^{*\,p}_t) \leq C\,E\left(\int_0^t \Delta^{*\,p}_s\,ds\right) + C\,t\,2^{-np/2}.}$$

Corollary By Gronwall's lemma, $E(\Delta^{*\,p}_t) \leq C\,T\,2^{-np/2}\,e^{CT}$, so $X^{(n)}_t \to X_t$ uniformly on $[0,T]$ almost surely.

Proof We can write

$$\Delta_t = \int_0^t \left\{\sigma\left(X(2^{-n}[2^n s])\right) - \sigma\left(X^{(n)}(2^{-n}[2^n s])\right)\right\} dW_s$$

$$+ \int_0^t \left\{b\left(X(2^{-n}[2^n s])\right) - b\left(X^{(n)}(2^{-n}[2^n s])\right)\right\} ds$$

$$+ \int_0^t \left\{\sigma(X_s) - \sigma\left(X(2^{-n}[2^n s])\right)\right\} dW_s$$

$$+ \int_0^t \left\{b(X_s) - b\left(X(2^{-n}[2^n s])\right)\right\} ds$$

Then

$$E(\Delta^{*\,p}_t) \leq C\,E\left[\int_0^t \Delta^{*\,p}_s\,ds\right] + C\,E\left[\left(\int_0^t \left|\sigma(X_s) - \sigma\left(X(2^{-n}[2^n s])\right)\right|^2 ds\right)^{p/2}\right]$$

$$+ C\,E\int_0^t \left|b(X_s) - b\left(X(2^{-n}[2^n s])\right)\right|^p ds$$

$$\leq C\,E\left[\int_0^t \Delta^{*\,p}_s\,ds\right] + C\,E\left[\int_0^t \left|X_s - X\left(2^{-n}[2^n s]\right)\right|^p ds\right]$$

So we have to bound the final term. But

$$E\,|X_s - X_0|^p \leq C\left\{E\left(\int_0^s |b(X_u) - b(X_0)|\,du\right)^p + E\left[\left(\int_0^s |\sigma(X_u) - \sigma(X_0)|^2\,du\right)^{p/2}\right]\right\}$$

Calculus of variations gives

$$0 = D_{p_j}\psi(0, x_0, p_0) - D_{x_j}\varphi(x_0)$$

$$0 = D_{x_j}\psi - D_{p_j}\left(D_t\psi + \dot{x}_k D_{x_k}\psi + \dot{p}_k D_{p_k}\psi \right) \qquad \text{along path}$$

$$0 = (D_{p_j}\psi)(T, x_T, p_T)$$

$$\leq C\left(s^p + s^{p/2} \right)$$

so for all $s \leq 1$ we deduce the bound $C s^{p/2}$. $\square$

(5) Let's go back to the action in the form

$$\varphi(x_0) + \int_0^T \psi(t, x_t, p_t)\, dt$$

which we minimise at $x = x^*$, which we find by calculus of variations. Now let's perturb x^* to $x^* + \xi$ and look at the change in the action. The first-order parts all vanish, so we're left with second order

$$\tfrac{1}{2} D_i D_j \varphi(x_0) \xi_0^i \xi_0^j + \int_0^T \left\{ \tfrac{1}{2} \xi^i \xi^j D_{x_i} D_{x_j} \psi + \xi^i \dot{\xi}^j D_{p_j} D_{x_i} \psi + \tfrac{1}{2} \dot{\xi}^i \dot{\xi}^j D_{p_i} D_{p_j} \psi \right\} dt$$

The bit in the curly brackets is what's giving the Gaussian structure. We can express it as

$$\tfrac{1}{2} \left(\dot{\xi} + (D_{pp}\psi)^{-1} (D_{px}\psi)\xi \right) \cdot D_{pp}\psi \left(\dot{\xi} + (D_{pp}\psi)^{-1} (D_{px}\psi)\xi \right)$$

$$+ \tfrac{1}{2} \xi \cdot \left(D_{xx}\psi - (D_{xp}\psi)(D_{pp}\psi)^{-1}(D_{px}\psi) \right) \xi$$

and the first piece can be interpreted as some SDE action piece, the second as an additional exponential-quadratic contribution to the density.

(6) This isn't wrong but there appears to be a more efficient way to handle things. Let's write

$$Q(\xi) \equiv \tfrac{1}{2} D_{ij} \varphi(x_0) \xi_0^i \xi_0^j + \int_0^T \left\{ \tfrac{1}{2} \xi^i \xi^j D_{x_i x_j} \psi + \xi^i \dot{\xi}^j D_{x_i p_j} \psi + \tfrac{1}{2} \dot{\xi}^i \dot{\xi}^j D_{p_i p_j} \psi \right\} dt$$

$$\equiv \tfrac{1}{2} D_{ij} \varphi(x_0) \xi_0^i \xi_0^j + \int_0^T \left\{ \tfrac{1}{2} \xi^i A_t^{ij} \xi_t^j + \xi_t^i B^{ij} \dot{\xi}_t^j + \tfrac{1}{2} \dot{\xi}_t^i q_{ij}(t) \dot{\xi}_t^j \right\} dt$$

for the quadratic functional of ξ which characterises the Gaussian distribution of the leading-order perturbation. Now suppose we have some symmetric matrix function of t, θ_t, such that $\theta_T = 0$. Then

$$Q(\xi) = Q(\xi) + \tfrac{1}{2} \xi_0 \theta_0 \xi_0 + \left[\tfrac{1}{2} \xi_t \theta_t \xi_t \right]_0^T$$

$$= \tfrac{1}{2} \xi_0 \left(D^2\varphi(x_0) + \theta_0 \right) \xi_0 + \int_0^T \left\{ \tfrac{1}{2} \xi_t A_t \xi_t + \xi_t B_t \dot{\xi}_t + \tfrac{1}{2} \dot{\xi}_t q_t \dot{\xi}_t + \tfrac{1}{2} \xi_t \dot{\theta}_t \xi_t + \xi_t \theta_t \dot{\xi}_t \right\} dt$$

$$D_{x_i} D_{x_\ell} \psi = - D_i q_{kj} (p_j - \mu_j)(D_\ell \mu_k) + q_{kj} (D_i \mu_j)(D_\ell \mu_k) - q_{kj} (p_j - \mu_j) D_i D_\ell \mu_k$$

$$+ \tfrac{1}{2} (p_k - \mu_k)(D_i D_\ell q_{kj})(p_j - \mu_j)$$

$$- D_\ell q_{kj} (p_j - \mu_j) D_i \mu_k$$

So the quadratic form inside the integral is

$$\tfrac{1}{2}\,\dot{\xi}_t q_t \dot{\xi}_t + \xi_t (B_t + \theta_t)\dot{\xi}_t + \tfrac{1}{2}\,\xi_t (A_t + \dot{\theta}_t)\xi_t$$

$$= \tfrac{1}{2}\,(\dot{\xi}_t + K_t \xi_t)\, q_t\, (\dot{\xi}_t + K_t \xi_t)$$

where $K_t = q_t^{-1}(B_t^T + \theta_t)$ <u>provided</u>

$$\boxed{A_t + \dot{\theta}_t = K_t^T q_t K_t = (B_t + \theta_t)\, q_t^{-1}\,(B_t^T + \theta_t)}$$

This gives us an ODE for θ to be solved with the BC $\theta_T = 0$! Once we have this, we can conclude that the perturbation solves

$$\boxed{d\xi_t = -K_t \xi_t\, dt + q_t^{-\frac{1}{2}}\, dW_t}$$

The covariance of ξ_t can be obtained by integrating up the SDE... !!
More simply, we have

$$d\,\xi\xi^T \doteq \left(-K\xi\xi^T - \xi\xi^T K^T + q^{-1}\right)dt$$

so if V_t is the covariance at time t, we find

$$\boxed{\dot{V}_t = -KV_t - V_t K^T + q^{-1}}$$

<u>14/6/10</u> We have $\psi(t,x,p) = \tfrac{1}{2}\,(p - \mu(t,x))\cdot q(t,x)(p - \mu(t,x))$, and so
$D_{p_j}\psi = q_{jk}(p_k - \mu_k)$. Thus in the ODE to be solved, we find

$$\begin{cases} D_t D_{p_j}\psi = (D_t q_{jk})(p_k - \mu_k) - q_{jk} D_t \mu_k \\[2ex] D_{x_i} D_{p_j}\psi = (D_{x_i} q_{jk})(p_k - \mu_k) - q_{jk} D_{x_i} \mu_k \\[2ex] D_{p_i} D_{p_j}\psi = q_{ij} \end{cases}$$

$$D_{x_i}\psi = -q_{kj}(p_j - \mu_j)(D_i \mu_k) + \tfrac{1}{2}\,(p_k - \mu_k)\, D_{x_i} q_{kj}\, (p_j - \mu_j)$$

Investment and corporate structure again (1/6/10)

Suppose that the price X_t of the product of a factory evolves as

$$dX_t = X_t(\sigma\, dW_t + \mu\, dt)$$

and that the shareholders of some firm have initially Q_0. This gets invested risklessly until time T, at which time the firm enters production. The cost of a single factory at time T is $I e^{rT}$, and the shareholders have $Q_0 e^{rT}$ at that time. The shareholders issue debt and use this together with their own cash $Q_0 e^{rT}$ to buy some number a of factories, to be determined.

The debt pays a coupon $c_0 e^{rT}$ up until some stopping time $S \geq T$ to be determined. At the stopping time S, the value to the debt-holders is $A_D X_S$, and to the shareholders is $A_Q X_S$.

The value of the debt at the time of issuance is (per factory)

$$E_T\left[\int_T^S c_0 e^{rT} e^{-r(u-T)}\,du + e^{-r(S-T)} A_D X_S\right]$$

$$= E_T\left[\int_T^S c_0 e^{rT} e^{-r(u-T)}\,du + e^{-r(S-T)} A_D \left(\frac{X_S}{X_T}\right) X_T\right]$$

$$= e^{rT} E_T\left[\int_T^S c_0 e^{-r(u-T)}\,du + e^{-r(S-T)} A_D \left(\frac{X_S}{X_T}\right)\tilde{X}_T\right]$$

where $\tilde{X}_t \equiv e^{-rt} X_t$. Likewise, the value of equity at the time of issuance is (per factory)

$$E_T\left[\int_T^S (1-\kappa) e^{-r(u-t)} \left(X_u - c_0 e^{rT}\right) du + e^{-r(S-T)} A_Q X_S\right]$$

$$= e^{rT} E_T\left[\int_T^S (1-\kappa)\, e^{-r(u-t)} \left(\tilde{X}_T \frac{X_u}{X_T} - c_0\right) du + e^{-r(S-T)} A_Q \left(\frac{X_S}{X_T}\right)\tilde{X}_T\right]$$

We shall suppose that

$$S = \inf\left\{t > T : \frac{X_t}{X_T} < b\right\}$$

for some barrier level b to be determined. Then the value of debt at issuance is of the form

$$e^{rT} d(\tilde{X}_T, b)$$

and the value of the firm when it invests is $e^{rT} q(\tilde{X}_T, b)$. The number a of units of the factory purchased satisfies

$$a\, I e^{rT} = Q_0 e^{rT} + a e^{rT} d(\tilde{X}_T, b)$$

so that
$$a = Q_0 / \left(I - d(\tilde{X}_T, b)\right)$$

Thus at the moment of investment the value of equity will be
$$a\, e^{rT} q(\tilde{X}_T, b) - e^{rT} Q_0$$

so the time-0 expected value (in excess of Q_0) will be
$$E\left[a\, q(\tilde{X}_T, b) - Q_0 \right]$$

which we just have to maximize over the choice of the investment level $\tilde{X}_T$ and the action trigger b.

Write
$$\mathcal{L} = \tfrac{1}{2}\sigma^2 x^2 \frac{d^2}{dx^2} + \mu x \frac{d}{dx}, \qquad \tilde{\mathcal{L}} = \mathcal{L} - rx\frac{d}{dx}$$

Then
$$\left.\begin{aligned} d(\tilde{x}, b) &= \frac{c_0}{r} + b^{\alpha}\left[A_D\, b\tilde{x} - \frac{c_0}{r}\right] \\ q(\tilde{x}, b) &= (1-\tau)\left[\frac{\tilde{x}}{r-\mu} - \frac{c_0}{r}\right] + \frac{c_0(1-\tau)}{r(1+\alpha)}\, b^{\alpha} \end{aligned}\right\}$$

where b is related to c_0 by
$$b\tilde{x} = \frac{\alpha(1-\tau)(r-\mu)c_0}{r(1+\alpha)\left(1 - \tau - (r-\mu)A_D\right)} \equiv K c_0$$

rather as we got on p 9.

Least-action filtering: an example (3/6/10)

If we go back to the story where we are trying to value an asset on the basis of infrequent observation of prices, we tried to explain the observed price Y as a BM with drift plus an independent OU process:

$$dY_t = dX_t + dz_t$$
$$= (\mu dt + \sigma_X dW_t^X) - \lambda z_t dt + \sigma \, dW_t$$

but if we don't know the parameters μ, λ these must be estimated too. So let's propose as a state vector

$$x = \begin{pmatrix} \mu \\ X \\ f^{-1}(\lambda) \end{pmatrix} \equiv \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

where $f : \mathbb{R} \to (0,\infty)$ is a homeomorphism ($f(x) = \exp(x)$, say). Then we have

$$\begin{cases} dx_1 = \sigma_1 \, dW^1 \\ dx_2 = x_1 \, dt + \sigma_2 \, dW^2 \\ dx_3 = \sigma_3 \, dW^3 \\ dY = (x_1 - f(x_3)(Y - x_2))dt + \sigma_4 \, dW^4 + \sigma_2 \, dW^2 \end{cases}$$

where we assume σ_4 is known, and σ_1, σ_3 are known and small, σ_2 known.

~~Then the action functional is~~ NOT:

$$\cancel{\psi(x,p) = \tfrac{1}{2}\sigma_4^{-2}\left(\dot{Y} + f(x_3)(Y-x_2) - x_1\right)^2 + \tfrac{1}{2}(p_3/\sigma_3)^2 + \tfrac{1}{2}\left(\frac{p_2 - x_1}{\sigma_2}\right)^2 + \tfrac{1}{2}\left(\frac{p_1}{\sigma_1}\right)^2}$$

and

$$\cancel{D_x \psi = \begin{pmatrix} -\sigma_4^{-2}\left(\dot{Y} + f(x_3)(Y-x_2) - x_1\right) - (p_2 - x_1)/\sigma_2^2 \\ -\sigma_4^{-2}\left(\dot{Y} + f(x_3)(Y_2 - x_2) - x_1\right) f(x_3) \\ -\sigma_4^{-2} f'(x_3)(Y - x_2)\left(\dot{Y} + f(x_3)(Y_2 - x_2) - x_1\right) \end{pmatrix}, \quad D_p \psi = \begin{pmatrix} p_1/\sigma_1^2 \\ (p_2 - x_1)/\sigma_2^2 \\ p_3/\sigma_3^2 \end{pmatrix}}$$

– this was what it was without the $+\sigma_2 dW^2$ in last equation

In fact,

$$\boxed{\psi(x,p) = \frac{1}{2}\left(\frac{p_1}{\sigma_1}\right)^2 + \frac{1}{2}\left(\frac{p_2 - x_1}{\sigma_2}\right)^2 + \frac{1}{2}\left(\frac{p_3}{\sigma_3}\right)^2 + \frac{1}{2}\left(\frac{p_4 - p_2 + f(x_3)(x_4 - x_2)}{\sigma_4}\right)^2}$$

Thus

$$D_x\psi = \begin{pmatrix} -(p_2-x_1)/\sigma_2^2 \\ -f(x_3)\{p_4-p_2+f(x_3)(Y-x_2)\}/\sigma_4^2 \\ (x_4-x_2)f'(x_3)\{p_4-p_2+f(x_3)(Y-x_2)\}/\sigma_4^2 \end{pmatrix}$$

$p_4 \equiv \dot{Y}$

$x_4 \equiv Y$

$$D_p\psi = \begin{pmatrix} p_1/\sigma_1^2 \\ (p_2-x_1)/\sigma_2^2 - \dfrac{p_4-p_2+f(x_3)(Y-x_2)}{\sigma_4^2} \\ p_3/\sigma_3^2 \end{pmatrix}$$

$$D_{xx}\psi = \begin{bmatrix} 1/\sigma_2^2 & 0 & 0 \\ 0 & f(x_3)^2/\sigma_4^2 & \dfrac{f'(x_3)(p_2-p_4)}{\sigma_4^2} - \dfrac{2f(x_3)f'(x_3)}{\sigma_4^2}(Y-x_2) \\ 0 & -\dfrac{f'(x_3)}{\sigma_4^2}\left[p_4-p_2+2(Y-x_2)f(x_3)\right] & \dfrac{(p_4-p_2)(x_4-x_2)f''(x_3)}{\sigma_4^2} + \dfrac{(x_4-x_2)^2}{\sigma_4^2}(f''f+f'^2)(x_3) \end{bmatrix}$$

$$D_{px}\psi = \begin{bmatrix} 0 & 0 & 0 \\ -\dfrac{1}{\sigma_2^2} & f(x_3)/\sigma_4^2 & \dfrac{-f'(x_3)(Y-x_2)}{\sigma_4^2} \\ 0 & 0 & 0 \end{bmatrix}$$

$$D_{pp}\psi = \begin{bmatrix} 1/\sigma_1^2 & 0 & 0 \\ 0 & 1/\sigma_2^2 + 1/\sigma_4^2 & 0 \\ 0 & 0 & 1/\sigma_3^2 \end{bmatrix}$$

Develop

$$E\left[e^{-\theta X_\tau - \lambda \tau}\right] = \int_0^{a/\varepsilon} \frac{(\alpha+\beta)e^{-\theta((1+\varepsilon)x-a)}}{e^{\alpha(a-\varepsilon x)} - e^{-\beta(a-\varepsilon x)}} e^{-\beta x} \left(\frac{1-e^{-(\alpha+\beta)(a-\varepsilon x)}}{1-e^{-(\alpha+\beta)a}} \right)^{1/\varepsilon} dx$$

$$= \int_0^{a/\varepsilon} (\alpha+\beta) \exp\left[-\theta((1+\varepsilon)x-a) - \beta x - \alpha(a-\varepsilon x)\right] \frac{\left(1-e^{-(\alpha+\beta)(a-\varepsilon x)}\right)^{\frac{1}{\varepsilon}-1}}{\left(1-e^{-(\alpha+\beta)a}\right)^{1/\varepsilon}} dx$$

$a - \varepsilon x = v$

$$= \frac{1}{\varepsilon} \int_0^{a} (\alpha+\beta) \exp\left[-\theta\left(\frac{1+\varepsilon}{\varepsilon}(a-v) - a\right) - \frac{\beta}{\varepsilon}(a-v) - \alpha v\right] \frac{\left(1-e^{-(\alpha+\beta)v}\right)^{\frac{1}{\varepsilon}-1}}{\left(1-e^{-(\alpha+\beta)a}\right)^{1/\varepsilon}} dv$$

$$= \frac{1}{\varepsilon} \int_0^{a} (\alpha+\beta) \exp\left[-\frac{\theta a}{\varepsilon} + \frac{\theta(1+\varepsilon)}{\varepsilon} v + \frac{\beta}{\varepsilon} v - \frac{\beta a}{\varepsilon} - \alpha v\right] \frac{\left(1-e^{-(\alpha+\beta)v}\right)^{\frac{1}{\varepsilon}-1}}{\left(1-e^{-(\alpha+\beta)a}\right)^{1/\varepsilon}} dv$$

