Submitted to PTRF

(52)

INTERACTING BROWNIAN PARTICLES AND THE WIGNER LAW

by

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Abstract: In this paper, we study interacting diffusing particles governed by the stochastic differential equations $dX_j(t) = \sigma_n dB_j(t) - \nabla_j \phi_n(X_1, \ldots, X_n)dt,\ \ j = 1, 2, \ldots, n$. Here the B_j are independent Brownian motions in $\mathbb{R}^d$, and $\phi_n(x_1, \ldots, x_n) = \alpha_n \sum\sum_{i \neq j} V(x_i - x_j) + \theta_n \sum_i U(x_i)$. The potential V has a singularity at 0 strong enough to keep the particles apart, and the potential U serves to keep the particles from escaping to infinity. Our interest is in the behaviour as the number of particles increases without limit, which we study through the empirical measure process. We prove tightness of these processes in a very general setting, and then take the special case of $d = 1, V(x) = -\log |x|, U(x) = x^2/2$ where it is possible to prove uniqueness of the limiting evolution and deduce that a limiting measure-valued process exists. This process is deterministic, and converges to the Wigner law as $t \to \infty$. Some information on the rates of convergence is derived, and the case of a Cauchy initial distribution is analysed completely.

Keywords and phrases: interacting Brownian particles, Wigner semi-circle law, empirical measure process, weak convergence.

1991 AMS subject classifications: Primary 60K35, 60F05, 60H10; Secondary 62E20.

1. Introduction

We are going to consider in this paper systems of interacting Brownian particles in $\mathbb{R}^d$ (but, very soon, $\mathbb{R}$) governed by

(1) $$dX_j(t) = \sigma_n dB_j(t) - \nabla_j \phi_n(X_1, \ldots, X_n)dt, \quad j = 1, 2, \ldots, n,$$

where the B_j are independent Brownian motions in $\mathbb{R}^d$, and the interaction potential ϕ_n is of the form

$$\phi_n(x_1, \ldots, x_n) = \alpha_n \sum\sum_{i \neq j} V(x_i - x_j) + \theta_n \sum_i U(x_i).$$

* Supported by SERC grant number GR/H 00444

Such equations have been studied by many others, going back at least to McKean [3]; see Sznitman [8] for an overview and many other references. Our interest here is in examples where the interaction potential $V(x-y)$ blows up in some way as $x \to y$, but is otherwise reasonably smooth. The external potential U will also be assumed to be well-behaved. The singularity of V destroys most of the analysis done on nicer examples. For one thing, questions of existence and uniqueness of solutions to the n-particle system (1) become nontrivial, unless it is possible to prove that particles never meet. *Does the equation* (1) *have some kind of "strong law" limit behaviour as the number of particles* $n \to \infty$? Such a limit process would have to be characterised in terms of the limit of the empirical measure processes,

$$\mu_t^n \equiv \frac{1}{n}\sum_{j=1}^n \delta_{X_j(t)}.$$

Elementary calculations with Itô's formula (assuming that there exists a pathwise-unique strong solution to (1) with the property that the particles never collide) give

$$\text{(2)} \qquad d\langle \mu_t^n, f\rangle = \frac{\sigma_n}{n}\sum_{j=1}^n \nabla f(X_j)dB_j + \frac{\sigma_n^2}{2}\langle \mu_t^n, \Delta f\rangle dt$$

$$-\frac{1}{n}\sum_{j=1}^n \nabla f(X_j)\{2\alpha_n \sum_{r\neq j}\nabla V(X_j - X_r)\}dt - \theta_n\langle \mu_t^n, \nabla f\cdot \nabla U\rangle dt,$$

where $\langle \mu, f\rangle \equiv \int f(x)\mu(dx)$, and $f \in C_b^2(\mathbf{R}^d)$. Making now the assumption that

$$V(x) = V(-x), \;\; x \in \mathbf{R}^d,$$

we see that

$$\text{(3)} \;\; d\langle \mu_t^n, f\rangle = dM_t^n - n\alpha_n\left(\int\int_{\{x\neq y\}}(\nabla f(x) - \nabla f(y))\nabla V(x-y)\mu_t^n(dx)\mu_t^n(dy)\right)dt$$

$$-\theta_n\langle \mu_t^n, \nabla f\cdot\nabla U\rangle dt + \frac{\sigma_n^2}{2}\langle \mu_t^n, \Delta f\rangle dt,$$

where $dM_t^n \equiv n^{-1}\sigma_n\sum_{j=1}^n \nabla f(X_j)dB_j$ is (the differential of) a continuous martingale, with

$$\frac{d[M^n]_t}{dt} = \frac{\sigma_n^2}{n^2}\sum_{j=1}^n |\nabla f(X_j)|^2 .$$

It is now tempting to conjecture that if $\theta_n = \theta$, $\alpha_n = \alpha/2n$, and $\sigma_n^2 \to \sigma^2 \ (n \to \infty)$, then the empirical measure processes (μ_t^n) should converge to a measure-valued process (μ_t) satisfying

$$(4) \qquad d\langle \mu_t, f\rangle = -\frac{\alpha}{2}\left(\int\int_{\{x\neq y\}} (\nabla f(x) - \nabla f(y))\nabla V(x-y)\mu_t(dx)\mu_t(dy)\right) dt$$

$$-\theta\langle \mu_t, \nabla f \cdot \nabla U\rangle dt + \frac{\sigma^2}{2}\langle \mu_t, \Delta f\rangle dt,$$

for all $f \in C_b^2(\mathrm{R}^d)$. The quadratic variation of the martingales M^n disappears in the limit, leaving an entirely *deterministic* evolution equation. There are, of course, numerous questions to be answered before this heuristic can become a proof:

(5.i) *Is the finite particle process well defined?*
(5.ii) *Is the double integral on the right-hand side of* (4) *well defined?*
(5.iii) *Is the family* $\{(\mu_t^n)_{t\geq 0}; n = 1, 2, \ldots\}$ *tight?*

and very importantly

(5.iv) *Does* (4) *characterise* (μ_t) *uniquely?*

The way we have written the double intergral in (4) helps to answer (5.ii), in the sense that if the singularity of V at 0 is not too bad, then it will be neutralised by the term $\nabla f(x) - \nabla f(y)$. The issue of tightness is not too hard in practice, but, as with the martingale-problem method applied to multidimensional diffusions (see Stroock & Varadhan [7]), it is the uniqueness assertion which is the hard work. We have as yet no general results; this paper is devoted to the study of one particular example. This example is in dimension $d = 1$, taking

$$(6) \quad V(x) = -\log|x|, \ U(x) = \frac{1}{2}x^2, \ \alpha_n = \frac{\alpha}{2n} > 0, \ \theta_n = \theta > 0, \ \sigma_n = \left(\frac{2\alpha}{n}\right)^{1/2},$$

so that (1) reduces to

$$(7) \qquad dX_j = \sqrt{\frac{2\alpha}{n}}dB_j + \frac{\alpha}{n}\sum_{r\neq j}\frac{dt}{X_j - X_r} - \theta X_j dt.$$

This random motion of particles with electrostatic repulsion and linear restoring force arises in a natural way in the study of the eigenvalues of a randomly-diffusing symmetric matrix; see McKean [3], Dyson [2], Norris, Rogers & Williams [4], and Pauwels & Rogers [5], p. 254. The eigenvalues of a randomly-diffusing symmetric matrix obey the equation (7) with $\theta = 0$. The case $\theta > 0$ just corresponds to a matrix diffusion of Ornstein-Uhlenbeck processes, rather than Brownian motions.

This interacting SDE has recently been studied by Chan[1], who uses different methods to establish the main result under certain additional assumptions. We were unable to follow the argument provided in a number of places. The approach adopted here is more concrete, in that it exploits special features of the example studied, and makes clear the parts of the argument which will not go through in a general setting.

The plan of the rest of the paper is as follows. In §2, we prove that (under suitable conditions on the coefficients) provided the particles start at different places, they never meet; this is established for general C^2 external potential U and the logarithmic potential V. The main result of §3 is that tightness of $\{(\mu_t^n)_{t\geq 0} : n \in \mathbb{N}\}$ is equivalent to the much simpler task of showing tightness of each of the real-valued processes $(\langle \mu_t^n, f_j \rangle)_{t \geq 0}$ for some suitable sequence (f_j) of test functions. From this we deduce the tightness of the empirical processes. The most interesting part of the current work is in §§4-5, where we exploit the special features of the problem. In §4, we take $U(x) = x^2/2$, $f(x) = (x-z)^{-1}$, where $z \in \mathbb{H} \equiv \{w \in \mathbb{C} : \text{ Im } w > 0\}$, and we use (4); a little algebra yields the PDE

$$\begin{cases} \frac{\partial}{\partial t} M_t(z) = (\alpha M_t(z) + \theta z) \frac{\partial}{\partial z} M_t(z) + \theta M_t(z) \\ M_0(z) = \int \frac{\mu_0(dv)}{v-z} \end{cases}$$

where

$$M_t(z) \equiv \int \frac{\mu_t(dv)}{v-z}.$$

It is the analysis of this which provides the proof of uniqueness (5.iv). In §5, we pursue the analysis further to complete the main result of this paper:

THEOREM 1. *Assume the explicit form* (6) *for the potential* ϕ.

(i) *The limiting measure-valued process*

$$(\mu_t)_{t\geq 0} = \text{w-lim}_{n\to\infty} (\mu_t^n)_{t \geq 0}$$

exists and is the unique continuous probability-measure-valued process satisfying

$$\langle \mu_t, f \rangle = \langle \mu_0, f \rangle - \frac{\alpha}{2} \int_0^t ds \Big(\int \int \frac{f'(x) - f'(y)}{x-y} \mu_s(dx) \mu_s(dy) \Big) - \theta \int_0^t ds \langle \mu_s, U' f' \rangle.$$

(ii) *As* $t \to \infty$,

$$\mu_t \Rightarrow \mu_W,$$

where μ_W *is the Wigner semicircle law with density*

$$\frac{\theta}{\pi\alpha}\left(\frac{2\alpha}{\theta} - x^2\right)^{\frac{1}{2}} I_{(|x| \le \sqrt{2\alpha/\theta})}.$$

We then exploit the explicit form for the solution to the PDE for M to obtain more detailed information on the rate of convergence in Theorem 1, under the assumption that μ_0 has a finite first moment. Finally, we compute $M_t(z)$ explicitly when μ_0 is a Cauchy law. One interesting feature is that μ_t has no moments for any $t > 0$, even though the limit law μ_W is compactly supported.

2. Non-collision of particles.

Let us assume from now on that we are in dimension $d = 1$, and that the interaction potential is $V(x) = -\log|x|$. We have the following result.

PROPOSITION 1. *Suppose that the* $X_j(0), j = 1, \ldots, n$ *are distinct, that* U *is* C^2, *and that the processes* X_j *are defined by* (1), *at least up to the stopping time*

$$T = \inf\{t : X_i(t) = X_j(t) \text{ for some } i \neq j\}.$$

If

$$4\alpha_n \geq \sigma_n^2,$$

then $\mathbf{P}(T = \infty) = 1$.

