A feature of dynamic programming (20/8/13)

(i) Suppose we are trying to control a finite-state Markov chain, where the control variable a is continuous, and the rewards $g_j(a)$ in state j, and the jump intensities $q_{jk}(a)$ depend on a in a smooth fashion. If we think of an infinite-horizon problem, the dynamic-programming equation looks like

$$0 = \sup_a \Big[g_j(a) - \rho V_j + \sum_k q_{jk}(a) V_k \Big].$$

We'll write $a = (a_1, \dots, a_N)$ for a (Markov) policy, with a_j the action in state j. If we use Markov policy a, then the value function $V(a)$ arising will solve

$$0 = g(a) - (\rho - Q(a)) V(a). \qquad (1)$$

Now if a^* is the optimal policy, we know that $\partial V_j / \partial a_m = 0$ at $a = a^*$ for all j, for all m. So if we differentiate (1), we have at $a = a^*$ that

$$0 = \frac{\partial g_j(a^*)}{\partial a_m} + \sum \frac{\partial q_{j\ell}(a^*)}{\partial a_m} V_\ell(a^*) \qquad \forall j, m$$

Notice also that $\partial g_j / \partial a_m = 0 = \partial q_{j\ell} / \partial a_m$ unless $j = m$, because the rewards and jump intensities in state j depend only on the action chosen in state j. Hence the first-order condition is simply

$$\boxed{0 = \frac{\partial g_j}{\partial a_j} + \sum_\ell \frac{\partial q_{j\ell}}{\partial a_j} V_\ell(a^*) \qquad \forall j}$$

Any good? Basically, this is just the optimality condition satisfied at each j, rather obviously...

(ii) What would happen in the context of a contract? Suppose that $a = (a_1, \dots, a_N)$ denotes the actions of the agent, $p = (p_1, \dots, p_N)$ the actions of the principal. Once p has been chosen, the agent solves

$$0 = \sup_a \Big[g(a,p) + \{ Q(a,p) - \rho \} V \Big].$$

Suppose that the principal has objective

$E\left[\int_0^\infty e^{-\gamma t} G(X_t; a_t, p_t)\,dt\right]$ so that his value is

$$\left(\gamma - Q(a,p)\right)^{-1} G(\cdot\,; a, p).$$

Suppose he proposes to try some differentiable trajectory (p_t) for p; can we readily work out the agent's response $a^*(p_t)$? Let's denote by $V(\cdot\,;t)$ the agent's value at time t. Then we know that

$$0 = g(a^*(p_t), p_t) + \left(Q(a^*(p_t), p_t) - \rho\right) V(\cdot\,;t)$$

and that $a^*(p)$ is optimal given p. So if we now differentiate with respect to t we should have

$$0 = \dot{p}_t\left[\frac{\partial g}{\partial p}(a^*(p), p) + \frac{\partial Q}{\partial p}(a^*(p), p)\, V(\cdot\,;t)\right]$$

$$+ Q(a^*(p_t), p_t)\, \dot{V}(\cdot\,;t)$$

which is a simple first-order ODE for V which we could solve given the driver (p_t) provided we could deduce the path $a^*(p_t)$. But this we could get by looking at the FOC for the agent

$$\boxed{\frac{\partial g}{\partial a}(a^*(p), p) + \frac{\partial Q}{\partial a}(a^*(p), p)\, V(\cdot\,;t) = 0}$$

So provided it's relatively easy to solve these for actions a, we could express $a^*(p)$ as a function of V and p, and go from there.

Merton problem with liquidity again (3/9/13)

When we look at the Merton problem with liquidity (see WN XXXV, pp 71 ~ 73), it seems that quite a bit of care is required over the boundary conditions.

So suppose that we have discretized the Y-motion onto a grid $y_0 = 0 < y_1 < \dots < y_N < y_{N+1}$ where we think of $\{y_1, \dots, y_N\}$ as the set where we may choose controls, creating a Q-matrix of jump rates, with (for the moment) absorption at y_0, y_{N+1}. How should we handle the behaviour at y_0, y_{N+1}?

(a) Behaviour at y_0 We assume that immediately the chain hits y_0, the agent is compelled to sell ΔH units of the stock at the least favourable rate, so he realizes cash $\Delta H(1-\beta)$; we assume that this is one of the values $\{y_1, \dots, y_N\}$.

(b) Behaviour at y_{N+1}. We assume that once the process leaves $\{y_1, \dots, y_N\}$ by a jump to y_{N+1} then there will be no changes allowed to H until Y has got back into the interior set $\mathcal{Y}_0 \equiv \{y_1, \dots, y_N\}$. Up until that time, consumption will happen at rate $\gamma_M Y_t$. Let τ be the time of first return to $\mathcal{Y}_0$. Then we need to know about

$$g(y) = E^y\left[\int_0^\tau e^{-\tilde\rho t} U(\gamma_M Y_t)\, dt + e^{-\tilde\rho \tau} K\right]$$

where $K \equiv g(y_N)$. Clearly g solves $(\tilde\alpha \equiv \alpha + \gamma_M)$

$$0 = \tfrac{1}{2}\sigma^2 y^2 g'' - \tilde\alpha y g' - \tilde\rho g + U(\gamma_M y)$$

and if the auxiliary quadratic $Q(t) \equiv \frac{1}{2}\sigma^2 t(t-1) - \tilde\alpha t - \tilde\rho$ has roots $-a_- < 0 < a_+$ we shall have solution

$$g(y) = A_0 \left(y/y_N\right)^{-a_-} - \frac{U(\gamma_M)}{Q(1-R)}\, y^{1-R}$$

So the matching condition is

$$g(y_N) \equiv K = A_0 - \frac{U(\gamma_M y_N)}{Q(1-R)} = A_0 - B_0 \quad \text{, say}$$

Accordingly,

$$g(y_{N+1}) = A_0 \underbrace{\left(y_N/y_{N+1}\right)^{a_-}}_{\equiv p_0} - \underbrace{\frac{U(\gamma_M y_N)}{Q(1-R)}}_{\equiv B_0} \underbrace{\left(y_N/y_{N+1}\right)^{R-1}}_{\equiv p_1}$$

$$= p_0 (K + B_0) - p_1 B_0$$

$$= p_0\, g(y_N) + (p_0 - p_1) B_0 .$$

Thus the linear equation at y_N will be

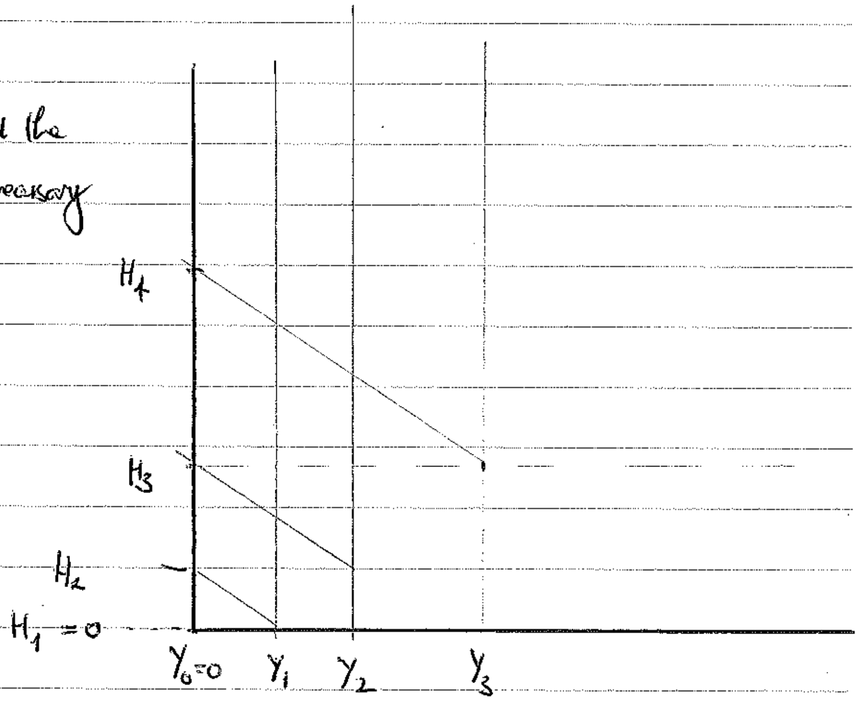
$$0 = U(c_N) - \tilde{\rho}\, g_N + q_{N,N-1}\, g_{N-1} + q_{NN}\, g_N + q_{N,N+1} \{ p_0 g_N + (p_0 - p_1) B_0 \}$$

$$= U(c_N) + q_{N,N+1} (p_0 - p_1) B_0 + q_{N,N-1}\, g_{N-1}$$

$$+ g_N \{ q_{N,N} + p_0\, q_{N,N+1} \} - \tilde{\rho}\, g_N$$

Further numerical stuff (9/9/13)

For setting up the H-grid and the Y-grid, it has at some time been necessary to do the following



We shall require H_k, Y_k to be increasing sequences, and we also want for some $q > 1$, and $a > 0$ that

$$\begin{cases} (H_{k+1} - H_k)(1-\beta) = Y_k & (k \geq 1) \\ H_k = b + a q^{k-2} & (k \geq 2) \end{cases}$$

This gives us
$$\begin{cases} a(1-\beta) q^{k-2} (q-1) = Y_k & (k \geq 2) \\ (a + b)(1-\beta) = Y_1 \end{cases}$$

So the condition $Y_1 < Y_2$ requires $a(q-1) > (a+b)$, so we can ensure this by selecting

$$(a+b) = a(q-1)/2 .$$

Towards the joint law of (I, X, S)? (17/9/13)

(i) Suppose m is the law of $(I_\tau, X_\tau, S_\tau, \sigma_\tau)$, where X is a symmetric simple random walk on $\mathbb{Z}$, $I_k = \inf_{n \le k} X_n$, $S_k = \sup_{n \le k} X_n$ and σ is the signature. Thus m is a prob'y measure on $(-\mathbb{Z}^+) \times \mathbb{Z} \times \mathbb{Z}^+ \times \{-1, +1\}$. As before, we are assuming that $\tau < \infty$ a.s. Let's write

$$m^{\pm}_{ab} = m(I = a, S = b, \sigma = \pm 1), \quad m_{ab} = m^+_{ab} + m^-_{ab}$$

and

$$\bar{m}^{\pm}_{ab} = m(X : I = a, S = b, \sigma = \pm 1), \quad \bar{m}_{ab} = \bar{m}^+_{ab} + \bar{m}^-_{ab}$$

Finally, let $(a \le 0 \le b$, not both zero$)$

$$\pi(a,b) = P(H_a < H_b < H_{a-1}) = P(H_b \le \tau, I(H_b) = a),$$

$$\pi(b,a) = P(H_b < H_a < H_{b+1}) = P(H_a \le \tau, S(H_a) = b),$$

$$\left.\begin{aligned}
\pi(a,b,b+1) &= P(H_a < H_b < H_{b+1} < H_{a-1}) \\
\pi(a,b,a-1) &= P(H_a < H_b < H_{a-1} < H_{b+1}) \\
\pi(b,a,a-1) &= P(H_b < H_a < H_{a-1} < H_{b+1}) \\
\pi(b,a,b+1) &= P(H_b < H_a < H_{b+1} < H_{a-1})
\end{aligned}\right\} \quad \boxed{(ab \ne 0)}$$

(ii) We have the equations $(ab \ne 0)$

$$\left.\begin{aligned}
\pi(a,b) &= \pi(a,b,b+1) + m^+_{ab} + \pi(a,b,a-1) \\
b\,\pi(a,b) &= (b+1)\pi(a,b,b+1) + \bar{m}^+_{ab} + (a-1)\pi(a,b,a-1) \\
\pi(b,a) &= \pi(b,a,a-1) + m^-_{ab} + \pi(b,a,b+1) \\
a\,\pi(b,a) &= (a-1)\pi(b,a,a-1) + \bar{m}^-_{ab} + (b+1)\pi(b,a,b+1)
\end{aligned}\right\}$$

Solving these for $\pi(\cdot,\cdot,\cdot)$ we obtain (for $ab \ne 0$)

$$\left.\begin{aligned}
(b-a+2)\pi(a,b,b+1) &= (b-a+1)\pi(a,b) - \bar{m}^+_{ab} + (a-1)m^+_{ab} \\
(b-a+2)\pi(b,a,b+1) &= \pi(b,a) - \bar{m}^-_{ab} + (a-1)m^-_{ab}
\end{aligned}\right\}$$

whence

$$\boxed{(b-a+2)\pi(a,b+1) = (b-a+1)\pi(a,b) + \pi(b,a) - \bar{m}_{ab} + (a-1)m_{ab}} \quad (1)$$

and symmetrically

$$\boxed{(b-a+2)\pi(b,a-1) = (b-a+1)\pi(b,a) + \pi(a,b) + \bar{m}_{ab} - (b+1)\, m_{ab}} \qquad (2)$$

These are interesting because we don't have to split according to σ!! Rewriting (1) in a more interpretable form, we have

$$\boxed{\pi(a,b+1) = \frac{b-a+1}{b-a+2}\pi(a,b) + \frac{1}{b-a+2}\pi(b,a) - E\left[\frac{X-a+1}{b-a+2} : I=a, S=b\right]}$$

To interpret this, we see that the expectation is

$$P\left[H_a, H_b \le \tau < H_{b+1} < H_{a-1}\right]$$

so if we take this over to the LHS we get

$$\pi(a,b+1) + E\left[\frac{X-a+1}{b-a+2} : I=a, S=b\right]$$

$$= P\left[H_a < H_{b+1} \le \tau \wedge H_{a-1}\right] + P\left[H_a, H_b \le \tau < H_{b+1} < H_{a-1}\right]$$

$$= P\left[H_a < H_b < H_{b+1} \le \tau \wedge H_{a-1}\right] + P\left[H_a < H_b \le \tau < H_{b+1} < H_{a-1}\right]$$

$$+ P\left[H_b < H_a < H_{b+1} \le \tau \wedge H_{a-1}\right] + P\left[H_b < H_a \le \tau < H_{b+1} < H_{a-1}\right]$$

$$= P\left[H_a < H_b \le \tau,\ H_{b+1} < H_{a-1}\right] + P\left[H_b < H_a \le \tau,\ H_{b+1} < H_{a-1}\right]$$

$$= \pi(a,b)\,\frac{b-a+1}{b-a+2} + \pi(b,a)\,\frac{1}{b-a+2}.$$

(iii) An interesting thing to note is that if we add (1) and (2) we find that

$$\boxed{\pi(a,b+1) + \pi(b,a-1) = \pi(a,b) + \pi(b,a) - m_{ab}} \qquad (3)$$

If there is a time when $I=a$, $S=b$, and these are not the final values of I, S, there will be a time when $I=a$, $S=b+1$ <u>or</u> $I=a-1$, $S=b$. This is natural, but is a consequence of (1), (2)

(iv) Notice also that (1), (2) could be used to calculate recursively the entire π array, since $\pi(0,1) = \pi(0,-1) = \frac{1}{2}(1-m_{00})$, so we can start, and move on to build up π along the lines $b+|a| = k$ for $k=2,3,\ldots$ We can

Conversely recover m_{ab}, $\bar{m}_{ab}$ from the π array – provided we have extended our knowledge from the region $ab \neq 0$ to the edges.

(v) How does it look at the edges?

We have for $b>0$

$$\frac{1}{1+b} = P(H_b < H_{-1})$$
$$= P(\tau < H_b < H_{-1}) + P[H_b \leq \tau < H_{-1}] + P[H_b < H_{-1} \leq \tau]$$
$$= E\left[\frac{X+1}{b+1} : I=0, S<b\right] + \pi(0,b) + \sum_{k\geq b} \pi(k,-1)$$

which is harder to reduce. But if we use the OST argument which led to (1)(2) we see for $b \geq 1$

$$\begin{cases} \pi(0,b) = \pi(0,b+1) + m_{0b} + \pi(b,-1) \\ b\,\pi(0,b) = (b+1)\pi(0,b+1) + \bar{m}_{0b} - \pi(b,-1) \end{cases}$$

whence

(4) $$\boxed{(b+1)\pi(0,b) = (b+2)\pi(0,b+1) + E[X+1 : I=0, S=b]} \qquad (b \geq 1)$$

Also, we know

(5) $$\boxed{\pi(0,1) = \tfrac{1}{2}(1-m_{00}) = \pi(0,-1)}$$

so with the notational convention $\boxed{\pi(0,0)=1}$ the validity of (4) extends also to $b=0$. Hence by summation

$$1-(b+1)\pi(0,b) = \sum_{0\leq \beta < b} E[X+1 : I=0, S=\beta]$$

so

$$\boxed{(b+1)\pi(0,b) = 1 - E[X+1 : I=0, S<b]} \qquad (b \geq 1)$$

Similarly, for $a \leq -1$

$$\boxed{(1-a)\pi(0,a) = 1 - E[1-X : S=0, I>a]}$$

and recursively

$$\boxed{(1-a)\pi(0,a) = (2-a)\pi(0,a-1) + E[1-X : S=0, I=a]} \qquad (a \leq -1)$$

Therefore if we make the convention

$$(5) \qquad \boxed{\pi(a,0) = \pi(b,0) = 0 \qquad \text{for } a<0<b}$$

the conclusion is that (1), (2) hold not just for $ab \neq 0$, but also for $a=0$, $b>0$ in the case of (1), and for $b=0$, $a<0$ in the case of (2).

Does (1) also hold for $b=0$, $a<0$? Using the notational convention (5), what this would require is that

$$(2-a)\pi(a,1) = \pi(0,a) - E[X-a+1 : S=0, I=a]$$

Now

$$\pi(a,1) = P[H_a < H_1 < H_{a-1}, \ H_1 \leq \tau]$$

$$= P[H_a < H_1 \leq \tau < H_{a-1}] + P[H_a < H_1 < H_{a-1} \leq \tau]$$

and

$$\frac{\pi(0,a)}{2-a} = P[H_a \leq \tau, \ S(H_a)=0, \ H_1 < H_{a-1}]$$

$$E\left[\frac{X-a+1}{2-a} : S=0, I=a\right] = P[H_1 < H_{a-1}, \ H_a \leq \tau < H_1]$$

$$= P[H_a \leq \tau < H_1 < H_{a-1}]$$

So if we combine,

$$\pi(a,1) + E\left[\frac{X-a+1}{2-a} : S=0, I=a\right] = P[H_a < H_1 \leq \tau < H_{a-1}] + P[H_a < H_1 < H_{a-1} \leq \tau]$$
$$+ P[H_a \leq \tau < H_1 < H_{a-1}]$$

$$= P[H_a < H_1 < H_{a-1}, \ H_a \leq \tau]$$

$$= \frac{\pi(0,a)}{2-a}$$

So now we know that (1) holds also for $b=0$, $a<0$. So by a symmetric argument, we have <u>that</u> (1) <u>and</u> (2) <u>hold for all</u> $a \leq 0 \leq b$ <u>except</u> $a=b=0$.

Thus we use (5) to get us started, and (1), (2) to reconstruct the entire array π from knowledge of m.