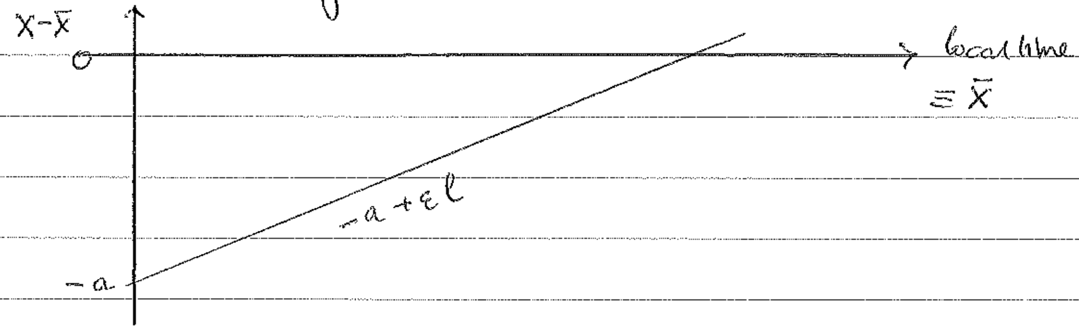
$v = \frac{-1}{\alpha+\beta} \log t$

$$= \frac{1}{\varepsilon} \int_{\exp(-(\alpha+\beta)a)}^{1} \exp\left[\frac{-(\theta+\beta)a}{\varepsilon} - \left(\frac{\theta(1+\varepsilon)}{\varepsilon} + \frac{\beta}{\varepsilon} - \alpha\right) \frac{\log t}{\alpha+\beta}\right] \frac{(1-t)^{1/\varepsilon - 1}}{t\left(1-e^{-(\alpha+\beta)a}\right)^{1/\varepsilon}} dt$$

$$= \frac{\frac{1}{\varepsilon} \exp\left\{-\frac{(\theta+\beta)a}{\varepsilon}\right\}}{\left(1-e^{-(\alpha+\beta)a}\right)^{1/\varepsilon}} \int_{\exp(-(\alpha+\beta)a)}^{1} (1-t)^{1/\varepsilon - 1} \, t^{-(\theta+\beta)(1+\varepsilon)/\varepsilon(\alpha+\beta)} dt$$

Converging stops (7/10/06)

This is an example in the trading-to-stops genre, where we use a stop that rises relative to the running max



We stop at $T \equiv \inf\{t : X_t < -a + (1+\varepsilon)\bar{X}_t\}$. As on p49 of WN XXX, if we let $-\alpha < 0 < \beta$ be roots of $\frac{1}{2}\sigma^2 z^2 + \mu z - \lambda = 0$, and set

$A = \{$excursions which are λ-marked before hit 0 or $-a\}$

$B = \{$excursions which get to $-a$ before λ-mark$\}$

then

$$n(A) = \frac{\beta e^{\alpha a} + \alpha e^{-\beta a} - \alpha - \beta}{e^{\alpha a} - e^{-\beta a}}$$

$$n(B) = \frac{\alpha+\beta}{e^{\alpha a} - e^{-\beta a}}$$

$$\Rightarrow n(A \cup B) = \frac{\beta e^{\alpha a} + \alpha e^{-\beta a}}{e^{\alpha a} - e^{-\beta a}} \equiv \nu(a)$$

Thus

$$P(\bar{X} \text{ reaches } t \text{ before stopping excursion}) \equiv \bar{F}(t)$$

$$= \exp\left[-\int_0^t \nu(a - \varepsilon s)\, ds \right]$$

$$= \exp\left[+\alpha t - \frac{1}{\varepsilon} \log\left(\frac{e^{(\alpha+\beta)a} - 1}{e^{(\alpha+\beta)(a-\varepsilon t)} - 1} \right) \right]$$

$$= e^{+\alpha t} \left\{ \frac{e^{(\alpha+\beta)a} - 1}{e^{(\alpha+\beta)(a-\varepsilon t)} - 1} \right\}^{-1/\varepsilon} = e^{-\beta t} \left(\frac{e^{(\alpha+\beta)(a-\varepsilon t)} - e^{-(\alpha+\beta)\varepsilon t}}{e^{(\alpha+\beta)(a-\varepsilon t)} - 1} \right)^{-1/\varepsilon}$$

Thus

$$E\left[e^{-\theta X_T - \lambda T} \right] = \int_0^{a/\varepsilon} e^{-\theta((1+\varepsilon)x - a)} \frac{\alpha+\beta}{e^{\alpha(a-\varepsilon x)} - e^{-\beta(a-\varepsilon x)}} \bar{F}(x)\, dx$$

... only numerically?

Spherically symmetric distributions in $\mathbb{R}^d$ (8/6/10)

1) What's the surface area A_d and volume V_d of unit ball in $\mathbb{R}^d$? We have

$$1 = \int e^{-|x|^2/2} \frac{dx}{(2\pi)^{d/2}} = (2\pi)^{-d/2} A_d \int_0^\infty e^{-r^2/2} r^{d-1}\, dr$$

$$= (2\pi)^{-d/2} A_d \int_0^\infty e^{-z} (2z)^{(d-2)/2}\, dz$$

$$= (2\pi)^{-d/2} A_d \, 2^{(d-2)/2} \, \Gamma(d/2)$$

$$\therefore \quad \boxed{A_d = (2\pi)^{d/2} \, 2^{-(d-2)/2} / \Gamma(d/2)}$$

Similarly, $V_d = \int_0^1 A_d r^{d-1}\, dr = A_d / d$.

(2) Consider a spherically-symmetric density $\propto \min(1, r^{-d-3})$, which has second moments.

$$\int_0^\infty r^{d-1} A_d \min(1, r^{-d-3})\, dr = V_d + A_d \int_1^\infty r^{-4}\, dr = V_d + \tfrac{1}{3} A_d$$

So the normalization is

$$\boxed{\varphi(r) = \left(V_d + \tfrac{1}{3} A_d\right)^{-1} \min\left(1, r^{-d-3}\right)}$$

and $E\,|X|^2 = \int_0^\infty r^{d+1} A_d \min(1, r^{-d-3})\, dr \, (V_d + \tfrac{1}{3} A_d)^{-1}$

$$= \left(\frac{A_d}{d+2} + A_d\right) \Big/ \left(\frac{A_d}{d} + \tfrac{1}{3} A_d\right)$$

$$= \frac{d+3}{d+2} \Big/ \frac{d+3}{3d} = \frac{3d}{d+2}.$$

(3) Suppose we have a ball of radius b hidden somewhere in $[0,1]^d$; the probability that a randomly-chosen point hits the ball is $\approx b^d V_d$, so if we want this to be p_c then

$$\log b \simeq \frac{1}{d} \left(\log p_c - \log V_d\right)$$

(4) Suppose we have some spherically-symmetric reference density φ_0 which gets scaled to $\lambda^{-d} \varphi_0(|x|/\lambda)$. If we demand that for some $\varepsilon > 0$

$$\lambda^{-d} \varphi_0(b/\lambda) \geq \varepsilon \, \lambda^{-d} \varphi_0(0)$$

this is saying that within the ball of radius b the density is a significant multiple of the maximum density (assuming φ is decreasing)

For the Gaussian, this says

$$-\frac{b^2}{2\lambda^2} \geq \log \varepsilon \quad , \text{ equivalently, } \quad \lambda \geq \frac{b}{\sqrt{2}} \Big/ \sqrt{\log 1/\varepsilon}$$

For the polynomial tail, we'll have

$$\left(\frac{b}{\lambda}\right)^{-d-3} \geq \varepsilon$$

That is

$$\lambda \geq b \; \varepsilon^{1/(d+3)}$$

Either way, the scaling parameter λ grows proportional to b, but it doesn't grow ridiculously fast with dimension.

(5) However, this isn't the issue*: the issue is whether we get impoverishment. If we think about the distance of the random points in $[0,1]^d$ from the hidden target then for R the distance, for small x,

$$P(R \leq x) \approx x^d V_d \quad \therefore \quad P(V_d R^d \leq y) \approx y,$$

and thus the values $V_d R_d^d$ are roughly $U[0,1]$ near zero. So the order statistics have means $1/N$, $2/N$, --- and thus

$$R^{(j)} \approx \left(j/NV_d\right)^{1/d}$$

Now we want to have that the difference in log likelihood for the most likely and second most likely point should be $O(1)$; thus for a Gaussian density, we'll want (with scaling λ) the difference $(R^{(2)})^2/2\lambda - (R^{(1)})^2/2\lambda$ is about

$$\frac{1}{2\lambda^2}\left(2^{2/d}-1\right)\left(\frac{1}{NV_d}\right)^{2/d} \approx 1$$

so that

$$\lambda \approx \left(\frac{2^{2/d}-1}{2}\right)^{1/2} (NV_d)^{-1/d}$$

What does that tell us about the scaling?

* Not the only issue, any rate; we don't want likelihoods of $(y \,|\, X^{(i)}_{t-1})$ to be very small either.

Trading to stops: introducing risk aversion (12/6/10)

Returning to the earlier stories about trading to stops, One thing that doesn't work so well is that if the drift were negative, then it still a good idea to push the lower stop down to $-\infty$, because the loss grows linearly with a, but the factor $E(e^{-\lambda T};\ T = H_a)$ dies exponentially with a. Some sort of risk aversion could be a good way to deal with this. So let's use a utility

$$U(x) = 1 - e^{-\gamma x}$$

and then do

$$\varphi = E\left[e^{-\lambda T}\, U(X_T - c)\right] + \varphi\, E\left[e^{-\lambda T}\right]$$

so that

$$\varphi = \frac{E\left[e^{-\lambda T}\, U(X_T - c)\right]}{1 - E\left[e^{-\lambda T}\right]} = \frac{E\left[e^{-\lambda T}\right] - E\left[e^{-\lambda T - \gamma X_T + \gamma c}\right]}{1 - E\left[e^{-\lambda T}\right]}$$

Thus in order to evaluate any particular stopping rule T, we need to be able to come up with an expression for

$$f(\lambda, \gamma) \equiv E\, \exp(-\lambda T - \gamma X_T).$$

In all the instances of interest, this is available. It may even be a bit differentiable ... To preclude infinite lower barriers, we shall want $\gamma > \alpha$

It appears hard to get any examples where you preclude infinite lower barriers, yet keep a positive mean value... but the trick is that you have to do the Bayesian version of the problem, where you have some big negative values of μ with small probability. This will generate some exponential aversion I believe.

Some thoughts on a seminar by David Elworthy (16/6/10)

(1) David Elworthy was looking at a question where you have a Markov process X on statespace $\mathfrak{X}$ with semigroup (P_t), a function $\varphi : \mathfrak{X} \to \mathcal{Y}$, and you find that $\varphi(X_t) = Y_t$ is Markovian because of the Dynkin criterion:

$$P_t \Phi = \Phi Q_t$$

where (Q_t) is the semigroup of Y and Φ is the kernel from $\mathfrak{X}$ to $\mathcal{Y}$. What he was interested in was how you filter X from Y, in the case where X, Y were diffusions in some manifolds.

(2) Seems like the issues here are Markov process issues - the diffusion stuff is not essential. Suppose for simplicity that $\mathfrak{X} = \mathcal{Z} \times \mathcal{Y}$ is a product space, and the function φ is projection onto $\mathcal{Y}$: $\varphi(z,y) = y$. Then I claim that if there is a reference measure μ on $\mathcal{Y}$ and ν on $\mathcal{Z}$ such that transition densities and RCDs have densities, the hypothesis is equivalent to

$$p_t((z,y);(z',y')) = q_t(y,y')\, k_t(z,y,y';z')$$

where $k_t(z,y,y';\cdot)$ is the density onto ν of the RCD for Z_t given $Z_0 = z$, $Y_0 = y$ and $Y_t = y'$.

So if you want

$$E\left[\prod_{j=0}^{n} f_j(Z_{t_j}) \,\Big|\, Y_{t_i} = y_i,\ i = 0,\dots,n\right]$$

$$= \int \pi_0(dz_0)\, f_0(z_0) \int k_{s_1}(z_0, y_0, y_1; z_1)\, f_1(z_1)\, \nu(dz_1) \dots$$

$$\int k_{s_n}(z_{n-1}, y_{n-1}, y_n; z_n)\, f_n(z_n)\, \nu(dz_n)$$

where $s_k = t_k - t_{k-1}$. Thus conditional on the observed path of Y, Z is Markovian.

Trading to stops with GBM (30/6/10)

Suppose that we invest in a log-brownian asset $S_t = S_0 \exp[\sigma W_t + (\mu - \frac{1}{2}\sigma^2)t]$ up to some stopping time at which we come out, receiving θS_τ ($\theta < 1$ fixed) then decide to consume a fraction λ of the wealth, reinvesting the remainder into the asset, and repeating the trade. Suppose that we take as the objective

$$\varphi(w_0) \equiv E\left[\sum_{n \geq 1} e^{-\rho T_n} U(w_n \theta \lambda)\right]$$

where $U'(x) = x^{-R}$, and w_n is the available wealth at time T_n before losses. Thus

$$w_{n+1} = w_n \theta (1-\lambda) \, . \, S(T_{n+1})/S(T_n)$$

Clearly there is a scaling property: $\varphi(w_0) = A\, U(w_0)$ for some $A > 0$ to be determined. By Strong Markov property, we get ($S_0 = 1$ for simplicity)

$$\varphi(w_0) = E\left[e^{-\rho T_1} U((S_{T_1}/S_0) w_0 \theta \lambda)\right] + E\left[e^{-\rho T_1} \varphi(w_0\, \theta(1-\lambda) S_{T_1}/S_{T_0})\right]$$

$$= U(w_0) E\left[e^{-\rho T_1} (S_{T_1} \theta\lambda)^{1-R}\right] + A\, U(w_0)\, E\left[e^{-\rho T_1} (\theta(1-\lambda) S_{T_1})^{1-R}\right].$$

Hence

$$A\left\{1 - E\left(e^{-\rho T_1} S_{T_1}^{1-R}\right)(\theta(1-\lambda))^{1-R}\right\} = E\left(e^{-\rho T_1} S_{T_1}^{1-R}\right)(\theta\lambda)^{1-R}$$

Thus

$$A = \frac{(\theta\lambda)^{1-R} E\left[e^{-\rho T_1} S_{T_1}^{1-R}\right]}{1 - (\theta(1-\lambda))^{1-R} E\left(e^{-\rho T_1} S_{T_1}^{1-R}\right)}$$

For a well-posed problem we would require $E\left[e^{-\rho t} S_t^{1-R}\right] \to 0$ if $0 < R < 1$; for $R > 1$, it seems to be ill posed always, because you wait a very long time until the asset rises to a high level, and not suffer if the asset falls, just by not acting then.

Thoughts on a presentation by Harrison Hong (3/7/10)

1) Here's a simple model which Harrison proposes. There's an asset which will deliver a random payment X at time 2, where $X \sim N(0, 1/\tau_0)$. At time 1, a bivariate signal $S = X + \varepsilon$ is observed where agent i thinks $\varepsilon \sim N(0, V_i)$ $i = 1, 2$. Both agents see the signal before they trade. There is net supply Q of the asset, and short sales constraint. Trading is allowed at time 0 before the signals are seen, then at time 1. What is the equilibrium for this model?

2) At time 1, having seen S, agent i thinks

$$\begin{pmatrix} X \\ S \end{pmatrix} \sim N\left(0, \begin{pmatrix} 1/\tau_0 & 1/\tau_0 \\ 1/\tau_0 & 1/\tau_0 + V_i \end{pmatrix}\right)$$

so (J is matrix of ones.)

$$\hat{X}_i = E^i[X|S] = \underset{\sim}{1} \cdot (J + \tau_0 V_i)^{-1} S$$

where $\underset{\sim}{1}$ is the vector of ones, and the conditional variance of $X|S$ is (according to agent i)

$$v_i = \frac{1}{\tau_0}\left(1 - \underset{\sim}{1} \cdot (J + \tau_0 V_i)^{-1} \underset{\sim}{1}\right).$$

If we assume agent i is CARA, with coefficient γ_i of absolute risk aversion, then agent i's demand for the asset will be

$$q_i = \frac{1}{\gamma_i v_i}(\hat{X}_i - p)^+$$

if the price is p. If both agents hold some of the asset, market clearing gives

$$Q = q_1 + q_2 = \frac{\hat{X}_1}{\gamma_1 v_1} + \frac{\hat{X}_2}{\gamma_2 v_2} - p\left(\frac{1}{\gamma_1 v_1} + \frac{1}{\gamma_2 v_2}\right)$$

So

$$\boxed{p = \pi_1 \hat{X}_1 + \pi_2 \hat{X}_2 - \lambda Q}$$

$\pi_i \propto 1/\gamma_i v_i$, $\pi_1 + \pi_2 = 1$, $\lambda = \left(\frac{1}{\gamma_1 v_1} + \frac{1}{\gamma_2 v_2}\right)^{-1}$

and

$$q_1 = \frac{1}{\gamma_1 v_1}\left\{\pi_2(\hat{X}_1 - \hat{X}_2) + \lambda Q\right\} \equiv \frac{1}{\gamma_1 v_1}\left\{\pi_2 \Delta + \lambda Q\right\}$$

$$q_2 = \frac{1}{\gamma_2 v_2}\left\{\pi_1(\hat{X}_2 - \hat{X}_1) + \lambda Q\right\} \equiv \frac{1}{\gamma_2 v_2}\left\{-\pi_1 \Delta + \lambda Q\right\}$$

where $\Delta \equiv \hat{X}_1 - \hat{X}_2$.

So by seeing where q_1 or q_2 goes negative, we see that there are three regimes:

(i) $\underline{\Delta < -\lambda Q/\pi_2}$: $q_1 = 0$, $q_2 = Q$, $p = \hat{X}_2 - \gamma_2 v_2 Q$

(ii) $\underline{-\lambda Q/\pi_2 \le \Delta \le \lambda Q/\pi_1}$: $q_1 = \frac{1}{\gamma_1 v_1}\{\pi_2 \Delta + \lambda Q\}$, $q_2 = \frac{1}{\gamma_2 v_2}\{-\pi_1 \Delta + \lambda Q\}$,

$$p = \pi_1 \hat{X}_1 + \pi_2 \hat{X}_2 - \lambda Q$$

(iii) $\underline{\frac{\lambda Q}{\pi_1} < \Delta}$: $q_1 = Q$, $q_2 = 0$, $p = \hat{X}_1 - \gamma_1 v_1 Q$.

We now want to calculate for each of the cases

$$E_i\left[-\exp(-\gamma_i q_i (X-p)) \mid S\right]$$

(i) We get

$$E_2\left[\exp(-\gamma_2 q_2 (X-p)) \mid S\right] = \exp\{-\tfrac{1}{2}\gamma_2^2 Q^2 v_2\}$$

(ii) $E_i\left[\exp\{-\gamma_i q_i (X-p)\} \mid S\right] = \exp\{-\tfrac{1}{2}\gamma_i^2 q_i^2 v_i\}$

after some calculations.