Proof. Consider

$$\begin{aligned} d(\phi_n(X_t)) &= D_j\phi_n(X_t)dX_j + \frac{1}{2}\sigma_n^2 \sum_j \Delta_j\phi_n(X_t)dt \\ &= D_j\phi_n(X_t)\sigma_n dB_j + \left\{ \frac{\sigma_n^2}{2} \sum_j \Delta_j\phi_n(X_t) - \sum_j D_j\phi_n(X_t)^2 \right\} dt. \end{aligned}$$

Since U is C^2, we have

$$\begin{aligned} D_j\phi_n &= -2\alpha_n \sum_{r \neq j} \frac{1}{x_j - x_r} + \theta_n U'(x_j), \\ \Delta_j\phi_n &= 2\alpha_n \sum_{r \neq j} \frac{1}{(x_j - x_r)^2} + \theta_n U''(x_j), \end{aligned}$$

so that the drift term is

$$
\begin{aligned}
&\frac{\sigma_n^2}{2}\sum_j \Delta_j\phi_n - \sum_j D_j\phi_n(X)^2 \\
&\quad = \alpha_n\sigma_n^2\sum\sum\nolimits_{r\neq j}\frac{1}{(x_j-x_r)^2} + \frac{\sigma_n^2\theta_n}{2}\sum_j U''(x_j) \\
&\qquad - \sum_j\left\{2\alpha_n\sum_{r\neq j}\frac{1}{x_j-x_r} - \theta_n U'(x_j)\right\}^2 \\
&\quad = \alpha_n\sigma_n^2\sum\sum\nolimits_{r\neq j}\frac{1}{(x_j-x_r)^2} - (2\alpha_n)^2\sum\sum\nolimits_{r\neq j}\frac{1}{(x_j-x_r)^2} \\
&\qquad - (2\alpha_n)^2\sum\sum\sum\nolimits_{j,p,r \text{ distinct}}\frac{1}{(x_j-x_r)(x_j-x_p)} \\
&\qquad + 4\alpha_n\theta_n\sum_j U'(x_j)\sum_{r\neq j}\frac{1}{x_j-x_r} - \theta_n^2\sum_j U'(x_j)^2 + \frac{\sigma_n^2\theta_n}{2}\sum_j U''(x_j) \\
&= (\alpha_n\sigma_n^2 - 4\alpha_n^2)\sum\sum\nolimits_{r\neq j}\frac{1}{(x_j-x_r)^2} + 2\alpha_n\theta_n\sum\sum\nolimits_{r\neq j}\frac{U'(x_j)-U'(x_r)}{x_j-x_r} \\
&\qquad\qquad -\theta_n^2\sum_j U'(x_j)^2 + \frac{\sigma_n^2\theta_n}{2}\sum_j U''(x_j),
\end{aligned}
$$

because the triple sum vanishes, as one sees by permuting the indices j, p, and r in the sum. The condition

$$4\alpha_n \geq \sigma_n^2,$$

ensures that the drift term is bounded above. If it were the case that $T < \infty$, then we would have $\lim_{t\uparrow T}\phi_n(X_t) = +\infty$, and so the martingale part of $\phi_n(X_t)$ would have to explode to $+\infty$ as $t \uparrow T$. This forces $\liminf_{t\uparrow T}\phi_n(X_t) = -\infty$; see, for example, Corollary IV.34.13 of Rogers & Williams [6]. But this is impossible if ϕ_n is bounded below, which we could always arrange by modifying U outside a large compact set. The conclusion follows. □

3. Tightness.

In this section, we prove the tightness of the family $\{(\mu_t^n)_{t\geq 0} : n \in \mathbb{N}\}$, and that any limiting measure-valued process satisfies the evolution equation (4).

Let us pick bounded C^∞ functions $f_j : \mathbb{R} \to \mathbb{C}$ $(j = 1, 2, \ldots)$ with the property

$$\langle \mu, f_j\rangle = \langle \mu', f_j\rangle \text{ for all } j \;\Rightarrow\; \mu = \mu', \tag{8}$$

and pick a C^∞ function $f_0 : \mathbb{R} \to [1, \infty)$ with the properties

$$f_0(x) = f_0(-x),\ f_0(x) \uparrow \infty \text{ as } x \uparrow \infty,\ x \in \mathbb{R}^+.$$

We are going to consider the Polish space

$$S = \{ \text{ probabilities } \mu \text{ on } \mathbb{R} : \langle \mu, f_0 \rangle < \infty \},$$

with the topology given by the metric

$$d(\mu, \mu') \equiv \sum_{n \geq 0} 2^{-n} (1 \wedge | \langle \mu - \mu', f_n \rangle |),$$

obtained by identifying

$$S \cong \prod_{j=0}^{\infty} S_j$$

$$\mu \mapsto (\langle \mu, f_j \rangle)_{j=0}^{\infty}$$

where $S_0 = \mathbb{R}, S_j = \mathbb{C}, (j \geq 1)$. Notice that the topology on S is stronger than the topology of weak convergence; if $d(\mu_n, \mu) \to 0$, then

$$\langle \mu_n, f_0 \rangle \to \langle \mu, f_0 \rangle,$$

so the sequence (μ_n) is tight. Therefore it contains a convergent subsequence, whose limit point μ^* must satisfy $\langle \mu^*, f_j \rangle = \langle \mu, f_j \rangle, \forall j \geq 0$; thus $\mu_* = \mu$, and $\mu_n \Rightarrow \mu$.
Now we shall consider the Polish space

$$C(\mathbb{R}^+, S) \cong \prod_{j=0}^{\infty} C(\mathbb{R}^+, S_j) \equiv \prod_{j=0}^{\infty} \tilde{S}_j \equiv \tilde{S}.$$

(This is Polish, because it is the countable product of Polish spaces, which is Polish). The empirical measure process $(\mu_t^n)_{t \geq 0}, n = 1, 2, \ldots$ which we consider will be random elements of $\tilde{S}$.