(vi) As a side discussion, but not unrelated, let's for the moment just consider SSRW without any stopping and define for $a < 0 < b$

$$F_{ab} = \{ H_a < H_b < H_{a-1} \}, \quad F_{ba} = \{ H_b < H_a < H_{b+1} \}.$$

Then we have

$$p_{ab} \equiv P(F_{ab}) = \frac{b}{(b-a)} \cdot \frac{1}{b-a+1}$$

$$p_{ba} \equiv P(F_{ba}) = \frac{-a}{(b-a)} \cdot \frac{1}{b-a+1}$$

If we consider $a < a' < 0 < b' < b$, then

$$P(F_{ab} \cap F_{a',b'}) = \frac{b'}{(b'-a')(b'-a'+1)} \cdot \frac{b-b'}{(b-a)(b-a+1)}$$

$$P(F_{ab} \cap F_{b'a'}) = \frac{-a'}{(b'-a')(b'-a'+1)} \cdot \frac{b-a'}{(b-a)(b-a+1)} \qquad (\text{ok if } a=a')$$

$$P(F_{ba} \cap F_{a'b'}) = \frac{b'}{(b'-a')(b'-a'+1)} \cdot \frac{b'-a}{(b-a)(b-a+1)}$$

$$P(F_{ba} \cap F_{b'a'}) = \frac{-a'}{(b'-a')(b'-a'+1)} \cdot \frac{a'-a}{(b-a)(b-a+1)}$$

Similarly, for $a < 0 < b' < b$ we shall have

$$P(F_{ab'} \cap F_{ab}) = \frac{b' \quad (b'-a+1)}{(b'-a)(b'-a+1)(b-a+1)} = \frac{b'}{(b'-a)(b-a+1)}$$

$$P(F_{b'a} \cap F_{ba}) = 0.$$

Suppose now we introduce the notation $\xi_{ab+1} = E\left[\frac{X-a+1}{b-a+2} : I=a, S=b\right]$, $\xi_{ba-1} = E\left[\frac{b+1-X}{b-a+2} : S=b, I=a\right]$, then our basic recursions (1), (2) read

$$\boxed{\begin{aligned} \pi(a, b+1) &= \frac{b-a+1}{b-a+2}\pi(a,b) + \frac{1}{b-a+2}\pi(b,a) - \xi_{ab+1} \\ \pi(b, a-1) &= \frac{b-a+1}{b-a+2}\pi(b,a) + \frac{1}{b-a+2}\pi(a,b) - \xi_{ba-1} \end{aligned}} \quad (7)$$

We need to understand this system of difference equations. Notice that the probabilities p_{ab}, p_{ba} solve the homogeneous system. The recursions would allow us to express $\pi(a, b+1)$ in terms of $\pi(a', b')$ for $a' < b' \in [a, b)$ and the various ξ values. What weight is attached to $\pi(a', b')$ in the expression for $\pi(a, b+1)$? It has to be $P(F_{a,b+1} | F_{a'b'})$!!

So we get

$$P\left(F_{a\,b+1} \mid F_{ab'}\right) = \frac{b'-a+1}{b-a+2} \qquad (0 < b' \le b,\ a \le 0)$$

$$P\left(F_{a,b+1} \mid F_{a'b'}\right) = \frac{b-b'+1}{(b-a+1)(b-a+2)} \qquad (a < a' \le 0 < b' \le b)$$

$$P\left(F_{a,b+1} \mid F_{b'a'}\right) = \frac{b-a'+1}{(b-a+1)(b-a+2)} \qquad (a \le a' < 0 \le b' \le b)$$

$$P\left(F_{b,a-1} \mid F_{ba'}\right) = \frac{b-a'+1}{b-a+2} \qquad (a \le a' < 0,\quad b \ge 0)$$

$$P\left(F_{b,a-1} \mid F_{b'a'}\right) = \frac{a'-a+1}{(b-a+1)(b-a+2)} \qquad (a < a' < 0 \le b' < b)$$

$$P\left(F_{b,a-1} \mid F_{a'b'}\right) = \frac{b'-a+1}{(b-a+1)(b-a+2)} \qquad (a \le a' < 0 < b' \le b)$$

If you keep careful track of the corner where $b + |a| \le 2$, I reckon you end up with

(8)
$$\boxed{\begin{aligned}\pi(a, b+1) = &-\sum_{0 < b' \le b+1} \frac{b'-a+1}{b-a+2}\,\xi_{ab'} - \sum\sum_{\substack{a < a' \le 0 < b' \le b \\ b'+|a'|>1}} \frac{b-b'+1}{(b-a+1)(b-a+2)}\,\xi_{a'b'} \\ &- \sum\sum_{\substack{a \le a' < 0 \le b' \le b \\ b'+|a'|>1}} \frac{b-a'+1}{(b-a+1)(b-a+2)}\,\xi_{b'a'} \\ &+ \frac{b+1}{(b-a+1)(b-a+2)}\left(1 - m_{00}\right)\end{aligned}}$$

The corresponding expression for $\pi(b, a-1)$ follows by similar reasoning, giving

(9)
$$\pi(b,a-1) = -\sum_{0>a'\geq a-1} \frac{b-a'+1}{b-a+2}\,\xi_{ba'} - \sum\sum_{\substack{a\leq a'<0\leq b'<b \\ |a'|+b'>1}} \frac{a'-a+1}{(b-a+1)(b-a+2)}\,\xi_{b'a'}$$
$$-\sum\sum_{a\leq a'\leq 0<b'\leq b} \frac{b'-a+1}{(b-a+1)(b-a+2)}\,\xi_{a'b'}$$
$$+\frac{1-a}{(b-a+1)(b-a+2)}\left(1-m_{00}\right)$$

(vii) <u>How shall this be interpreted?</u> Let's just look at (8) for now. The first sum on the RHS is

$$-\sum_{0<b'\leq b+1} \frac{b'-a+1}{b-a+2}\, E\left[\frac{X-a+1}{b'-a+1} : I=a,\ S=b'-1\right]$$

$$= -E\left[\frac{X-a+1}{b-a+2} : I=a,\ S\leq b\right] = -P\left[H_{b+1}<H_{a-1},\ H_a\leq\tau<H_{a-1},\ H_{b+1}>\tau\right]$$

The second term on the RHS, the first double sum, is

$$-\sum\sum_{a<a'\leq 0<b'\leq b} E\left[\frac{(b-b'+1)(X-a'+1)}{(b-a+2)(b-a+1)(b'-a'+1)} : I=a',\ S=b'-1\right]$$

$$= -\sum\sum_{a<a'\leq 0<b'\leq b} E\left[\frac{(b-S)(X-I+1)}{(b-a+2)(b-a+1)(S-I+2)} : I=a',\ S=b'-1\right]$$

$$= -E\left[\underbrace{\frac{X-I+1}{S-I+2}}_{P(\text{get to } S+1 \text{ before } I-1)}\cdot\underbrace{\frac{b-S}{b-a+1}}_{P(\text{from } S+1,\text{ reach } a \text{ before } b+1)}\cdot\underbrace{\frac{1}{b-a+2}}_{P(\text{from } a,\text{ reach } b+1 \text{ before } a-1)} : I>a,\ S<b\right]$$

The third term, the second double sum, is

$$-\sum\sum_{a\leq a'<0\leq b'\leq b} E\left[\frac{(b'+1-X)(b-a'+1)}{(b-a+1)(b-a+2)(b'-a'+1)} : I=a'+1,\ S=b'\right]$$

$$= -\sum_{a\leq a'<0\leq b'\leq b} E\left[\frac{S+1-X}{S-I+2}\cdot\frac{b-I+2}{b-a+1}\cdot\frac{1}{b-a+2} : I=a'+1,\ S=b'\right]$$

$$= -E\left[\underbrace{\frac{S+1-X}{S-I+2}}_{P(\text{escape }[I,S]\text{ at }I-1)}\cdot\underbrace{\frac{b-I+2}{b-a+1}}_{P(\text{from }I-1,\text{ hit }a\text{ before }b+1)}\cdot\underbrace{\frac{1}{b-a+2}}_{P(\text{from }a,\text{ hit }b+1\text{ before }a-1)} : I>a,\ S\le b\right]$$

So if we rearrange (8) we have

$$\frac{b+1}{(b-a+1)(b-a+2)}(1-m_{00})$$

$$= \pi(a,b+1) + E\left[\frac{X-a+1}{b-a+2} : I=a, S\le b\right]$$

$$+ E\left[\frac{X-I+1}{S-I+2}\cdot\frac{b-S}{b-a+1}\cdot\frac{1}{b-a+2} : I>a, S<b\right]$$

$$+ E\left[\frac{S+1-X}{S-I+2}\cdot\frac{b-I+2}{b-a+1}\cdot\frac{1}{b-a+2} : I>a, S\le b\right]$$

$$= \pi(a,b+1) + E\left[\frac{X-a+1}{b-a+2} : I=a,\ S\le b\right]$$

$$+ E\left[\frac{b+1-X}{(b-a+1)(b-a+2)} : I>a,\ S\le b\right]$$

when simplified. Everything in this equality apart from $\pi(a,b+1)$ can be calculated from m, so it is a necessary condition that when we calculate the expression for $\pi(a,b+1)$, each $a\le 0$, $b\ge 0$, we get a probability.

A question of Julien Guyon (1/10/13)

(i) Julien asked me about this when I visited Bloomberg, 13/9/13. Suppose you have some (high-dimensional) diffusion with generator $\mathcal{L}$ and an option which pays $g(X_T)$ at time T. Suppose that g takes both signs, and that the option is subject to counterparty default; if the counterparty defaults while the option value is negative (ie you are owed money) then you suffer a loss. If we just assume that defaults happen at rate $\beta > 0$ constant, then the option value $u(t,x)$ should satisfy

$$(*) \qquad \frac{\partial u}{\partial t} + \mathcal{L}u + \beta u^- = 0, \qquad u(T,\cdot) = g(\cdot)$$

How would you solve this?

(ii) Julien + Pierre Henry-Labordère were trying to approximate u^- by some polynomial, and then interpret this as a McKean-Vlasov evolution with branchings; the merit of this is that one could consider a direct simulation of a branching diffusion without need for simulations within simulations. The down side is that it is hard to do any analysis; the polynomial approximation can only be estimated if we know that the solution u is bounded, but to get such bounds we would need to get a grip on expectations of terms of both signs, and this looks hard to me.

(iii) An approach which looked possible was to interpret (*) as a control problem:

$$\sup_{0 \le y \le \beta} \left[\frac{\partial u}{\partial t} + \mathcal{L}u - yu \right] = 0, \qquad u(T,\cdot) = g(\cdot)$$

So what we would have is $u(t,X_t)\exp(-\int_0^t y_s\,ds)$ is a supermartingale, and a martingale under optimal control y. The value at time 0 would be

$$u(0,X_0) = \sup_{0 \le y \le \beta} E\left[\exp(-\int_0^T y_s\,ds)\, g(X_T) \,\middle|\, X_0 \right]$$

which we could bound by duality:

$$u(0,X_0) \le \inf_M E\left[\sup_y \left\{ \exp(-\int_0^T y_s\,ds)\, g(X_T) - \int_0^T y_s\,dM_s \right\} \,\middle|\, X_0 \right]$$

As usual, the question is to find a good M. I thought it might be a good idea to use the Martingale which comes from the default-free option.

(I,X,S) equation counting (2/10/13)

(i) We've seen that if we were given the joint law of (I,X,S) we could deduce $\pi(a,b)$, $\pi(b,a)$ for all $a \leq 0 \leq b$, but if we had deduced those, could we rebuild the martingale? The problem in effect is to decide how we step forward one; we are at $S=X=b$, $I=a$, τ has not yet happened; what are the prob's that at the next step we are at $b+1$, $a-1$, stopped in $[a,b]$? Reconsidering the notation of the (I,X,S,σ) analysis, we have relations

(1) $$\pi(a,b) = \pi(a,b,b+1) + P(I=a, S=b, \sigma=+) + \pi(a,b,a-1)$$

(2) $$\pi(b,a) = \pi(b,a,a-1) + P(I=a, S=b, \sigma=-) + \pi(b,a,b+1)$$

(3) $$b\,\pi(a,b) = (b+1)\pi(a,b,b+1) + E[X : I=a, S=b, \sigma=+] + (a-1)\pi(a,b,a-1)$$

(4) $$a\,\pi(b,a) = (a-1)\pi(b,a,a-1) + E[X : I=a, S=b, \sigma=-] + (b+1)\pi(b,a,b+1)$$

with the further conditions

(5) $$\pi(a,b+1) = \pi(a,b,b+1) + \pi(b,a,b+1)$$

(6) $$\pi(b,a-1) = \pi(b,a,a-1) + \pi(a,b,a-1)$$

(7) $$P(I=a, S=b, \sigma=+) + P(I=a, S=b, \sigma=-) = P(I=a, S=b)$$

(8) $$E[X : I=a, S=b, \sigma=+] + E[X : I=a, S=b, \sigma=-] = E[X : I=a, S=b]$$

This gives a system of eight linear equations in the eight unknowns $\pi(a,b,b+1)$, $\pi(a,b,a-1)$, $\pi(b,a,a-1)$, $\pi(b,a,b+1)$, $P(I=a, S=b, \sigma=\pm)$ and $E[X : I=a, S=b, \sigma=\pm]$. The linear system is singular; <u>the kernel has dimension 2</u>.

(ii) As with (I,X,S,σ) we may now propose to insert a barrier at

$$\theta = E[X \mid I=a, S=b]$$

where the probability that the barrier is present is q_+ on an excursion down, q_- on an excursion up. This then gives

(9) $$\pi(a,b,b+1) = \pi(a,b)\left[(1-q_+)\frac{b-a+1}{b-a+2} + q_+\frac{b-\theta}{b-\theta+1}\right]$$

(10) $$\pi(a,b,a-1) = \pi(a,b)(1-q_+)/(b-a+2), \qquad \pi(b,a,b+1) = \pi(b,a)(1-q_-)/(b-a+2),$$

(11) $$\pi(b,a,a-1) = \pi(b,a)\left[(1-q_-)\frac{b-a+1}{b-a+2} + q_-\frac{\theta-a}{\theta-a+1}\right]$$

so that the four previously unknown probabilities have become just two: does this render the system non singular?

In fact, the answer is "No"; numerics show there is still a one-dimensional kernel. But this is still not the whole story, because if we do this construction we would have

$$(12)\begin{cases} P(I=a, S=b, \sigma=+) = \pi_{ab}\, q_+/(b+1-\theta) \\ P(I=a, S=b, \sigma=-) = \pi_{ba}\, q_-/(\theta-a+1) \end{cases}$$

and $E[X : I=a, S=b, \sigma=\pm] = \theta\, P(I=a, S=b, \sigma=\pm)$. There are now just two free variables, $q_\pm$, and eight equations to satisfy. It turns out from this construction that equations (1)–(4) will always hold, so the issue revolves around the last four equations.

Thus if M denotes the 8×8 matrix of the original equations, A denotes the 8×2 matrix which expresses the 8 unknowns in terms of $q_\pm$, and if $B = MA$, then we find that the first four rows of B are all zero, and the last four rows, a 4×2 matrix, has rank 1.

We may simplify (9), (11):

$$(13)\begin{cases} \pi(a,b,b+1) = \pi(a,b)\left[\dfrac{b-a+1}{b-a+2} - q_+\dfrac{\theta-a+1}{(b-a+2)(b-\theta+1)}\right] \\[2ex] \pi(b,a,a-1) = \pi(b,a)\left[\dfrac{b-a+1}{b-a+2} - q_-\dfrac{b+1-\theta}{(b-a+2)(\theta-a+1)}\right] \end{cases}$$

Suppose we have been able to choose $q_\pm$ so that when we calculate $P(I=a, S=b, \sigma=\pm)$ using (12) we find that equation (7) holds. Then from (13) we find (invoking (7) from the bottom of p9) that (5)–(6) will automatically hold. Moreover, from our construction, $E(X : I=a, S=b, \sigma=\pm 1) = \theta P(I=a, S=b, \sigma=\pm 1)$ so if we have found that (7) holds we automatically get (8) as well.

So the conclusion is: provided we can pick $q_\pm$ so that (7) holds, where $P(I=a, S=b, \sigma=\pm)$ are given in terms of $q_\pm$ by (12), then all of the eight equations will hold.

This can be done if and only if

$$\frac{\pi_{ab}}{b+1-\theta} + \frac{\pi_{ba}}{\theta-a+1} \geqslant P(I=a, S=b)$$

$$\Leftrightarrow \quad \pi_{ab}\frac{\theta-a+1}{b+1-\theta} + \pi_{ba} \geqslant (\theta-a+1)P(I=a,S=b) = (b-a+2)\xi_{a,b+1}$$

$$= (b-a+1)\pi_{ab} + \pi_{ba} - (b-a+2)\pi(a,b+1)$$

(14) $$\Leftrightarrow \quad \boxed{\pi_{ab}\frac{b-\theta}{b+1-\theta} \leqslant \pi(a,b+1)}$$

By looking at (3) $-\theta\times(1)$ we obtain

$$(b-\theta)\pi_{ab} - (b-\theta+1)\pi(a,b,b+1) = E\left[X-\theta : I=a, S=b, \sigma=+\right] + (a-1-\theta)\pi(a,b,a-1)$$

so

(15) $$(b-\theta)\pi_{ab} - (b-\theta+1)\pi(a,b+1) = E\left[X-\theta : I=a, S=b, \sigma=+\right] + (a-1-\theta)\pi(a,b,a-1) - (b-\theta+1)\pi(b,a,b+1)$$

Similarly, (4) $-\theta\times(2)$ gives us

$$(a-\theta)\pi(b,a) = (b+1-\theta)\pi(b,a,b+1) + E\left[X-\theta : I=a, S=b, \sigma=-\right] + (a-1-\theta)\pi(b,a,a-1)$$

(16) $$= (b+1-\theta)\pi(b,a,b+1) - E\left[X-\theta : I=a, S=b, \sigma=+\right] + (a-1-\theta)\pi(b,a,a-1)$$

Combining (15) and (16) tells us that

$$\boxed{(b-\theta)\pi_{ab} - (b-\theta+1)\pi(a,b+1) = (\theta-a)\pi(b,a) - (\theta-a+1)\pi(b,a-1)}$$

and the condition we require is that this expression should be $\leqslant 0$.

Is this a necessary condition, or does it need to be imposed?

Approaches to the two-sided exit problem (12/10/13)

(i) If we fix some interval $[a,b]$, $a<0<b$, and let $\tau = \inf\{t : X_t \notin [a,b]\}$ the aim is to find

$$\pi(dx) = E\left[e^{-c\tau} : X_\tau \in dx\right]$$

the first exit distribution from the interval, where X is a Lévy process, which we could for simplicity suppose is a compound Poisson process with phase-type jump distribution. If we set

$$m(dx) = \int_0^\infty e^{-ct} P(X_t \in dx, \tau > t)\, dt$$

and $r_c(dx) = \int_0^\infty e^{-ct} P(X_t \in dx)\, dt$ then it is clear that

$$\boxed{r_c = m + \pi * r_c}$$

Taking Fourier transforms we get $\hat{r}_c(i\theta) = \hat{m}(i\theta) + \hat{r}_c(i\theta)\hat{\pi}(i\theta)$, so this gives us

$$(1) \qquad \boxed{\frac{1-\hat{\pi}(i\theta)}{c-\psi(i\theta)} = \hat{m}(i\theta)}$$

and $\hat{m}(z) \equiv \int e^{zx} m(dx)$ is an entire function of finite type. We also have that

$$m * \mu = \pi \qquad \text{outside } [a,b]$$

so that

$$m * \mu = \pi + \nu$$

for some non-negative measure ν on $[a,b]$. If the total intensity of all jumps is $\lambda = \mu(\mathbb{R})$, then $\psi(i\theta) = \hat{\mu}(i\theta) - \lambda$, and we have $\hat{\pi} = \hat{m}\hat{\mu} - \hat{\nu}$. We could alternative work with ν rather than π; if we do so, we have

$$1-\hat{\pi} = 1 - \hat{m}\hat{\mu} + \hat{\nu} = \hat{m}(c+\lambda-\hat{\mu})$$

$$\Rightarrow \quad 1 + \hat{\nu} = \hat{m}\,(c+\lambda)$$

or again,

$$\boxed{\hat{m} = \frac{1}{c+\lambda} + \frac{\hat{\nu}}{c+\lambda}}$$

which we understand by noticing that for the compound Poisson story, the measure m has an atom $(c+\lambda)^{-1}$ at zero corresponding to no jump before the exponential time. If there is a jump, you go to a multiple of ν in effect.

$$b_{jk} = \int_{-\infty}^{\infty} \frac{\hat{\mu}(t)' \left(1 - \cos tL\right)}{\pi' \left(t - \frac{2\pi k}{L}\right)\left(t - \frac{2\pi j}{L}\right)} \, dt$$

(ii) We could alternatively think in terms of compensating

$$A_t = I_{\{t \geq \tau\}} \; h(X_\tau) e^{-c\tau}$$

for some nice test function h. This would be compensated by

$$\int_0^{t \wedge \tau} e^{-cs} \tilde{h}(X_s) ds$$

where

$$\tilde{h}(x) = \int_{[a,b]^c} h(y) \mu(dy - x)$$

So if $f(x) = E^x[h(X_\tau) e^{-c\tau}]$, we get $e^{-ct} f(X_t)$ is a martingale if stopped at τ, so

$$-cf(x) + \int \mu(dy)\{f(x+y) - f(x)\} = 0 \qquad \forall x \in [a,b]$$

or again

$$\boxed{(c+\lambda) f(x) = \int_{[a,b]} f(z) \mu(dz - x) + \tilde{h}(x)} \qquad (x \in [a,b])$$

In this case, we have $(c+\lambda) f(x) = (Kf)(x) + \tilde{h}(x)$ for some integral kernel K, and we could express f as the limit of some recursive scheme $f^{(n+1)} = (Kf^{(n)} + \tilde{h})/(c+\lambda)$.