(iii) is analogous to (i)

Overall then, in all cases

$$\boxed{E_i\left[\exp\{-\gamma_i q_i (X-p)\} \middle| S\right] = \exp\{-\tfrac{1}{2}\gamma_i^2 v_i q_i^2\}}$$

3) At time 0, agent i will choose to hold θ_i units of the stock, where once again $\theta_i \ge 0$ will be demanded. The price p_0 will have to be paid, and p_0 will need to be chosen to clear the market. But in order to do that we have to know what the value of agent i's objective would be if he started off with θ_i units of stock, so we must calculate

$$E_i \exp(-\gamma_i q_i (X-p) - \gamma_i \theta_i p)$$

$$= E_i\, E_i\left[\exp(-\gamma_i q_i (X-p) - \gamma_i \theta_i p) \mid S\right]$$

$$= E_i\, E_i\left[\exp\{-\tfrac{1}{2}\gamma_i^2 v_i q_i^2 - \gamma_i \theta_i p\} \mid S\right]$$

$$= E_i\Big[\exp(-\tfrac{1}{2}\gamma_i^2 v_i q_i^2)\, E_i[\exp(-\gamma_i\theta_i p)\,/\Delta]\Big] \qquad (*)$$

So for this, we need to know the conditional law of $\hat{X}_1$ given Δ. Notice that $\hat{X}_i = z_i^T S$, so if $w \equiv z_1 - z_2$, we have that for agent i

$$\begin{pmatrix}\hat{X}_1\\ \Delta\end{pmatrix} \sim N\Big(0,\ (z_1\, w)^T\big(\tfrac{1}{\tau_0}+V_i\big)(z_1\, w)\Big)$$

Thus we have $(\hat{X}_1/\Delta) \sim N(a_i\Delta,\ b_i)$ where a_i, b_i can be obtained from the covariance matrix. Accordingly, the conditional expectation can be handled in the three cases

(i) if $\Delta < -\lambda Q/\pi_2$, $p = -\Delta + \hat{X}_1 - \gamma_2 v_2 Q$

$$E_i[\exp(-\gamma_i\theta_i p)/\Delta]$$

$$= \exp\Big[-\gamma_i\theta_i(-\Delta - \gamma_2 v_2 Q) - \gamma_i\theta_i a_i\Delta + \tfrac{1}{2}(\gamma_i\theta_i)^2 b_i\Big]$$

(ii) if $-\lambda Q/\pi_2 \leq \Delta \leq \lambda Q/\pi_1$, we have $p = -\pi_2\Delta + \hat{X}_1 - \lambda Q$, so

$$E_i[\exp(-\gamma_i\theta_i p)/\Delta]$$

$$= \exp\Big\{-\gamma_i\theta_i(-\pi_2\Delta - \lambda Q) - \gamma_i\theta_i a_i\Delta + \tfrac{1}{2} b_i(\gamma_i\theta_i)^2\Big\}$$

(iii) if $\Delta > \lambda Q/\pi_1$ we have $p = \hat{X}_1 - \gamma_1 v_1 Q$, so we get

$$E_i[\exp(-\gamma_i\theta_i p)/\Delta]$$

$$= \exp\Big\{\gamma_i\theta_i\gamma_1 v_1 Q - \gamma_i\theta_i a_i\Delta + \tfrac{1}{2} b_i(\gamma_i\theta_i)^2\Big\}$$

4) Returning to the calculation of $(*)$, we have that agent i thinks that $\Delta \sim N(0, c_i)$ so the expectation in $(*)$ can be calculated in three pieces, depending on the interval in which Δ lies.

(i) $$I_- \equiv E_i\Big[\exp(-\tfrac{1}{2}\gamma_i^2 v_i q_i^2)E_i[\exp(-\gamma_i\theta_i p)/\Delta] : \Delta < -\lambda Q/\pi_2\Big]$$

$$= \exp\{-\tfrac{1}{2}\gamma_i^2 v_i q_i^2\}\exp\{\gamma_i\theta_i\gamma_2 v_2 Q + \tfrac{1}{2} b_i(\gamma_i\theta_i)^2\}$$
$$E_i\Big[\exp(-\gamma_i\theta_i(a_i - 1)\Delta) : \Delta < -\lambda Q/\pi_2\Big]$$

$$= \exp\left[-\tfrac{1}{2}\gamma_i^2 v_i q_i^2 + \gamma_i\theta_i\gamma_2 v_2 Q + \tfrac{1}{2}b_i(\gamma_i\theta_i)^2 + \frac{c_i\nu_i^2}{2}\right]\Phi\left(\frac{c_i\nu_i - \lambda Q/\pi_2}{\sqrt{c_i}}\right)$$

where $\nu_i \equiv \gamma_i\theta_i(a_i - 1)$

(ii) In the middle region, assuming $i=1$ to begin with, we have

$$(*) = E_1\left[\exp\left(-\tfrac{1}{2}\gamma_1^2 v_1\left(\frac{\pi_2\Delta + \lambda Q}{\gamma_1 v_1}\right)^2 + \gamma_1\theta_1\lambda Q + \tfrac{1}{2}(\gamma_1\theta_1)^2 b_1 - \tilde{\nu}_1\Delta\right) : \frac{-\lambda Q}{\pi_2} \leq \Delta \leq \frac{\lambda Q}{\pi_1}\right]$$

where $\tilde{\nu}_1 \equiv \gamma_1\theta_1(a_1 - \pi_2)$

$$= E_1\left[\exp\left(\tfrac{1}{2}b_1(\gamma_1\theta_1)^2 + \gamma_1\theta_1\lambda Q - \frac{(\pi_2\Delta + \lambda Q)^2}{2v_1} - \tilde{\nu}_1\Delta\right) : -\frac{\lambda Q}{\pi_2} \leq \Delta \leq \frac{\lambda Q}{\pi_1}\right]$$

$$= \exp\left[\tfrac{1}{2}b_1(\gamma_1\theta_1)^2 + \gamma_1\theta_1\lambda Q\right]\int_{-\frac{\lambda Q}{\pi_2}}^{\frac{\lambda Q}{\pi_1}} \exp\left\{-\frac{(\pi_2 x + \lambda Q)^2}{2v_1} - \tilde{\nu}_1 x - \frac{x^2}{2c_1}\right\}\frac{dx}{\sqrt{2\pi c_1}}$$

$$\left[\frac{1}{\tilde{v}_1} = \frac{\pi_2^2}{v_1} + \frac{1}{c_1},\quad \tilde{b}_1 = \left(\tilde{\nu}_1 + \frac{\lambda Q\pi_2}{v_1}\right)\tilde{v}_1\right]$$

$$= \exp\left\{\tfrac{1}{2}b_1(\gamma_1\theta_1)^2 + \gamma_1\theta_1\lambda Q\right\}\exp\left\{-\frac{(\lambda Q)^2}{2v_1} + \frac{\tilde{b}_1^2}{2\tilde{v}_1}\right\}.$$

$$\int_{-\frac{\lambda Q}{\pi_2}}^{\lambda Q/\pi_1} \exp\left\{-\frac{(x + \tilde{b}_1)^2}{2\tilde{v}_1}\right\}\frac{dx}{\sqrt{2\pi c_1}}$$

$$= \sqrt{\frac{\tilde{v}_1}{c_1}}\exp\left[\tfrac{1}{2}b_1(\gamma_1\theta_1)^2 + \gamma_1\theta_1\lambda Q - \frac{(\lambda Q)^2}{2v_1} + \frac{\tilde{b}_1^2}{2\tilde{v}_1}\right].$$

$$\left\{\Phi\left(\frac{\lambda Q/\pi_1 + \tilde{b}_1}{\sqrt{\tilde{v}_1}}\right) - \Phi\left(\frac{-\lambda Q/\pi_2 + \tilde{b}_1}{\sqrt{\tilde{v}_1}}\right)\right\}$$

Analogously for agent 2, if $1/\tilde{v}_2 \equiv \frac{\pi_1^2}{v_2} + 1/c_2$, $\tilde{b}_2 = \left(\tilde{\nu}_2 - \lambda Q\pi_1/v_2\right)\tilde{v}_2$, $\tilde{\nu}_2 = \gamma_2\theta_2(a_2 - \pi_2)$, we get

$$\sqrt{\frac{\tilde{v}_2}{c_2}}\exp\left[\tfrac{1}{2}b_2(\gamma_2\theta_2)^2 + \gamma_2\theta_2\lambda Q - \frac{(\lambda Q)^2}{2v_2} + \frac{\tilde{b}_2^2}{2\tilde{v}_2}\right].$$

$$\left\{\Phi\left(\frac{\lambda Q/\pi_1 + \tilde{b}_2}{\sqrt{\tilde{v}_2}}\right) - \Phi\left(\frac{-\lambda Q/\pi_2 + \tilde{b}_2}{\sqrt{\tilde{v}_2}}\right)\right\}$$

(iii) The third contribution is like the first: we get

$$I_+ \equiv E_i\Big[\exp\big(-\tfrac{1}{2}\gamma_i^2 v_i q_i^2\big)\, E_i\big[\exp(-\gamma_i\theta_i p)\,\big|\,\Delta\big] : \Delta > \lambda Q/\pi_1\Big]$$

$$= \exp\Big\{-\tfrac{1}{2}\gamma_i^2 v_i q_i^2 + \gamma_i\theta_i\gamma_1 v_1 Q + \tfrac{1}{2}\gamma_i^2\theta_i^2 b_i + c_i\kappa_i^2/2\Big\}\, \bar{\Phi}\left(\frac{\lambda Q\pi_1 + c_i\kappa_i}{\sqrt{c_i}}\right)$$

where $\kappa_i = \gamma_i\theta_i a_i$

5) Assembling all of this, we have an expression for

$$E_i\big[\exp\big(-\gamma_i q_i(X-p) - \gamma_i\theta_i p\big)\big]$$

where p is the function of S, and q_i is the function of S which correspond to the time-1 market clearing prices and quantities. The agents have to pay the time-0 price p_0 in order to buy their desired θ_i units of stock, so each agent attempts to

$$\max_{0\le\theta\le Q} \; -E_i\big[\exp\big(-\gamma_i q_i(X-p) - \gamma_i\theta_i p + \gamma_i\theta_i p_0\big)\big]$$

(Notice that $p_0 < 0$ is to be expected, since we are buying a risky zero-mean CE.)
Writing $F_i(\theta)$ for $E_i\big[\exp\{-\gamma_i q_i(X-p) - \gamma_i\theta p\}\big]$ we shall want for optimality

$$\frac{d\log F_i}{d\theta} = -\gamma_i p_0 \qquad (i=1,2)$$

so for market clearing we'll require

$$\gamma_2\,\frac{d\log F_1}{d\theta}(\theta) - \gamma_1\,\frac{d\log F_2}{d\theta}(Q-\theta) = 0.$$

Of course, we may get an endpoint of the interval.

Some curious stylized facts about asset returns (28/7/10)

1) Suppose that Y_t is logreturn of some asset on day t. What's rather striking is that if you look at $\Delta Y_t = Y_t - Y_{t-1}$, calculate the ACF of ΔY_t for various lags, what you find is that the autocovariance at lag 1 is about $-\frac{1}{2}$ and not very large for all lags > 1. This seems to be true for a very wide range of assets, and it holds if you scale out volatility or not.

What could be explaining this?

2) The first thing I thought of was

$$\begin{cases} Y_{t+1} = \mu_{t+1} + \eta_{t+1} \\ \mu_{t+1} = \mu_t + \varepsilon_{t+1} \end{cases}$$

with $\eta_t \sim N(0, \sigma_\eta^2)$, $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$ all independent. Then we would have

$$\Delta Y_t = Y_t - Y_{t-1} = \eta_t - \eta_{t-1} + \varepsilon_t$$

and so $E[\Delta Y_t^2] = 2\sigma_\eta^2 + \sigma_\varepsilon^2$, $E[\Delta Y_t \Delta Y_{t-1}] = -\sigma_\eta^2$

$$\Rightarrow -\operatorname{corr}(\Delta Y_t, \Delta Y_{t-1}) \equiv -\gamma_1 = \frac{\sigma_\eta^2}{2\sigma_\eta^2 + \sigma_\varepsilon^2}.$$

So if we had

$$\boxed{\sigma_\varepsilon \ll \sigma_\eta}$$

this would explain why $-\gamma_1$ should be close to $1/2$ ...

(3) ... but not why $-\gamma_1$ should be larger than $1/2$, which the data certainly does show. Could we explain this by supposing that ε_t, η_t are correlated? A few calculations show that this can't explain the phenomenon:

$$v_0 \equiv E[\Delta Y_t^2] = 2\sigma_\eta^2 + 2\rho\sigma_\eta\sigma_\varepsilon + \sigma_\varepsilon^2$$

$$v_1 \equiv E[\Delta Y_t \Delta Y_{t-1}] = -\sigma_\eta^2 - \rho\sigma_\eta\sigma_\varepsilon$$

$$\Rightarrow -\gamma_1 = \frac{\sigma_\eta^2 + \rho\sigma_\varepsilon\sigma_\eta}{2\sigma_\eta^2 + 2\rho\sigma_\eta\sigma_\varepsilon + \sigma_\varepsilon^2} < \frac{1}{2}$$

So perhaps we might try correlation across neighbouring periods?

$$\Delta Y_t = \eta_t - \eta_{t-1} + \varepsilon_t$$

$$E\left[\Delta Y_t \Delta Y_{t-1}\right] = E\left[\left(\eta_t - \eta_{t-1} + \varepsilon_t\right)\left(\eta_{t-1} + \varepsilon_{t-1} - \eta_{t-2}\right)\right]$$

(4) Maybe η_t correlated with ε_{t+1}?

$$E[\Delta Y_t^2] = 2\sigma_\eta^2 + \sigma_\varepsilon^2 - 2\rho\sigma_\eta\sigma_\varepsilon$$

$$E[\Delta Y_t \Delta Y_{t-1}] = -\sigma_\eta^2 + \rho\sigma_\eta\sigma_\varepsilon$$

$$\Rightarrow \quad -\gamma_1 = \frac{\sigma_\eta^2 - \rho\sigma_\eta\sigma_\varepsilon}{2\sigma_\eta^2 - 2\rho\sigma_\eta\sigma_\varepsilon + \sigma_\varepsilon^2} < \frac{1}{2}$$

So this doesn't work

(5) Maybe η_t correlated with ε_{t-1}?

$$E[\Delta Y_t^2] = 2\sigma_\eta^2 + \sigma_\varepsilon^2$$

$$E[\Delta Y_t \Delta Y_{t-1}] = \rho\sigma_\eta\sigma_\varepsilon - \sigma_\eta^2$$

$$\therefore \quad \boxed{-\gamma_1 = \frac{\sigma_\eta^2 - \rho\sigma_\eta\sigma_\varepsilon}{2\sigma_\eta^2 + \sigma_\varepsilon^2}}$$

which could explain it: $\boxed{-\gamma_1 > \frac{1}{2} \Leftrightarrow \rho < -\sigma_\varepsilon / 2\sigma_\eta}$

Since we expect typically that $\sigma_\varepsilon \ll \sigma_\eta$, this would require a little negative correlation between ε_{t-1} and η_t to give $-\gamma_1 > \frac{1}{2}$ ... no problem with that.

Therefore the issue will be with estimating σ_ε, σ_η and ρ. A simple thing that could be done would be to assume we can ignore σ_ε^2 in $E[\Delta Y_t^2]$, but this might be a bit inaccurate

(6) I tried out simulating the baseline story of ε, η independent, and looked at the sorts of magnitudes you get for γ_1: it was broadly quite similar to what I observed empirically: $-\gamma_1 \in (.45, .55)$, empirical very similar, perhaps a little higher... <u>maybe all that comes out of this is that GBM is not too bad a model</u>...

Some thoughts on contracting (1/8/10)

1) I suggested to Takashi that we might consider a very simple contracting problem where the outcome $X \sim N(a, 1)$, where a is the agent's effort, and the principal wants to max $E\, U_p(X - \varphi(X))$ subject to the agent's participation constraint $\sup_a E\, U_A(\varphi(X)) = t$. We can formulate this as

$$\sup_\varphi \int F(x, \varphi(x), a)\, dx$$

subj to

$$\sup_a \int G(x, \varphi(x), a)\, dx = t,$$

where $F(x, \varphi(x), a) \equiv U_p(x - \varphi(x)) f(x|a)$, $G(x, \varphi(x), a) = U_A(\varphi(x) - c(a)) f(x|a)$ say, where $c(a)$ is cost of effort a.

2) Suppose $\varphi(t, \cdot)$ is the optimal contract for reservation utility level t, which cause agent to use action $a_t \equiv a(t)$. Suppose now that the principal alters $\varphi(t, \cdot)$ to $\varphi(t, \cdot) + \psi(\cdot)\Delta t$ in such a way as to result in raising the agent's utility to $t + \Delta t$, causing optimal a_t to modify to $a_t + \Delta a$. To leading order, the change in the principal's objective is

$$\Delta t \int \Big\{ F_\varphi(x, \varphi(t,x), a_t)\, \psi(x) + F_a(x, \varphi(t,x), a_t) \frac{\Delta a}{\Delta t} \Big\} dx$$

which he wants to maximise subject to

$$\int G_\varphi(x, \varphi(t,x), a_t)\, \psi(x)\, dx = 1$$

(because the change of the agent's objective with a is, to leading order, 0, since a_t was optimal). The only issue is concerning Δa. But since a_t was optimal, we know that

$$\int G_a(x, \varphi(t,x), a_t)\, dx = 0$$

so when we perturb to $t + \Delta t$ the same must remain true. To leading order,

$$0 = \int \Big\{ G_{a\varphi}(x, \varphi(t,x), a_t)\, \psi(x) + G_{aa}(x, \varphi(t,x), a_t) \frac{\Delta a}{\Delta t} \Big\} dx$$

$$\Rightarrow \boxed{\frac{\Delta a}{\Delta t} = - \frac{\int G_{a\varphi}(x', \varphi, a)\, \psi(x')\, dx'}{\int G_{aa}(x', \varphi, a)\, dx'}}$$

The principal's objective is therefore

$$\int \psi(x) \left\{ F_\varphi(x, \varphi(t,x), a_t) - \frac{G_{a\varphi}(x, \varphi(t,x), a_t)}{\int G_{aa}(x', \varphi(t,x'), a_t)\,dx'} \cdot \int F_a(x', \varphi(t,x'), a_t)\,dx' \right\} dx$$

which has to be maximised subject to $\int G_\varphi(x, \varphi, a)\psi(x)\,dx = 1$. By considering the Lagrangian form, we see we have to have for some λ

$$F_\varphi(x, \varphi(t,x), a_t) - G_{a\varphi}(x, \varphi(t,x), a_t) \cdot \frac{\int F_a(x', \varphi(t,x'), a_t)\,dx'}{\int G_{aa}(x', \varphi(t,x'), a_t)\,dx'} = \lambda G_\varphi(x, \varphi(t,x), a_t)$$

and

$$\boxed{\frac{da}{dt} = - \frac{\int G_{a\varphi}(x, \varphi(t,x), a_t)\,dx}{\int G_{aa}(x, \varphi(t,x), a_t)\,dx}}$$

If we set $\theta_t \equiv \dfrac{\int F_a(x, \varphi(t,x), a_t)\,dx}{\int G_{aa}(x, \varphi(t,x), a_t)\,dx}$

then we have to obtain $\varphi(t, \cdot)$ by solving

$$F_\varphi(x, y, a_t) - G_{a\varphi}(x, y, a_t)\theta_t = \lambda G_\varphi(x, y, a_t)$$

which would determine $y = \varphi(t,x)$ given the constants θ_t, λ, which are themselves specified via

$$\boxed{\begin{aligned} \lambda &= \lambda \int G_\varphi(x, \varphi, a)\,dx = \int (F_\varphi - \theta G_{a\varphi})(x, \varphi(t,x), a_t)\,dx \\ \theta_t &= \int F_a(x, \varphi, a)\,dx \Big/ \int G_{aa}(x, \varphi, a)\,dx \end{aligned}}$$

3) Takashi asks a good question: suppose agent sees some signal Y which the agent doesn't; how could this be included?