If $\pi_j : \tilde{S} \to \tilde{S}_j$ is the j^{th} co-ordinate projection, we have the following simple result.

LEMMA 1. *A family $(m_\alpha)_{\alpha \in A}$ of probabilities on $\tilde{S}$ is tight if and only if, for each j, $(m_\alpha \circ \pi_j^{-1})_{\alpha \in A}$ is tight on $\tilde{S}_j$.*
Proof. "$\Rightarrow$". If $K \subset \tilde{S}$ is compact, then

$$K_j \equiv \pi_j(K) = \{ y \in S_j : (x_0, \ldots, x_{j-1}, y, x_{j+1}, \ldots) \in K \text{ for some } x_i \}$$

is compact, and

$$K \subset \prod_j K_j.$$

Thus if $m_\alpha(K) \geq 1 - \epsilon, \forall \alpha$, we have $m_\alpha(\prod_j K_j) \geq 1 - \epsilon, \forall \alpha$. So, for every $\alpha \in A$,

$$1 - \epsilon \leq m_\alpha(\prod_j K_j) \leq m_\alpha((\prod_{r \neq j} \tilde{S}_r) \times K_j) = (m_\alpha \circ \pi_j^{-1})(K_j),$$

so that $(m_\alpha \circ \pi_j^{-1})_{\alpha \in A}$ is tight.
"$\Leftarrow$". If we take $K_j \subset \tilde{S}_j$ compact, $(m_\alpha \circ \pi_j^{-1})(K_j) \geq 1 - 2^{-j-1}\epsilon$, and set $K = \prod_j K_j$, we shall have

$$m_\alpha(K^c) = m_\alpha(\bigcup_{j \geq 0}(K_j^c \times \prod_{r \neq j} \tilde{S}_r)) \leq \sum_{j \geq 0}(m_\alpha \circ \pi_j^{-1})(K_j^c) = \epsilon. \qquad \square$$

The conclusion therefore is that in order to deduce the tightness of the sequence of measure-valued process $\{(\mu_t^n)_{t \geq 0}; n \in \mathbb{Z}^+\}$, it is sufficient to prove that for each j the continuous real-valued processes

$$(\langle \mu_t^n, f_j \rangle)_{t \geq 0}, \quad n \in \mathbb{Z}^+$$

are tight. This is a lot easier!

From (2) and (3) we have (using the fact that $V(x) = -\log|x|$)

$$d\langle \mu_t^n, f \rangle = \frac{\sigma_n}{n} \sum_{j=1}^n f'(X_j) dB_j + \langle \mu_t^n, \frac{1}{2}\sigma_n^2 f'' - \theta_n U' f' \rangle dt$$
$$+ n\alpha_n dt \int\int_{\{x \neq y\}} \frac{f'(x) - f'(y)}{x - y} \mu_t^n(dx) \mu_t^n(dy),$$

and the final term is

$$\left\{ n\alpha_n \int\int \frac{f'(x) - f'(y)}{x - y} \mu_t^n(dx) \mu_t^n(dy) - \alpha_n \langle \mu_t^n, f'' \rangle \right\} dt,$$

so that

$$d\langle \mu_t^n, f \rangle = \frac{\sigma_n}{n} \sum_j f'(X_j) dB_j + \langle \mu_t^n, (\frac{1}{2}\sigma_n^2 - \alpha_n) f'' - \theta_n U' f' \rangle dt$$
$$+ n\alpha_n dt \int\int \frac{f'(x) - f'(y)}{x - y} \mu_t^n(dx) \mu_t^n(dy).$$

Now let us adopt the assumptions (6) concerning the parameters: $\theta_n = \theta, \alpha_n = \alpha/2n, \sigma_n = (2\alpha/n)^{1/2}, \forall n$. This ensures non-collision of particles, by Proposition 1. Let us also assume from now on that

$$f_j, f_j'', U'f_j' \text{ are bounded for all } j \in \mathbb{Z}^+.$$

Thus we have

$$\langle \mu_t^n, f_j \rangle - \langle \mu_0^n, f_j \rangle - \int_0^t \left(\frac{\alpha}{2} \int\int \frac{f_j'(x) - f_j'(y)}{x-y} \mu_s^n(dx)\mu_s^n(dy) \right) + \int_0^t \theta \langle \mu_s^n, U'f_j' \rangle ds$$

$$(9) \qquad = \sqrt{\frac{2\alpha}{n^3}} \int_0^t \sum_{r=0}^n f_j'(X_r(s))dB_r(s) + \int_0^t \langle \mu_s^n, (\frac{\sigma_n^2}{2} - \alpha_n) f_j'' \rangle ds.$$

Now the laws of the processes on the right-hand side of (9) are easily shown to be tight; using next the assumption that $U'f_j'$ and f_j'' are bounded, the tightness of $\{(\langle \mu_t^n, f_j \rangle)_{t \geq 0} : n \in \mathbb{Z}^+\}$ follows from this for every $j \geq 1$. Tightness also follows for $j = 0$ if we have

$$\langle \mu_0^n, f_0 \rangle \to \text{ finite limit } (n \to \infty).$$

So let us suppose that the initial distributions μ_0^n have the property $\langle \mu_0^n, f_0 \rangle \leq K$ for some K, for all n. For given μ_0, we could always find μ_0^n and f_0 to satisfy this and the other conditions, and this then gives the tightness for $j = 0$ also.