Alternatively, we could think of the interval as $[0, L]$, and write f as a Fourier series

$$f(x) = \sum_{k \in \mathbb{Z}} a_k \exp\left(\frac{2\pi i k}{L} x\right)$$

Then

$$(c+\lambda) a_k L = \int_0^L \exp\left(-\frac{2\pi i k}{L} x\right)\left(\int_{[0,L]} f(z) \mu(dz - x) + \tilde{h}(x)\right) dx$$

$$= \sum_{j \in \mathbb{Z}} a_j \int_0^L e^{-2\pi i k x/L} \int_0^L e^{2\pi i j z/L} \mu(dz - x) dx$$

$$+ \int_0^L e^{-2\pi i k x/L} \tilde{h}(x) dx$$

If we set

$$b_{kj} \equiv \int_0^L e^{-2\pi i k x/L} \int_0^L e^{2\pi i j z/L} \mu'(z - x) dz dx$$

$$= \int_{-L}^{0} e^{-2\pi i k x/L} \int_{-L}^{0} e^{2\pi i j z/L} \mu'(z-x)\, dz\, dx$$

$$= \int_{0}^{L} e^{2\pi i k x/L} \int_{0}^{L} e^{-2\pi i j z/L} \mu'(x-z)\, dx\, dz$$

$$= b_{jk}$$

so the matrix b is symmetric. Likewise,

$$\bar{b}_{kj} = \int_0^L e^{2\pi i k x/L} \int_0^L e^{-2\pi i j z/L} \mu'(z-x)\, dx\, dz$$

$$= \int_{-L}^{0} e^{-2\pi i k x/L} \int_{-L}^{0} e^{+2\pi i j z/L} \mu'(x-z)\, dx\, dz$$

$$= \int_0^L e^{-2\pi i k x/L} \int_0^L e^{2\pi i j z/L} \mu'(x-z)\, dx\, dz$$

$$= b_{-k,-j}$$

(iii) If we go back to the WH factorization of Markov chains, we expect that

$$\hat{\pi}(s) = e^{bs} w_+ (-s-Q_+)^{-1} q_+ + e^{as} w_- (s-Q_-)^{-1} q_-$$

for some weights $w_\pm$ where we have

$$\psi(s) = -q_0 - u_-(Q_- - s)^{-1} q_- - u_+(Q_+ + s)^{-1} q_+$$

(see p 32 of WN $\overline{\underline{XXXIV}}$). Eigenvalues of the matrix WH factorization are characterized by $c = \psi(s)$. If we inspect (1) again, we see that poles of $c - \psi(i\theta)$ will cancel out with poles in the numerator $1-\hat{\pi}(i\theta)$, and zeros of the denominator will become poles of $\hat{m}(i\theta)$, so the poles of $\hat{m}(i\theta)$ are the roots of $c = \psi(s)$. What we expect is that

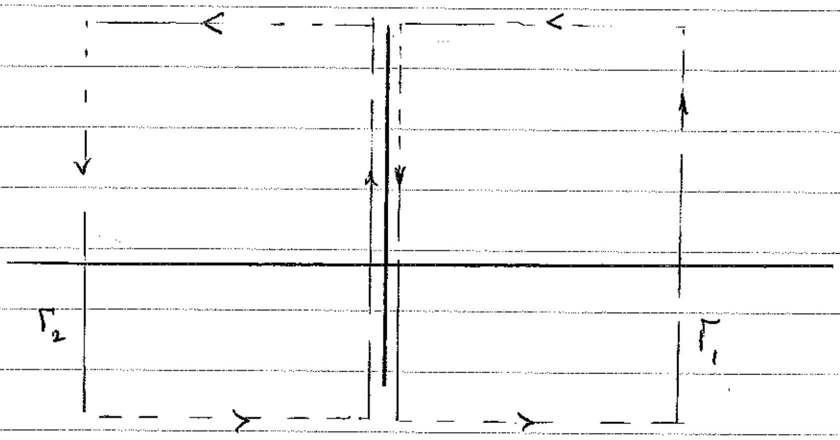
$$m(dx) = \sum \alpha_j^+ e^{-\lambda_j^+ x} dx + \sum \alpha_j^- e^{-\lambda_j^- x} dx + \frac{1}{c+\lambda} \delta_{\{0\}}(dx)$$

(2) so $$\boxed{\hat{m}(s) = \sum \alpha_j^+ \frac{e^{(s-\lambda_j^+)b} - e^{(s-\lambda_j^+)a}}{s-\lambda_j^+} + \sum \alpha_j^- \frac{e^{(s-\lambda_j^-)b} - e^{(s-\lambda_j^-)a}}{s-\lambda_j^-} + \frac{1}{c+\lambda}}$$

Inspecting this, you might be led to guess that for $Re(s) = 0$

$$\hat{m}(s) \overset{?}{=} \oint_{\Gamma_1 + \Gamma_2} \frac{e^{b(s-z)} - e^{a(s-z)}}{s-z} \frac{dz}{2\pi i\,(c - \psi(z))} + \frac{1}{c+\lambda}$$

where



and since $\{e^{b(s-z)} - e^{a(s-z)}\}/(s-z) = \int_a^b e^{t(s-z)}\,dt$ has no singularity at $z = s$, we could just roll this all into

$$\boxed{\hat{m}(s) \overset{?}{=} \lim_{N\to\infty} \oint_{\Gamma_N} \frac{e^{b(s-z)} - e^{a(s-z)}}{s-z} \frac{dz}{2\pi i\,(c-\psi(z))}} + \frac{1}{c+\lambda}$$

where Γ_N is contour where $x = \pm N$, $y = \pm N$.

Of course, this just a <u>guess</u>: we could include any analytic weighting function in the contour integral and the desired form (2) would still appear...

(iv) To pin this down a bit better, let's obtain as much information as we can about the occupation measure m in the case of matrix WH, using the notation of WN $\underline{\underline{XXXIV}}$, pp 31–33. For a test function g, we define

$$h(\varphi, \xi) = E\left[\int_0^\tau \exp(-cA_s)\, g(\varphi_s)\, I_{\{\xi_s = 0\}}\, ds \;\middle|\; \varphi_0 = \varphi,\ \xi_0 = \xi\right]$$

and observe that

$$\exp(-cA_t)\, h(\varphi_t, \xi_t) + \int_0^t \exp(-cA_s)\, g(\varphi_s)\, I_{\{\xi_s = 0\}}\, ds$$

is a martingale up til the first time φ exits $[a, b]$. Hence from Itô we obtain

$$0 = Qh + v\, h' - c\, I_{\{\xi=0\}} h + g\, I_{\{\xi=0\}}$$

So if $h = \begin{pmatrix} h_- \\ h_0 \\ h_+ \end{pmatrix}$ we see that

$$\begin{cases} u_- h_- + u_+ h_+ - (q_0 + c)\, h_0 + g(\varphi) = 0 \\ Q_+ h_+ + q_+ h_0 + h_+' = 0 \\ Q_- h_- + q_- h_0 - h_-' = 0 \end{cases}$$

So $h_0 = (u_- h_- + u_+ h_+ + g) / (q_0 + c)$, and

$$0 = \bar{Q} \begin{pmatrix} h_- \\ h_+ \end{pmatrix} + \begin{pmatrix} -h_-' \\ h_+' \end{pmatrix} + \frac{g(\varphi)}{q_0 + c} \begin{pmatrix} q_- \\ q_+ \end{pmatrix}$$

$$\Rightarrow \boxed{0 = V^{-1} \bar{Q}\, h + h' + \frac{g(\varphi)}{q_0 + c} \begin{pmatrix} -q_- \\ q_+ \end{pmatrix}}$$

by slight abuse of notation, meaning here $h = \begin{pmatrix} h_- \\ h_+ \end{pmatrix}$. So we have

$$0 = \frac{d}{d\varphi} \left(\exp(\varphi V^{-1} \bar{Q})\, h(\varphi) \right) + \exp(\varphi V^{-1} \bar{Q}) \begin{pmatrix} -q_- \\ q_+ \end{pmatrix} \cdot \frac{g(\varphi)}{q_0 + c}$$

$$= \frac{d}{d\varphi} \left(\exp(\varphi V^{-1} \bar{Q})\, h(\varphi) \right) - \exp(\varphi V^{-1} \bar{Q})\, (V^{-1} \bar{Q})\, 1\, \frac{g(\varphi)}{c}$$

if it helps. Hence we have

$$h(\varphi) = \exp(-\varphi V^{-1} \bar{Q}) \left[w + \int_0^{\varphi} \exp(s V^{-1} \bar{Q}) \begin{pmatrix} -q_- \\ q_+ \end{pmatrix} \frac{g(s)}{q_0 + c}\, ds \right]$$

where $h(\varphi, j) = 0$ if $j \in E^-$, $\varphi = a$; or if $j \in E^+$, $\varphi = b$. Abbreviate $V^{-1} \bar{Q} = M$ so we get

$$\boxed{h(\varphi) = \exp(-\varphi M) \left[w - \int_0^{\varphi} \exp(sM)\, M 1\, g(s) \frac{ds}{c} \right]}$$

with

$$\begin{cases} (0\ I)\, e^{-bM} \left(w - \int_0^b e^{sM} M 1\, g(s)\, ds/c \right) = 0 \\ (I\ 0)\, e^{-aM} \left(w + \int_a^0 e^{sM} M 1\, g(s)\, ds/c \right) = 0 \end{cases}$$

Going a bit further, if λ is an eigenvalue of $V^{-1}\bar{Q}$ with negative real part, then

$\exp(-\lambda\varphi_t - cA_t)\, f(\xi_t)$ is martingale ($-\lambda V f + \bar{Q} f = 0$, we see)

so we have the upward eigenfunctions F^+, and

$$F = \begin{pmatrix} F^- & \Pi^+ F^+ \\ \Pi^- F^- & F^+ \end{pmatrix} = S \begin{pmatrix} F^- & \cdot \\ \cdot & F^+ \end{pmatrix}, \quad S = \begin{pmatrix} I & \Pi^+ \\ \Pi^- & I \end{pmatrix}$$

with $G_+ F^+ = F^+ \Lambda^-$, $G_- F^- = -F^- \Lambda^+$ where $\Lambda^\pm$ is the diagonal matrix of eigenvalues of M with positive real part, so

$$\begin{pmatrix} -G_- & \cdot \\ \cdot & G_+ \end{pmatrix} = \begin{pmatrix} F^- & \cdot \\ \cdot & F^+ \end{pmatrix} \begin{pmatrix} \Lambda^+ & \cdot \\ \cdot & \Lambda^- \end{pmatrix} = S^{-1} M S.$$

So $M = S \begin{pmatrix} -G_- & \cdot \\ \cdot & G_+ \end{pmatrix} S^{-1}$ and the boundary conditions become

$$(\Pi^- \; I) \begin{pmatrix} e^{bG_-} & \cdot \\ \cdot & e^{-bG_+} \end{pmatrix} S^{-1} \left\{ w - \int_0^b S \begin{pmatrix} e^{-tG_-} & \cdot \\ \cdot & e^{tG_+} \end{pmatrix} \begin{pmatrix} -G_- & \cdot \\ \cdot & G_+ \end{pmatrix} S^{-1} 1\, g(t) \frac{dt}{c} \right\} = 0$$

$$(I \; \Pi^+) \begin{pmatrix} e^{aG_-} & \cdot \\ \cdot & e^{-aG_+} \end{pmatrix} S^{-1} \left\{ w + \int_a^0 S \begin{pmatrix} e^{-tG_-} & \cdot \\ \cdot & e^{tG_+} \end{pmatrix} \begin{pmatrix} -G_- & \cdot \\ \cdot & G_+ \end{pmatrix} S^{-1} 1\, g(t) \frac{dt}{c} \right\} = 0.$$

Agents interacting according to rules (16/10/13)

(i) Let's summarize and generalize some of the ideas I was working on with Lukas Gonon ($\overline{\text{WN XXXV}}$ p41 + others). What we have generally is that if S_t is the vector of observed prices on day t then agent j has a state vector on day t of the form

$$Z_t^j = \begin{pmatrix} S_t \\ x_t^j \end{pmatrix} = \begin{pmatrix} S_t \\ B_j\, x_{t-1}^j + F_j S_t \end{pmatrix}$$

and according to his model he thinks $Z_{t+1}^j \sim N(A_j Z_t^j, V_j)$, which is how things are going to look in any steady-state linear-Gaussian model.

He enters day t with cash φ_t^j and stock portfolio θ_t^j, and chooses his consumption c_t^j and new portfolio θ_{t+1}^j so that

$$\varphi_t^j + \theta_t^j \cdot S_t = c_t^j + \varphi_{t+1}^j + \theta_{t+1}^j \cdot S_t$$

determines his residual cash φ_{t+1}^j. His choices are made so as to optimize

$$E\Big[U(c_t) + \nu U\big((1+r)\varphi_{t+1} + \theta_{t+1}\cdot S_{t+1} \big) \Big]$$

$$= -e^{-\gamma c} - \nu \exp\Big\{ -\gamma\big((1+r)\varphi_{t+1} + \theta_{t+1}\cdot a_j Z_t^j \big) + \tfrac{1}{2}\gamma^2\, \theta_{t+1}\cdot v_j\, \theta_{t+1} \Big\}$$

where a_j is the first d rows of A_j (d stocks!) and v_j is the first $d\times d$ block of V_j. His optimization therefore is

$$-e^{-\gamma c} - \nu\exp\Big[-\gamma\big((1+r)(\varphi_t + \theta_t\cdot S_t - c_t - \theta_{t+1}\cdot S_t) + \theta_{t+1}\cdot a_j Z_t^j \big) + \tfrac{1}{2}\gamma^2\theta_{t+1}\, v_j\, \theta_{t+1} \Big]$$

leading to

$$\boxed{\theta_{t+1} = \gamma^{-1} v_j^{-1}\big(a_j Z_t^j - (1+r) S_t \big)}$$

The optimization over c is now straightforward. The challenge now is to find market-clearing S_t, r ...

To make the optimization over c completely explicit, we write the objective as a function of c as

$$-e^{-\gamma c} - k\, e^{\gamma(1+r)c}$$

where $k = \nu \exp\left[-\gamma(1+r)(\varphi_t + \theta_t \cdot S_t) + \frac{1}{2}(a_j Z_t^j - (1+r)S_t) v_j^{-1} (a_j Z_t^j - (1+r)S_t) \right]$

so optimizing over c gives

$$(2+r)\gamma c = -\log(k(1+r)).$$

Can we always get equilibrium solutions? Just one? If so, how does it all evolve? Anything like this in the literature?

(ii) Let's suppose that all the γ_j are the same, and that the aggregate supply of cash and of stocks remains constant, & see if it is clearer in that situation. The form of θ_{t+1}^j can be more usefully expressed as

$$\theta_{t+1}^j = \gamma^{-1} v_j^{-1} \left((m_j - r) S_t + n_j x_{t-1}^j \right).$$

so market clearing says

$$\gamma \bar{\theta} = \sum_{j=1}^{J} v_j^{-1} \left((m_j - r) S_t + n_j x_{t-1}^j \right)$$

$$\equiv (M - rQ) S_t + \xi_{t-1} \qquad \text{for short}$$

which gives $\boxed{S_t = (M - rQ)^{-1} (\gamma \bar{\theta} - \xi_{t-1})}$

If we sum over j the budget equation for each agent we learn that $\sum c^j = 0$. The optimality equation for c^j gives us

$$\log(\nu(1+r)) + (2+r)\gamma c^j = -\gamma(1+r)(\varphi_t^j + \theta_t^j \cdot S_t) + \tfrac{1}{2} \gamma^2 \theta_{t+1}^j v_j \theta_{t+1}^j$$

so summing over j leads to

$$\boxed{J \log(\nu(1+r)) = -\gamma(1+r)(\bar{\varphi} + \bar{\theta} \cdot S_t) + \tfrac{1}{2} \sum_{j=1}^{J} \left((m_j - r) S_t + n_j x_{t-1}^j \right) v_j^{-1} \left((m_j - r) S_t + n_j x_{t-1}^j \right)}$$

As $r \to \infty$, we'll have $S_t \to 0$, and $r S_t$ tends to a limit. So for large r, the RHS is like $-\gamma r \bar{\varphi}$ which tends to $-\infty$ as $r \to \infty$. Of course, the RHS may well have a few poles; if there was just one in $(0, \infty)$, we might fail to have a root, but usually it looks like there should be at least one...

A curious question (22/10/13)

Suppose you want to make a network of ~~resistors~~ resistors between nodes $i = 1, \ldots, N$, where you maintain node 1 at potential $V_1 = 1$, and node N at potential $V_N = 0$, with a view to making the heat dissipated along each edge constant: how would you do this? (So it's a question about underfloor heating for a bathroom!)

The shape of the network is given to you, all you can do is to choose the values R_{ij} of resistance along any edge (i,j). We will have the equations

$$V_i - V_j = x_{ij} R_{ij}$$

$$\sum_{j \sim i} x_{ij} = \xi_i \equiv \begin{cases} 1 & i = 1 \\ -1 & i = N \\ 0 & \text{else} \end{cases}$$

where x_{ij} is the current flowing from i to j. The heat dissipated on edge (i,j) is just $x_{ij}^2 R_{ij}$, so we shall insist that for some constant c we get

$$x_{ij}^2 = c/R_{ij}$$

We are implicitly assuming that the current flowing in/out is 1, so total energy dissipated is 1, so if there are K edges we want $c = 1/K$, and therefore

$$x_{ij}^2 R_{ij} = 1/K$$

Any way to solve this but numerically?

An interpretation of finite difference schemes (22/10/13)

Suppose we have a diffusion operator $\mathcal{L}$ and we want to solve the parabolic PDE

$$\dot{u} + \mathcal{L} u = 0 \quad , \quad u(T, \cdot) \text{ given.}$$

Then the classical method is to do a time grid $t_0 = 0 < t_1 < \dots < t_N = T$, and then solve

$$0 = \frac{u^{n+1} - u^n}{h} + \theta \mathcal{L} u^n + (1-\theta) \mathcal{L} u^{n+1}$$

where h is the timestep. If we have $\theta = 0$, then it's explicit:

$$u^n = u^{n+1} + h \mathcal{L} u^{n+1}$$

which we can understand as

$$u^n(x) = u^{n+1}(x) + \int_0^h \mathcal{L} u^{n+1}(x)\, ds$$

$$= E^x\left[u^{n+1}(X_h) \right] + E^x\left[\int_0^h \left\{ \mathcal{L} u^{n+1}(x) - \mathcal{L} u^{n+1}(X_s) \right\} ds \right]$$

$$= E^x\left[u^{n+1}(X_h) \right] + O(h^2)$$

if we knew that $\mathcal{L}(\mathcal{L} u^{n+1})$ were bounded. Likewise, the use of $\theta = 1$ gives us the fully implicit scheme ($\lambda \equiv 1/h$)

$$(\lambda - \mathcal{L}) u^n = \lambda u^{n+1}$$

so

$$u^n(x) = E^x\left[u^{n+1}(X_{T_\lambda}) \right]$$

where $T_\lambda \sim \exp(\lambda)$. Suppose now we take some $\varepsilon \in (0, h)$, set $\alpha = 1/(h - \varepsilon)$ and consider

$$E^x\left[u^{n+1}(X_{\varepsilon + T_\alpha}) \right] = E^x\left[\alpha R_\alpha u^{n+1}(X_\varepsilon) \right]$$

$$= \alpha R_\alpha u^{n+1}(x) + \varepsilon \mathcal{L}(\alpha R_\alpha u^{n+1})(x) + O(\varepsilon^2)$$

so if we set this equal to $u^n(x)$ and ignore the $O(\varepsilon^2)$ term, we shall learn

$$(\alpha - \mathcal{L}) u^n = \alpha u^{n+1} + \varepsilon \alpha \mathcal{L} u^{n+1}$$

which says

$$0 = \alpha(u^{n+1} - u^n) + \varepsilon\alpha \mathcal{L}u^{n+1} + \mathcal{L}u^n$$

$$= \alpha(u^{n+1} - u^n) + \frac{\varepsilon}{h-\varepsilon}\mathcal{L}u^{n+1} + \mathcal{L}u^n$$

$$\boxed{\therefore 0 = \frac{u^{n+1} - u^n}{h} + \frac{\varepsilon}{h}\mathcal{L}u^{n+1} + \left(1 - \frac{\varepsilon}{h}\right)\mathcal{L}u^n}$$

So this provides some sort of probabilistic explanation of what is going on in these finite difference schemes; to within error h^2 the value of $u^n(x)$ is $E^x[u^{n+1}(X_{\varepsilon+T_\alpha})]$.