A contracting example (2/10/08).

(1) Here's a question I put to Takashi. Suppose outcome $X \sim N(a, 1)$, where the action a is chosen by the agent. How do we solve the contracting problem

$$\max \int U_p(x - \varphi(x))\, f(x|a)\, dx$$

$$\text{s.t.} \quad \sup_a \int \{U_A(\varphi(x)) - c(a)\}\, f(x|a)\, dx = \underline{u}$$

(2) In the first question, suppose we allow only $a = a_0$ or $a = a_1$, with corresponding densities f_0, f_1. Let's decompose the problem a bit. Suppose the principal initially seeks the best φ which would induce agent to use action a_0. Then it's

$$\max \int U_p(x - \varphi(x))\, f_0(x)\, dx$$

$$\text{s.t.} \quad \underline{u} = \int \{U_A(\varphi(x)) - c_0\}\, f_0(x)\, dx \geq \int (U_A(\varphi(x)) - c_1)\, f_1(x)\, dx$$

Now we expect that the inequality will be satisfied strictly, so small perturbations of optimal φ will not affect the unfeasibility of action 1. So the problem would be

$$\max \int \{U_p(x - \varphi(x)) + \lambda(U_A(\varphi(x)) - c_0)\}\, f_0(x)\, dx$$

solved when

$$\frac{U_p'(x - \varphi(x))}{U_A'(\varphi(x))} = \lambda$$

This generates a solution $\varphi_\lambda(x)$ which is increasing with λ, since U_p, U_A are both concave. We therefore need to identify λ_0^* which is value of λ which makes the solution feasible.

$$\underline{u} = \int \{U_A(\varphi_{\lambda_0^*}(x)) - c_0\}\, f_0(x)\, dx$$

$$= \int \{U_A(\varphi_{\lambda_1^*}(x)) - c_1\}\, f_1(x)\, dx$$

by symmetric reasoning. I claim that the best thing for the principal to do is to choose $\lambda^* = \min\{\lambda_0^*, \lambda_1^*\}$ and use $\varphi_{\lambda^*}(\cdot)$ as the contract.

Why should this be correct? If $\lambda^* = \lambda^*_0 < \lambda^*_1$, then using the contracts φ_{λ^*} will result in utility $\underline{u}$ if agent uses a_0, but if he uses a_1, he will get

$$\int \{ U(\varphi_{\lambda^*_0}(x)) - c_1 \} f_1(x)\, dx \le \int \{ U(\varphi_{\lambda^*_1}(x)) - c_1 \} f_1(x)\, dx = \underline{u}$$

so this proves that it's best for the agent to pick a_0.

(3) This argument now works for any range of choices for a, not just a two-point set! We find $\lambda^*(a)$ to solve

$$\underline{u} = \int \{ U_A(\varphi_\lambda(x)) - c(a) \} f(x|a)\, dx$$

and find a^* to minimise $\lambda^*(a)$. This is optimal.

[But this assumes that perturbing φ doesn't change a, and this must be incorrect.]

(4) The argument also extends to the situation where the agent's objective is

$$\int U_A(\varphi(x) - c(a))\, f(x|a)\, dx$$

(i.e. the cost is inside the utility, arguably a more natural story)

(5) If $f(x|a) = \exp(-\frac{1}{2}(x-a)^2)(2\pi)^{-\frac{1}{2}}$, $U_P'(x) = e^{-\gamma x}$, $U_A'(x) = e^{-\beta x}$ then we find after a few calculations that $\lambda^*(a)$ satisfies

$$\frac{\beta}{\gamma+\beta} \log \lambda^*(a) = \frac{1}{2}\beta a^2 - \frac{\beta\gamma a}{\beta+\gamma} + \frac{1}{2}\frac{(\beta\gamma)^2}{(\beta+\gamma)^2} - \log(-\beta\bar{u})$$

Minimising over a gives best a is $a = \gamma/(\beta+\gamma)$ [This assumes the second form of the objective, as given in (4) above]

Stylized facts of asset-returns again (5/8/10)

(i) We have seen that it makes sense to model log-returns $Y_t = \log(P_t/P_{t-1})$ as

$$\left.\begin{aligned} Y_t &= \mu_t + \eta_t \\ \mu_t &= \mu_{t-1} + \varepsilon_t \end{aligned}\right\} \qquad \eta_t \text{ correlated with } \varepsilon_{t-1}$$

Let's follow through the KF analysis of this. To express the correlation of η_t with ε_{t-1}, we can write $\eta_t = \alpha\varepsilon_{t-1} + \xi_t$ where ξ is indept of the ε's. Then we have

$$Y_t = \mu_t + \alpha\varepsilon_{t-1} + \xi_t = \mu_t + \alpha(\mu_{t-1} - \mu_{t-2}) + \xi_t$$
$$\mu_t = \mu_{t-1} + \varepsilon_t$$

Thus we have the <u>three</u>-dimensional state vector $x_t = (\mu_t, \mu_{t-1}, \mu_{t-2})^T$,

$$x_t = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} x_{t-1} + \begin{pmatrix} \varepsilon_t \\ 0 \\ 0 \end{pmatrix} \equiv A x_{t-1} + z_t$$

with $Y_t = Cx_t + \xi_t$, $C = (1, \alpha, -\alpha)$.

(ii) Suppose that

$$(x_t \mid \mathcal{Y}_t) \sim N(\hat{x}_t, V_t)$$

so that then $\qquad [M_t \equiv AV_tA^T + \Sigma_z]$

$$\begin{pmatrix} x_{t+1} \\ Y_{t+1} \end{pmatrix} \Bigg| \mathcal{Y}_t \sim N\left(\begin{pmatrix} A\hat{x}_t \\ CA\hat{x}_t \end{pmatrix}, \begin{pmatrix} M_t & M_tC^T \\ CM_t & CM_tC^T + \sigma_\xi^2 \end{pmatrix} \right).$$

Therefore

$$\left.\begin{aligned} \hat{x}_{t+1} - A\hat{x}_t &= M_tC^T \frac{(Y_{t+1} - CA\hat{x}_t)}{CM_tC^T + \sigma_\xi^2} \\ V_{t+1} &= M_t - \frac{M_tC^TCM_t}{\sigma_\xi^2 + CM_tC^T}. \end{aligned}\right\}$$

(iii) If we have limiting forms V, M for V_t, M_t, then there's a steady-state form for the updating:

$$\boxed{\begin{aligned}\hat{x}_{t+1} &= \left(I - \frac{MC^TC}{\sigma_\xi^2 + CMC^T}\right)A\hat{x}_t + \frac{MC^T}{\sigma_\xi^2 + CMC^T}\, Y_{t+1} \\ &= K\hat{x}_t + G\,Y_{t+1}\end{aligned}}$$

Notice: $K1 + G1 = 1$, $K\begin{pmatrix}0\\0\\1\end{pmatrix} = 0$, so K has a zero eigenvalue.

I tried to find some closed-form expressions for M, V using Maple, but nothing seemed to simplify at all.

Black-Scholes:

$$C(K) = S_0\, \bar{\Phi}(a - \sigma\sqrt{T}) - e^{-rT} K\, \bar{\Phi}(a) \quad , \qquad a = \frac{1}{\sigma\sqrt{T}}\left(\log\left(\frac{K}{S_0}\right) - rT + \frac{1}{2}\sigma^2 T\right)$$

$$C'(K) = -e^{-rT}\, \bar{\Phi}(a)$$

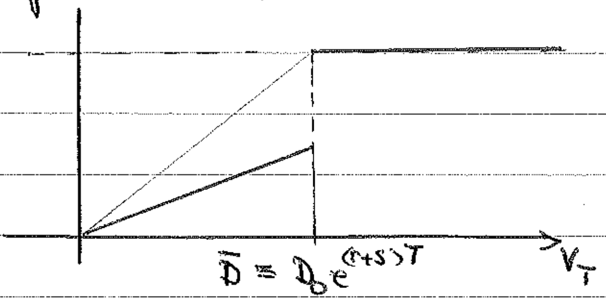
Comments on a paper of Jurek & Stafford (2/9/10)

(1) Suppose V_t is the value at time t of a firm's assets. A borrower wants to buy a unit of stock at time 0 by putting in cash (equity) Q_0, and borrowing $D_0 = V_0 - Q_0$. Debt is repayable at T, and the lender charges a spread s. How should we calculate s?

Suppose that if the firm ends in default (i.e. $V_T < D_0 e^{(r+s)T}$) then there is recovery fraction ρ.

Then

$$D_T = \begin{cases} D_0 e^{(r+s)T} & \text{if } V_T \geq D_0 e^{(r+s)T} \\ \rho V_T & \text{else} \end{cases}$$



Thus the value of debt is expressed as a digital $(1-\rho) D_0 e^{(r+s)T} I_{\{V_T \geq D_0 e^{(r+s)T}\}}$ plus a constant $\rho \bar{D} = \rho D_0 e^{(r+s)T}$ minus ρ puts with strike $\bar{D}$.

All of these should be easily calculated from the call price function, or the put price function. So if $C(K) = E[(V_T - K)^+ e^{-rT}]$, we have

$$C(K) - P(K) = V_0 - e^{-rT} K$$

and $C'(K) = -e^{-rT} P(K < V_T)$. Thus the time-0 value of the debt is

$$e^{-rT} \rho \bar{D} - (1-\rho) \bar{D} C'(\bar{D}) - \rho P(\bar{D})$$

and the spread has to be adjusted to make this equal to the initial borrowing D_0.

[Note that none of this supposes particular asset dynamics.]

(2) In such a situation, the firm's equity is worth $C(\bar{D})$ at time 0, and $(V_T - \bar{D})^+$ at time T. If someone wants to buy one unit of the stock by putting in his own cash to the value of q_0, and borrowing remainder $d_0 = C(\bar{D}) - q_0$ at overall rate of interest R, then the value repaid to the lender at time T is

$$\min\{ d_0 e^{RT}, (V_T - \bar{D})^+ \}$$

so we need to select R so as to equate

$$d_0 = e^{-rT} E\left[d_0 e^{RT} \wedge (V_T - \bar{D})^+ \right] = C(\bar{D}) - C(d_0 e^{RT} + \bar{D})$$

(3) Now the story gets more BS-like. Suppose there's a market asset

$$d\bar{V} = \bar{V}(\sigma_M d\bar{W} + r\,dt)$$

and that other assets are correlated therewith:

$$dV = V(\sigma(\beta d\bar{W} + \beta' dW) + r\,dt), \qquad \beta^2 + \beta'^2 = 1$$

The idea is that an individual corporate bond at time T will be worth $\bar{D} I_{\{V_T \geq \bar{D}\}} + \rho V_T I_{\{V_T < \bar{D}\}}$, but that the CDO pool of bonds will be worth

$$E\left[\bar{D} I_{\{V_T \geq \bar{D}\}} + \rho V_T I_{\{V_T < \bar{D}\}} \,\middle|\, \bar{W}_T\right]$$

at time T.

Now $\log(V_T/V_0) = \sigma(\beta\bar{W}_T + \beta' W_T) + (r - \frac{1}{2}\sigma^2)T$, so

$$V_T = V_0 \exp(\sigma\beta\bar{W}_T) \exp(\sigma\beta' W_T + (r - \tfrac{1}{2}\sigma^2)T)$$

so $E\left[(V_T - K)^+ \,\middle|\, \bar{W}_T = w\right]$

$$= E\left[\left(V_0 \exp(\sigma\beta w - \tfrac{1}{2}\sigma^2\beta^2 T)\, e^{\sigma\beta' W_T - \frac{1}{2}\sigma^2\beta'^2 T} e^{rT} - K\right)^+ \,\middle|\, \bar{W}_T = w\right]$$

$$= e^{rT} C_{BS}\left(V_0 e^{\sigma\beta w - \frac{1}{2}\sigma^2\beta^2 T}, K, \sigma\beta', T, r\right).$$

We need to calculate

$$E\left[\bar{D} I_{\{V_T \geq \bar{D}\}} + \rho V_T I_{\{V_T < \bar{D}\}} \,\middle|\, \bar{W}_T = w\right]$$

$$= E\left[-\rho(\bar{D} - V_T)^+ + \rho\bar{D} + (1-\rho)\bar{D} I_{\{V_T \geq \bar{D}\}} \,\middle|\, \bar{W}_T = w\right]$$

$$= E\left[\rho V_T - \rho(V_T - \bar{D})^+ + (1-\rho)\bar{D} I_{\{V_T \geq \bar{D}\}} \,\middle|\, \bar{W}_T = w\right]$$

This can be evaluated quite explicitly. Call this function $\Psi(w)$

(4) If we want to work out tranche spreads, then we need to think what the pool is worth at time T, viz. $\Psi(W_T)$, always $\leq \bar{D}$. If we want to do the $[a, b]$ tranche (where $0 < a < b < 1$) then what you receive for your initial D_0 is

$$\left\{\frac{1}{b-a}\left(\Psi(W_T) - (1-b)\bar{D}\right)^+\right\} \wedge \bar{D}$$

A simple model coming from a question of Ezequiel Antar (5/9/10)

1) This is just a two-period story, where all randomness comes from $X \sim N(0,V)$.
Agent j is CARA(γ_j) and is exposed to baseline risk $b_j \cdot X$, $\bar{b} = \sum_j b_j$.
The market only allows certain positions θ to be taken:

$$\theta = M\varphi$$

where the number of rows of M is at least the number of columns, and M is of full rank.
The idea is to work out the equilibrium and see what changes if we enlarge the market to include various (zero-net supply) financial assets. Assume that

$$a = \sum_j \theta_j = M\alpha$$

is the total supply.

2) Agent j's problem is

$$\max_{\varphi_j} E - \exp\{-\gamma_j\{(M\varphi_j)\cdot(X-p) + b_j\cdot X\}\}$$

$$= \max - \exp\{\gamma_j(M\varphi_j)\cdot p + \tfrac{1}{2}\gamma_j^2(M\varphi_j + b_j)\cdot V(M\varphi_j + b_j)\}$$

equivalently,

$$\min \tfrac{1}{2}\gamma_j^2(M\varphi + b_j)\cdot V(M\varphi + b_j) + \gamma_j(M\varphi)\cdot p$$

Calculus $\Rightarrow$

$$\gamma_j M^T V(M\varphi_j + b_j) = -M^T p$$

$$\boxed{\varphi_j = -\gamma_j^{-1}(M^TVM)^{-1}M^Tp - (M^TVM)^{-1}M^TVb_j}$$

Write $K \equiv (M^TVM)^{-1}$ for brevity. Market clearing gives us

$$\alpha = -\Gamma^{-1}KM^Tp - KM^TV\bar{b}$$

$$\therefore \boxed{M^Tp = -\Gamma K^{-1}\alpha - \Gamma M^TV\bar{b}}$$

and hence $\varphi_j = +\gamma_j^{-1}\Gamma\alpha + KM^TV(\gamma_j^{-1}\Gamma\bar{b} - b_j)$

$$= \pi_j(\alpha + KM^TV\bar{b}) - KM^TVb_j \qquad (\pi_j = \gamma_j^{-1}\Gamma)$$

The minimized quadratic for agent j turns out (after some routine but lengthy calculations) to be

$$\tfrac{1}{2}\gamma_j^2\Big[\, b_j^T V b_j - \{\pi_j\,(K^{-1}\alpha + M^T V \bar b) - M^T V b_j\} \cdot K \{\pi_j\,(K^{-1}\alpha + M^T V \bar b) - M^T V b_j\}\,\Big]$$

3) Let's specialize to $M = \binom{I}{0}$, $X = \binom{Y}{Z}$, so that Y is the vector of initially traded contingent claims. In this case, we have initially that the prices of assets Y are given by

$$p^Y = -\Gamma V_{YY}\,\alpha - \Gamma\,(V_{YY}\;\; V_{YZ})\,\bar b$$

If we now change the story to allow trading of all assets, then we get prices of the original assets become unchanged.

How about the values of the agents? For agent j, the minimized quadratic is

$$\tfrac{1}{2}\gamma_j^2\Big[\, b_j^T V b_j - \big(\pi_j K^{-1}\alpha + M^T V(\pi_j \bar b - b_j)\big)\cdot K\,\big(\pi_j K^{-1}\alpha + M^T V(\pi_j \bar b - b_j)\big)\Big]$$

and the bit that might change when we introduce financial derivatives is

$$\big(\pi_j K^{-1}\alpha + M^T V(\pi_j \bar b - b_j)\big)\cdot K\,\big(\pi_j K^{-1}\alpha + M^T V(\pi_j b - b_j)\big)$$

$$= \pi_j^2\,\alpha\cdot K^{-1}\alpha + 2\pi_j\,\alpha^T M^T V x + x^T V M K M^T V x \qquad \Big[\, x = \pi_j b - b_j \text{ for short} \Big]$$

$$= \pi_j^2\, a^T V a + 2\pi_j\, a^T V x + x^T V M K M^T V x.$$

The first two terms are not altered when we add the financial assets. If we consider the problem

$$\min\; (x - M\varphi)\cdot V(x - M\varphi)$$

the solution is

$$x\cdot Vx - x\cdot V M K M^T V x$$

If we enlarge the approximating space, the L^2-norm of the approximation decreases so by going from $M = \binom{I}{0}$ to $M = I$ we will increase $x\cdot VMKM^TVx$, and hence we will decrease agent j's minimized quadratic – so everyone is better off!

Market Selection: more remarks (7/9/10)

1) Suppose ζ, ζ' are two strictly positive semimartingales which we use to generate pricing operators $(\pi_t)_{t\geq 0}$, $(\pi'_t)_{t\geq 0}$ for cash flows $(c_s)_{s\geq t}$ by

$$\pi_t(c) = \frac{1}{\zeta_t} E_t\left[\int_t^\infty \zeta_s c_s \, ds \right]$$

analogously for π'_t. When would we consider π, π' asymptotically the same? Seems to me that a reasonable definition of $\pi_t \sim \pi'_t$ (notation for (π_t) asymptotically equivalent to (π'_t)) would be that the following two conditions hold

(i) for all $t \geq t_0(\omega)$, $\{c \geq 0 : \pi_t(c) < \infty\} = \{c \geq 0 : \pi'_t(c) < \infty\} \equiv \mathcal{A}_t$

(ii) for $t \geq t_0(\omega)$ we have

$$\sup_{\substack{|c| \leq 1 \\ c \in \mathcal{A}_t}} \frac{\pi_t(c)}{\pi'_t(c)} \to 1, \qquad \sup_{\substack{|c| \leq 1 \\ c \in \mathcal{A}_t}} \frac{\pi'_t(c)}{\pi_t(c)} \to 1.$$

Proposition. $\pi_t \sim \pi'_t$ iff there exist ~~positive~~ adapted $(\alpha_t), (\beta_t)$ such that

(i)$'$ for all $t \geq t_0(\omega)$ $\quad \alpha_t \leq \zeta_s / \zeta'_s \leq \beta_t \quad \forall s \geq t$

(ii)$'$ $\quad \alpha_t / \beta_t \to 1 \quad$ a.s.