4. Convergence to the limit process when $n \to \infty$.

In this section, we show the weak convergence of μ^n when $n \to \infty$, to a measure-valued process satisfying the evolution equation (4). By the tightness, we have at least that $\mu^n \Rightarrow \mu$ along a subsequence. Any such limit process $\mu = (\mu_t; t \in \mathbb{R}^+)$ satisfies

$$(10) \quad \langle \mu_t, f_j \rangle = \langle \mu_0, f_j \rangle + \frac{\alpha}{2} \int_0^t \int\int \frac{f_j'(x) - f_j'(y)}{x-y} \mu_s(dx)\mu_s(dy)ds - \theta \int_0^t \langle \mu_s, U'f_j' \rangle ds.$$

Now let us take up the remaining assumption of (6), that $U(x) = x^2/2$, and set

$$f_j(x) = \frac{1}{x - z_j},$$

where the z_j run through $(\mathbb{Q} \times \mathbb{Q}) \cap \mathbb{H}$. Then

$$M_t(z) \equiv \int \frac{\mu_t(dv)}{v - z}, \quad z \in \mathbb{H}$$

obeys

$$M_t(z) = M_0(z) + \alpha \int_0^t ds \int\int \frac{\mu_s(dx)\mu_s(dy)}{(x-z)(y-z)^2} + \theta \int_0^t ds \int \frac{x}{(x-z)^2}\mu_s(dx),$$

firstly for $z \in (\mathbb{Q} \times \mathbb{Q}) \cap \mathbf{H}$ from (10), then for every $z \in \mathbf{H}$ by continuity. Simple manipulations show now that M satisfies the PDE:

$$\text{(11)} \qquad \begin{cases} \frac{\partial}{\partial t} M_t(z) = (\alpha M_t(z) + \theta z)\frac{\partial}{\partial z} M_t(z) + \theta M_t(z) \\ M_0(z) = \int \frac{\mu_0(dv)}{v-z} \end{cases}$$

If we can show that (11) has a unique solution, then by property (8), we actually prove the convergence of μ^n - this time not only up to subsequences! - and thereby part (i) of Theorem 1.

We first need a lemma.

LEMMA 2. *For every $z \in \mathbf{H}, \lambda > 0$ and $\sigma > 0$, there exists a unique $r = r(\lambda, \sigma, z)$ such that*

$$z = \lambda r - \sigma \int \frac{\mu_0(dv)}{v - r} \equiv \lambda r - \sigma M_0(r).$$

Proof. (Uniqueness) If $r_1 \neq r_2 \in \mathbf{H}$ satisfy

$$\text{(12)} \qquad z = \lambda r_j - \sigma \int \frac{\mu_0(dv)}{v - r_j}, \quad (j = 1, 2),$$

then

$$\lambda(r_1 - r_2) = \sigma \int \mu_0(dv) \left(\frac{1}{v - r_1} - \frac{1}{v - r_2} \right) = \sigma \int \mu_0(dv) \frac{r_1 - r_2}{(v - r_1)(v - r_2)},$$

implying

$$\int \frac{\mu_0(dv)}{(v - r_1)(v - r_2)} = \frac{\lambda}{\sigma}.$$

Writing $r_j = a_j + ib_j, (j = 1, 2), b_j > 0$, we deduce that

$$\text{(13)} \qquad \int \frac{(v - a_1)(v - a_2) - b_1 b_2}{| v - r_1 |^2 | v - r_2 |^2} \mu_0(dv) = \frac{\lambda}{\sigma};$$

$$(14) \qquad \int \frac{b_1(v-a_2)+b_2(v-a_1)}{\mid v-r_1\mid^2\mid v-r_2\mid^2}\mu_0(dv)=0.$$

It follows from (14) that

$$(15) \qquad \int \frac{v}{\mid v-r_1\mid^2\mid v-r_2\mid^2}\mu_0(dv)=\frac{a_2b_1+a_1b_2}{b_1+b_2}\int \frac{\mu_0(dv)}{\mid v-r_1\mid^2\mid v-r_2\mid^2},$$

and since

$$(v-a_1)(v-a_2)-b_1b_2=\mid v-r_2\mid^2+(a_2-a_1)(v-a_2)-b_2^2-b_1b_2,$$

we obtain from (13) that

$$\begin{aligned}
\frac{\lambda}{\sigma} &= \int \frac{\mu_0(dv)}{\mid v-r_1\mid^2}+(a_2-a_1)\int \frac{v\mu_0(dv)}{\mid v-r_1\mid^2\mid v-r_2\mid^2} \\
&\quad -(a_2(a_2-a_1)+b_2^2+b_1b_2)\int \frac{\mu_0(dv)}{\mid v-r_1\mid^2\mid v-r_2\mid^2} \\
&\overset{(15)}{=} \int \frac{\mu_0(dv)}{\mid v-r_1\mid^2}+\frac{(a_2-a_1)(a_2b_1+a_1b_2)}{b_1+b_2}\int \frac{\mu_0(dv)}{\mid v-r_1\mid^2\mid v-r_2\mid^2} \\
&\quad -(a_2(a_2-a_1)+b_2^2+b_1b_2)\int \frac{\mu_0(dv)}{\mid v-r_1\mid^2\mid v-r_2\mid^2} \\
&\overset{(12)}{=} \frac{\lambda}{\sigma}-\frac{\mathrm{Im}z}{\sigma b_1}-\frac{b_2}{b_1+b_2}[(a_2-a_1)^2+(b_1+b_2)^2]\int \frac{\mu_0(dv)}{\mid v-r_1\mid^2\mid v-r_2\mid^2} \\
&< \frac{\lambda}{\sigma},
\end{aligned}$$

which is a contradiction.