Finite difference schemes for Markovian evolutions (8/11/13)

(i) Suppose we try to find $V(t,x) = E[\varphi(X_T) \mid X_t = x]$ for some $\varphi \in \mathcal{D}(\mathcal{L})$ where $\mathcal{L}$ is the generator of the Markov process X. Then we know that

$$0 = \frac{\partial V}{\partial t} + \mathcal{L}V$$

and we may attempt to discretize this onto some time grid and solve recursively

$$0 = \frac{V^{n+1} - V^n}{\Delta t} + \theta \mathcal{L} V^n + (1-\theta)\mathcal{L}V^{n+1}$$

where $\theta \in [0,1]$ is fixed. Write $\lambda = 1/\Delta t$ so that this says

$$\lambda V^{n+1} + (1-\theta)\mathcal{L}V^{n+1} = \lambda V^n - \theta \mathcal{L} V^n \equiv \theta(\alpha - \mathcal{L})V^n$$

where $\alpha = \lambda/\theta$. Applying $(\alpha - \mathcal{L})^{-1}$ throughout leads to (V^{n+1} is known, V^n to be found)

$$\theta V^n + (1-\theta)V^{n+1} = \alpha(\alpha - \mathcal{L})^{-1} V^{n+1}.$$

We have that

$$\boxed{V^n = (\alpha - \mathcal{L})^{-1}\left(\alpha + \frac{1-\theta}{\theta}\mathcal{L}\right) V^{n+1}}$$

which we want to compare to $P_{\Delta t} V^{n+1}$. We have

$$(\alpha - \mathcal{L})^{-1}\left(\alpha + \frac{1-\theta}{\theta}\mathcal{L}\right) = \alpha R_\alpha - \frac{1-\theta}{\theta} + \frac{1-\theta}{\theta}\alpha R_\alpha$$

$$= \frac{1}{\theta}\alpha R_\alpha - \frac{1-\theta}{\theta}$$

So $(\alpha - \mathcal{L})^{-1}\left(\alpha + \frac{1-\theta}{\theta}\mathcal{L}\right) f - P_{\Delta t} f$

$$= \frac{1}{\theta}\int_0^\infty \alpha e^{-\alpha s} P_s f \, ds - \frac{1-\theta}{\theta} f - P_{\Delta t} f$$

Now suppose that $\boxed{f \in \mathcal{D}(\mathcal{L}^2)}$ so that we have

$$\| P_t f - f - t\mathcal{L}f \| = \left\| \int_0^t (P_s \mathcal{L} f - \mathcal{L} f)\, ds \right\|$$

$$= \left\| \int_0^t \left(\int_0^s P_u \mathcal{L}^2 f \, du \right) ds \right\|$$

$$\leq \| \mathcal{L}^2 f \| \; \tfrac{t^2}{2}$$

Thus we shall have

$$\left\| (\alpha - \mathcal{L})^{-1} \left(\alpha + \tfrac{1-\theta}{\theta} \mathcal{L} \right) f - P_{\Delta t} f \right\|$$

$$\leq \left\| \tfrac{1}{\theta} \int_0^\infty \alpha e^{-\alpha t} (f + t \mathcal{L} f) \, dt - \tfrac{1-\theta}{\theta} f - (f + \Delta t \, \mathcal{L} f) \right\|$$

$$+ \left\| \int_0^\infty \alpha e^{-\alpha s} \tfrac{s^2}{2} \, ds \right\| . \| \mathcal{L}^2 f \| / \theta$$

$$+ \| \mathcal{L}^2 f \| \; \Delta t^2 / 2$$

$$= \| \mathcal{L}^2 f \| \left\{ \frac{1}{\alpha^2} \cdot \frac{1}{\theta} + \frac{1}{2} (\Delta t)^2 \right\}$$

$$= \| \mathcal{L}^2 f \| \, (\Delta t)^2 \left\{ \theta + \tfrac{1}{2} \right\}$$

Therefore if $M \equiv (\alpha - \mathcal{L})^{-1} \left(\alpha + \frac{1-\theta}{\theta} \mathcal{L} \right)$ is the operator of the FD approximation, and we let $F^n \equiv P_{\Delta t} F^{n+1}$, $F^N = \varphi$, be the true sequence, then

$$\delta_n \equiv \| V^n - F^n \| = \| M V^{n+1} - P_{\Delta t} F^{n+1} \|$$

$$\leq \| (M - P_{\Delta t}) F^{n+1} \| + \| M (V^{n+1} - F^{n+1}) \|$$

$$\leq (\Delta t)^2 (\theta + \tfrac{1}{2}) \| \mathcal{L}^2 F^{n+1} \| + \| M \| . \delta_{n+1}$$

$$\leq (\Delta t)^2 (\theta + \tfrac{1}{2}) \| \mathcal{L}^2 \varphi \| + \| M \| . \delta_{n+1}$$

So we get a bound $\text{Const } \Delta t$ on δ_0, so as we let the time spacing tend to zero we get convergence of V^0 to the true value.

(ii) There is also going to be a discretization going on in the space variable. Let's see what we can say about this.

Suppose we put down a finite sequence $x_1, \dots, x_N$ of points, and now let $(g_k)_{k=1}^N$ be a partition of unity:

$$\boxed{0 \le g_k \le 1, \quad \sum g_k = 1, \quad g_k(x_j) = \delta_{jk}}$$

and we shall suppose $g_k \in \mathcal{D}(\mathcal{L})$ for all k. Then we have the following.

<u>Proposition</u>

$$q_{jk} \equiv (\mathcal{L} g_k)(x_j) \qquad (j,k = 1,\dots,N)$$

defines a Q-matrix.

<u>Proof</u> Clearly $\sum_k q_{jk} = 0$, so what's needed is to prove that $q_{jk} \ge 0$ for $j \ne k$. If this was false, that is, $q_{jk} < 0$ for some $j \ne k$, we would start the Markov process at x_j and run until $\tau = \inf\{t : \mathcal{L} g_k(X_t) \ge 0\}$. Then we consider the martingale

$$M_t \equiv g_k(X_t) - \int_0^t \mathcal{L} g_k(X_s)\,ds$$

stopped at $\tau \wedge n$ and we see that

$$E\, g_k(X_{\tau\wedge n}) = E\left[\int_0^{\tau\wedge n} \mathcal{L} g_k(X_s)\,ds\right]$$

Assuming we have a FD process, and $\mathcal{L} g_k$ is continuous, the LHS must be ≥ 0, the RHS < 0 ※. □

<u>Remark.</u> This looks very exciting, but doesn't necessarily help without careful selection of the g_k. For example, if X was a diffusion, and $g_k = 1$ in a neighbourhood of x_k, then all the q_{kj} will be zero, which doesn't proxy effectively for the process X.

(iii) Here is an observation that might turn out to be useful.

<u>Proposition.</u> Suppose that $f \in \mathcal{D}(\mathcal{L}^2)$. Then

$$\| \lambda R_\lambda f - (2\lambda R_{2\lambda})^2 f \| \le \| \mathcal{L}^2 f \| / 4\lambda^2$$

<u>Proof</u> We write

If we have $\frac{1}{\lambda} + \frac{1}{\mu} = \frac{1}{\alpha}$ then

$$\| \alpha R_\alpha f - \lambda R_\lambda \mu R_\mu f \| \leq \| \mathcal{L}^2 f \| \cdot \frac{1}{\lambda \mu}$$

$$\lambda R_\lambda f - (2\lambda R_{2\lambda})^2 f$$

$$= \int_0^\infty \lambda e^{-\lambda t} P_t f \, dt - \int_0^\infty (2\lambda)^2 t e^{-2\lambda t} P_t f \, dt$$

$$= \int_0^\infty \lambda e^{-\lambda t} \int_0^t P_s \mathcal{L} f \, ds \, dt - \int_0^\infty (2\lambda)^2 t e^{-2\lambda t} \int_0^t P_s \mathcal{L} f \, ds \, dt$$

$$= \int_0^\infty P_s \mathcal{L} f \left\{ e^{-\lambda s} - e^{-2\lambda s}(1 + 2\lambda s) \right\} ds$$

$$= \int_0^\infty (P_s \mathcal{L} f - \mathcal{L} f) \left\{ e^{-\lambda s} - e^{-2\lambda s}(1 + 2\lambda s) \right\} ds$$

$$= \int_0^\infty \left(\int_0^s P_u \mathcal{L}^2 f \, du \right) \left(e^{-\lambda s} - e^{-2\lambda s}(1 + 2\lambda s) \right) ds$$

$$= \int_0^\infty P_u \mathcal{L}^2 f \cdot \frac{1}{\lambda} e^{-2\lambda u} \left(e^{\lambda u} - 1 - \lambda u \right) du$$

whence

$$\| \lambda R_\lambda f - (2\lambda R_{2\lambda})^2 f \| \leq \| \mathcal{L}^2 f \| \cdot \frac{1}{\lambda^2} \cdot \underbrace{\int_0^\infty e^{-2t} (e^t - 1 - t) \, dt}_{1 - \frac{1}{2} - \frac{1}{4} = \frac{1}{4}}.$$

$\square$

If it helps, there is probably a similar bound on

$$\| \lambda R_\lambda f - (a\lambda R_{a\lambda})(b\lambda R_{b\lambda}) f \| \leq \text{const} \, \| \mathcal{L}^2 f \| / \lambda^2$$

if $1/a + 1/b = 1$.

If we do an approximation, we may imagine this as replacing f with

$$\tilde{f} = \sum_{k=1}^N f(x_k) \, g_k$$

and then we would hope to see that $\lambda R_\lambda f$ and $\lambda R_\lambda \tilde{f}$ are close, at least at the points x_k.

How could this go wrong? If f was equal to 1 at all x_k, but fell down to 0 rapidly as you moved away from the x_k, the two things could be very different. Similarly, if one of the g_k was very spiked at x_k, then $\tilde{f}$ only looks like

$f(x_k)$ very close to x_k, otherwise it could be quite different. Put together, these two observations would suggest that we want something like a modulus of continuity for f and for the g_k — but this has to be expressed in terms of the Markovian fundamentals ... or maybe we would want a bound on $\|\mathcal{L}f\|$?

What might work better as an approximation to f would be to take

$$\boxed{\tilde{f} = \sum_k \big(f(x_k) + \varepsilon \mathcal{L} f(x_k) \big)(I - \varepsilon \mathcal{L}) g_k} \qquad (*)$$

where $\varepsilon = 1/\lambda$, $\lambda R_\lambda = (I - \varepsilon \mathcal{L})^{-1}$. In that case

$$\lambda R_\lambda \tilde{f} = (I - \varepsilon \mathcal{L})^{-1} \tilde{f}$$

$$= \sum \{ f(x_k) + \varepsilon \mathcal{L} f(x_k) \} \, g_k$$

so if we look at this at the point x_k we find

$$\lambda R_\lambda \tilde{f}(x_k) = f(x_k) + \varepsilon \mathcal{L} f(x_k)$$

Now

$$\lambda R_\lambda f = \int_0^\infty \lambda e^{-\lambda t} P_t f \, dt$$

$$= \int_0^\infty \lambda e^{-\lambda t} \{ f + \int_0^t P_s \mathcal{L} f \, ds \} \, dt$$

$$= f + \int_0^\infty e^{-\lambda s} P_s \mathcal{L} f \, ds$$

$$= f + \int_0^\infty e^{-\lambda s} \{ \mathcal{L} f + \int_0^s P_u \mathcal{L}^2 f \, du \} \, ds$$

$$= f + \varepsilon \mathcal{L} f + \int_0^\infty \frac{1}{\lambda} e^{-\lambda u} P_u \mathcal{L}^2 f \, du$$

whence

$$\| \lambda R_\lambda f - f - \varepsilon \mathcal{L} f \| = \| \int_0^\infty \frac{1}{\lambda} e^{-\lambda u} P_u \mathcal{L}^2 f \, du \|$$

$$\leq \lambda^{-2} \| \mathcal{L}^2 f \|.$$

So the conclusion from this is that if we make an approximation $\tilde{f}$ to f as at $(*)$

then we shall have

$$\boxed{\left|(\lambda R_\lambda \tilde{f})(x_k) - (\lambda R_\lambda f)(x_k)\right| \leq \frac{1}{\lambda^2} \|\mathcal{L}^2 f\|}$$

So at the grid points we care about, we find that replacing f by $\tilde{f}$ has caused an error of at most $\|\mathcal{L}^2 f\| \varepsilon^2$, which is usably small.

An optimal stopping question (30/12/13)

There was a paper of Thomas Bruss (Ann Prob $\underset{\sim}{28}$, 1384-1391, 2000) where the following question was discussed. Suppose that $X_1, \dots, X_N$ are independent $\{0,1\}$-valued random variables, and let $\sigma = \sup\{k : X_k = 1\}$. Can we determine the stopping time τ which maximizes $P(\tau = \sigma)$?

If we take V_k = value if we have not stopped by time k and $X_k = 1$

U_k = value if we have not stopped by time k and $X_k = 0$

then ($p_k \equiv P(X_k = 1) \equiv 1 - q_k$) we have $V_N = 1$, $U_N = 0$, and for $k < N$

$$\begin{cases} V_k = \max\left\{ \bar{Q}_k,\ p_{k+1} V_{k+1} + q_{k+1} U_{k+1} \right\} \\ U_k = p_{k+1} V_{k+1} + q_{k+1} U_{k+1} \end{cases}$$

where $\bar{Q}_k \equiv \prod_{j=k+1}^{N} q_j$. We deduce that $V_k \geq U_k$, and $U_k \geq U_{k+1}$, together with

$$V_k = \max\left[\bar{Q}_k, U_k \right] \qquad (k < N)$$

Now the sequence $\bar{Q}_k$ is non-decreasing, U_k is non-increasing, so there has to be a crossover, and the optimal rule is to stop at the first $k \geq k^*$ for which $X_k = 1$, for some k^*. We would then have for $k \geq k^* - 1$ that

$$U_k = p_{k+1} \bar{Q}_{k+1} + q_{k+1} U_{k+1}$$

so if we set $x_k \equiv U_k / \bar{Q}_k$ we shall obtain ($r_k \equiv p_k / q_k$)

$$x_k = r_{k+1} + x_{k+1} \qquad (k \geq k^* - 1)$$

$$\Rightarrow \quad x_k = \sum_{j=k+1}^{N} r_j \quad \text{for } (k^* - 1 \leq k < N)$$

Then
$$k^* = \inf\left\{ k : \bar{Q}_k \geq \bar{Q}_k \sum_{j=k+1}^{N} r_j \right\}$$
$$= \inf\left\{ k : 1 \geq \sum_{j=k+1}^{N} r_j \right\}$$

– this is the same as Thomas gets.

Cts time KF story from 34:16

$$dX = AX\,dt + dM \qquad dM\,dM^T = \Sigma_{XX}\,dt$$
$$dY = CX\,dt + dN \qquad dM\,dN^T = \Sigma_{XY}\,dt$$
$$dN\,dN^T = \Sigma_{YY}\,dt$$

If $d\nu = dY - C\hat{X}_t\,dt$, we have

$$d\hat{X} = A\hat{X}\,dt + H\,d\nu$$

where $V_t = \widehat{(XX^T)}_t - \hat{X}_t\hat{X}_t^T$ solves

$$\dot{V} = VA^T + AV + \Sigma_{XX} - (VC^T + \Sigma_{XY})\Sigma_{YY}^{-1}(CV + \Sigma_{YX})$$

and $\qquad H\Sigma_{YY} = VC^T + \Sigma_{XY}.$

[Non-centred modifications in pencil]

Back to the interacting agents story (3/1/14)

(i) Return to the story on p.23, but let's instead cast it as a standard KF problem. So for now let's just see what happens for a single agent, who sees the observation process Y_t, which he believes is driven by hidden process X, where

$$X_t = AX_{t-1} + \varepsilon_t + b \qquad E\,\varepsilon\varepsilon^T = \Sigma_{XX}$$
$$Y_t = CX_t + \eta_t \qquad E\,\varepsilon\eta^T = \Sigma_{XY}$$
$$E\,\eta\eta^T = \Sigma_{YY}.$$

Then if $(X_t | \mathcal{Y}_t) \sim N(\hat{X}_t, V_t)$ the usual story leads us to the recursions

$$\hat{X}_{t+1} = A\hat{X}_t + K(Y_{t+1} - CA\hat{X}_t) + (I - KC)b$$

and $$K = (QC^T + \Sigma_{XY})(CQC^T + C\Sigma_{XY} + \Sigma_{YX}C^T + \Sigma_{YY})^{-1}$$

with $Q \equiv AVA^T + \Sigma_{XX}$, along with the steady state equation for V:

$$V = Q - K(CQ + \Sigma_{YX}).$$

(ii) Now we interpret Y as the price vector S. Agent enters period t with cash φ_t', holdings θ_t of the stock, and receives income δ_t (in cash). Then chooses consumption c_t and next portfolio θ_{t+1} which determines the cash to be left in the bank via the budget equation

(1) $$w_t \equiv \varphi_t' + \theta_t \cdot S_t + \delta_t = c_t + \varphi_{t+1} + \theta_{t+1} \cdot S_t \qquad [\varphi_t' \equiv \varphi_t(1+r_t)]$$

The agent thinks that

$$S_{t+1} = CX_{t+1} + \eta_{t+1} \sim N(CA\hat{X}_t + Cb,\ \Sigma_{YY} + C\Sigma_{XY} + \Sigma_{YX}C^T + CQC^T)$$

$$\equiv N(a\hat{X}_t + \tilde{b},\ v) \qquad \tilde{b} \equiv Cb$$

for simplicity. We propose objective $[U(x) \equiv -\exp(-\gamma x)]$

$$U(c) + \nu\, E\, U((1+r)\varphi_{t+1} + \theta_{t+1} \cdot S_{t+1})$$

$$= -e^{-\gamma c} - \nu \exp\left[-\gamma((1+r)\varphi_{t+1} + \theta_{t+1} \cdot (a\hat{X}_t + \tilde{b})) + \tfrac{1}{2}\gamma^2\, \theta_{t+1} \cdot v\, \theta_{t+1}\right]$$

$$= -e^{-\gamma c} - \nu \exp\left[-\gamma(1+r)(w_t - c_t) - \gamma\theta_{t+1}(a\hat{X}_t + \tilde{b} - (1+r)S_t) + \tfrac{1}{2}\gamma^2\, \theta_{t+1} \cdot v\, \theta_{t+1}\right]$$

When we optimize over θ_{t+1} we get

(2) $$\boxed{\theta_{t+1} = \gamma^{-1} v^{-1} \left(a \hat{X}_t + \tilde{b} - (1+r) S_t\right)}$$

In terms of c, the optimized objective becomes

$$-e^{-\gamma c} - v \exp[\gamma(1+r)c - k]$$

(3) where $\boxed{k \equiv \gamma(1+r)w_t + \tfrac{1}{2}\left(a\hat{X}_t + \tilde{b} - (1+r)S_t\right)\cdot v^{-1}\left(a\hat{X}_t + \tilde{b} - (1+r)S_t\right).}$ The optimization over c happens at

(4) $$\boxed{(2+r)\gamma c = k - \log(v(1+r)).}$$

[k is quadratic in $(1+r)$ ... but S_t also depends on r.]