Proof If these two conditions hold, then if $t \geq t_0$ it is clear that the sets of c for which $\pi_t(c)$, $\pi'_t(c)$ are finite will be the same, and that the second requirement holds. So we just need to prove the necessity.

This uses a little result, that if $Q \ll P$, $dQ/dP = Z$, and Z is not a.s. bounded (ie. $P(Z > t) > 0 \ \forall t$) then there is a random variable which has finite P-expectation, but infinite Q-expectation.

If we write $\zeta_{t,s} \equiv \zeta_s / \zeta_t$ for $s \leq t$, the requirement that eventually the two pricing operators have the same domain implies that ultimately $\zeta_{t,s} / \zeta'_{t,s}$ and $\zeta'_{t,s} / \zeta_{t,s}$ are bounded, which is condition (i)$'$. Notice that if $0 < \alpha_t \leq \zeta_s / \zeta'_s \leq \beta_t < \infty \quad \forall s \geq t$, then this property holds for all later t, using the same α_t, β_t if necessary. However, we may find the best α_t, β_t by setting

$$\tilde{\beta}_t = \operatorname{ess\,inf} \left\{ b : E_t \int_t^\infty I_{\{\zeta_{s,t} / \zeta'_{s,t} \geq b\}} e^{-(s-t)} ds = 0 \right\}$$

Likewise, we define

$$\tilde{\alpha}_t = \operatorname{ess\,sup}\left\{ a : E_t\left[\int_t^\infty e^{-s} I_{\{\zeta_{t,s}/\zeta'_{t,s} \le a\}}\, ds\right] = 0 \right\}$$

Then clearly $\tilde{\alpha}_t \le \tilde{\beta}_t$. Suppose that $\tilde{\beta}_t = 1+\lambda > 1$, and now we set $b \equiv 1+\lambda/2 > 1$, then define

$$c_s = I_{\{\zeta_{t,s}/\zeta'_{t,s} \ge b\}} \frac{e^{-(s-t)}}{1+\zeta_{t,s}+\zeta'_{t,s}},$$

which is bounded, and is in $\mathcal{A}_t$. So we have

$$\pi_t(c) = E_t\left[\int_t^\infty \zeta_{t,s}\, c_s\, ds\right] \ge b\, E_t\left[\int_t^\infty \zeta'_{t,s}\, c_s\, ds\right] = b\, \pi'_t(c).$$ This now shows that

$$\sup_{|c|\le 1,\, c\in\mathcal{A}_t} \frac{\pi_t(c)}{\pi'_t(c)} \ge b = 1+\tfrac{\lambda}{2} > 1.$$

As we assume (ii) holds, it must be that for large enough t, $\tilde{\beta}_t < 1+\lambda$. So it follows that $\tilde{\beta}_t \to 1$. Similarly, $\tilde{\alpha}_t \to 1$ a.s. Hence

$$\frac{\tilde{\beta}_t}{\tilde{\alpha}_t} \to 1 \quad \text{a.s.}$$

Now we set $\beta_t = \tilde{\beta}_t \zeta_t/\zeta'_t$, $\alpha_t = \tilde{\alpha}_t \zeta_t/\zeta'_t$ and we have

$$\alpha_t \le \zeta_s/\zeta'_s \le \beta_t \qquad \text{for } s \ge t,$$

and $$\beta_t/\alpha_t = \tilde{\beta}_t/\tilde{\alpha}_t \longrightarrow 1 \quad \text{a.s.}$$

(2) To get somewhere with the issue of starvation because of different beliefs, we need to exclude other causes of starvation; different ρ_j, different U_j. Even when we do this we can get starvation because of different initial allocations. To rule this out for <u>all</u> possible δ (equivalently, all possible ζ) we need to have for each $\lambda > 1$

$$\inf_{x>0} I(\lambda x)/I(x) > 0, \quad \text{equivalently,} \quad \inf_{x>0} I(2x)/I(x) \equiv \gamma > 0.$$

If this holds, then starvation of agent 1, i.e. $\frac{c_t^1}{c_t^1+c_t^2} \to 0$ a.s., <u>implies</u> $\frac{\Lambda_t^1}{\Lambda_t^2} \xrightarrow{\text{as}} 0$

The converse need not hold, but will if we have

$$\sup_{x>0} I(2x)/I(x) < 1.$$

(3) Can we characterise going broke (i.e. $w_t^i/(w_t^1+w_t^2) \to 0$) in terms of the Λ_t^i? No: if you look at the condition, it is clear that going broke must depend on δ_t also. It could be interesting to try to build an example with some Λ^1, Λ^2, and two different $\delta, \tilde{\delta}$, with the property that you go broke with δ, but not with $\tilde{\delta}$. It's rather a fluke that the influence of δ in the question of starvation just cancels out.

Are we actually doing any more than Blume + Easley, ... ?? Maybe the examples, and the result on price impact is all there is.

<u>A question of Sergei Foss (7/9/10)</u>

1) Suppose that X_i are independent random variables with common light-tailed law: $E e^{\lambda X_1} = e^{\psi(\lambda)} < \infty \ \forall \lambda \in \mathbb{R}$, and suppose given a light-tailed $\mathbb{Z}^+$-valued RV N; $E e^{\lambda N} < \infty \ \forall \lambda \in \mathbb{R}$. Sergei asks: "Is it the case that all exponential moments of $S = X_1 + \cdots + X_N$ exist?" where of course we do <u>not</u> assume independence of N and the X_i.

2) It seems to me the answer must be "Yes". [~~First notice that for any RV Y, and event A with $P(A) \le p$,~~

$$E[e^Y : A] \le \int_{\bar{F}_Y^{-1}(p)}^{\infty} e^y \, dy$$

though this is actually not needed]. We estimate

$$E\left[\exp\{\lambda(X_1 + \cdots + X_N)\} : N = n\right]$$

$$\le E\left(\exp 2\lambda(X_1 + \cdots + X_N) : N = n\right)^{\frac{1}{2}} \sqrt{P(N=n)}$$

$$\le \left\{E \exp 2\lambda(X_1 + \cdots + X_n)\right\}^{\frac{1}{2}} \sqrt{P(N=n)}$$

$$= \exp\left(\tfrac{n}{2}\psi(2\lambda)\right) \sqrt{P(N=n)}$$

$$\left[= \exp\left(\tfrac{n}{2}\psi(2\lambda)\right) \sqrt{P(N \ge n) - P(N \ge n+1)}\right]$$

$$\le \exp\left(\tfrac{n}{2}\psi(2\lambda)\right) \sqrt{P(N \ge n)}.$$

Now since all exponential moments of N exist, we have for any $\beta > 0$

$$P(N \ge n) \le e^{-\beta n} \, E e^{\beta N}$$

So we pick β so large that $\frac{1}{2}\beta > \frac{1}{2}\psi(2\lambda)$, and the sum

$$\sum_{n \ge 0} E\left[\exp\{\lambda(X_1 + \cdots + X_N)\} : N = n\right] = E e^{\lambda S} < \infty.$$

Explicit solution of a very simple contracting problem (15/9/10)

1) Suppose that $U_p(x) = -\frac{1}{\gamma_p}\exp(-\gamma_p x)$, $U_A(x) = -\frac{1}{\gamma_A}\exp(-\gamma_A x)$, $c(a) = k a^2$, and we have

$$\underline{P}: \quad \max_{\varphi} \int U_p(x - \varphi(x)) \, f(x|a)\, dx$$

$$\underline{A}: \quad \max_{a} \int \{U_A(\varphi(x)) - c(a)\} \, dx \, f(x|a) \geq \underline{u}$$

where only actions $a=0$, $a=1$ are available, $f(x|a) = \exp(-\frac{1}{2}(x-a)^2)/\sqrt{2\pi}$.
We shall also insist that $\underline{u} + k < 0$, else action 1 would never be used.

2) If we knew for certain which action A would use, we have a simple Lagrangian argument which tells us that

$$\exp(\lambda) = \frac{U_p'(x - \varphi(x))}{U_A'(\varphi(x))} = \exp\{-\gamma_p(x-\varphi) + \gamma_A \varphi\}$$

Hence $$\varphi(x) = \frac{\lambda + \gamma_p x}{\gamma_A + \gamma_p}$$

and we have to identify values λ_0, λ_1 which satisfy the participation constraint when $a = 0, 1$. For $a=0$, we get $[\, \sigma \equiv \gamma_A \gamma_p / (\gamma_A + \gamma_p) \,]$

$$-\frac{1}{\gamma_A} \exp\left(-\gamma_A \frac{\lambda_0}{\gamma_A + \gamma_p} + \frac{1}{2}\sigma^2\right) = \underline{u}$$

$$\Rightarrow \quad -\frac{\gamma_A \lambda_0}{\gamma_A + \gamma_p} + \frac{1}{2}\sigma^2 = \log(-\gamma_A \underline{u})$$

For $a=1$, we get similarly

$$-\frac{1}{\gamma_A} \exp\left\{-\gamma_A \frac{\lambda_1 + \gamma_p}{\gamma_A + \gamma_p} + \frac{1}{2}\sigma^2\right\} = \underline{u} + k$$

$$\Rightarrow \quad -\frac{\gamma_A}{\gamma_A + \gamma_p}\lambda_1 - \sigma + \frac{1}{2}\sigma^2 = \log(-\gamma_A(\underline{u} + k))$$

Hence $$\boxed{\frac{\gamma_A}{\gamma_A + \gamma_p}(\lambda_1 - \lambda_0) = \log\left(\frac{\underline{u}}{\underline{u} + k}\right) - \sigma}$$

There is a critical value $k^* = -\underline{u}(1-e^{-\sigma}) > 0$ such that if $k < k^*$ the smaller value of λ is λ_1, otherwise λ_0.

3) Is this the solution? For each action a, we've identified a Lagrange multiplier λ_a and a contract φ_a, and the claim is that one of the φ_a has to be optimal. This is because if $\bar\varphi$ were the optimal contract, and $\bar a$ was the action which agent was going to use if offered $\bar\varphi$, then $\varphi_{\bar a}$ is the best contract for the principal when agent uses $\bar a$.

If the principal offers φ_{a^*} and agent uses $a \neq a^*$, then

$$\int \{U_A(\varphi_{a^*}(x)) - c(a)\} f(x|a)\, dx < \int \{U_A(\varphi_a(x)) - c(a)\} f(x|a)\, dx = \underline{u}$$

since $\varphi_{a^*}(\cdot) < \varphi_a(\cdot)$ (we chose a^* by minimising λ_a — we assume that the minimising a is unique). Thus if P uses φ_{a^*} agent will certainly pick $a = a^*$. If the principal uses φ_b, then

$$\sup_a \int \left(U_A(\varphi_b(x)) - c(a)\right) f(x|a)\, dx$$

$$> \sup_a \int \left(U_A(\varphi_{a^*}(x)) - c(a)\right) f(x|a)\, dx$$

$$\geq \int \left(U_A(\varphi_{a^*}(x)) - c(a^*)\right) f(x|a^*)\, dx = \underline{u}$$

So the agent gets more value than $\underline{u}$. So φ_b cannot be optimal.

Some thoughts on a theory of opportunities (11/10/10)

(1) Here's a very simple first approach to an idea where economic agents' decisions are not to do with microadjustments of continuous variables, but rather a sequence of 0/1 choices at random times at which opportunities arise.

Suppose that an agent is receiving an income stream of $\varepsilon\, dt$, but at the times of a Poisson process of rate λ he is offered the chance to make an investment of size K which will generate a random increase in ε, but also commit him to paying back $rK\, dt$ forever, thereby reducing his income stream. All income is consumed; questions of investing etc are left aside for now.

(2) If the agent has conventional von Neumann-Morgenstern preferences, then his value function $V(\cdot)$ satisfies

$$V(\varepsilon) = E\left[\int_0^{\tau} e^{-\rho s} U(\varepsilon)\, ds + e^{-\rho\tau} E\, V(\varepsilon_{\tau+})\right]$$

$$= \frac{U(\varepsilon)}{\rho+\lambda} + \frac{\lambda}{\lambda+\rho} E\, V(\varepsilon_{\tau+})$$

Now we need to understand the final term:

$$E\, V(\varepsilon_{\tau+}) = \int \max\left\{ V(\varepsilon), \int V(\varepsilon+x) F(dx|K) \right\} \mu(dK)$$

where μ is the law of K, and $F(\cdot|K)$ is the conditional law of the change in income stream.

(3) A lot of this is looking hard to carry forward. Perhaps we might instead think about multi-objective decision making (I want a well paid job, but I don't want to have to do a lot of travelling, I don't want a job in an expensive part of the country...) where maybe we have to allow some trading off of the different criteria against each other.

Some basic calculations for local regression (15/10/10)

In my attempts with Nora to do American-style option pricing, we propose to represent the value function at any given time as

$$V(x) \approx \sum_{j=1}^{J} w_j \, \varphi_j(x)$$

where the $\varphi_j(\cdot)$ are suitable 'local' basis functions. In the situation where the underlying process is $BM(\mathbb{R}^d)$, a natural class of such basis functions could be of the form

$$\varphi(x) = \exp\left(-\tfrac{1}{2}k\,|x-a|^2\right)$$

or $\quad \varphi(x) = \exp\left(-\tfrac{1}{2}k\,|x-a|^2 + b\cdot x\right)$

or $\quad \varphi(x) = \exp\left(-\tfrac{1}{2}k\,|x-a|\right) x$

The key thing we need is to calculate $P_t\varphi(z)$ so as to facilitate the expectation step of the DP calculation. Let's write $k = 1/v$ so we're parametrizing by variance. We want to calculate

$$\int f(x)\varphi(x)\,dx$$

where $f(x) = \exp(-|x-z|^2/2t)(2\pi t)^{-d/2}$, $\varphi(x) = \exp(-\tfrac{1}{2}|x-a|^2/v + b\cdot x)$. This we can evaluate by taking Fourier transforms.

$$\int f(x)\varphi(x)\,dx = (2\pi)^{-d}\int \hat{f}(\theta)\,\hat{\varphi}(-\theta)\,d\theta$$

$$= (2\pi)^{-d}(2\pi v)^{d/2}\int \exp\left(i\theta\cdot z - \tfrac{1}{2}|\theta|^2 t\right)\exp\left((b\cdot i\theta)\cdot a + \tfrac{1}{2}|b\cdot i\theta|^2 v\right)d\theta$$

$$= \left(\frac{v}{2\pi}\right)^{d/2}\int \exp\left[-\tfrac{1}{2}|\theta|^2(t+v) + i\theta(z-a-bv) + b\cdot a + \tfrac{1}{2}|b|^2 v\right]d\theta$$

$$= \left(\frac{v}{t+v}\right)^{d/2}\exp\left\{b\cdot a + \tfrac{1}{2}v|b|^2\right\}\exp\left\{-\tfrac{1}{2}|z-a-bv|^2/(t+v)\right\}$$

$$\boxed{= \left(\frac{v}{t+v}\right)^{d/2}\exp\left\{-\frac{|z-a|^2}{2(t+v)} + \frac{b\cdot(ta+vz)}{t+v} + \frac{tv|b|^2}{2(t+v)}\right\} = E^z\left\{e^{-\frac{1}{2}|X_t-a|^2/v + b\cdot X_t}\right\}}$$

after a little rearranging.

Market selection: a sketched example (15/10/10)

(i) Writing up the paper with Katsumasa, I had made the assertion that whether agent 1 goes broke or not cannot be decided on the basis of the LR martingales (Λ^j_t) alone, but also involves what δ is up to. While I'm fairly sure this is true, it doesn't seem to be very easy to make an example.

(ii) Let's just stick with CRRA utilities, so we see

$$e^{\rho t}\Lambda^j_t (c^j_t)^{-R} = \nu_j \zeta_t$$

$\Rightarrow$ $c^j_t = \pi^j_t \delta_t$ where $\pi^j_t = (\Lambda^j_t/\nu_j)^{1/R} \Big/ \sum_i (\Lambda^i_t/\nu_i)^{1/R}$, and altogether

$$\delta_t = \zeta_t^{-1/R}\, e^{-\rho t/R} \sum_j (\Lambda^j_t/\nu_j)^{1/R}.$$

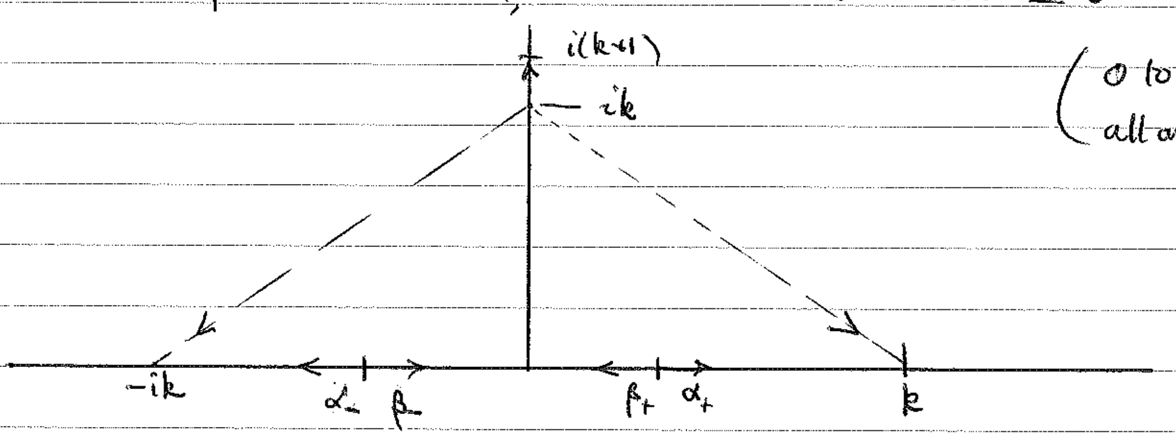
The wealths are

$$w^i_t = \frac{1}{\zeta_t} E_t\Big[\int_t^\infty \delta_s \zeta_s \pi^i_s\, ds\Big] = \frac{1}{\zeta_t} E_t\Big[\int_t^\infty e^{-\rho s}\delta_s^{1-R}\Big(\sum(\Lambda^j_s/\nu_j)^{1/R}\Big)^R \pi^i_s\, ds\Big]$$

$$= \frac{1}{\zeta_t} E_t\Big[\int_t^\infty e^{-\rho s}\underbrace{\delta_s^{1-R}\Big(\sum(\Lambda^j_s/\nu_j)^{1/R}\Big)^{R-1}}_{=\lambda_s\text{, say.}} (\Lambda^i_s)^{1/R}\, ds\Big]\,\nu_i^{-1/R}$$

The idea is to make λ_s into a change-of-measure martingale (which tells us what δ to be taking) in such a way as to give the desired result.