(Existence) Let $z_0 \in \mathsf{H}$ be fixed and let

$$\phi(r)\equiv z_0+\sigma\int \frac{\mu_0(dv)}{v-r}, \quad r\in \mathsf{H}.$$

So $\phi:\mathsf{H}\to\mathsf{H}$. We only have to show that $\lambda r-\phi(r)$ has a zero in H. If not, we first notice that, when $0<\mathrm{Im}(r)<\mathrm{Im}(z_0)/2\lambda$,

$$\mathrm{Im}(\lambda r-\phi(r))\leq -\mathrm{Im}(z_0)/2;$$

on the other hand, when $\mathrm{Im}(r)\geq \mathrm{Im}(z_0)/2\lambda$ and $r\to\infty$, $\phi(r)$ remains bounded, so $(\lambda r-\phi(r))^{-1}$ converges to 0. Therefore, $(\lambda r-\phi(r))^{-1}$ is bounded analytic in H. Let

$$h(r) \equiv \frac{1}{\lambda r - \phi(r)}, \quad r \in \mathbf{H}.$$

It follows that, for every $r \in \mathbf{H}$ and every $\epsilon > 0$,

$$h(r + i\epsilon) = \int_{-\infty}^{+\infty} \frac{dx}{\pi} \frac{\mathrm{Im}h(x + i\epsilon)}{x - r}. \tag{16}$$

For $x \in \mathbf{R}$ and $\epsilon < \mathrm{Im}(z_0)/2\lambda$, the imaginary part of $\lambda(x+i\epsilon)-\phi(x+i\epsilon)$ is smaller than $-\mathrm{Im}(z_0)/2$, which implies that, $h(x + i\epsilon)$ has positive imaginary part. Consequently, we have from (16) that $\mathrm{Im}h(r+i\epsilon) > 0$ for all $r \in \mathbf{H}$, and then, as $0 < \epsilon < \mathrm{Im}(z_0)/\lambda$ is arbitrary, $h(r) \equiv (\lambda r - \phi(r))^{-1} \in \mathbf{H}$. Thus $\phi(r) - \lambda r \in \mathbf{H}$ for all $r \in \mathbf{H}$, which is absurd. □

It is time to show the uniqueness of the solution to (11). Suppose $M_t(z)$ is a solution to (11). If we define

$$N \equiv M + \frac{\theta}{\alpha} z,$$

then re-expressing (11) yields

$$\begin{cases} \frac{\partial}{\partial t} N = \alpha N \frac{\partial N}{\partial z} - \frac{\theta^2}{\alpha} z \\ N_0(z) = \int \frac{\mu_0(dv)}{v-z} + \frac{\theta}{\alpha} z \end{cases} \tag{17}$$

Consider the following differential equation:

$$\dot{z}_t = -\alpha N_t(z_t), \tag{18}$$

with some given starting value $z_0 \in \mathbf{H}$. Then there exists a unique solution, defined up to the time when z_t first exits $\mathbf{H}$. Notice that $\mathrm{Im}(z_t)$ is decreasing. Differentiating with respect to t, we get that

$$\begin{aligned} \ddot{z}_t &= -\alpha \dot{N}_t(z_t) - \alpha \dot{z}_t \frac{\partial N}{\partial z} \\ &= -\alpha^2 N \frac{\partial N}{\partial z} + \theta^2 z_t + \alpha^2 N \frac{\partial N}{\partial z} \\ &= \theta^2 z_t, \end{aligned}$$

implying that

$$\begin{aligned} z_t &= A\cosh\theta t + B\sinh\theta t \\ &= z_0\cosh\theta t - \frac{\alpha}{\theta}N_0(z_0)\sinh\theta t \\ &= z_0 e^{-\theta t} - \frac{\alpha}{\theta}M_0(z_0)\sinh\theta t. \end{aligned}$$

By Lemma 2, given any $t \geq 0$, and any $\omega \in \mathsf{H}$, there is a unique choice of $z_0 = z_0(t,\omega), z_t = \omega$. Thus

$$N_t(\omega) = -\frac{1}{\alpha}\dot{z}_t = -\frac{\theta}{\alpha}z_0(t,\omega)\sinh\theta t + N_0(z_0(t,\omega))\cosh\theta t, \tag{19}$$

proving the uniqueness of the solution N to (17), hence the uniqueness of the solution M. The first part of Theorem 1 is now proved.