(iii) The interesting part of the story is that we have that $\hat{X}_t$ depends on S_t, so if we introduce superscripts to denote the agents, we have

(4') $$\hat{X}_t^j = (I - K^j C^j) A^j \hat{X}_{t-1}^j + K^j S_t + (I - K^j C^j) b^j$$

and so

$$\theta_{t+1}^j = (\gamma^j v^j)^{-1} \left\{ (a^j K^j - (1+r)) S_t + a^j (I - K^j C^j) A^j \hat{X}_{t-1}^j + a^j (I - K^j C^j) b^j + \tilde{b}^j \right\}.$$

Using market clearing,

(5) $$\bar{\theta}_{t+1} = \sum_{j=1}^{J} (\gamma^j v^j)^{-1} \left\{ (a^j K^j - (1+r)) S_t + a^j (I - K^j C^j) A^j \hat{X}_{t-1}^j + a^j (I - K^j C^j) b^j + \tilde{b}^j \right\}$$

$$\equiv (M - rL) S_t + \sum_{j=1}^{J} B^j \hat{X}_{t-1}^j + H$$

Clearing of the consumption market gives the equations

(6) $$(2+r)\bar{c}_t \equiv \sum_j (2+r) c_t^j = -\sum_{j=1}^{J} \gamma^j \log(v_j(1+r)) + \sum_{j=1}^{J} k^j/\gamma^j$$

What would aggregate consumption be? If the money supply in period t is m_t, then we'd have

(7) $$(1+r_t) m_{t-1} + \bar{\delta}_t = \bar{c}_t + m_t$$

where r_t is the rate of interest prevailing from time $t-1$ to time t. Or is this the correct story? Makes sense if m_t denotes total household savings (deposits)

A story for fund management (11/2/14)

(i) How would we try to tell a story for a fund manager, who will typically have convex (or at least somewhere convex) incentives?

Suppose the level x_t of the fund will evolve as

$$dx_t = x_t \left[r\,dt + \pi_t (dX_t - r\,dt) \right]$$

where π_t is the chosen control, and $dX_t = \sigma\, dW_t + \mu\, dt$ is the returns process.

The AUM w_t will be affected by how well the fund has been doing, and also possibly by how risky the manager's investment has been. So let's suppose

$$\frac{dw_t}{w_t} = \frac{dx_t}{x_t} + \alpha (x_t - x_0)\, dt - \tfrac{1}{2} \lambda |\sigma^T \pi_t|^2\, dt$$

Then the value function

$$V(t, x, w) = \sup E\left[\varphi(Y_T) \mid x_t = x,\ w_t = w \right]$$

is the manager's objective, where we may simply put

$$Y_T = \delta w_T + \gamma (x_T - x_0)^+ w_T$$

This is of course an over-simplification – the performance element will be a path-dependent random variable, but it approximates the objective.

(ii) Now

$$d \log\left(\tfrac{w}{x}\right) = \alpha (x_t - x_0)\, dt - \tfrac{1}{2} \lambda |\sigma^T \pi_t|^2\, dt$$

so

$$w_t = (w_0 / x_0)\, x_t \exp\left[\int_0^t \left\{ \alpha(x_s - x_0) - \tfrac{1}{2} \lambda |\sigma^T \pi_s|^2 \right\} ds \right]$$

so the final objective can be expressed solely in terms of x:

$$\max E\left[\varphi\left(\frac{w_0}{x_0}\, x_T \exp\left\{ \int_0^T \left(\alpha(x_s - x_0) - \tfrac{1}{2} \lambda |\sigma^T \pi_s|^2 \right) ds \right\} \times \left(\delta + \gamma (x_T - x_0)^+ \right) \right) \right]$$

Anything doing?

A little story for modelling asset returns (28/2/14)

There was a little story I was telling about the returns for some equities that I needed to work through for a talk at the TGM. Stock returns (once scalified) are not independent across assets, and some rough exploratory data analysis suggests that one factor might capture it. So we try a model where the vector Y_t of returns on day t are given by

$$Y_t = z_t C + \varepsilon_t$$

where $z_t = \alpha z_{t-1} + \eta_t$. Suppose covariance of ε is $\Sigma_\varepsilon = \sigma_\varepsilon^2$. Vector C is fixed and known.

If $(z_t | y_t) \sim N(m_t, v_t)$, then

$$\begin{pmatrix} z_{t+1} \\ Y_{t+1} \end{pmatrix} \Big| y_t \sim N\left(\begin{pmatrix} \alpha m_t \\ \alpha m_t C \end{pmatrix}, \begin{pmatrix} q & q C^T \\ q C & q C C^T + \Sigma_\varepsilon \end{pmatrix} \right)$$

where $q = \alpha^2 v_t + \Sigma_\eta$. We need $(q CC^T + \Sigma_\varepsilon)^{-1} = \sigma_\varepsilon^{-1}(I + bb^T)\sigma_\varepsilon^{-1}$, where $b = \sqrt{q}\, \sigma_\varepsilon^{-1} C$. Thus

$$(q CC^T + \Sigma_\varepsilon)^{-1} = \sigma_\varepsilon^{-1}\left(I - \frac{bb^T}{1+|b|^2}\right)\sigma_\varepsilon^{-1}$$

and the Kalman gain matrix is

$$K = q C^T (q CC^T + \Sigma_\varepsilon)^{-1}$$

$$= \sqrt{q}\, b^T \left(I - \frac{bb^T}{1+|b|^2}\right)\sigma_\varepsilon^{-1}$$

$$= \frac{\sqrt{q}}{1+|b|^2} b^T \sigma_\varepsilon^{-1}$$

The new variance is $v_{t+1} = q - Kq C = q - \frac{|b|^2 q}{1+|b|^2}$

$$= \frac{q}{1+|b|^2}$$

$$= \frac{q}{1 + q\,|\sigma_\varepsilon^{-1} C|^2}$$

Doesn't seem to work very well though...

Possibly interesting class of distributions (13/3/14)

Suppose we consider a distribution over $\mathbb{R}^n$ with density

$$f(x) \propto \exp\left(-\tfrac{1}{2}(x-\mu)\cdot V^{-1}(x-\mu)\right)\left\{a + (x-b)\cdot M(x-b)\right\} \Big/ (2\pi)^{n/2}\sqrt{\det V}$$

for PDS matrices V, M, $a>0$, $\mu, b \in \mathbb{R}^n$. What's the MGF of this? We need to calculate

$$\int e^{z\cdot x - \frac{1}{2}(x-\mu)\cdot V^{-1}(x-\mu)}\left(a + (x-b)\cdot M(x-b)\right)\frac{dx}{(2\pi)^{n/2}\sqrt{\det V}}$$

$$= e^{\mu z + \frac{1}{2} z V z}\int \exp\left[-\tfrac{1}{2}(x-\mu-Vz)V^{-1}(x-\mu-Vz)\right]\left(a + (x-b)\cdot M(x-b)\right)\frac{dx}{(2\pi)^{n/2}\sqrt{\det V}}$$

$$y = V^{-\frac{1}{2}}(x-\mu-Vz)$$

$$= e^{\mu z + \frac{1}{2} z V z}\int e^{-|y|^2/2}\left(a + (V^{\frac{1}{2}}y+\mu+Vz-b)\cdot M(V^{\frac{1}{2}}y+\mu+Vz-b)\right)\frac{dy}{(\sqrt{2\pi})^n}$$

$$= e^{\mu z + \frac{1}{2} z V z}\int e^{-|y|^2/2}\left\{a + (y-c)\cdot \tilde{M}(y-c)\right\}\frac{dy}{(2\pi)^{n/2}}$$

$$\tilde{M} \equiv V^{\frac{1}{2}} M V^{\frac{1}{2}}, \quad c \equiv V^{-\frac{1}{2}}(b-\mu-Vz)$$

$$= e^{\mu z + \frac{1}{2} z V z}\left\{a + c\cdot\tilde{M}c + \operatorname{tr}\tilde{M}\right\}$$

Thus we can find the correct normalization by setting $z=0$; the MGF is

$$\boxed{\varphi(z) = \frac{e^{\mu z + \frac{1}{2} z V z}\left\{a + \operatorname{tr}(MV) + (b-\mu-Vz)\cdot M(b-\mu-Vz)\right\}}{a + \operatorname{tr}(MV) + (b-\mu)\cdot M(b-\mu)}}$$

If we add an independent MVN to such a RV, it remains in the class, and this might be useful; such a density could be the density of a 'big move'...

This example is treated in the book of Gittins, Glazebrook + Webber

Suppose we have a BM $dX_t = \sigma dW_t + \mu_t dt$, where μ is an OU process.
When we filter $\hat{\mu}$ is again an OU process so we'd have the problem

$$V(\hat{\mu}) = \sup_T E\left[\int_0^T \rho e^{-\rho t} (\hat{\mu}_t - \lambda)\, dt \;\middle|\; \hat{\mu}_0 = \hat{\mu}\right]$$

and to solve this we'd have $V \geq 0$, and

$$-\rho V + \tfrac{1}{2}\sigma^2 V'' - \beta x V' + \rho(x - \lambda) \leq 0$$

For very large x, we expect

$$V(x) \simeq \int_0^\infty \rho e^{-\rho t} (x e^{-\beta t} - \lambda)\, dt = \frac{\rho x}{\rho + \beta} - \lambda$$

so should be quick to solve as an optimal stopping problem. The only issue is that we would have to do a search on λ for a given $\hat{\mu}_0$ but it is probably quite fast.

A nice bandit example (20/3/14)

(i) Suppose we have independent processes

$$dX^i_t = dW^i_t + a^i dt$$

where the a^i are constant but unknown with a Normal distribution. At any time, we can choose which process to operate, and our overall reward will be

$$\int_0^\infty \rho e^{-\rho s} a^{i(s)} ds$$

where $i(s)$ is the index used at time s.

(ii) When we do Bayesian learning about a (for now drop the index) we shall have posterior mean $\hat{\alpha}_t$ and posterior precision τ_t, where

$$d\hat{\alpha}_t = \frac{d\hat{W}_t}{\tau_t}, \qquad d\tau_t = dt$$

(iii) The Gittins index for process i in state (α, τ) is

$$\boxed{G_i(\alpha,\tau) = \sup_T \frac{E\left[\int_0^T \rho e^{-\rho s} \hat{\alpha}_s \, ds \,\middle|\, \hat{\alpha}_0 = \alpha, \tau_0 = \tau\right]}{E\left[1 - e^{-\rho T} \,\middle|\, \hat{\alpha}_0 = \alpha, \tau_0 = \tau\right]}}$$

and the famous theorem says that at any time you should operate the process whose Gittins index is currently highest.

(iv) To try to work out the Gittins index, it's enough to find

$$\sup_T E\left[\int_0^T \rho e^{-\rho s}(\hat{\alpha}_s - \lambda) ds \,\middle|\, \hat{\alpha}_0 = \alpha, \tau_0 = \tau\right] \equiv V(\alpha,\tau)$$

for any λ. We know that $V \geq 0$, and that while $V > 0$ it has to satisfy

$$\frac{1}{2\tau^2} V_{\alpha\alpha} - \rho V + V_\tau + \rho(\alpha - \lambda) = 0$$

If we can solve the problem for $\lambda = 0$, with solution $V^0(\alpha,\tau)$, then for general λ the solution will be $V^0(\alpha - \lambda, \tau)$; so we may suppose $\lambda = 0$. Further, if we write $\tilde{V} \equiv V(\alpha,\tau) - \alpha$, then the equations

$$\boxed{\frac{1}{2\tau^2}\tilde{V}_{\alpha\alpha} - \rho\tilde{V} + \tilde{V}_\tau = 0, \qquad \tilde{V} \geq -\alpha}$$

Also, as $\tau \to \infty$, we gain certainty about a, so $V(\alpha, \infty) = \alpha^+$. This gives us a fairly straightforward parabolic PDE to solve (numerically)

[Irritatingly, D+V only consider the big case $R=1$].

Davis-Vellekoop example (28/3/14)

(i) Harry Zheng drew my attention to this example, which he & Michael Monoyios are working on. The original version of D+V tried to prove the form of the solution by looking at the dual problem with a restricted form for the candidate state-price densities, but they found (not surprisingly) a duality gap, so at some level there remains work to be done. The model is an almost trivial variation of the standard Merton problem: the agent's wealth dynamics are

$$dw_t = rw_t\,dt + \theta_t(\sigma\,dW_t + (\mu - r)dt) + (a\,I_{\{t \le \tau\}} - c_t)\,dt$$

where $a > 0$ is known and fixed, $\tau \sim \exp(\eta)$ independent of the rest. The objective is the usual one

$$\sup E\left[\int_0^\infty e^{-\rho t}\,U(c_t)\,dt \,\Big|\, w_0 = w\right] = V(w).$$

(ii) If we write $v(w) = \gamma_M^{-R}\,U(w)$, where U is CRRA for $R > 0$, $R \neq 1$, then the HJB is

$$0 = \sup\left[U(c) - \rho V + (rw + \theta(\mu - r) + a - c)V' + \tfrac{1}{2}\sigma^2\theta^2 V'' + \eta(v - V)\right]$$

because the function v is just the Merton value for the standard problem $a = 0$. The nice interpretation is that you earn income at rate $a > 0$ until you get thrown out of your job at time τ.

As usual, we get optimality conditions

$$U'(c) = V', \qquad \theta = -\frac{(\mu - r)}{\sigma^2}\,\frac{V'}{V''},$$

and then

$$\boxed{0 = \tilde{U}(V') - \rho V + rwV' + aV' - \tfrac{1}{2}\kappa^2\, V'^2/V'' + \eta(v - V)}$$

Now if we use dual variables $z = V'$, $J = V - wz$ we find the dual equation

$$\boxed{0 = \tilde{U}(z) + \eta\,v(-J') - (\rho + \eta)J + (\rho - r + \eta)zJ' + \tfrac{1}{2}z^2\kappa^2 J'' + az}$$

The nonlinear term $v(-J')$ makes this difficult.

(iii) Boundary behaviour. The primal problem can be dealt with numerically using policy improvement, but we do have to specify the boundary behaviour and this could be fiddly. Alternatively, we might try to solve the dual equation, just treating it as a non-linear ODE, but determining initial conditions is likely to be an issue.

There are various rather obvious comparisons. If we were given $a\tau$, the total of our earnings, at time 0, then we would do better, so

$$V(w) \le \int_0^\infty \eta e^{-\eta t} v(w+at)\, dt.$$

Clearly we also have $V(w) \ge v(w)$. Another obvious statement is that if we just collected all of our earnings in a tin, and then added them to our active wealth at time τ, trading with active wealth at all times, then we will get

$$\int_0^\infty \eta e^{-\eta t} \; E\left[\int_0^t e^{-\rho s} U(c_s)\,ds + e^{-\rho t} v(w_t + at)\right] dt$$

$$= \int_0^\infty \eta e^{-\eta t} \Big\{ v(w_0) + E\left[e^{-\rho t}\big(v(w_t+at) - v(w_t)\big)\right] dt$$

$$= v(w) + \int_0^\infty \eta\, e^{-(\eta+\rho)t}\; E\big(v(w_t+at) - v(w_t)\big)\, dt$$

For large w, we have

$$1 \le \frac{V(w)}{v(w)} \le \int_0^\infty \eta e^{-\eta t} \frac{v(w+at)}{v(w)}\, dt$$

$$= \int_0^\infty \eta e^{-\eta t} \left(1 + \frac{at}{w}\right)^{1-R} dt \;\to 1 \text{ by}$$

monotone cgce.

So this suggests that $V'(w)w + (R-1)V(w) \approx 0$ for very large w.

Small w has an upper bound

$$\frac{\gamma_M^{-R}}{1-R} \int_0^\infty \eta e^{-\eta t} (w+at)^{1-R}\, dt \qquad t = ws/a$$

$$= \frac{\gamma_M^{-R}}{1-R} \int_0^\infty \eta\, e^{-\eta ws/a}\, w^{1-R} (1+s)^{1-R}\, w\frac{ds}{a}$$

$$\sim \frac{\gamma_M^{-R}}{1-R}\; \frac{w^{2-R}}{a} \int_0^\infty \eta\, (1+s)^{1-R}\, ds \qquad \text{if } \boxed{R>2}$$

$$= -\eta \gamma_M^{-R}\; w^{2-R} / a(1-R)(2-R)$$

On the other hand, if $R<2$, we do the asymptotics as

$$\frac{\gamma_M^{-R}}{1-R}\int_0^\infty \eta e^{-\eta t}(w+at)^{1-R}\,dt \to \frac{\gamma_M^{-R}}{1-R}\int_0^\infty \eta e^{-\eta t}(at)^{1-R}\,dt$$

which is finite.

As for a lower bound, we could consume our income at rate pa for some $p\in(0,1)$ up until time τ, putting the residue into cash, and we'd achieve at least

$$\int_0^\infty \eta e^{-\eta t}\left[\int_0^t U(pa)e^{-\rho s}ds + e^{-\rho t}v(w+qat)\right]dt$$

$$= \frac{U(pa)}{\eta+\rho} + \int_0^\infty \eta e^{-(\eta+\rho)t}v(w+qat)\,dt$$

As before, if $R>2$ the second integral $\sim w^{2-R}$ and therefore dominates. If not, we have for $R<2$ there will be a finite limit. This leads one to suspect that

$$V(w)\sim w^{2-R} \qquad (w\downarrow 0)$$

for $R>2$, and perhaps $V(w)\sim \text{const}+\text{const}\,w^{2-R}$ for $R<2$?

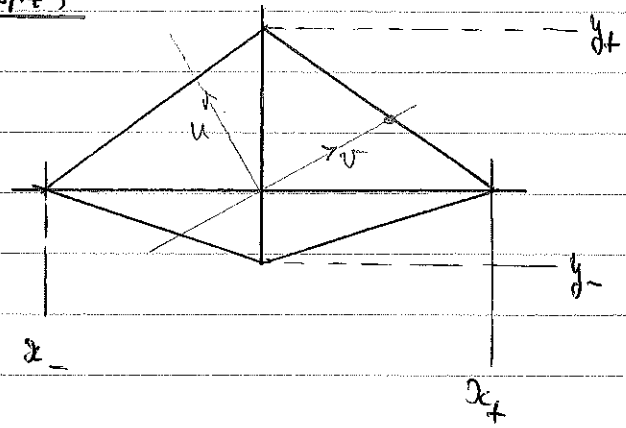
If we think then that $V(w)\sim B\frac{w^{1-R}}{(1-R)}$ for large w, by substituting back into the HJB we get an equation for B

$$0 \doteq \tilde{U}(Bw^{-R}) - \rho B\,U(w) + rBw^{1-R} + \frac{\kappa^2}{2R}B + \eta\left(\frac{\gamma_M^{-R}}{1-R}-\frac{B}{1-R}\right)w^{1-R}$$

$$\therefore 0 \doteq -\frac{B^{1-1/R}}{1-1/R} - \frac{\rho B}{1-R} + rB + \frac{\kappa^2 B}{2R} + \eta\frac{\gamma_M^{-R}-R}{1-R}$$

Should help identify B. But there's still a fair bit of effort needed on the BCs at $w=\infty$ —

Discretizing a diffusion in 2 dimensions (29/3/14)



Suppose we have four grid points on a rectangular grid, neighbours of $(0,0)$.

We are given a covariance matrix $a = a(0,0)$ and a drift, and we want to make a Markov chain discretization.

The idea here is to calculate the eigenvectors of a; and then use the fact that the diffusion is made up of independent components in the two eigendirections.