(iii) Here's a possible construction, based on a Markov chain on $\mathbb{Z} \cup i\mathbb{N} \subseteq \mathbb{C}$.



(0 to 1, −1, i all at rate 1)

There are jumps to nearest neighbours on each ladder, and from ik to $\pm k$. The Reference probability thinks that $\beta_\pm > \alpha_\pm$ and jumps from ik to $\pm k$ happen with intensity 2^{-k}, jumps ik to $i(k+1)$ with intensity 1, so that in the reference probability we find that eventually the process climbs the imaginary ladder to infinity.

Agent 1 thinks jumps ik to k come at rate 2^k, jumps ik to $-k$

or to $i(k+1)$ come at rate 1. He also thinks $\alpha_+ > \beta_+$, $\alpha_- < \beta_-$, so that for him escape to $+\infty$ is certain. Agent 2 is a mirror image of agent 1.

For simplicity, it may help to assume $\alpha_+ + \beta_+ = \alpha_- + \beta_-$, so that the additive functional contributions to the LR martingales are the same. Looks like the smart thing to do is to imagine that agent 1 thinks

$$\alpha_+ = \beta_- = 2, \qquad \alpha_- = \beta_+ = 1$$

for then if the process jumps down from the imaginary axis and makes it back to zero, then the likelihood contributions for both agents for that piece of path are <u>exactly the same</u> [Careful! we shall have to say that when we enter 0 we always go up the imaginary ladder, intensity of jump 0 to i = 1]

Thus if we consider

$$\Lambda^1_t / \Lambda^2_t \qquad \text{(note: } \Lambda^1_t = \Lambda^2_t \text{ any time we're in } i\mathbb{N}\text{)}$$

at some time when we are at $n > 0$, having last jumped from $i\mathbb{N}$ at position k, and having since made m steps down, j steps up, where $k + j - m = n$, then the likelihood ratio Λ^1/Λ^2 is coming from the ratio of the jump rates:

$$\frac{\Lambda^1}{\Lambda^2} = \frac{2^k \cdot 1^m \cdot 2^j}{1 \cdot 2^m \cdot 1^j} = 2^{k+j-m} = 2^n$$

Thus if we set it up so that one of the dividend processes makes the probability into $\mathbb{P}^1$, when we are at ik the calculation

$$\frac{1}{\lambda_t} \mathbb{E}^1_t \left[\int_t^\infty e^{-\rho s} \lambda_s \left(\frac{\Lambda_s^1}{\Lambda_t^1}\right)^{1/R} ds \right]$$

will give something $\simeq 2^{k/R}$ (since we are about to jump to k very soon!) whereas the same calculation for Λ^2 will give something $O(1)$

Modelling agent choice by opportunities (18/10/10)

1) The idea here is to simplify the story for agent choice; at the times of some Poisson process, agents get opportunities to invest, which they either take or leave. We'll tell a simplified story where output Y_t is given as

$$Y_t = f_0(K_t) = c_t + s_t + \delta K_t \qquad \equiv f(K_t) + \delta K_t$$

so that we assume that all depreciation on capital is paid off before the remaining output is split between consumption + saving. Thus K_t changes only when an opportunity to invest comes along. The evolution of the bank account x_t is

$$\dot{x}_t = s_t + \varphi(x_t)$$

where $\varphi(x) = r_L x I_{\{x>0\}} + r_B x I_{\{x<0\}}$ with $0 < r_L \leq r_B$.
We insist

$$x_t \geq -K_t \qquad \forall t.$$

2) When an opportunity comes along, it has a cost κ, and a distribution $F(\cdot|\kappa)$ of possible increases in K. Once the gain ΔK in K is observed (immediately after κ is paid, let's assume for simplicity), the agent chooses a constant rate s of saving, so that we imagine $c = f(K+\Delta K) - s$ for ever after.
Let's suppose the agent wants to

$$\max \int_0^\infty e^{-\rho t} g(c_t)\, h(x_t)\, dt$$

so that the thing we care about is

$$\int_0^\infty e^{-\rho t} h(x_t)\, dt$$

when $\dot{x}_t = s + \varphi(x_t)$. This ODE can be solved piecewise. If the initial condition x_0 is non-negative, then

$$x_t = e^{r_L t}\left(x_0 + (s/r_L)(1-e^{-r_L t})\right)$$

If $x_0 < 0$, then

$$x_t = e^{r_B t}\left(x_0 + (s/r_B)(1-e^{-r_B t})\right) \qquad \text{for } t \leq \tau,$$

where τ is the time the solution hits 0. Explicit solution in general is not possible, but if we assume

$$\boxed{h(x) = x + K,}$$

Then for starting point $x_0 > 0$ we can calculate

$$\int_0^\infty e^{-\rho t} h(x_t)\,dt = \frac{x_0 + s/\rho}{\rho - r_L}$$

For starting point $x_0 < 0$, we get

$$\int_0^\tau e^{-\rho t} h(x_t)\,dt = \frac{s}{\rho r_B(\rho - r_B)} \left\{ r_B(1 - e^{-\rho\tau}) - \rho(1 - e^{-r_B \tau}) \right\}$$

where

$$x_0 + s/r_B = \left(\frac{s}{r_B}\right) e^{-r_B \tau}$$

Hence $$\boxed{\int_0^\tau e^{-\rho t} h(x_t)\,dt = \frac{s(1 - e^{-\rho\tau})}{\rho(\rho - r_B)} + \frac{x_0}{\rho - r_B}}$$

(3) 28/10/10

Let's go into discrete time, and suppose a linear production function:

$$c_t + s_t = A K_t$$

Suppose that in any period, we initially get output, then we get investment opportunity (if any) which requires investment of θK_t and changes K_t to $Z K_t$, where θ may be random, and the Z is random and can depend on θ. Suppose also a CRRA investor maxing $E\left[\sum_{t \geq 0} \beta^t u(c_t)\right]$. Then the value function is of the form

$$V(x, K; \theta) = K^{1-R} V(x/K, 1; \theta) \equiv K^{1-R} v(\xi; \theta)$$

where $\xi \equiv x/K \geq -1$.

If you enter period t with (x_t, K_t), and you decide to do the opportunity, you get

$$\begin{cases} x_{t+1} = (1 + r_t(x_t'))\, x_t' & \text{with } x_t' = x_t - \theta K_t + s_t \\ K_{t+1} = Z K_t & \\ c_t = A K_t - s_t & \end{cases}$$

(so if $\theta = 0$, we could by convention suppose $Z = 1$). So the DP equation is

$$V(x, K; \theta) = \sup_s \left\{ u(AK - s) + \beta E\left[(ZK)^{1-R} v\left(\frac{1 + r(\xi')}{Z} \xi' ; \tilde{\theta} \right) \right] \right\}$$

where $\xi' \equiv \xi - \theta + q$, $\quad q \equiv \Delta/K$;

$$= K^{1-R} \sup_q \left\{ u(A-q) + \beta E\left[Z^{1-R} v\left(\frac{(1+r(\xi'))\xi'}{Z} ; \tilde{\theta} \right) \right] \right\}$$

and $\tilde{\theta}$ is the opportunity you get next period, assumed independent of all other periods. Thus

$$\boxed{v(\xi;\theta) = \sup_q \left\{ u(A-q) + \beta E\left[Z^{1-R} v\left(\frac{1+r(\xi')}{Z}\xi' ; \tilde{\theta} \right) \right] \right\}}$$

where the law of Z will depend on θ. Only numerics?

(4) Back to an earlier notion: Suppose your cash and capital only change at times when you get an opportunity, but that there may be opportunities to sell (and thereby decrease capital) as well as opportunities to buy.

Keep to CRRA investor, and suppose that the output function is linear: $Y_t = AK_t$ (no depreciation). Save only what is needed to keep cash constant:

$$x_{t+1} \equiv (1+r(x_t))(x_t + \Delta_t) = x_t \quad \Rightarrow \quad \Delta_t = \frac{-r x_t}{1+r}$$

so that $q_t = AK_t + \frac{r x_t}{1+r}$. When an opportunity comes along, we have to invest ($a>0$) or disinvest ($a<0$) an amount aK_t of cash, which then changes K_t to ZK_t, where Z is random + unknown, law depending on a.

The DP story becomes $(\xi \equiv x/K)$

$$K^{1-R} v(\xi) = V(x,K) = \max\left\{ u\left(AK + \frac{rx}{1+r}\right) + \beta \int F(dz|a)\, V(x - aK, zK),\ u\left(AK + \frac{rx}{1+r}\right) + \beta V(x,K) \right\}$$

$\Rightarrow$

$$\boxed{v(\xi) = \max\left\{ u\left(A + \frac{r\xi}{1+r}\right) + \beta \int z^{1-R} v\left(\frac{\xi - a}{z}\right) F(dz|a),\ u\left(A + \frac{r\xi}{1+r}\right) + \beta v(\xi) \right\}}$$

[Of course, we have to mix RHS over a ...]

Some thoughts on a dynamic contracting problem (29/10/10)

(1) Takashi is interested in a paper of DeMarzo et al where there is production and an agent. The system is given by

$$\begin{cases} dA_t = \sigma\, dX_t + a_t \mu\, dt \\ dY_t = K_t\,(dA_t - c(i_t)\,dt) \\ dK_t = K_t\,(i_t - \delta)\,dt \end{cases}$$

where X is BM, $0 \le a_t \le 1$ is the agent's effort-level, $\mu > 0$. The function c is increasing and strictly convex, representing costs of investment. The agent chooses action (a_t) and has objective

$$E^a\left[\int_0^\tau e^{-\gamma s}\,(dU_s + \mu\lambda(1-a_s)\,K_s\,ds)\right]$$

where the increasing process U is the cumulative wage paid to the agent, τ is the time that the principal stops, and $\lambda > 0$. The principal's objective is

$$E^a\left[\int_0^\tau e^{-rt}\,(dY_t - dU_t) + e^{-r\tau}\,\ell\, K_\tau\right]$$

where $\ell > 0$ is a terminal valuation of capital. The principal looks to make the best contract he can which will ensure the agent always works: $\boxed{a_t \equiv 1}$

(2) Let's write $\varepsilon_t \equiv 1 - a_t$ and choose as reference measure $\mathbb{P}^0$ the law you get if $a_t \equiv 1$: $dA_t = \sigma\, dX_t + \mu\, dt$ under $\mathbb{P}^0$, where X is a $\mathbb{P}^0$-BM. If the agent chooses to slack, so $\varepsilon \not\equiv 0$, then we have measure $\mathbb{P}^\varepsilon$ where

$$\Lambda_t^\varepsilon \equiv \left.\frac{d\mathbb{P}^\varepsilon}{d\mathbb{P}^0}\right|_{\mathcal{F}_t} \quad \text{solves} \quad d\Lambda_t^\varepsilon = \Lambda_t^\varepsilon\left(-\frac{\varepsilon_t \mu}{\sigma}\right) dX_t .$$

Now the agent's objective is

$$E^0\left[\int_0^\tau e^{-\gamma s}\Lambda_s^\varepsilon\,(dU_s + \lambda\varepsilon_s K_s\,ds)\right]$$

$$= E^0\left[\int_0^\tau e^{-\gamma s}\Lambda_s^\varepsilon\,dU_s - \int_0^\tau e^{-\gamma s}K_s\,\lambda\sigma\, d\Lambda_s^\varepsilon\, dX_s\right]$$

$$= E^0\left[\int_0^\tau e^{-\gamma s}\Lambda_s^\varepsilon\,dU_s - \int_0^\tau \lambda\sigma\, e^{-\gamma s}K_s\, d(\Lambda_s^\varepsilon X_s)\right]$$

$$= E^0\left[\int_0^\tau e^{-\gamma s}\Lambda_s^\varepsilon\,dU_s - \Big[\lambda\sigma\, e^{-\gamma s}K_s\Lambda_s^\varepsilon X_s\Big]_0^\tau + \int_0^\tau \Lambda_s^\varepsilon X_s\, e^{-\gamma s}K_s\,(i_s - \delta - \gamma)\,\lambda\sigma\, ds\right]$$

$$= E^0\left[\Lambda_\tau^\varepsilon\left\{-\lambda\sigma K_\tau X_\tau e^{-\gamma\tau} + \int_0^\tau e^{-\gamma s}\,(dU_s + \lambda\sigma X_s K_s\,(i_s - \delta - \gamma)\,ds)\right\}\right]$$

$$= E^0\left[\Lambda_\tau^\varepsilon\left\{\int_0^\tau e^{-\gamma s}\,(dU_s - \lambda\sigma K_s\,dX_s)\right\}\right]$$

Where this gets us is that we have separated the effect of the agent's choice, Λ^ε_τ, from the term in $\{-\}$ which is entirely up to the principal to choose. So if we write $\{\}$ as

$$\int_0^\tau e^{-\gamma t}\left(dU_t + \lambda\sigma K_t X_t\,(r_t-\delta-\gamma)dt\right) - \lambda\sigma e^{-\gamma\tau} K_\tau X_\tau \equiv Q_\tau$$

$$= b + \int_0^\tau \tilde{H}_s\, dX_s$$

by the Brownian integral representation, then the agent's objective is

$$b + E\left[\int_0^\tau \tilde{H}_s\, d\Lambda^\varepsilon_s\, dX_s\right] = b + E\left[\int_0^\tau \tilde{H}_s \,\frac{-\mu\varepsilon_s}{\sigma}\, ds\right]$$

So in order that $\varepsilon \equiv 0$ should be optimal, we'll insist that $\boxed{\tilde{H} \geq 0.}$

Notice also that we have more simply the agent's objective is

$$Q_\tau = \int_0^\tau e^{-\gamma t}\, dU_t - \lambda\sigma \int_0^\tau e^{-\gamma s} K_s\, dX_s$$

So if we set

$$M_t \equiv E\left[\int_0^\tau e^{-\gamma s}\, dU_s \,\Big|\, \mathcal{F}_t\right] = b + \int_0^t H_s\, dX_s$$

then $\tilde{H}_t \equiv H_t - \lambda\sigma e^{-\gamma t} K_t \geq 0$, we insist. As a piece of notation, set

$$H_t \equiv e^{-\gamma t} h_t, \text{ so we must have } \boxed{h_t \geq \lambda\sigma K_t.}$$

(3) This effectively removes the agent from the optimization; provided the principal offers a contract where $h_t \geq \lambda\sigma K_t$, the agent always works.

Now let's introduce

$$z_t \equiv E_t\left[\int_t^\tau e^{-\gamma(s-t)}\, dU_s\right]$$

which is the agent's target at time t. The principal's value at any time should depend only on K_t and z_t, so we may write it as $V(K_t, z_t)$. We have that

$$e^{-rt} V(K_t, z_t) + \int_0^t e^{-rs}(dY_s - dU_s) \equiv N_t$$

must be a martingale, and we also have that

$$e^{-\gamma t} z_t = M_t - \int_0^t e^{-\gamma s}\, dU_s$$

Using these together, we can do an Itô expansion of N

$$dN_t \doteq e^{-rt}\Big[-rV\,dt + K(i-\delta)V_K\,dt + V_z\,dz + \tfrac{1}{2}V_{zz}h^2dt + K(\mu - c(i))dt - dU \Big]$$

$$\doteq e^{-rt}\Big[\{ -rV + (i-\delta)K V_K + V_z \gamma z + \tfrac{1}{2}h^2 V_{zz} + K(\mu - c(i)) \} dt - (1+V_z)\,dU \Big]$$

Usual MPOC gives

$$\begin{cases} 1+V_z \geq 0 \quad , & dU > \text{ only when } 1+V_z = 0 \\ \sup_{h \geq \lambda\sigma K,\ i} \Big[-rV - \delta K V_K + \gamma z V_z + \tfrac{1}{2}h^2 V_{zz} + \mu K + iKV_K - c(i)K \Big] = 0 \end{cases}$$

(4) Notice a scaling property: for $\alpha > 0$, $V(\alpha K, \alpha z) = \alpha V(K,z)$, so that $V(K,z) = K v(x)$, $x \equiv z/K$, and we get

$$\boxed{\begin{aligned} & 1 + v' \geq 0 \\ & \sup_{i,\ q \geq \lambda\sigma} \Big[-rv - \delta(v - xv') + \gamma x v' + \tfrac{1}{2}q^2 v'' + \mu + i(v - xv') - c(i) \Big] = 0 \end{aligned}}$$

Now for the sup to be bounded, we shall need $\boxed{v'' \leq 0}$ and then $q = \lambda\sigma$. If we set $\tilde{c}(s) \equiv \inf_y \{ c(y) + ys \}$, we expect that V is increasing with K, and so $V_K = v - xv' > 0$, and we see finally

$$\boxed{\begin{aligned} & v' \geq -1 \\ & -(r+\delta)v + (\gamma+\delta)xv' + \tfrac{1}{2}(\lambda\sigma)^2 v'' + \mu - \tilde{c}(xv' - v) = 0 \end{aligned}}$$

with $v(0) = \ell$, and if $x^* \equiv \inf\{x : v'(x) = -1\}$, then get C^2 condition at x^*.