5. Convergence of the limit process to the Wigner law.

We have proved that $(\mu_t^n)_{t\geq 0} \Rightarrow (\mu_t)_{t\geq 0}$ where $(\mu_t)_{t\geq 0}$ is characterised as the unique solution to (10). We shall prove in this section that, when $t \to \infty$,

$$\mu_t \Rightarrow \mu_W,$$

where μ_W is the Wigner law with density

$$\frac{\theta}{\pi\alpha}\left(\frac{2\alpha}{\theta} - x^2\right)^{\frac{1}{2}} I_{(|x|\leq\sqrt{2\alpha/\theta})}.$$

By Lemma 2, for every $z \in \mathsf{H}$ and $t \geq 0$, there exists a unique $z_0 = z_0(t,z)$ such that

$$z = e^{-\theta t}z_0 - \frac{\alpha}{\theta}M_0(z_0)\sinh\theta t. \tag{20}$$

Recall

$$N = M + \frac{\theta}{\alpha}z.$$

Developing (19) yields

$$\begin{aligned} N_t(z) &= -\frac{\theta}{\alpha}z_0(t,z)\sinh\theta t + N_0(z_0(t,z))\cosh\theta t \\ &= \frac{\theta}{\alpha}z_0(t,z)e^{-\theta t} + M_0(z_0(t,z))\cosh\theta t \\ &= \frac{\theta}{\alpha}(z + \frac{\alpha}{\theta}M_0(z_0(t,z))\sinh\theta t) + M_0(z_0(t,z))\cosh\theta t, \end{aligned} \tag{21}$$

from which immediately

$$M_t(z) = e^{\theta t} M_0(z_0(t,z)).$$

Fixing z, let $e^{-\theta t} z_0(t,z) = a(t) + ib(t)$. Then from (20), $b(t) \geq \mathrm{Im}(z) > 0$, for every $t \geq 0$. Moreover,

$$\mathrm{Im}(z) \geq b(t) - \frac{\alpha}{\theta} \frac{\sinh \theta t}{e^{\theta t}} \frac{1}{b(t)} \geq b(t) - \frac{\alpha}{\theta b(t)},$$

implying that

$$b(t) \leq \mathrm{Im}(z) + \sqrt{\alpha/\theta}.$$

Thus

$$0 < \mathrm{Im}(z) \leq b(t) \leq \mathrm{Im}(z) + \sqrt{\alpha/\theta}.$$

We also claim that

$$\sup_{t \geq 0} \mid a(t) \mid < +\infty.$$

Indeed, if there existed $t_n \uparrow \infty$ such that $a(t_n) \to \infty$, then by dominated convergence, we would have

$$\sinh(\theta t_n) \int \frac{\mu_0(dv)}{v - z_0} \to 0.$$

Therefore

$$\mathrm{Re}(z) = a(t_n) - \frac{\alpha}{\theta} \mathrm{Re}\left(\sinh(\theta t_n) \int \frac{\mu_0(dv)}{v - z_0} \right) \to \infty,$$

which would be absurd. So, for some sequence $t_n \to \infty$, we have $a(t_n) \to a, b(t_n) \to b > 0$. By dominated convergence, writing $\omega \equiv a + ib$, we obtain that

$$z = \omega + \frac{\alpha}{2\theta\omega}.$$

The unique solution to this quadratic which lies in $\mathbf{H}$ is given by

$$\omega = \frac{z + \sqrt{z^2 - \frac{2\alpha}{\theta}}}{2} = \frac{\alpha}{\theta} \frac{1}{z - \sqrt{z^2 - \frac{2\alpha}{\theta}}}.$$

The uniqueness of (a, b) shows the convergence of $a(t)$ and $b(t)$ (not only up to subsequences, now). Again because $b(t) = \mathrm{Im}(e^{-\theta t} z_0(t,z)) \geq \mathrm{Im}(z) > 0$, by dominated convergence, we have that, when $t \to \infty$,

$$M_t(z) = e^{\theta t} \int \frac{\mu_0(dv)}{v - z_0(t,z)} \to -\frac{1}{\omega} = \frac{\theta}{\alpha}\left(\sqrt{z^2 - \frac{2\alpha}{\theta}} - z\right) = \int \frac{\mu_{\mathrm{W}}(dv)}{v - z}.$$

Finally, to deduce that $\mu_t \Rightarrow \mu_W$, consider $\{\mu_t\}_{t\geq 0}$ as a family of measures on the compact space $[-\infty, +\infty]$. If ν is any limit probability measure on $[-\infty, +\infty]$, then it follows from what we have just proved that

$$\int_{]-\infty,+\infty[} \frac{\mu_{\mathrm{W}}(dv)}{v - z} = \int_{[-\infty,+\infty]} \frac{\mu_{\mathrm{W}}(dv)}{v - z} = \int_{[-\infty,+\infty]} \frac{\nu(dv)}{v - z} = \int_{]-\infty,+\infty[} \frac{\nu(dv)}{v - z},$$

for every $z \in \mathsf{H}$, which, by property (8), implies that $\mu_{\mathrm{W}} = \nu$. The proof of Theorem 1 is completed.

Example. Let us analyse completely the only situation where one can calculate anything in a usable closed form: the case of a Cauchy initial distribution. In this case,

$$M_0(z) = \frac{-1}{z + \omega},$$

where $\omega \equiv a + ib \in \mathsf{H}$ is fixed, and the integral curve of the vector field (18) started at $z \in \mathsf{H}$ is (see (21))

$$F_t(z) \equiv F(t,z) \equiv z e^{-\theta t} + \frac{\alpha \sinh \theta t}{\theta} \frac{1}{z + \omega}, \tag{22}$$

which we may invert explicitly: for $\zeta \in \mathsf{H}$,

$$F_t^{-1}(\zeta) = \frac{1}{2}\left[\zeta e^{\theta t} - \omega + \left\{(\omega + \zeta e^{\theta t})^2 - \frac{2\alpha}{\theta}(e^{2\theta t} - 1)\right\}^{1/2}\right].$$