Suppose that $v = [v_1, v_2]$ is the e-vector pointing into the first orthant. We want to find $p \in [0,1]$ and $t > 0$ such that

$$t\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} p x_+ \\ q y_+ \end{pmatrix} \qquad q = 1-p$$

which we get if

$$\frac{v_1}{v_2} = \frac{p x_+}{(1-p) y_+} \quad \text{i.e.} \quad \boxed{p = \frac{v_1 y_+}{v_1 y_+ + v_2 x_+}}$$

and then

$$\boxed{t = \frac{x_+ y_+}{v_1 y_+ + v_2 x_+}}$$

For the opposite (negative) orthant, we just use x_-, y_- in place of x_+, y_+ in those same expressions, and for the other orthants we have the other e-vector

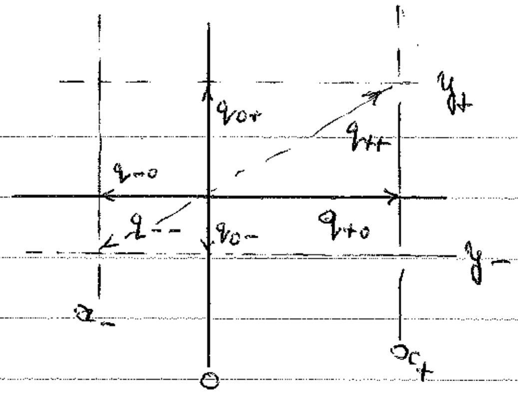
$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -v_2 \\ v_1 \end{pmatrix}$$

and by similar calculations, $tu = \begin{pmatrix} p x_- \\ q y_+ \end{pmatrix}$ if

$$\boxed{p = \frac{u_1 y_+}{u_1 y_+ + u_2 x_-}, \quad t = \frac{x_- y_+}{u_1 y_+ + u_2 x_-}}$$

In practice, for nonzero correlation this seems not to work ... – it's perfect for the zero-correlation case though.

Tried various things: jumps to corners, non-positive intensities, and nothing works when there's non-zero correlation ... maybe need FEM?



One more thing to try:

Suppose drift is zero, and covariance is

$$a = \begin{bmatrix} a_{xx} & a_{xy} \\ a_{yx} & a_{yy} \end{bmatrix}$$

with $a_{xy} \geq 0$. Can we make jumps with non-negative intensities as shown so as to match the effect of the generator on x^2, y^2, xy?

Set $q_{x+} = q_{+0} + q_{++}$, $q_{x-} = q_{-0} + q_{--}$

If we want

$$a_{xx} = q_{x+} x_+^2 + q_{x-} x_-^2$$
$$a_{yy} = q_{y+} y_+^2 + q_{y-} y_-^2$$
$$a_{xy} = q_{++} x_+ y_+ + q_{--} x_- y_-$$

along with the positivity condition $q_{++} \leq q_{x+} \wedge q_{y+}$, $q_{--} \leq q_{x-} \wedge q_{y-}$ can this be done?

In general, no. If $x_\pm = \pm 1$, $y_\pm = \pm 1$, the equations we have are

$$\begin{cases} a_{xx} = q_{x+} + q_{x-} \\ a_{yy} = q_{y+} + q_{y-} \\ a_{xy} = q_{++} + q_{--} \end{cases}$$

and we would want to be able to get $a_{xy}^2 = a_{xx} a_{yy}$ with non-negative jump rates; this would be the condition

$$q_{++}^2 + 2 q_{++} q_{--} + q_{--}^2 = q_{x+} q_{y+} + q_{x+} q_{y-} + q_{x-} q_{y+} + q_{x-} q_{y-}$$

If we had $q_{++} = q_{x+} < q_{y+}$, $q_{--} = q_{y-} < q_{x-}$, we shall see that LHS < RHS.

Seems to be completely impossible... what did Kushner + Dupuis find?

If $a_{ii} \geq \sum_{j \neq i} |a_{ij}|$ for each i, you can do something

Connecting the interacting agents to a bit more of an economy (4/4/14)

Return to the story + notation of pp 35-36. We enter period t with each agent holding his own pf of stocks and cash balance at the bank. If we are told the rate r_{t+1} of interest that will apply to cash balances from period t to $t+1$, we can deduce

- all stock prices S_t from (5)
- updated values of $\hat{x}_t^j$ from (4')
- values of c_t^j from (3), (4), and hence individual choices φ_{t+1}^j of cash balances to hold into period $t+1$
- from that, the aggregate quantity m_t of cash held by all agents on deposit.

So what? Let's try to tell a story where there is a quantity Q_t of productive capital on day t, and that $\bar{\delta}_t = \varepsilon Q_t$ (it's not particularly important to know how this gets divided among the agents). We'll suppose that Q_t changes as

$$Q_t = \beta Q_{t-1} + \lambda I_{t-1}$$

where I_t is the level of investment capital at time t, where

$$I_t = \nu I_{t-1} + \alpha (m_t - m_{t-1})$$

The idea is that more cash in the bank (m_t) results in increased loans to industry, with a multiplier α, and this then gets used in investment, but this is not available immediately for productive activity.

In this story, the interest rate is a control.

HJB:

$$0 = -\rho f + R f^{1-1/R} + r(1-R) f - (\lambda - \varepsilon)\mu f'(\mu)$$

$$+ \tfrac{1}{2}\varepsilon^2\sigma^2 f''(\mu) + \frac{(1-R)\left((\mu - r) f + \sigma^2 \varepsilon f'(\mu)\right)^2}{2\sigma^2 R f(\mu)}$$

Asset dynamics with trend (14/4/14)

(i) Here's a story with echoes of the HR model which tries to capture the idea of trend in asset prices. Suppose then that an asset has dynamics

$$\begin{cases} dS_t = S_t \; dX_t \\ dX_t = \sigma \, dW_t + \mu_t \, dt \\ \mu_t = \varepsilon \int_{-\infty}^{t} \lambda e^{\lambda(s-t)} (X_t - X_s) \, ds \end{cases}$$

In this case, we shall have

$$d\mu_t + \lambda \mu_t \, dt = \varepsilon \, dX_t = \varepsilon \sigma dW_t + \varepsilon \mu_t \, dt$$

so we see that

$$\boxed{d\mu_t = \varepsilon \sigma \, dW_t - (\lambda - \varepsilon) \mu_t \, dt}$$

so that μ_t is an OU process; and μ is observable.

(ii) Suppose we do a standard investment/consumption story:

$$dw_t = r w_t \, dt + \theta_t (dX_t - r dt) - c_t \, dt$$

with CRRA felicity U. We expect that the value will be of the form $V(w, \mu) = U(w) f(\mu)$ for some f to be found from the HJB equations

$$0 = \sup \Big[-\rho V + U(c) + (rw + \theta(\mu - r) - c) V_w + \tfrac{1}{2} \sigma^2 \theta^2 V_{ww} - (\lambda - \varepsilon) \mu V_\mu$$
$$+ \theta \sigma^2 \varepsilon V_{w\mu} + \tfrac{1}{2} \varepsilon^2 \sigma^2 V_{\mu\mu} \Big]$$

$$= \sup U(w) \Big[-\rho f + z^{1-R} + (r + y(\mu - r) - z)(1-R) f + \tfrac{1}{2} \sigma^2 y^2 (-R)(1-R) f$$
$$- (\lambda - \varepsilon) \mu f' + \sigma^2 \varepsilon y (1-R) f' + \tfrac{1}{2} \varepsilon^2 \sigma^2 f'' \Big]$$

with optimizing

$$\boxed{\begin{aligned} z &= c/w = f^{-1/R} \\ y &= \theta/w = \frac{(\mu - r) f + \sigma^2 \varepsilon f'}{\sigma^2 R f} \end{aligned}}$$

Trading a fund in the presence of taxes (14/4/14)

This story would say that wealth evolves as

$$dw_t = rw_t\,dt + \theta_t(\sigma dW_t + (\mu - r)dt) - \varepsilon\, d\bar{w}_t - \lambda c_t\, dt$$

where $\lambda > 1$ to account for taxes which have to be paid on distributed wealth and the $-\varepsilon d\bar{w}_t$ is there to account for the capital gains taxes to be paid (here, $\bar{w}_t \equiv \sup_{u \le t} w_u$). With the usual CRRA von Neumann-Morgenstern preferences, how does it look?

We expect from scaling that $V(aw, a\bar{w}) = a^{1-R}\,V(w,\bar{w})$ for any $a \geq 0$, so we can express it as

$$V(w,\bar{w}) = \bar{w}^{1-R} f(w/\bar{w})$$

for some f to be found from the HJB equations

$$0 = \sup\left[-\rho V + U(c) + (rw + \theta(\mu - r) - \lambda c)\,V_w + \tfrac{1}{2}\sigma^2\theta^2 V_{ww}\right]$$

$$-\varepsilon V_w + V_{\bar{w}} = 0 \qquad \text{when} \quad w = \bar{w}$$

So when we use the scaling form of the problem, we shall see $(x \equiv w/\bar{w})$

$$0 = \sup \bar{w}^{1-R}\Big[-\rho f + U(z) + (rx + y(\mu - r) - \lambda z) f'(x) + \tfrac{1}{2}\sigma^2 y^2 f''(x)\Big]$$

$$0 = -\varepsilon f(x) + (1-R)f(x) - x f'(x) \qquad \text{when} \quad x = 1.$$

The optimality conditions are $z^{-R} = \lambda f'(x)$, $y = -(\mu - r)f'/\sigma^2 f''$, so we get

$$0 = -\rho f + \tilde{U}(\lambda f') + rx f' - \frac{\kappa^2}{2}\frac{f'^2}{f''}.$$

Properties of the solution of the Merton problem (22/4/14)

1) The study which Keiichi did of short selling of a stock suggests that we may want to know about the number of different units of the stocks which we hold as time evolves. In the Merton problem solution, we find

$$dw_t = w_t\{ r\,dt + \pi_M\cdot(\sigma\,dW_t + (\mu-r)dt) - \gamma\,dt\}$$

where $\gamma = \gamma_M$ if there is running consumption, $\gamma=0$ if it is a terminal wealth question. Solving this, we get

$$W_t = w_0 \exp\left[\ \pi_M\cdot\sigma\ W_t + \alpha t\right] = w_0 \exp\left[R^{-1}\kappa\cdot W_t + \alpha t\right]$$

where $\alpha = r + \pi_M\cdot(\mu-r) - \gamma - \frac{1}{2}|\sigma^T\pi_M|^2$

$$= r + \frac{|\kappa|^2}{2R^2}(2R-1) - \gamma$$

(see (1.31) in OI). The value of the j^{th} stock comes across as

$$S^j_t = S^j_0 \exp\left[\ e_j\cdot(\sigma\ W_t + \mu\ t) - \tfrac{1}{2}a_{jj}\,t\right] \qquad (a\equiv\sigma\sigma^T)$$

so the <u>number</u> of units of stock j held will be

$$\boxed{\nu^j_t = \frac{\pi^j_M w_t}{S^j_t} = \frac{\pi^j_M w_0}{S^j_0}\exp\left[\ (R^{-1}\kappa - e_j\cdot\sigma)W_t + (\alpha - \mu^j + \tfrac{1}{2}a_{jj})t\right]}$$

which is (of course) a GBM.

2) What we might imagine is that the lender of the stock will recall it once the number ν^j_t of stocks he's supposed to hold drops too low. If we write $\underline{\nu}_t = (\inf\{\nu^j_s : s\le t\}/\nu^j_0)$ then we'd suppose that recall is triggered once $\underline{\nu}$ falls below some independent $U(0,1)$ barrier; this now leads to a nice question, which can be summarized as follows.

We suppose that the short seller observes the value S^j_t of the stock sold short, but <u>not</u> the wealth process of the lender. So now what we have in effect is a two-dimensional correlated BM (X_t, Y_t) with drift where we observe $\mathcal{Y}_t = \sigma(Y_s : s \le t)$ but not X; we now want to determine the dual previsible projection of

$$\underline{X}_t \equiv \inf\{ X_s : s \le t \}.$$

This would allow us to determine the recall intensity of the shorted stock.

Actually, what we will in fact want is to evaluate

$$E\left[\int_0^\tau \varphi_s \, ds \right]$$

where φ is some $\mathcal{Y}$-previsible process, $\tau \equiv \inf\{ t : \exp(\underline{X}_t) < U \}$ where U is an independent $U[0,1]$. What we care about then is

$$E\left[\int_0^\infty \varphi_s \, I_{\{s < \tau\}} \, ds \right] = E\left[\int_0^\infty \varphi_s \, I_{\{\exp(\underline{X}_s) > U\}} \, ds \right]$$

$$= E\left[\int_0^\infty \varphi_s \exp(\underline{X}_s) \, ds \right]$$

Obtaining the distribution of $(\bar{X}_t \mid \mathcal{Y}_t)$ is almost certainly futile.

Option pricing by transforms (24/4/14)

(i) Suppose we have an asset whose vol is determined by a Markov chain ξ with jump intensity matrix Q. Then the log price (assuming $r=0$) evolves as

$$dx_t = \sigma(\xi_t)\, dW_t - \tfrac{1}{2}\sigma(\xi_t)^2\, dt$$

and if we want to price a put option with strike $K \equiv e^k$ we need to calculate

$$E\Big[\big(K - \exp(W(A_T) - \tfrac{1}{2}A_T)\big)^+\Big]$$

where $A_t = \int_0^t \sigma(\xi_s)^2\, ds$. If we set $X \equiv W(A_T) - \frac{1}{2}A_T$, then we have

$$\varphi(z) \equiv E\big[\exp(zX)\big] = \exp\big(T(Q+\tilde{z}V)\big)\mathbb{1} \qquad (z \in \mathbb{C})$$

where $V \equiv \operatorname{diag} \sigma_k^2$, $\tilde{z} \equiv z(z-1)/2$. Thus we know (an expression for) the characteristic function of X.

Next, for $\varepsilon > 0$, we notice that

$$\int_{-\infty}^{\infty} e^{i\theta x} e^{\varepsilon x} (e^k - e^x)^+ dx = \int_{-\infty}^{k} e^{(i\theta+\varepsilon)x}(e^k - e^x)\, dx$$

$$= e^{(1+\varepsilon+i\theta)k} \Big/ (\varepsilon+i\theta)(\varepsilon+1+i\theta)$$

so the FT of the put price function can also be discovered, up to knowing how to let $\varepsilon \downarrow 0$. Accordingly, we can in principle evaluate the option prices by Plancherel.

(ii) Notice that the option price will be (by Plancherel)

$$\int \varphi(i\theta) \cdot e^{(1+\varepsilon+i\theta)k} \left(\frac{1}{\varepsilon+i\theta} - \frac{1}{\varepsilon+1+i\theta}\right) \frac{d\theta}{2\pi}$$

where $\varphi(z) = \pi_0 \cdot \exp\big(T(Q + \frac{1}{2}z(z-1)V)\big)\mathbb{1}$. We can split the price as

$$e^k \int \varphi(i\theta)\, e^{(\varepsilon+i\theta)k} \frac{d\theta}{(\varepsilon+i\theta)2\pi} - \int \varphi(i\theta)\, e^{(1+\varepsilon+i\theta)k} \frac{d\theta}{(1+\varepsilon+i\theta)2\pi}$$

If we differentiate the integrals w.r.t. k, we shall get $\int \varphi(i\theta)\, e^{(\varepsilon+i\theta)k}\, d\theta/2\pi$, but this isn't going anywhere new...

Alternative: seems like we may want to deal also with non-centred processes. Then we get some changes (we don't really need to add a constant term to the observations, because we'd just subtract it off again)

$$X_t = A X_{t-1} + b + \varepsilon_t$$ for some non-zero b

Updating:

$$\hat{X}_t = A\hat{X}_{t-1} + b + K\left(Y_t - CA\hat{X}_{t-1} - Cb\right)$$

Steady-state K, V exactly as before.

$$Y_{t+1} = C(AX_t + b + \varepsilon_{t+1}) + \eta_{t+1}$$

$$\sim N\left(C(A\hat{X}_t + b),\ \tilde{v}\right)$$

so agent thinks

$$S_{t+1} \equiv (1,0)\,Y_{t+1} \sim N\left(a\hat{X}_t + \omega,\ v\right)$$ $\omega = (1,0)\,Cb$

Desired portfolio

$$\theta_{t+1} = (\gamma v)^{-1}\left(a\hat{X}_t + \omega - (1+r)S_t\right)$$

Production, investment + the stock market (29/4/14)

(i) We will develop the story on pp 35–36, and p 46. So there will be a single risky asset (the stock market) with price S_t on day t, and a total output $\bar{\delta}_t$ on day t which for now we'll think of in cash (though later we may want to work with consumption good. Thus the observable on day t will be

$$Y_t = \begin{pmatrix} S_t \\ \bar{\delta}_t \end{pmatrix} \qquad \text{(two-dimensional)}$$

and agent j (superscript suppressed for now) believes that

$$\begin{cases} Y_t = C X_t + \eta_t \\ X_t = A X_{t-1} + \varepsilon_t \end{cases}$$

for some hidden state process X_t. The agent therefore performs a Kalman filtering and assuming steady state thinks that $(X_t \mid \mathcal{Y}_t) \sim N(\hat{X}_t, V)$ where

$$\hat{X}_t - A\hat{X}_{t-1} = K(Y_t - CA\hat{X}_{t-1})$$

with K, V as on p35.

(ii) On day t, the agent looks ahead to day $t+1$ and thinks

$$\begin{aligned} Y_{t+1} &= CAX_t + C\varepsilon_{t+1} + \eta_{t+1} \\ &\sim N\left(CA\hat{X}_t,\ CAVA^TC^T + C\Sigma_{XX}C^T + C\Sigma_{XY} + \Sigma_{YX}C^T + \Sigma_{YY}\right) \\ &\equiv N(\tilde{a}\hat{x}_t, \tilde{v}) \quad \text{, say} \end{aligned}$$

Therefore the agent thinks that

$$\boxed{S_{t+1} \equiv (1,0)\, Y_{t+1} \sim N(a\hat{X}_t, v)}$$

where $a = (1,0)\tilde{a}$, $v = \tilde{v}_{00}$. As on pp 35–36 the desired portfolio will be

$$\boxed{\theta_{t+1} = (\gamma v)^{-1}\left(a\hat{X}_t - (1+r)S_t\right)}$$

and desired consumption

$$\boxed{(2+r)\gamma c = k - \log(v(1+r)), \quad k = \gamma(1+r)w_t + \tfrac{1}{2}\gamma^2\, \theta_{t+1} \cdot v\, \theta_{t+1}.}$$

$$\hat{X}_t = A\hat{X}_{t-1} + b + K(Y_t - CA\hat{X}_{t-1} - Cb)$$

$$\theta_{t+1} = (\gamma v)^{-1}\left[a K Y_t + a(I-KC)A\hat{X}_{t-1} + a(I-KC)b - (1+r)S_t + \omega\right]$$

$$= (\gamma v)^{-1}\left[(ak_0 - 1 - r)S_t + ak_1\bar{\delta}_t + \omega + a(I-KC)(A\hat{X}_{t-1} + b)\right]$$

$$\bar{\theta}_{t+1} = \sum_{j=1}^{J}(\gamma_j v_j)^{-1}\left[(a^j k_0^j - 1 - r)S_t + a^j k_1^j \bar{\delta}_t + a^j(I-K^jC^j)A^j\hat{X}^j_{t-1}\right]$$
$$+ \sum_{j=1}^{J}(\gamma_j v_j)^{-1}a^j(I-K^jC^j)b^j + \Sigma(\gamma_j v_j)^{-1}\omega^j$$

{The final term involving b is constant, so in effect just alters the net supply, if you want to think of it like that}

(iii) Now $\hat{X}_t = K(Y_t - CA\hat{X}_{t-1}) + A\hat{X}_{t-1}$, so the demand θ_{t+1} for the index depends on S_t, $\bar{\delta}_t$, and $\hat{X}_{t-1}$, in fact,

$$\theta_{t+1} = (\gamma v)^{-1}\Big[\, aKY_t + a(I-KC)A\hat{X}_{t-1} - (1+r)S_t \,\Big]$$

$$= (\gamma v)^{-1}\Big[\,(ak_0 - 1 - r)S_t + ak_1\bar{\delta}_t + a(I-KC)A\hat{X}_{t-1}\,\Big]$$

where $K = [k_0, k_1]$. Now market clearing gives (with agent labels reinstated)

$$\boxed{\begin{aligned} \bar{\theta}_{t+1} &= \sum_{j=1}^{J}(\gamma_j v_j)^{-1}\Big[(a^j k_0^j - 1 - r)S_t + a^j k_1^j \bar{\delta}_t + a^j(I - K^j C^j)A^j\hat{X}^j_{t-1}\Big] \\ &\qquad + \sum(\gamma_j v_j)^{-1}\{a^j(I-K^jC^j)b^j + w^j\} \\ &= (M - rL)S_t + g\,\bar{\delta}_t + \sum_{j=1}^{J}(\gamma_j v_j)^{-1}a^j(I-K^jC^j)A^j\hat{X}^j_{t-1} \end{aligned}}$$

where

$$M \equiv \sum_{j=1}^{J}(\gamma_j v_j)^{-1}(a_j k_0^j - 1), \qquad L \equiv \sum_{j=1}^{J}(\gamma_j v_j)^{-1}$$

$$g = \sum_{j=1}^{J}(\gamma_j v_j)^{-1}a^j k_1^j$$

(iv) As previously, we shall have productive capital Q_t, investment I_t evolving as

$$I_t = \psi I_{t-1} + \alpha(m_t - m_{t-1})$$

$$Q_t = \beta Q_{t-1} + \lambda I_{t-1}$$

with

$$\bar{\delta}_t = \varepsilon_0 Q_t .$$

A question from Matheus Grasselli (12/5/14)

(i) Matheus has been looking at the De Long-Shleifer-Summers-Waldman paper on the equilibrium of noise traders + informed traders, and asks whether there is an analogue in continuous time. Here is one possibility, stated for J+1 agents – the case J=1 is the analogue of DSSW.