(5) Could we likewise solve the problem if we don't insist that the contract always makes the agent work? let's see. We have

$$\begin{cases} dA_t = \sigma dX_t + \mu\,dt & \text{under } \mathbb{P}^0, \text{ where } X \text{ is a } \mathbb{P}^0\text{-BM} \\ dK_t = K_t(i_t - \delta)\,dt \\ dY_t = K_t\{dA_t - c(i_t)dt\} \end{cases}$$

With the same notation as previously the agent's residual value z_t at time t is

$$\zeta_t = E_t^{\varepsilon}\left[\int_t^{\tau} e^{-\gamma(s-t)}\,(dU_s + \lambda\mu\varepsilon_s K_s ds)\right]$$

$$= e^{\gamma t}\, E_t^{0}\left[\Lambda_\tau^{\varepsilon}\int_t^{\tau} e^{-\gamma s}(dU_s + \lambda\mu\varepsilon_s K_s ds)\right]/\Lambda_t^{\varepsilon}$$

$$= e^{\gamma t}\, E_t^{0}\left[\Lambda_\tau^{\varepsilon}\int_t^{\tau} e^{-\gamma s} dU_s - \int_t^{\tau} \lambda\sigma K_s e^{-\gamma s}\, d\langle\Lambda_s^{\varepsilon}, X_s\rangle\right]/\Lambda_t^{\varepsilon}$$

$$= e^{\gamma t} E_t^{\varepsilon}\left[\int_t^{\tau} e^{-\gamma s} dU_s\right] - e^{\gamma t} E_t^{0}\left[\int_t^{\tau} \lambda\sigma K_s e^{-\gamma s}\, d(\Lambda^{\varepsilon} X)\right]/\Lambda_t^{\varepsilon}$$

$$= e^{\gamma t} E_t^{\varepsilon}\left[\int_t^{\tau} e^{-\gamma s}(dU_s - \lambda\sigma K_s dX_s)\right]$$

$$\Rightarrow\ e^{-\gamma t}\zeta_t = E_t^{\varepsilon}\left[\int_0^{\tau} e^{-\gamma s}(dU_s - \lambda\sigma K_s dX_s)\right] - \int_0^{t} e^{-\gamma s}(dU_s - \lambda\sigma K_s dX_s)$$

Now we propose an integral representation

Scale with K?

$$M_\tau = \int_0^{\tau} e^{-\gamma s} dU_s = b_0 + \int_0^{\tau} e^{-\gamma s} K_s h_s\, dX_s$$

so that

$$e^{-\gamma t}\zeta_t = E_t^{\varepsilon}\left[\int_0^{\tau} e^{-\gamma s} K_s (h_s - \lambda\sigma) dX_s\right] - \int_0^{t} e^{-\gamma s}(dU_s - \lambda\sigma K_s dX_s) + b_0$$

We expect that everything scales linearly with K, and that the key variable of interest is $\boxed{\xi_t \equiv \zeta_t / K_t}$ which we expect will solve an autonomous SDE:

$$\boxed{d\xi_t = a(\xi_t)\, dX_t + b(\xi_t)\, dt - dL_t}$$

where L will be local-time-like. Also, since $E^{\varepsilon}\left[\int_0^{\tau} e^{-\gamma s} K_s (h_s - \lambda\sigma)\, dX_s\right]$
$= E^0 \int_0^{\tau} e^{-\gamma s} K_s (h_s - \lambda\sigma)(-\varepsilon_s \mu/\sigma)\, ds$, the optimal strategy for the agent will be to take

$$\varepsilon_t = I_{\{h_t < \lambda\sigma\}}$$

so we expect $h_t = \varphi(\xi_t)$, $\varepsilon_t = \varepsilon(\xi_t)$. Let's go further and suppose that L is local time of ξ^*, and that ξ gets reflected down from ξ^*, with $\boxed{dU_t = K_t\, dL_t}$.
We also expect $i_t = i(\xi_t)$. Now consider

$$M_t = E_t^0\left[\int_0^{\tau} e^{-\gamma s} K_s\, dL_s\right] = \int_0^{t} e^{-\gamma s} K_s\, dL_s + e^{-\gamma t} K_t\, \psi(\xi_t)$$

where $\psi(\xi) = E^0\left[\int_0^{\tau} e^{-\gamma s - \delta s + \int_0^s i_u du}\, dL_s \,\middle|\, \xi_0 = \xi\right]$ solves

$$\boxed{\mathcal{L}^0\psi - (\gamma + \delta - i)\,\psi = 0, \qquad \psi'(\xi^*) = -1}$$

$\mathcal{L}^0 = \frac{1}{2}a(\xi)^2 D^2 + b(\xi) D$, $\mathcal{L}^{\varepsilon} = \frac{1}{2}a(\xi)^2 D^2 + \tilde{b}(\xi) D$, $\tilde{b}(\xi) \equiv b(\xi) - a\varepsilon\mu/\sigma$.

Doing Ito on M tells us that

$$\boxed{h_t \equiv \varphi(\xi_t) = a(\xi_t)\,\psi'(\xi_t).}$$

In particular, we do $\varepsilon_t = 1$ only where $a\psi'(\xi) < \lambda\sigma$. Now let's look at z_t (which we recall is $K_t \xi_t$). We have

$$z_t e^{-\gamma t} = E_t^{\varepsilon}\left[\int_0^{\tau} e^{-\gamma s} K_s (h_s - \lambda\sigma)\, dX_s\right] - \int_0^t e^{-\gamma s} K_s (dL_s - \lambda\sigma\, dX_s)$$

$$= E_t^{\varepsilon}\left[\int_t^{\tau} e^{-\gamma s} K_s (h_s - \lambda\sigma)\, dX_s\right] + \int_0^t e^{-\gamma s} K_s \big((h_s - \lambda\sigma)\, dX_s - dL_s\big)$$

$$= e^{-\gamma t} K_t \underbrace{E_0^{\varepsilon}\left[\int_0^{\tau} e^{-\gamma u - \delta u + \int_0^u i_v\, dv} (h_u - \lambda\sigma)\, dX_u\right]}_{= \tilde{\psi}(\xi_t)} + \int_0^t e^{-\gamma s} K_s \big((h_s - \lambda\sigma)\, dX_s - dL_s\big)$$

where $\tilde{\psi}$ solves

$$\mathcal{L}^{\varepsilon}\tilde{\psi} - (\gamma + \delta - i)\tilde{\psi} + (h - \lambda\sigma)\,\mu/\sigma = 0$$

Now do Ito on this:

$$e^{-\gamma t}\{-\gamma z_t\, dt + dz_t\} = e^{-\gamma t} K_t \{-\gamma \tilde{\psi}\, dt + (i - \delta)\tilde{\psi}\, dt + d\xi\, \tilde{\psi}' + \tfrac{1}{2} a^2 \tilde{\psi}''\, dt\}$$
$$+ e^{-\gamma t} K_t \{(h_t - \lambda\sigma)\, dX_t - dL_t\}$$

Recalling that $z = K\xi$, we get

$$(*) \quad \boxed{\begin{aligned} d\xi - (\gamma + \delta - i)\xi\, dt = {} & -(\gamma + \delta - i)\tilde{\psi}\, dt + \tilde{\psi}'\, d\xi + \tfrac{1}{2} a^2 \tilde{\psi}''\, dt \\ & + (h_t - \lambda\sigma)\, dX - dL \end{aligned}}$$

with $\tilde{\psi}'(\xi^*) = 0$. The other thing is that

$$N_t \equiv e^{-\gamma t} z_t + \int_0^t e^{-\gamma s} K_s (dL_s + \lambda\mu\varepsilon_s\, ds) \quad \text{is a } \mathbb{P}^{\varepsilon}\text{-martingale}$$

so

$$dN_t = e^{-\gamma t} K_t \left[d\xi_t - (\gamma + \delta - i)\xi\, dt + dL + \lambda\mu\varepsilon_t\, dt\right]$$

$$= e^{-\gamma t} K_t \left[a(\xi)\left(dX^{\varepsilon} - \tfrac{\varepsilon\mu}{\sigma}\, dt\right) + b(\xi)\, dt - (\gamma + \delta - i)\xi\, dt + \lambda\mu\varepsilon_t\, dt\right]$$

Therefore

$$\boxed{b(\xi) - a(\xi)\varepsilon(\xi)\,\mu/\sigma - (\gamma + \delta - i(\xi))\xi + \lambda\mu\,\varepsilon(\xi) = 0}$$

Matching up terms in $(*)$ tells us that

$$\boxed{a\,(1 - \tilde{\psi}') = h - \lambda\sigma}$$

$$b - \tilde{\gamma}\xi \equiv b - (\gamma + \delta - i)\xi = \frac{a\varepsilon\mu}{\sigma} - \lambda\mu\varepsilon$$

Trading to stops: some variants (8/12/10)

Here are a couple of variants of the trading to stops questions which Nora came up with. The story is that the reset times are considered to be times when not only do you take profits but you also may review your investment choice. When the drift is randomized, the story we currently tell corresponds to playing for ever with the same asset. But we could do

(a) Each time you come out, you go into an independent copy of the previous asset. Then

$$\varphi = \sum_j p_j \left\{ E^{\mu_j}\left\{ e^{-\rho T_1} U(X_{T_1}-c) \right\} + E^{\mu_j} e^{-\rho T_1} \right\} \varphi$$

(b) You keep playing the asset until you make a loss. Then you switch to an independent asset, statistically the same. Then we shall have

$$\varphi = \sum_j p_j \left(h_j + \psi_j \varphi \right)$$

where $h_j = E^{\mu_j}\left[\sum_{n=1}^{\nu} e^{-\rho T_n} U(X_{T_n} - X_{T_{n-1}} - c) \right]$

$$\psi_j = E^{\mu_j}\left[e^{-\rho T_\nu} \right]$$

where ν is the first index where you go out at the lower end. We get

$$h_j = E^{\mu_j}\left[e^{-\rho T_1} U(X_{T_1}-c) \right] + E^{\mu_j}\left[e^{-\rho T_1} : X_{T_1} = b \right] h_j$$

$$\psi_j = E^{\mu_j}\left[e^{-\rho T_1} ; X_{T_1} = a \right] + E^{\mu_j}\left[e^{-\rho T_1} : X_{T_1} = b \right] \psi_j$$

This gives φ quite explicitly.

$$\therefore \psi_j = E^{\mu_j}\left[e^{-\rho T_1} \right] - E^{\mu_j}\left[e^{-\rho T_1} : X_{T_1} = b \right] (1-\psi_j)$$

Another little tale we could tell would be a proper Bayesian analysis of the stopping problem if we suppose $dX_t = dW_t + \mu\, dt$, where $\mu \sim N(\hat\mu_0, 1/\tau_0)$ is the assumed prior for μ. If $\hat\mu_t$ is the time-t MLE, then we have

$$\hat\mu_t = \frac{\tau_0 \hat\mu_0 + X_t}{\tau_0 + t}, \quad dX_t = d\hat W_t + \hat\mu_t\, dt, \qquad d\hat\mu_t = \frac{d\hat W_t}{\tau_0 + t}$$

as the evolution. If we get stopping reward $\tilde{g}(t,x)$ then we can re-express this as $g(\tau,\hat{\mu})$ and then we look for a value $V(\tau,\mu)$ which will solve

$$V_\tau + \frac{1}{2\tau^2} V_{\mu\mu} = 0 \quad \text{in continue region}$$

$$V = g \quad \text{in stop region.}$$

Notice also that if we set $v(s,\mu) = V(-1/s,\mu)$, we shall have $(s<0)$

$$v_s + \tfrac{1}{2} v_{\mu\mu} = 0$$

so this is just the heat equation in $(-\infty,0] \times \mathbb{R}$, which is certainly easy to do numerically if no other way!

Investing in opportunities (10/12/10)

(1) Let's suppose an agent with initial cash x_0, initial capital K_0 gets an opportunity to invest a units of cash into some new project which will generate aZ units of capital (Z is random). At time 1, the capital produces a total of $A(K_0+aZ)$ units of consumption good, where $A>0$ is fixed and known. He may also trade ZCBs at time 0, at price B. Thus his optimization is

$$\max_{\substack{\alpha=0,a\\ \theta}} \left\{ U(c_0) + \beta\, E\, U(c_1) \right\}$$

subj to $c_0 = x_0 - \alpha - \theta B$, $c_1 = A(K_0+\alpha Z) + \theta$. The optimality condition is

$$B\, U'(x_0 - \alpha - \theta B) = \beta\, E\, U'(A(K_0+\alpha Z)+\theta)$$

Suppose $U(x) = -\gamma^{-1}\exp(-\gamma x)$ for simplicity, and $E\, e^{-\theta Z} = e^{\psi(\theta)}$. Then we have

$$-\gamma(x_0-\alpha-\theta B) + \log B = \log\beta - \gamma(AK_0+\theta) + \psi(\gamma\alpha A)$$

which gives

$$\boxed{\gamma\theta(B+1) = \log(\beta/B) + \gamma(x_0-\alpha-AK_0) + \psi(\gamma\alpha A)}$$

Substituting this into the objective gives

$$-\frac{1+B}{\gamma}\exp\left\{-\gamma(x_0-\alpha-\theta B)\right\}$$

So the action $\alpha=0,a$ must be chosen so as to minimise

$$\alpha+\theta B = \frac{B}{\gamma(B+1)}\left(\log(\beta/B) + \gamma(x_0-AK_0)\right) + \frac{B}{\gamma(B+1)}\left(-\gamma\alpha+\psi(\gamma\alpha A)\right) + \alpha$$

$$= \text{const} + \frac{B}{\gamma(B+1)}\psi(\gamma\alpha A) + \frac{\alpha}{B+1}$$

So we decide to invest iff

$$\boxed{B\,\psi(\gamma\alpha A) + \gamma\alpha < 0.}$$

Suppose $Z \sim N(\mu,\sigma^2)$, so $\psi(\theta) = -\theta\mu + \frac{1}{2}\sigma^2\theta^2$. The condition is

$$0 > \gamma\alpha AB\left(\mu + \tfrac{1}{2}\sigma^2\gamma\alpha A\right) + \gamma\alpha$$

iff $AB(\mu + \frac{1}{2}\sigma^2\gamma\alpha A) + 1 < 0$ iff $\boxed{\frac{1}{AB} < \mu - \frac{1}{2}\sigma^2\gamma\alpha A}$

This behaves sensibly; as γ increases, or a increases, you get more unlikely to invest. Similarly, if B increases, you get more likely to invest.

(2) Now let's explore equilibria. Suppose the bond is in zero net supply. Agents have cles of ARA γ_j, initial cash x_j, capital K_j, and productivity A_j. Let

$$I_j = \begin{cases} 1 & \text{if } \mu - \frac{1}{2}\sigma^2 a_j \gamma_j A_j > 1/BA_j \\ 0 & \text{if not} \end{cases}$$

Then market clearing is

$$0 = \sum \gamma_j^{-1} \log(\beta_j/B) + \sum(x_j - A_j K_j) + \sum I_j \left(\psi(\gamma_j a_j A_j) - \gamma_j a_j \right)$$

$$= -\frac{\log B}{\Gamma} + \underbrace{\sum \gamma_j \log \beta_j + \sum(x_j - A_j K_j)}_{K\text{, a constant}} + \sum I_j \left(-\mu a_j \gamma_j A_j + \tfrac{1}{2}\sigma^2 (\gamma_j A_j a_j)^2 - \gamma_j a_j \right)$$

$$= -\frac{\log B}{\Gamma} + K + \sum I_j \gamma_j a_j A_j \left(-\mu + \tfrac{1}{2}\sigma^2 a_j A_j \gamma_j - \frac{1}{A_j} \right)$$

Thus the market clearing condition is

$$\Gamma^{-1} \log B = K - \sum I_j \underbrace{\left\{ \mu - \tfrac{1}{2}\sigma^2 a_j \gamma_j A_j - \frac{1}{A_j} \right\}}_{\text{always positive if } I_j > 0} \gamma_j a_j A_j$$

The LHS increase with B from $-\infty$ to 0. The RHS decreases with B. It's possible that $B > 1$ may be needed for equilibrium but that's not impossible a priori.

We may fail to get an equilibrium value for B, because of the discontinuity. Somehow we need to deal with this; if EU is the criterion, we will always go for invest or no-invest ... could we do a randomized choice? Not with EU ... so perhaps we need to be looking at Machina or Kreps-Porteus??

Dynamic contracting again (16/12/10)

① Let's come back to the situation studied by DeMarzo, Fishman, He + Wang. The dynamics of capital K_t, output Y_t and return A_t are given by

$$(1)\quad \begin{cases} dA_t = \sigma dX_t + \mu(1-\varepsilon_t)\,dt \\ dY_t = K_t\,(dA_t - c(i_t)\,dt) \\ dK_t = K_t\,(i_t - \delta)\,dt \end{cases}$$

where $0 \le \varepsilon_t \le 1$ is a rate of slacking, controlled by agent, and $i_t = I_t/K_t$ is investment rate per unit of capital, $c(i) \ge i$ is convex, $\delta > 0$, $\mu > 0$ constants. The principal chooses a wage process dU_t non-decreasing and pays this to the agent. The agent's objective is

$$\max\ E\Big[\int_0^\tau e^{-\gamma s}\,(dU_s + \lambda\mu\varepsilon_s K_s\,ds)\Big]$$

where $\lambda \in [0,1]$, $\gamma > r > 0$ and the principal aims to optimise

$$E\Big[\int_0^\tau e^{-rs}\,(dY_s - dU_s) + e^{-r\tau}\,\ell\, K_\tau\Big],$$

where the termination time τ is available for the principal to choose.

② We shall understand the agent's choice of control as the choice of a measure. Thus we take as reference measure $\mathbb{P}^0$ under which

$$dA_t = \sigma\, dX_t + \mu\, dt$$

with X a $\mathbb{P}^0$-Brownian motion, and when agent picks control ε we switch to measure $\mathbb{P}^\varepsilon$, where

$$(2)\quad \Lambda^\varepsilon_t \equiv \frac{d\mathbb{P}^\varepsilon}{d\mathbb{P}^0}\Big|_{\mathcal{F}_t} \quad \text{solves} \quad d\Lambda^\varepsilon_t = \Lambda^\varepsilon_t\Big(-\frac{\mu\varepsilon_t}{\sigma}\Big)\,dX_t,$$

and

$$dX^\varepsilon_t \equiv dX_t + \frac{\mu\varepsilon_t}{\sigma}\,dt \quad \text{is a } \mathbb{P}^\varepsilon\text{-Brownian motion.}$$

Then the agent's objective, employing control ε, is

$$E^\varepsilon \int_0^\tau e^{-\gamma s}\,(dU_s + \lambda\mu\varepsilon_s K_s\,ds)$$

$$= E^0\Big[\Lambda^\varepsilon_\tau \int_0^\tau e^{-\gamma s}\,dU_s + \Lambda^\varepsilon_\tau \int_0^\tau e^{-\gamma s}\lambda\mu\varepsilon_s K_s\,ds\Big]$$

$$= E^0\Big[\int_0^\tau \Lambda^\varepsilon_s e^{-\gamma s}\,dU_s + \int_0^\tau \Lambda^\varepsilon_s\,\lambda\mu\varepsilon_s K_s e^{-\gamma s}\,ds\Big]$$

$$= E^0\Big[\Lambda^\varepsilon_\tau \int_0^\tau e^{-\gamma s}\,dU_s - \Lambda^\varepsilon_\tau \int_0^\tau \lambda\sigma e^{-\gamma s} K_s\,dX_s\Big]$$

$$(3)\quad = E^0\Big[\Lambda^\varepsilon_\tau \int_0^\tau e^{-\gamma s}\,(dU_s - \lambda\sigma K_s\,dX_s)\Big]$$

The key point here is that $\int_0^\tau e^{-\gamma s}(dU_s - \lambda\sigma K_s dX_s)$ is entirely under the control of the principal, Λ^ε_τ is under the control of the agent. This separation is key to the solution.

(3) A contract Φ is a triple $\Phi = (U, i, \tau)$, and given Φ the agent chooses ε to achieve

$$Q(\Phi) \equiv \sup_{0 \le \varepsilon \le 1} E^\varepsilon \int_0^\tau e^{-\gamma s}(dU_s + \lambda\mu\varepsilon_s K_s\, ds)$$

If the agent has reservation value z, then the value to the principal with initial capital K_0 is

$$V(K_0, z) = \sup\left\{ E^\varepsilon\left[\int_0^\tau e^{-rs}(dY_s - dU_s) + e^{-r\tau}\ell K_\tau\right] : Q(\Phi) = z,\ \varepsilon \text{ optimises for } \Phi \right\}$$

Linearity of dynamics and objective make clear that for any $\alpha > 0$ $V(\alpha K_0, \alpha z) = \alpha V(K_0, z)$, so $V(K_0, z) = K_0\, v(\xi) \equiv K_0\, v(z/K)$ for some function v to be discovered.

(4) Let's suppose that the principal chooses what the residual value process ξ_t shall be; then we see what the optimal ε is for the agent; then we adjust the definition of ξ to be most favourable to the principal. Showing that this is optimal remains to be done.