Recalling that

$$\dot{F}(t,z) = -\alpha N_t(F(t,z)),$$

we see from (22) that

$$N_t(F(t,z)) = \frac{\theta z}{\alpha} e^{-\theta t} - \frac{\cosh \theta t}{z + \omega} = \frac{\theta}{\alpha} F(t,z) - \frac{e^{\theta t}}{z + \omega},$$

implying that for $\zeta \in \mathsf{H}$,

$$\begin{aligned} N_t(\zeta) &= \frac{\theta}{\alpha}\zeta - \frac{2e^{\theta t}}{\zeta e^{\theta t} + \omega + [(\zeta e^{\theta t} + \omega)^2 - 2\alpha\theta^{-1}(e^{2\theta t} - 1)]^{1/2}} \\ &= \frac{\theta}{\alpha}\zeta - \frac{\zeta e^{\theta t} + \omega - [(\zeta e^{\theta t} + \omega)^2 - 2\alpha\theta^{-1}(e^{2\theta t} - 1)]^{1/2}}{2\alpha\theta^{-1}\sinh\theta t}. \end{aligned}$$

Our principal interest is in the behaviour of the imaginary part of this on the real axis. Writing ϵ for $e^{-\theta t}$, we have for $x \in \mathbb{R}$,

$$\begin{aligned}\operatorname{Im} N_t(x) &= \operatorname{Im} M_t(x) \\ &= \operatorname{Im} \frac{\theta}{\alpha}\frac{[(x+\omega\epsilon)^2 - 2\alpha\theta^{-1}(1-\epsilon^2)]^{1/2} - \omega\epsilon}{1-\epsilon^2}\end{aligned}$$

The asymptotics of this can be computed:

$$\begin{aligned}\operatorname{Im} M_t(x) &= \epsilon\theta\alpha^{-1}[\mid x \mid (x^2 - 2\alpha\theta^{-1})^{-1/2} - 1]\operatorname{Im}\omega + o(\epsilon) \qquad &(x^2 > 2\alpha\theta^{-1}) \\ &= \theta\alpha^{-1}(2\alpha\theta^{-1} - x^2)^{1/2} - \epsilon\,\theta\alpha^{-1} x \operatorname{Re}\omega + o(\epsilon) \qquad &(x^2 < 2\alpha\theta^{-1}).\end{aligned}$$

Another interesting feature of this is that for $t > 0$ fixed, the asymptotics in x are given by

$$\operatorname{Im} M_t(x) \sim \operatorname{Im}(\omega e^{-\theta t})/x^2, \qquad (\mid x \mid \to \infty),$$

so that for any $t > 0$, μ_t *has no moments*, even though the limit law μ_W is compactly supported!

The very explicit form of the solution to the evolution equation (18) allows us to make precise statements about the large t asymptotics. If

$$F_t(z) \equiv F(t,z) \equiv z e^{-\theta t} - \alpha\theta^{-1}\sinh\theta t M_0(z)$$

is the flow map of the solution, then we see that

$$N_t(F_t(z)) = -\frac{1}{\alpha}\dot{F}(t,z) = \frac{\theta}{\alpha}F(t,z) + M_0(z)e^{\theta t};$$

so if we can get the asymptotics of the inverse to the flow map, we can find the asymptotics of $N_t(\cdot)$ and therefore of $M_t(\cdot)$. Let us now assume

$$\int \mid x \mid \mu_0(dx) < \infty, \int x\mu_0(dx) = c. \tag{23}$$

Setting $\epsilon = e^{-\theta t}$, let us define

$$\begin{aligned}\zeta(\epsilon, z) &\equiv F(t, e^{\theta t} z) \\ &= z + \frac{\alpha}{2\theta}(1-\epsilon^2)\int \frac{\mu_0(dx)}{z - \epsilon x}\end{aligned}$$

and observe that, because of assumption (23), ζ is C^1 in a neighbourhood of $(0, z_0)$ for any $z_0 \in \mathbb{H}$. Considering the map

$$(\epsilon, z) \mapsto (\epsilon, \zeta(\epsilon, z)) \equiv \Phi(\epsilon, z),$$

this map is C^1, with differential at $(0, z_0)$ equal to

$$\begin{pmatrix} 1 & 0 \\ \frac{\alpha c}{2\theta z_0^2} & 1 - \frac{\alpha}{2\theta z_0^2} \end{pmatrix}$$

So by the inverse function theorem, the derivative of the inverse map at $(0, \zeta_0)$ is

$$[1 - \alpha/2\theta z_0^2]^{-1} \begin{pmatrix} 1 - \frac{\alpha}{2\theta z_0^2} & 0 \\ -\frac{\alpha c}{2\theta z_0^2} & 1 \end{pmatrix}$$

where $\zeta_0 = \zeta(0, z_0) = z_0 + \alpha/2\theta z_0$. Thus

$$\Phi^{-1}(\epsilon, \zeta_0) = (0, z_0) + \left(1, -\frac{\alpha c}{2\theta z_0^2 - \alpha}\right)\epsilon + o(\epsilon).$$

Therefore

$$\begin{aligned} N_t(\zeta_0) &= \frac{\theta}{\alpha}\zeta_0 + \int \frac{\mu_0(dx)}{z_0 - \frac{\alpha c \epsilon}{2\theta z_0^2 - \alpha} - \epsilon x} + o(\epsilon) \\ &= \frac{\theta}{\alpha}\zeta_0 + \frac{1}{z_0} + \epsilon \frac{2\theta c}{2\theta z_0^2 - \alpha} + o(\epsilon) \\ &= N_\infty(\zeta_0) + \epsilon \frac{2\theta c}{2\theta z_0^2 - \alpha} + o(\epsilon), \end{aligned}$$

where

$$N_\infty(\zeta) = \theta\alpha^{-1}(\zeta^2 - 2\alpha/\theta)^{1/2}.$$

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