(ii) Suppose there are agents of J+1 types, the proportion of type j being μ_j, $j=0,\dots,J$. Agent j believes that the dividend process of the one stock evolves as

$$d\delta_t = \sigma dW_t + (\sigma a_j - \beta\delta_t)\,dt$$

where $\beta \geq 0$, $\sigma > 0$, and the a_j are distinct. Agent 0 is in fact correct in his beliefs; the rest are not. The LR martingales for the agents are

$$\Lambda^j_t = \exp\left[(a_j - a_0)W_t - \tfrac{1}{2}(a_j - a_0)^2 t\right] \equiv \exp\left[\alpha_j W_t - \tfrac{1}{2}\alpha_j^2 t\right]$$

and the common state-price density ζ therefore has the property

$$\zeta_t \propto \Lambda^j_t e^{-\rho t} U'(c^j_t)$$

for each j, where we assume all agents have the same preferences $U'(x) = e^{-\gamma x}$. Taking logs,

$$\log\zeta_t = \text{const} + \alpha_j W_t - \tfrac{1}{2}\alpha_j^2 t - \rho t - \gamma c^j_t \qquad (*)$$

and so if we average with weights μ_j and invoke market clearing, we find

$$\log\zeta_t = \text{const} + \bar{\alpha} W_t - bt - \gamma\delta_t \qquad \left[b = \textstyle\sum \mu_j(\tfrac{1}{2}\alpha_j^2 + \rho)\right]$$

Thus the wealth of agent j at time t will be

$$w^j_t = \frac{1}{\zeta_t} E_t\left[\int_t^\infty \zeta_s c^j_s\,ds\right]$$

$$= E_t\left[\int_t^\infty \exp\{\bar{\alpha}(W_s - W_t) - b(s-t) - \gamma(\delta_s - \delta_t)\}\, c^j_s\,ds\right]$$

Now we have using (*) that for some constants A_j

$$c^j_t = A_j + (\alpha_j - \bar{\alpha})W_t + (b - \tfrac{1}{2}\alpha_j^2 - \rho)t + \gamma\delta_t$$

so the thing we have to do is evaluate (taking current time to be 0 for simplicity)

$$E\left[\int_0^\infty \exp\{\bar{\alpha} W_t - bt - \gamma\delta_t\}\left(\lambda_0 + \lambda_1 W_t + \lambda_2 t + \lambda_3 \delta_t\right) dt\right]$$

If we can evaluate $E\exp(\theta W_t + z\delta_t)$ we can get the general stuff by differentiation. But $\theta W_t + z\delta_t$ is Gaussian with mean $z[\bar{a}_0 + e^{-\beta t}(\delta_0 - \bar{a}_0)]$, variance

$$\theta^2 t + 2z\theta\sigma(1-e^{-\beta t})/\beta + z^2\sigma^2(1-e^{-2\beta t})/2\beta$$

(here, $\bar{a}_0 = \sigma a_0/\beta$). Hence

$$E\exp(\theta W_t + z\delta_t) = \exp\left[z(\bar{a}_0 + e^{-\beta t}(\delta_0 - \bar{a}_0)) + \tfrac{1}{2}\theta^2 t + z\theta\sigma(1-e^{-\beta t})\beta^{-1} + \tfrac{1}{2}z^2\sigma^2(1-e^{-2\beta t})/2\beta\right]$$

This is explicit, but the integral can't be done in closed form, except if $\beta = 0$. In this case, $E\,\delta_t = \delta_0 + \sigma a_0 t$, $\operatorname{var}(\theta W_t + z\delta_t) = t(\theta^2 + 2\theta\sigma z + \sigma^2 z^2)$, so

$$E\exp(\theta W_t + z\delta_t) = \exp\left[z(\delta_0 + \sigma a_0)t + \left(\tfrac{1}{2}\theta^2 + \theta\sigma z + \tfrac{1}{2}\sigma^2 z^2\right)t\right]$$

whence

$$E\left[W_t e^{\theta W_t + z\delta_t}\right] = (\theta + \sigma z)t\ e^{\{-\}}$$

$$E\left[\delta_t e^{\theta W_t + z\delta_t}\right] = (\delta_0 + \sigma a_0 + \theta\sigma + \sigma^2 z)t\ e^{\{-\}}$$

Notes on fluctuations of Lévy processes (14/5/14)

(i) For $a < 0 < b$ and a Lévy process (X_t) let's define

$$\tau_a = \inf\{t : X_t < a\}, \quad \tau_b = \inf\{t : X_t > b\}, \quad \tau_a^b = \tau_a \wedge \tau^b$$

We can similarly define

$$\pi_a(dx) = E\left[e^{-c\tau_a} : X(\tau_a) \in dx \right]$$

with π^b, π_a^b defined analogously replacing τ_a by τ^b, τ_a^b, respectively. Let's also define

$$m^b(dx) = E\left[\int_0^{\tau^b} e^{-ct} I_{\{X_t \in dx\}} dt \right]$$

with analogous definitions of $m_a(\cdot)$, $m_a^b(\cdot)$. What are we able to say about these measures?

(ii) There's an old identity which is already recorded in Nick Bingham's survey:

$$\psi_+(i\theta) \int_0^\infty \lambda e^{-\lambda b} E \exp[-c\tau^b + i\theta X(\tau^b)] \, db$$

$$= \int_0^\infty \lambda e^{-\lambda b} E\left[\exp(i\theta \bar{X}_T) : \bar{X}_T > b \right] db$$

$$= E\left[\exp(i\theta \bar{X}_T)(1 - e^{-\lambda \bar{X}_T}) \right]$$

$$= \psi_+(i\theta) - \psi_+(i\theta - \lambda)$$

which implies that

$$\boxed{\int_0^\infty \lambda e^{-\lambda b} \hat{\pi}^b(i\theta) \, db = 1 - \frac{\psi_+(i\theta - \lambda)}{\psi_+(i\theta)}} \qquad (1)$$

(iii) If we let $r_c(dx) = E\left[\int_0^\infty e^{-ct} P(X_t \in dx) \, dt\right]$, we have the Wiener-Hopf factorization:

$$c\hat{r}_c(i\theta) = \frac{c}{c - \psi(i\theta)} = \psi_+(i\theta)\psi_-(i\theta)$$

but also we have for any $b > 0$

$$\hat{r}_c(i\theta) = \hat{m}^b(i\theta) + \hat{\pi}^b(i\theta) \cdot \hat{r}_c(i\theta) \qquad (2)$$

by strong Markov at τ^b. Thus if we multiply by $\lambda e^{-\lambda b}$ and integrate, using (1) gives us

$$\boxed{\begin{aligned}\int_0^\infty \lambda e^{-\lambda b}\, \hat{m}^b(i\theta)\, db &= \hat{v}_c(i\theta)\, \frac{\psi_+(i\theta-\lambda)}{\psi_+(i\theta)} \\ &= \psi_+(i\theta-\lambda)\,\psi_-(i\theta)/c\end{aligned}}$$

So if we know the WH factors, we know the LT of $\hat{m}^b$, $\hat{\pi}^b$.

(iv) It seems that this not sufficient to help find the two-sided things. For example, we have for $x > b$

$$\pi^b(dx) = \pi_a^b(dx) + \int_{-\infty}^a \pi_a^b(dy)\, \pi^{b-y}(dx-y)$$

so multiplying by $e^{i\theta x}$ and integrating gives

$$\hat{\pi}^b(i\theta) = \hat{\pi}_a^b(i\theta) + \int_{-\infty}^a \pi_a^b(dy)\, e^{i\theta y}\, \hat{\pi}^{b-y}(i\theta)$$

$$= \hat{\pi}_a^b(i\theta) + \int_{-\infty}^0 \pi_a^b(dy)\, e^{i\theta y}\, \hat{\pi}^{b-y}(i\theta)$$

and the final integral doesn't go nicely if we integrate $\alpha e^{\alpha a}\,da$, nor if we integrate $\lambda e^{-\lambda b}\,db$

Merton illiquidity problem : changing the boundary conditions (14/5/14)

(i) In the Merton problem with illiquidity costs, when we work with the variables (Y,H) where Y is the cash denominated in units of the stock, and H is the number of units of the stock, we obtained the HJB equations

$$0=\sup_{c,h}\left[U(c)-\tilde{\rho}F+\tfrac{1}{2}\sigma^2Y^2F_{YY}-(h+hf(ch)+c+\alpha Y)F_Y+hF_H\right]$$

with various boundary conditions; at the upper value Y^* of Y we have some linear boundary condition

$$aF_Y+bF+c=0,$$

which can be handled by FEniCS, and at $Y=0$ we have a boundary condition

$$0=\beta F_Y-F_H$$

which FEniCS cannot handle. However, if we introduce new coordinates $x\geq 0$, $y\geq 0$, and set

$$\begin{cases} Y = x \\ H = \frac{y}{2}\left(1+\exp\left(-2x/\beta y\right)\right)\end{cases}$$

then set

$$G(x,y)=F(Y,H)=F\left(x,\tfrac{1}{2}y\left(1+\exp(-2x/\beta y)\right)\right)$$

we get

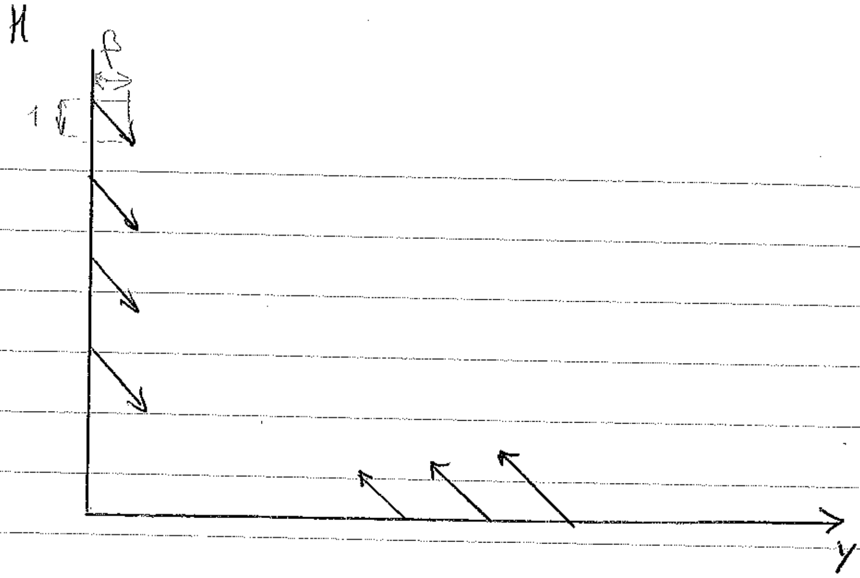
$$G_x = F_Y-\beta^{-1}\exp(-2x/\beta y)F_H$$

$$G_y=\tfrac{1}{2}\left[1+\exp\left(-2x/\beta y\right)+\frac{2x}{\beta y}\exp\left(-2x/\beta y\right)\right]F_H$$

from which we see that when $x=0$, $G_x=F_Y-\beta^{-1}F_H=0$, that is, we have obtained a normal reflection BC at $x=0$! At $x=Y^*$, so long as we have that $x\gg y$, we can use the approximation $G_x\simeq F_Y$, which renders the problem suitable for FEniCS.

(ii) To deal with this more completely, let's write $Z\equiv\exp\left(-2x/\beta y\right)$ and then notice that with

$$\lambda=\frac{4yZ}{\beta y+2xZ+\beta yZ},$$



In fact, we will need some similar bc at $H=0$ I think, in that we might want to say that we would be ready to do a little purchasing of stock here. If we said that at $Y=0$ we'd have to have

$$F_Y - F_H = 0$$

this would mean that when $Y=0$, we could purchase at no penalty - we get the stuff for the market price. Thus we may try some diffeomorphism

$$Y = \frac{x(y+mx)}{2y+mx} = x\left[\frac{1}{2} + \frac{mx/2}{2y+mx}\right]$$

$$H = \frac{y(x+by)}{2x+by} = y\left[\frac{1}{2} + \frac{by/2}{2x+by}\right]$$

Simple calculus gives us

at $x=0$ $\quad \frac{\partial Y}{\partial x} = \frac{1}{2}, \quad \frac{\partial H}{\partial x} = -\frac{1}{b}$

at $y=0$, $\quad \frac{\partial Y}{\partial y} = -\frac{1}{m}, \quad \frac{\partial H}{\partial y} = \frac{1}{2}$

So if we have

$$g(x,y) = F(Y(x,y), H(x,y))$$

and we take $m=2$, $b=2\beta$, we get
$$\begin{cases} \frac{\partial g}{\partial y}(x,y) = 0 & \text{at } y=0 \\ \frac{\partial g}{\partial x}(x,y) = 0 & \text{at } x=0 \end{cases}$$

So g solves some PDE with normal reflection at the boundaries!
[What about the boundary where H is high?]

$$\mu \equiv \frac{4y^2 Z^2}{\beta^2 y^2 (1+Z)^2 + 4\beta x y Z(1+Z) + 4x^2 Z^2}$$

we find that

$$\boxed{\begin{aligned} &G_{xx} + \lambda G_{xy} + \mu G_{yy} \\ &\quad = F_{yy}(Y,H) + \frac{yZ(1+Z)^2}{\beta^2 y^2 (1+Z)^2 + 4\beta x y Z(1+Z) + 4x^2 Z^2} F_H \end{aligned}}$$ [Maple]

which tells us what the diffusion matrix has to be in the new coordinates. In the same fashion, we can find

$$\boxed{G_x = (Z\, F_H - \beta F_Y)/\beta, \quad G_y = \frac{\beta y(1+Z) + 2xZ}{2\beta} F_H}$$

Assembling this allows us to rephrase the first derivatives of F in terms of the first derivatives of G.

(iii) An alternative reparametrization would be to take

$$H = \frac{y(x+\beta y)}{2x + \beta y}$$

which still has $H = y$ when $x = 0$, $H \to y/2$ $(x \to \infty)$, $\frac{\partial H}{\partial x} = -1/\beta$ at $x = 0$.

The advantage of this is that we can express y in terms of x, H by solving a quadratic:

$$\boxed{y = \frac{\beta H - x + \sqrt{(\beta H - x)^2 + 8\beta x H}}{2\beta}}$$

Then we obtain

$$\frac{\partial F}{\partial H} = \frac{\partial G}{\partial y} \frac{\beta y + 2x}{2\beta y + x - \beta H},$$

$$\frac{\partial F}{\partial y} = \frac{\partial G}{\partial x} + \frac{\partial G}{\partial y} \frac{2H - y}{2\beta y + x - \beta H},$$

$$\frac{\partial^2 F}{\partial y^2} = G_{xx} + 2G_{xy} \frac{2H-y}{2\beta y + x - \beta H} + G_{yy} \left(\frac{2H-y}{2\beta y + x - \beta H}\right)^2 + \ldots$$

$$+ \frac{(2H-y)}{(2\beta y+Y-\beta H)^2}\left(-2\beta y - 2Y - 2\beta H\right).$$

So this allows us to do a direct translation of the HJB PDE into the variables (x, y) !

Candidate models in diverse beliefs (19/5/14)

Return to the story on pp 52 - 53, with a view to proposing some interesting dynamic models. But first, we are dealing so far with centred processes - how would that get modified? Want

$$\begin{cases} \tilde{X}_{t+1} = A\tilde{X}_t + b + \varepsilon_{t+1} \\ \tilde{Y}_{t+1} = C\tilde{X}_{t+1} + \tilde{y} + \eta_{t+1} \end{cases}$$

If we define $X_t \equiv \tilde{X}_t - (I-A)^{-1} b$, $Y_t \equiv \tilde{Y}_t$, then $\left(\bar{y} \equiv \tilde{y} - C(I-A)^{-1} b \right)$

$$\begin{cases} X_t = AX_t + \varepsilon_{t+1} \\ Y_{t+1} = CX_{t+1} + \bar{y} + \eta_{t+1} \end{cases}$$

means we get back to the old story if we simply treat $Y_t - \bar{y}$ as the new observation.

Now for some candidate models.