So suppose that principal makes ξ an autonomous diffusion driven by X:

$$(4) \qquad \boxed{d\xi_t = g(\xi_t)\, dX_t + b(\xi_t)\, dt - dL_t}$$

where L is local time of ξ at $\bar{\xi}$, to be determined. Investment will be $i(\xi_t)$, and $dU_t = K_t\, dL_t$, as we shall see. If the agent chooses Λ^ε_t as the measure, we get

$$(5) \qquad \boxed{d\xi_t = g(\xi_t)\, dX^\varepsilon_t + b^\varepsilon(\xi_t)\, dt - dL_t}$$

where $b^\varepsilon(\xi_t) = b(\xi_t) - g(\xi_t)\mu\varepsilon_t/\sigma$. Now notice that

$$d(e^{-\gamma t}K_t) = e^{-\gamma t}K_t\,(i(\xi_t) - \delta - \gamma)\, dt \equiv e^{-\gamma t}K_t\, n(\xi_t)\, dt$$

where $n(\xi) \equiv i(\xi) - \gamma - \delta$, so if $N_t \equiv \int_0^t n(\xi_u)\, du$ we shall have for $0 \le t \le s$

$$\boxed{K_s e^{-\gamma s} = K_t e^{-\gamma t} \exp(N_s - N_t)}$$

Now the agent's value Z_t at time t will be

$$\xi_t K_t \equiv Z_t = E_t^\varepsilon \Big[\int_t^\tau e^{-\gamma(s-t)} (dU_s - \lambda\sigma K_s dX_s) \Big]$$

$$= E_t^\varepsilon \Big[\int_t^\tau e^{-\gamma(s-t)} K_s (dL_s - \lambda\sigma\, dX_s) \Big]$$

$$= e^{-N_t} K_t \; E_t^\varepsilon \Big[\int_t^\tau e^{N_s} (dL_s - \lambda\sigma\, dX_s) \Big]$$

so that

$$\xi_t e^{N_t} + \int_0^t e^{N_s} (dL_s - \lambda\sigma\, dX_s) \qquad \text{is a } \mathbb{P}^\varepsilon\text{-martingale}$$

so by Itô,

$$0 \doteq n(\xi_t)\xi_t\, dt + d\xi_t + dL_t - \lambda\sigma\, (dX_t^\varepsilon - \mu\varepsilon_t/\sigma\, dt)$$

$$= \{ \xi_t n(\xi_t) + b^\varepsilon(\xi_t) + \lambda\mu\varepsilon_t \}\, dt$$

so we find

(6) $$\boxed{0 = \xi n(\xi) + b(\xi) + \varepsilon (\lambda\mu - g\mu/\sigma)}$$

What this tells us is that if the agent is behaving optimally,

(7) $$\boxed{\varepsilon_t = I_{\{\lambda\sigma > g(\xi_t)\}}}$$

and

(8) $$\boxed{0 = \xi n(\xi) + b(\xi) + (\lambda\sigma - g(\xi))^+ \mu/\sigma}$$

This represents a constraint on the principal's choice of $b(\cdot), g(\cdot), i(\cdot)$.

(5) Now let's see what the principal is going to do. He gets value

$$V(K_t, \xi_t) = K_t v(\xi_t)$$

$$= E^\varepsilon \Big[\int_t^\tau e^{-r(s-t)} (dY_s - dU_s) + e^{-r(\tau-t)} \ell K_\tau \Big]$$

$$= E^\varepsilon \Big[\int_t^\tau e^{-r(s-t)} K_s (\underbrace{dA_s}_{-c(i_t)dt} - dL_s) + e^{-r(\tau-t)} \ell K_\tau \Big]$$

$$= e^{-N_t} K_t E^\varepsilon \Big[\int_t^\tau e^{N_s + (\gamma - r)(s-t)} (\underbrace{dA_s}_{-c(i_t)dt} - dL_s) + e^{N_\tau + (\gamma - r)(\tau - t)} \ell K_\tau \Big]$$

which tells us that

$$\boxed{e^{N_t+(\gamma-r)t}\,v(\xi_t) + \int_0^t e^{N_s+(\gamma-r)s}\,(dA_s - dL_s - c(i_s)ds) \text{ is a } \mathbb{P}^{\varepsilon}\text{-martingale.}}$$

So doing Itô gives

$$0 \doteq (n+\gamma-r)v(\xi)dt + v'(\xi)\,d\xi + \tfrac12 v''(\xi)\,g(\xi)^2dt + (\sigma dW_t + \mu dt) - c(i_t)dt - dL_t$$

from which we deduce $\boxed{v'(\bar\xi) = -1}$ and

(9) $$0 = \tfrac12 g^2 v'' + (n+\gamma-r)v + \mu - c(i) + \left(b - \tfrac{\varepsilon\mu}{\sigma}g\right)v' - \mu\varepsilon$$

when optimal policy is being used. Use the relation $0 = \xi n + b + (\lambda\sigma - g)^+\mu/\sigma$ to eliminate b from this:

$$0 = \tfrac12 g^2 v'' + (n+\gamma-r)v + \mu - c(i) + \left(-\xi n - (\lambda\sigma - g)^+\mu/\sigma - \tfrac{\varepsilon\mu}{\sigma}g\right)v' - \mu\varepsilon$$

$$= \tfrac12 g^2 v'' + (i-\delta-r)v + \mu - c(i) - I_{\{\lambda\sigma>g\}}(\mu\lambda v' + \mu) - \xi v' n$$

$$= \tfrac12 g^2 v'' + (v-\xi v')n - c(i) + \mu + (\gamma-r)v - \mu(1+\lambda v')I_{\{\lambda\sigma>g\}}$$

$$= \tfrac12 g^2 v'' - \mu(1+\lambda v')I_{\{\lambda\sigma>g\}} + i(v-\xi v') - c(i) - (\gamma+\delta)(v-\xi v') + (\gamma-r)v + \mu$$

Since we've optimized, we need to have always

(10) $$\boxed{0 = \sup_{i,g}\left[\tfrac12 g^2 v'' - \mu(1+\lambda v')I_{\{\lambda\sigma>g\}} + i(v-\xi v') - c(i) - (r+\delta)v + (\gamma+\delta)\xi v' + \mu\right]}$$

Maximizing over g shows we must have $v'' \le 0$, and since $v'(\bar\xi) = -1$ we shall have $v'(\xi) \ge -1$ always. Thus $1+\lambda v' \ge 1-\lambda \ge 0$. Looking at the max over g, we have

(11) $$\boxed{\begin{aligned} g &= 0 \quad \text{if} \quad -\tfrac12(\lambda\sigma)^2 v'' > \mu(1+\lambda v') \\ &= \lambda\sigma \quad \text{else} \end{aligned}}$$

Suppose $\tilde c(y) = \sup\{yx - c(x)\}$ so we get the equation

$$0 = \mu + (\gamma+\delta)\xi v' - (r+\delta)v + \tilde c(v-\xi v') + \tfrac12(\lambda\sigma)^2 v''\, I_{\{-\frac12(\lambda\sigma)^2 v'' > \mu(1+\lambda v')\}}$$
$$- \mu(1+\lambda v')\, I_{\{-\frac12(\lambda\sigma)^2 v'' \le \mu(1+\lambda v')\}}$$

If $c(x) = \frac{1}{2}qx^2$, then $\tilde{c}(y) = y^2/2q$

The ODE (B) when the second alternative applies is

$$0 = \mu + (\gamma+\delta)\xi v' - (r+\delta)v + \frac{(v-\xi v'-1)^2}{2q} - \beta\mu(1+\lambda v')$$

$$= \frac{\xi^2}{2q}(v')^2 + v'\left\{(\gamma+\delta)\xi - \frac{\xi(v-1)}{q} - \lambda\beta\mu\right\} + \mu - (r+\delta)v + \frac{(v-1)^2}{2q} - \beta\mu$$

$$\Rightarrow 0 = \frac{1}{2}\xi^2(v')^2 - v'\{\underbrace{\xi(v-1) + \lambda\beta\mu q - q(\gamma+\delta)\xi}_{\equiv B}\} + \underbrace{\left(\frac{1}{2}(v-1)^2 + \mu q - (r+\delta)qv - \beta\mu q\right)}_{\equiv C}$$

So the solution is

$$\frac{B - \sqrt{B^2 - 2\xi^2 C}}{\xi^2} = \frac{2C}{B + \sqrt{B^2 - 2\xi^2 C}}$$

If $c(x) = x + \frac{1}{2}qx^2$, then

$$\tilde{c}(y) = \sup_x\{yx - c(x)\} = (y-1)^2/2q$$

which may be more succinctly expressed as

(12) $$\boxed{0 = \mu + (\gamma+\delta)\xi v'(\xi) - (r+\delta)v + \tilde{c}(v - \xi v') + \max\left\{ \tfrac{1}{2}(\lambda\sigma)^2 v'', -\mu(1+\lambda v') \right\}}$$

DFHW get something very similar at their equation (18); the final term for them is simply $\frac{1}{2}(\lambda\sigma)^2 v''$, since they insist that the contract chosen should cause the agent always to make effort.

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(6) (7/1/11) When you try to do numerics here, it turns out that solving the first-order ODE given by taking the second alternative in the max in the HJB at the top of this page, then $v'' > 0$; in other words, the second alternative never applies, and the agent will have to work all the time. So although the question was different, the answer was not. Indeed, it's clear from looking at the principal's objective when $\varepsilon = 1$ that the $\int \cdots dY$ is a supermartingale, therefore losing value as it runs, and the termination reward $e^{-r\tau} \ell K_\tau$ is falling off as we wait, so you would shut down immediately. So this is rather disappointing.

But what if we supposed that $0 \le \varepsilon_t \le \beta$, so that the project has a positive (if $\beta < 1$) growth even when the agent is slacking??

In this case, the story runs as before down to equation (7), which now reads

$$\varepsilon_t = \beta\, I_{\{\lambda\sigma > g_t\}}$$

The HJB equation (10) changes to

$$0 = \sup_{i,g}\Big[\tfrac{1}{2} g^2 v'' - \beta\mu(1+\lambda v') I_{\{\lambda\sigma > g\}} + i(v - \xi v') - c(i) - (r+\delta)v + (\gamma+\delta)\xi v' + \mu \Big]$$

!

So the rule is

$$g = 0 \quad \text{if} \quad -\tfrac{1}{2}(\lambda\sigma)^2 v'' > \beta\mu(1+\lambda v')$$
$$= \lambda\sigma \quad \text{if not}$$

and

!

(13) $$\boxed{0 = \mu + (\gamma+\delta)\xi v' - (r+\delta)v + \tilde{c}(v - \xi v') + \max\left\{ \tfrac{1}{2}(\lambda\sigma)^2 v'', -\beta\mu(1+\lambda v') \right\}}$$

Some thoughts on a question of Ezequiel Antar (7/1/11)

Ezequiel is considering a situation where there are two agents with a priori exposures ξ^A, ξ^B at time 1, and able to trade in some market to create any gains-from-trade random variable $Z \in V$ at time 1. What are the Pareto-efficient risk transfers Y between them?

(i) Let's consider the generalisation of this to J agents, each with C^2 strictly concave utility satisfying the Inada conditions. The central planner problem is

$$\max \sum_{j=1}^{J} \lambda_j U_j(x_j + y_j) \qquad \text{st. } \sum y_j = 0$$

where $\lambda_j > 0$ and x_j are given. The FOCs here give

$$\lambda_j U_j'(x_j + y_j) = \alpha$$

for Lagr. multiplier α, so $(x_j + y_j) = I_j(\alpha/\lambda_j)$ and we must adjust α so that

$$\sum_{j=1}^{J} I_j(\alpha/\lambda_j) = \sum x_j \equiv X \text{, say.}$$

Thus the optimal α, and hence optimal $(x_j + y_j)$, depends on the sequence (x_j) only through $X \equiv \sum x_j$. The optimized value is also clearly a concave increasing function of X. Write

$$V(X) = \max\left\{ \sum_{j=1}^{J} \lambda_j U_j(z_j) : \sum z_j = X \right\}$$

The dual function is

$$\tilde{V}(q) = \sup_X \sup_{\sum z_j = X} \left\{ \sum_{j=1}^{J} \lambda_j U_j(z_j) - qX \right\}$$

$$= \sum_{j=1}^{J} \lambda_j \tilde{U}_j(q).$$

So the multi-agent problem becomes a single-agent problem.

(ii) So suppose a single agent has a priori exposure ξ, and may generate any $Z \in H$, a vector space, by trading. If his utility is V, then the optimization problem is

$$\sup_{Z \in H} E\left[V(\xi + Z) \right]$$

If Y is any EMM density, we have always

$$E\, V(\xi+Z) = E\left[V(\xi+Z) - YZ \right] \leq E\left[\tilde{V}(Y) + \xi Y \right]$$

so

$$\sup_{Z \in H} E\, V(\xi+Z) \leq \inf_{Y} E\left[\tilde{V}(Y) + \xi Y \right]$$

If we're lucky, there's no duality gap etc.

(iii) Ezequiel looks at the case where all agents are CARA, when it's easy to prove that $V(\cdot)$ is also CARA. So we're down to the single-agent problem where we have to optimize over the gains-from-trade process.

Another contracting type of question (10/1/11)

(i) We could consider a situation where the principal's wealth evolves as

$$dw_t = w_t \{ \sigma dW_t + (\mu - \theta_t) dt \} - c_t\, dt - q_t\, dt$$

where q_t is the wages paid to the agent and c_t is principal's consumption. The agent can slack (or steal) at rate $\theta_t dt$, which we suppose delivers him value equivalent to wages $\beta w_t \theta_t$, where $0 < \beta < 1$. We might suppose that the objectives are:

$$P: \quad \sup E\left[\int_0^\infty e^{-\rho s} U_p(c_s)\, ds \right] \qquad A: \quad \sup E\left[\int_0^\infty e^{-\gamma s} U_A(q_s + \beta \theta_s w_s)\, ds \right]$$

(ii) Let's suppose $U_p' = x^{-R_p}$, $U_A' = x^{-R_A}$. We suspect that the values to the agents depend only on w, and scale appropriately, so that

$$V_p(x) = \kappa_p U_p(x), \quad V_A(x) = \kappa_A U_A(x).$$

Then the MPOC would tell us

$$\left.\begin{aligned}
&\sup_\theta \left[-\gamma V_A + V_A' \{ x(\mu - \theta) - c - q \} + \tfrac{x^2}{2}\sigma^2 V_A'' + U_A(q + \beta \theta x) \right] = 0 \\
&\sup_{c,q} \left[-\rho V_p + V_p' \{ x(\mu - \theta) - c - q \} + \tfrac{x^2}{2}\sigma^2 V_p'' + U_p(c) \right] = 0
\end{aligned}\right\}$$

which becomes the conditions ($\tilde{c} \equiv c/x$, $\tilde{q} \equiv q/x$)

$$\left.\begin{aligned}
&\sup_\theta U_A(x) \left[-\gamma \kappa_A + (1-R_A)\{ \mu - \theta - \tilde{c} - \tilde{q} \} \kappa_A - R_A(1-R_A) \tfrac{\sigma^2}{2} \kappa_A + (\tilde{q} + \beta\theta)^{1-R_A} \right] = 0 \\
&\sup_{c,q} U_p(x) \left[-\rho \kappa_p + (1-R_p)\kappa_p \{ \mu - \theta - \tilde{c} - \tilde{q} \} - R_p(1-R_p) \tfrac{\sigma^2}{2} \kappa_p + \tilde{c}^{1-R_p} \right] = 0
\end{aligned}\right\}$$

The agent's FOC tells us

$$\kappa_A = \beta (\tilde{q} + \beta\theta)^{-R_A}$$

so that $\tilde{q} + \beta\theta = (\kappa_A/\beta)^{-1/R_A} \doteq (\beta/\kappa_A)^{1/R_A}$, a constant; at least if θ is at an interior point. It seems natural in the context of the problem to suppose $\theta \geq 0$ so the FOC should really say

$$\beta(\tilde{q} + \beta\theta)^{-R_A} - \kappa_A \leq 0, \quad \text{equal if } \theta > 0.$$

so

$$\tilde{q} + \beta\theta \geq (\beta/\kappa_A)^{1/R_A}, \quad \text{equal if } \theta > 0$$

We could write this as

$$\tilde{q} = (\beta/\kappa_A)^{1/R_A} - \beta\theta + z$$

where $z \geq 0$ is a slack variable, $z\theta = 0$. The principal's optimization now would give

$$\tilde{c}^{-R_P} = \kappa_P$$

and an optimization over θ which says

$$\max\{-\theta - \tilde{q}\} = \max\left\{-(1-\beta)\theta - z - (\beta/\kappa_A)^{1/R_A}\right\}$$

Clearly best is to take $\theta = z = 0$ but if we were to impose some reservation utility requirement it might be necessary to offer some $q > (\beta/\kappa_A)^{1/R_A}$.
But proceeding now with the assumption $\theta = z = 0$, we get

$$\begin{cases} 0 = -\gamma\kappa_A + (1-R_A)\kappa_A\left(\mu - \kappa_P^{-1/R_P}\right) - R_A(1-R_A)\frac{\sigma^2}{2}\kappa_A + \{1-\beta(1-R_A)\}\left(\frac{\beta}{\kappa_A}\right)^{1/R_A - 1} \\ 0 = -\rho\kappa_P + (1-R_P)\kappa_P\left\{\mu - (\beta/\kappa_A)^{1/R_A}\right\} - \frac{\sigma^2}{2}R_P(1-R_P)\kappa_P - R_P\kappa_P^{1-1/R_P} \end{cases}$$

which need to be solved simultaneously for κ_A, κ_P.

Interesting questions.

(1) 7/6/10: suggested Takashi might want to try finding general methods for solving contracting problems...

1/9/10 Ezequiel asks about the following. Suppose agent j has utility U_j in a 1-period model, and is exposed to risk Z_j. There are assets $X_1, \dots, X_k$ in supply $a_1, \dots, a_k$, so agent j will try to

$$\max E\, U_j\left(\theta_j \cdot (X - p) + Z_j \right)$$

where p is the equilibrium price. Now suppose that financial assets $Y_1, \dots, Y_m$ are introduced (zero net supply) so now the game is

$$\max E\, U_j\left(\theta_j \cdot (X-p) + \gamma_j \cdot (Y-q) + Z_j \right)$$

What is the effect on equilibrium? Agents can freely generate such contracts — what might be Pareto efficient choices of the Y_i??

7/9/10 Sergei Foss asks: suppose $X_1, X_2, \dots$ are IID nonnegative, $E\, e^{\lambda X_i} < \infty\ \forall \lambda$. Suppose also that the $\mathbb{Z}^+$-valued RV N has all exponential moments, but is not independent of the X_i. Does $X_1 + \dots + X_N$ have all exponential moments?

7/9/10 Martin Barlow asks: suppose we consider FX trading — can everyone be gaining? If at time T the FX rate is back at its initial value, does it follow that the aggregate gains of all agents amount to zero?

4/10/10 Tilman Sayer was talking to me about his attempts to do option pricing in a Heston SV model where there's correlation between the two BMs. I suggested he should probably try Crank-Nicolson as a first choice. Also, suggested that the SV SDE should be put in natural scale, with grid points placed so that the mean time to reach next grid point should be Δt.