(i) This is in effect saying that changes in dividends are OU, stock prices are some multiple of dividends. Precisely,

$$X_t \equiv \begin{pmatrix} x_t \\ z_t \end{pmatrix} = \begin{pmatrix} x_{t-1} + z_t + \varepsilon^1_t \\ z_t \end{pmatrix} \equiv \begin{pmatrix} x_{t-1} + \beta z_{t-1} \\ \beta z_{t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon^1_t + \varepsilon^2_t \\ \varepsilon^2_t \end{pmatrix}$$

and we observe

$$Y_t = \begin{pmatrix} S_t \\ \delta_t \end{pmatrix} = \begin{pmatrix} \lambda x_t \\ x_t \end{pmatrix} + \eta_t$$

So in this example,

$$A = \begin{pmatrix} 1 & \beta \\ 0 & \beta \end{pmatrix}, \quad C = \begin{pmatrix} \lambda & 0 \\ 1 & 0 \end{pmatrix}, \quad \Sigma_{xx} = \begin{pmatrix} v_1 + v_2 & v_2 \\ v_2 & v_2 \end{pmatrix}, \quad \Sigma_{yy} = \begin{pmatrix} \sigma^2_{\eta_1} & \cdot \\ \cdot & \sigma^2_{\eta_2} \end{pmatrix}$$

(ii) In this case, an OU process drives changes in S and δ. Take

$$X_t = \begin{pmatrix} x^1_t \\ x^2_t \\ z_t \end{pmatrix} = \begin{pmatrix} x^1_{t-1} + \lambda_1 z_t + \varepsilon^1_t \\ x^2_{t-1} + \lambda_2 z_t + \varepsilon^2_t \\ z_t \end{pmatrix} = \begin{pmatrix} 1 & 0 & \lambda_1\beta \\ 0 & 1 & \lambda_2\beta \\ 0 & 0 & \beta \end{pmatrix} X_{t-1} + \begin{pmatrix} \varepsilon^1_t + \lambda_1 \varepsilon^3_t \\ \varepsilon^1_t + \lambda_2 \varepsilon^3_t \\ \varepsilon^3_t \end{pmatrix}$$

We observe

$$Y_t = \begin{pmatrix} S_t \\ \delta_t \end{pmatrix} \equiv \begin{pmatrix} \alpha_1 x^1_t \\ \alpha_2 x^2_t \end{pmatrix} + \eta_t = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \end{pmatrix} X_t + \eta_t$$

so this time

$$A = \begin{pmatrix} 1 & 0 & \lambda_1\beta \\ 0 & 1 & \lambda_2\beta \\ 0 & 0 & \beta \end{pmatrix}, \quad C = \begin{pmatrix} \alpha_1 & \cdot & \cdot \\ \cdot & \alpha_2 & \cdot \end{pmatrix}, \quad \Sigma_{XX} = \begin{pmatrix} v_1 + \lambda_1^2 v_3 & \lambda_1\lambda_2 v_3 & \lambda_1 v_3 \\ \lambda_1\lambda_2 v_3 & v_2 + \lambda_2^2 v_3 & \lambda_2 v_3 \\ \lambda_1 v_3 & \lambda_2 v_3 & v_3 \end{pmatrix}$$

(iii) This story would exploit the idea that $S_{t-1} = E_{t-1}\left[\beta(\delta_t + S_t)\right]$, so we'd have $S_t = \frac{1}{\beta} S_{t-1} - \delta_t + \text{noise}$. This suggests we try

$$X_t = \begin{pmatrix} x^1_t \\ x^2_t \\ z_t \end{pmatrix} = \begin{pmatrix} \beta^{-1} x^1_{t-1} - x^2_t \\ x^2_{t-1} + z_t \\ z_t \end{pmatrix} = \begin{pmatrix} \beta^{-1} x^1_{t-1} - (x^2_{t-1} + \alpha z_{t-1}) \\ x^2_{t-1} + \alpha z_{t-1} \\ \alpha z_{t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon^1_t - \varepsilon^2_t - \alpha\varepsilon^3_t \\ \varepsilon^2_t + \alpha\varepsilon^3_t \\ \varepsilon^3_t \end{pmatrix}$$

and we see

$$Y_t \equiv \begin{pmatrix} S_t \\ \delta_t \end{pmatrix} = \begin{pmatrix} x^1_t \\ x^2_t \end{pmatrix} + \eta_t$$

so $C = \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \end{pmatrix}$, and

$$A = \begin{pmatrix} \beta^{-1} & -1 & -\alpha \\ 0 & 1 & \alpha \\ 0 & 0 & \alpha \end{pmatrix}, \quad \Sigma_{XX} = \begin{bmatrix} v_1 + v_2 + \alpha^2 v_3 & -(v_2 + \alpha^2 v_3) & -\alpha v_3 \\ \cdot\ \cdot & v_2 + \alpha^2 v_3 & \alpha v_3 \\ \cdot\ \cdot & \cdot\ \cdot & v_3 \end{bmatrix}.$$

Non centred versions of the above aren't so easy, because $A - I$ isn't invertible —.

Some thoughts on diffusion approximation (21/5/14)

(i) Talking to Lukas + Josef about the difference between the LAF story + the story of Andrew Stuart, Jochen Voss et al leads me to wonder whether the difference is to do with the topology on $C[0,1]$. If $x \in C[0,1]$, and we write $\Delta x_i^n \equiv x(i2^{-n}) - x((i-1)2^{-n})$, we could set

$$q_n(x) = \sum_{i=1}^{2^n} (\Delta x_i^n)^2$$

and then consider

$$\|x\|^2 = |x_0|^2 + \sum_{n \geq 0} 2^{-n} q_n(x).$$

This is a norm on $C[0,1]$, though probably we ought to restrict to $H^0 = \{x : \|x\| < \infty\}$ since it's not obvious that the norm is finite everywhere.

Notice that if x is a BM, we have $E\, q_n(x) = 1$ for all n, so almost every Brownian path is in H^0

Notice that if (x_n) is a Cauchy sequence in this norm, then we will certainly have $x_n(s)$ converges to a finite limit for each dyadic rational s. It seems that we don't necessarily stay in $C[0,1]$ if we take Cauchy sequences; the function $I_{[\frac12,1]}(\cdot)$ is a limit in this topology of continuous x with finite norm. Quite likely other nastier examples exist.

(ii) Suppose that we consider

$$\xi^n = \sum_{i=1}^{2^n} (\Delta B_i^n)^2$$

for BM (B_t). As we know, $E(\xi_n) = 1$, and ξ_n is measurable wrt $\mathcal{G}_n = \sigma(B(j2^{-n}),\ j = 0, \dots, 2^n)$. If we think about two Brownian increments y, z over intervals of length h, and consider their distribution given the sum, we have

$$\begin{pmatrix} y \\ z \\ y+z \end{pmatrix} \sim N\left(0, \begin{pmatrix} h & 0 & h \\ 0 & h & h \\ h & h & 2h \end{pmatrix}\right) \quad \text{so} \quad \left(\begin{matrix} y \\ z \end{matrix} \,\middle|\, y+z\right) \sim N\left(\begin{pmatrix} \frac{y+z}{2} \\ \frac{y+z}{2} \end{pmatrix}, h\begin{pmatrix} \frac12 & -\frac12 \\ -\frac12 & \frac12 \end{pmatrix}\right)$$

So $E\left[y^2 + z^2 \,\middle|\, y+z\right] = h + \frac{(y+z)^2}{2}$. Therefore

$$E\left[\xi_{n+1} \middle| \mathcal{G}_n\right] = \tfrac12 \xi_n + \tfrac12$$

so $E\left[2^{n+1}\xi_{n+1} \middle| \mathcal{G}_n\right] = 2^n \xi_n + 2^{n+1}/2 \Rightarrow \boxed{2^n(\xi_n - 1) \equiv M_n \text{ is martingale}}$

The martingale is bounded in L^1 but not in L^2.

$$P^x\left[\text{hit } y \mid \text{exit at } b\right] = \frac{(b-x)(y-a)}{(b-y)(x-a)} \wedge 1$$

$$P^x\left[\text{hit } y \mid \text{exit at } a\right] = \frac{(x-a)(b-y)}{(y-a)(b-x)} \wedge 1$$

So $$E^x\left[L^y_\infty \mid \text{exit at } b\right] = \frac{2(b-y)(y-a)}{b-a} \cdot P^x\left[\text{hit } y \mid \text{exit at } b\right]$$

Simulating CIR again (24/5/14).

This is a question which arose in discussion with Josef + Philipp Harms about simulating CIR. I did some stuff on this a while back with Ralf's group, but it never got carried through. Looking back, it seems there may be good ways to do this. The SDE is

$$dx_t = v\sqrt{x_t}\, dW_t + (\alpha - \beta x_t)\, dt$$

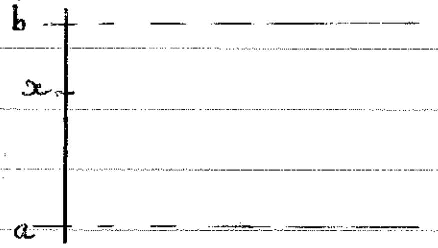
and the proposed approach is to find the scale function s and put the diffusion in natural scale: if $Y_t \equiv s(x_t)$

$$dY_t = s'(x_t)\, v\sqrt{x_t}\, dW_t \equiv \tilde{\sigma}(Y_t)\, dW_t.$$

Now we shall discretize the diffusion onto a y-grid. We know the exit probabilities, because they are Brownian, but what about mean exit times from an interval, conditional on what side you go out?

If we start from y, the expected local time at the first exit from $[a, b]$ is

$$E^y L^y_\tau = \frac{2(b-y)(y-a)}{b-a}$$



If we start from x, and condition on exit at b then

$$P^x(\text{hit } y \mid \text{exit at } b) = 1 \qquad \text{if } x \le y \le b$$

$$= \frac{b-x}{b-y} \cdot \frac{y-a}{b-a} \Big/ \frac{x-a}{b-a} \qquad \text{if } a \le y \le x$$

so the expected local time at y given you exit at b will be

$$E^y\left[L^y_\tau \mid \text{exit at } b\right] = \frac{2(b-y)(y-a)}{b-a} \qquad \text{if } x \le y \le b$$

$$= \frac{2(b-x)(y-a)^2}{(b-a)(x-a)} \qquad \text{if } a \le y \le x$$

Likewise,

$$E^y\left[L^y_\tau \mid \text{exit at } a\right] = \frac{2(b-y)(y-a)}{b-a} \qquad \text{if } a \le y \le x$$

$$= \frac{2(x-a)(b-y)^2}{(b-a)(b-x)} \qquad \text{if } x \le y \le b$$

$$I_\nu(z) \sim e^z / \sqrt{2\pi z}$$

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}}\; e^{-z}$$

for large z

$$q^\delta_t(x,y) = \frac{1}{2t}\left(\frac{y}{x}\right)^{\nu/2} \exp\left(-\frac{x+y}{2t}\right) I_\nu\left(\frac{\sqrt{xy}}{t}\right)$$

$$\nu = \delta/2 - 1$$

Then we will need to calculate the integrals

$$\int_a^b E^y\left[L^y_\tau \,|\, \text{exit at } b\right] \tilde{\sigma}(y)^2 \, dy$$

in order to know the mean time taken to exit the interval. It's rather ponderous, but seems unavoidable. We might attempt to do such calculations directly on the CIR process, but it's no easier, because you still need the scale function..

The calculations may be streamlined if we are able to get the indefinite integrals

$$\int_0^{\cdot} y^n \tilde{\sigma}(y)^2 \, dy \qquad \text{for } n = 0, 1, 2.$$

This approach seems to require a lot of precomputation, and also leads to a lot of time steps in the simulation. There might be a neater way to proceed. Let's have the SDE in the form

$$dX_t = \sigma \sqrt{X_t}\, dW_t + \mu(X_t)\, dt$$

where we use $\mu(x) = \alpha - \beta(x-c)^+$ and set $\alpha \equiv \sigma^2(1+\nu)/2$ for the index ν. When $\nu = 0$, this corresponds to BESQ(2), and for all $\nu < 0$ the process does hit 0. Let's have $\psi^\pm_\lambda$ as the increasing/decreasing eigenfunctions of the diffusion:

$$\tfrac{1}{2}\sigma^2 x f''(x) + \mu(x) f'(x) - \lambda f(x) = 0$$

Now in a neighbourhood of zero, the solutions are

$$x^{-\nu/2} K_\nu\left(2\sqrt{2\lambda x}/\sigma\right), \quad x^{-\nu/2} I_\nu\left(2\sqrt{2\lambda x}/\sigma\right)$$

and we have finite limits at zero of these two of $\nu < 0$:

$$\boxed{\begin{aligned} \lim_{x \downarrow 0} x^{-\nu/2} K_\nu\left(2\sqrt{2\lambda x}/\sigma\right) &= \frac{\Gamma(|\nu|)}{2} \left(\frac{2\lambda}{\sigma^2}\right)^{\nu/2} \\ \lim_{x \downarrow 0} x^{-\nu/2} I_\nu\left(\frac{2\sqrt{2\lambda x}}{\sigma}\right) &= \frac{1}{\Gamma(\nu+1)} \left(\frac{2\lambda}{\sigma^2}\right)^{\nu/2} \end{aligned}}$$

Or maybe we just go straight to what we want? If we use the drift

scipy.special.hyperu (a, b, x)
scipy.special.hyp1f1 (a, b, x)
scipy.special.hyp2f0 (a, b, x)

$\Big\}$ may help!

$\mu(x) = \alpha - \beta x$, then the solution is in terms of Kummer functions:

$$f(x) = M\left(\frac{\lambda}{\beta}, 1+\nu, \frac{2\beta x}{\sigma^2}\right) \qquad \left[(1+\nu) \equiv 2\alpha/\sigma^2\right]$$

or $f(x) = U(\lambda/\beta,\ 1+\nu,\ 2\beta x/\sigma^2)$

We have $U(a,b,z) \sim z^{-a}$ $(z\to\infty)$, $M(a,b,z) \sim \Gamma(b)\, e^{z} z^{a-b} / \Gamma(a)$ $(z \to \infty)$
and the alternative forms

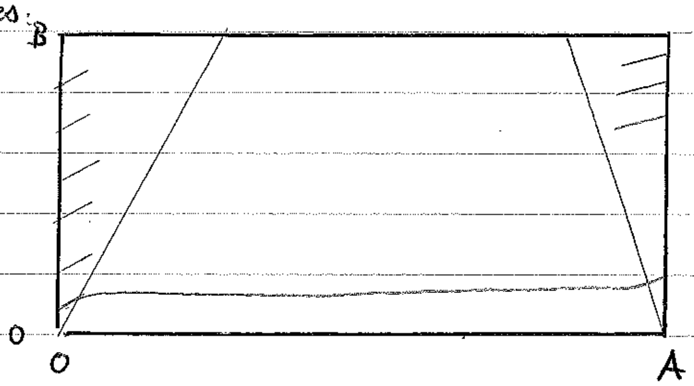
$$U(a,b,z) = z^{-a} \, {}_2F_0\,(a, 1+a-b, -1/z)$$

$$M(a,b,z) = {}_1F_1\,(a,b,z)$$

[However, seems that the thesis of Shao provides a very neat way to do CIR simulation, so this may be redundant]

Boundary conditions for the liquidity problem again. (4/6/14)

It seems that we want to find a diffeomorphism of a rectangle $[0,A] \times [0,B]$ with prescribed slopes on the edges:



So can we construct a (reasonably smooth) function $f: [0,A] \to \mathbb{R}^+$ which starts at $y>0$ has slope $a>0$ at $y=0$, has slope $b>0$ at $y=A$, is increasing and has $f(A) = \text{const.}\, y$?

What we could do is to take some positive $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$, with indefinite integral $\Phi(t) = \int_0^t \varphi(s)\,ds$, such that $\varphi(0)=1$, φ is supported in $[0, A/B]$ and now define

$$f'(t) = a\,\varphi(t/y) + b\,\varphi((A-t)/y), \quad f(0)=y$$

So we see $f' \geq 0$, $f'(0)=a$, $f'(A)=b$, and

$$f(t) = y + ay\,\Phi(t/y) + by\left(\Phi(\infty) - \Phi((A-t)/y)\right)$$

which gives us $f(0)=y$, $f(A) = y\left(1+(a+b)\Phi(\infty)\right)$.

In the new coordinate system, any point of f would have second coordinate equal to y.

Suppose we have $\varphi(t) = (1-\lambda t)^+$, so that $\Phi(s) = (s - \frac{1}{2}\lambda s^2) \wedge (1/2\lambda)$?

No, in fact

$$\Phi(s) = (s \wedge \tfrac{1}{\lambda}) - \tfrac{\lambda}{2}(s \wedge \tfrac{1}{\lambda})^2.$$

What we can do is to define

$$f(t,\eta) = \eta + a\eta\,\Phi(t/\eta) + b\eta\left(\Phi(\infty) - \Phi((A-t)/\eta)\right)$$

which is the y-coordinate on the constant η curve when $x=t$. Likewise we could determine the analogous curves $g(s,\xi)$, and given a point (x,y) we would like to find (ξ,η) such that

$$f(x,\eta) = y, \quad g(y,\xi) = x$$

This is the diffeomorphism. So we would also be interested in the inverse of this diffeomorphism, because we will want to work in (ξ, η) coordinates and have then to map back to (x, y). This we can do by noticing

$$g(f(x,\eta), \xi) = x \quad , \quad f(g(y,\xi), \eta) = y$$

Can this work though? We will get reflection in the direction of the axes in the (ξ, η) coordinate system, but the region won't be a rectangle, so it won't be normal reflection...

Asset dynamics with trend again (15/6/14)

Come back to the story on p 47 again. I will ultimately want the process X_t there to be the dividend process of a single stock in a CARA agent equilibrium, but let's look again at what we have:

$$dX_t = \sigma\, dW_t + \mu_t\, dt$$

$$d\mu_t = \varepsilon\sigma\, dW_t + \alpha\mu_t\, dt \qquad (\alpha \equiv \varepsilon - \lambda)$$

is a linear Gaussian system for $Z_t = \begin{pmatrix} X_t \\ \mu_t \end{pmatrix}$, evolving as

$$dZ_t = \sigma \begin{pmatrix} 1 \\ \varepsilon \end{pmatrix} dW_t + A Z_t\, dt$$

$A = uv^T$, $\quad u = \begin{pmatrix} 1 \\ \alpha \end{pmatrix}, v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. We can calculate

$$\exp(tA) = I + \frac{uv^T}{\alpha}(e^{\alpha t} - 1)$$

Therefore

$$\exp(-tA)Z_t - Z_0 = \int_0^t \sigma \exp(sA) \begin{pmatrix} 1 \\ \varepsilon \end{pmatrix} dW_s$$

$$= \sigma \int_0^t \left\{ \begin{pmatrix} 1 \\ \varepsilon \end{pmatrix} + \frac{e^{-\alpha s} - 1}{\alpha}\, \varepsilon \begin{pmatrix} 1 \\ \alpha \end{pmatrix} \right\} dW_s$$

$$= \sigma \begin{pmatrix} 1 - \varepsilon/\alpha \\ 0 \end{pmatrix} W_t + \sigma \begin{pmatrix} \varepsilon/\alpha \\ \varepsilon \end{pmatrix} \int_0^t e^{-\alpha s}\, dW_s$$

Thus if we leave aside the centring, we get the covariance from

$$Z_t = \sigma \exp(tA) \left[\begin{pmatrix} 1 - \varepsilon/\alpha \\ 0 \end{pmatrix} W_t + \begin{pmatrix} \varepsilon/\alpha \\ \varepsilon \end{pmatrix} \xi_t \right] \qquad \left(\xi_t = \int_0^t e^{-\alpha s}\, dW_s \right)$$

So if we want the covariance we calculate (assume $u \leq t$)

$$E\left[Z_u Z_t^T\right] = \sigma^2 \exp(uA) \Bigg[\left(1 - \frac{\varepsilon}{\alpha}\right)^2 \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix} + \frac{\varepsilon}{\alpha} \begin{pmatrix} 1-\varepsilon/\alpha & \alpha - \varepsilon \\ 0 & 0 \end{pmatrix} \frac{1 - e^{-\alpha u}}{\alpha}$$

$$+ \frac{1 - e^{-\alpha u}}{\alpha} \frac{\varepsilon}{\alpha} \begin{pmatrix} 1 - \varepsilon/\alpha & 0 \\ \alpha - \varepsilon & 0 \end{pmatrix} \frac{1 - e^{-\alpha u}}{\alpha} + \frac{\varepsilon^2}{\alpha^2} \begin{pmatrix} 1 & \alpha \\ \alpha & \alpha^2 \end{pmatrix} \frac{1 - e^{-2\alpha u}}{\alpha} \Bigg] \exp(tA^T)$$

If we want the covariance just of X_u, X_t, we obtain (Maple)

$$\boxed{\left(1 - \frac{\varepsilon}{\alpha}\right)^2 u + \frac{\varepsilon}{\alpha^2}(e^{\alpha t} + e^{\alpha u})(1 - e^{-\alpha u}) + \frac{\varepsilon^2}{2\alpha^3} \left\{ e^{\alpha t}(e^{\alpha u} - e^{-\alpha u})^2 - 2(e^{\alpha u} - 1) \right\}.}$$

Questions

1) In a stochastic optimal control problem it is often easier to characterize the dual value function than the primal; can we do a verification proof working from the dual value function?

2) Dym + McKean p21: if f is an entire function of finite type, then

$$f(z) = c_1 \exp(c_2 z)\, z^m \prod_{n\geq 1} \left(1 - z/\omega_n\right) \exp(z/\omega_n).$$

3) Jesper Andreasen says that if you just are given some call option prices $C(T_i, K_j)$ for finitely many strikes + expiries, then there is no known good way to find a function $\bar{C}$ which is nice + smooth, also a call price surface, which matches the given prices exactly at (T_i, K_j).

4) Another question from Jesper Andreasen (which he says Avellaneda has solved). Suppose you have a perpetual American option which pays on exercise the value $S_\tau - S_{\tau-a}$ where $a>0$ is known and fixed; how would you exercise it?

5) Nice talk in Aarhus by Kallsen on work with Mühle-Karbe. You have

$$dw_t = r w_t\, dt + \varphi_t\,(dS_t - rS_t\, dt) - \varepsilon S_t\, |d\varphi|_t - c_t\, dt$$

and try to optimise

$$E\left[\int_0^T U(s, c_s)\, ds + F(W_T)\right]$$

They get asymptotics, but the intuition wasn't very clear to me.