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Séminaire de probabilités (Strasbourg), tome 14 (1980), p. 332-342
<http://www.numdam.org/item?id=SPS_1980__14__332_0>

TIME-SUBSTITUTION BASED ON FLUCTUATING ADDITIVE FUNCTIONALS

(WIENER-HOPF FACTORIZATION FOR INFINITESIMAL GENERATORS)

by

L.C.G. Rogers and David Williams

1. This note is merely a first indication of how some of the ideas in the preceding paper [2] by Barlow, Rogers, and Williams (hereafter denoted by [BRW]), extend to Markov processes with 'continuous' state-space. We hope to publish a more detailed study soon. Unusual and interesting purely-analytic problems are posed by the work. However, our main purpose is to attempt to understand what is going on in the probabilistic aspects of the subject.

Our problem has considerable practical importance (but we can make no such claims for the results presented here!) Pure-mathematical technicalities are therefore avoided. We remark however that this work (though not today's examples) forces us to acknowledge the practical usefulness of branch-points, incursions, and other 'exotica' of the general theory. Vivent les hypothèses droites!

Here, we try to convey just a whiff of the flavour of things via two concrete examples. But, for the deepest concrete work done, and on a problem which *is* important, see McKean [5].

Note. We are aware that many of the results in the present paper may be obtained via the classical Wiener-Hopf methods described for example in Bingham [3]. That our methods are (in principle!) of much wider applicability is of course evident from [BRW].

Acknowledgement. We thank Professor J.F.C. Kingman for proving our conjecture that (4.5) $\Longleftrightarrow$ (4.7), and for allowing us to publish his fine proof.

2. Let X be a nice Markov process with state-space E. Let $\{R_\lambda\}$ be the resolvent of X, defined as usual, but now <u>for all complex</u> λ <u>with</u> $\Re(\lambda) > 0$, by

$$R_\lambda f(x) \equiv \underset{\sim}{E}^x \int_0^\infty e^{-\lambda t} f(X_t)dt.$$

(Here, f is a bounded complex-valued function on E. The symbol '$\equiv$' signifies 'is defined to equal'). We use Q to denote the 'natural' infinitesimal generator of X defined as follows. If g is a bounded (complex-valued) function on E, write $g \in D(Q)$ and $Qg = f$ is f is a bounded function on E such that for some (then every) λ with $\Re(\lambda) > 0$,

$$g = R_\lambda(\lambda g - f).$$

<u>Notes</u>.

(a) Q extends the classical strong generator of Hille-Yosida theory. Meyer uses a similar (but not identical) form of generator in [6].

(b) For $g \in D(Q)$, $f = Qg$ is defined only 'modulo a set of potential zero'. Two 'versions' f_1 and f_2 of f satisfy

$$\underset{\sim}{P}^x[\text{meas}\{t: f_1(X_t) \neq f_2(X_t)\} = 0] = 1, \qquad \forall x. \qquad \square$$

Let φ be a fluctuating perfect continuous additive functional of X; by this, we mean:

(i) $t \mapsto \varphi_t$ is continuous,

(ii) φ is $\{\mathcal{F}_t\}$ adapted,

(iii) $\varphi_{s+t} = \varphi_s + \varphi_t \circ \theta_s$, $\forall s$, $\forall t$.

The case when

(2.1) $$\varphi_t = \int_0^t V(X_s)ds$$

for some function $V: E \to \mathbb{R}$ is the most important. However, cases in which φ

involves local times, and cases where φ is not of finite variation, are also of interest. For $t \geq 0$, set

$$\tau_t^+ \equiv \inf\{s : \varphi_s > t\}.$$

A standard argument based on the strong Markov property of X shows that $\tilde{X}^+$, where $\tilde{X}_t^+ \equiv X(\tau_t^+)$, is a (strong) Markov process. For $c \geq 0$, we wish to calculate the transition function $\{\tilde{P}_c^+(t)\}$, where

$$\tilde{P}_c^+(t)f(x) \equiv \underset{\sim}{E}^x[\exp(-c\tau_t^+)f \circ X(\tau_t^+)],$$

or, equivalently, the resolvent $\{\tilde{R}_c^+(\lambda)\}$, or 'natural' generator $\tilde{Q}^+$, of $\{\tilde{P}_c^+(t)\}$. When $c = 0$, we suppress c from the notation; but note that

$$\tilde{P}^+(t)f(x) \equiv \underset{\sim}{E}^x[f \circ X(\tau_t^+); \tau_t^+ < \infty].$$

Amongst interesting probabilistic problems posed by this work is the following: what form of killing of $\tilde{X}^+$ is induced by killing X at rate c ?

3. Let φ be of the form (2.1), and suppose that E^+ is closed, where $E^+ \equiv \{x \in E : V(x) \geq 0\}$. By right-continuity of paths, $\tilde{X}^+$ lives in E^+.

Suppose first that $c > 0$, and regard c as fixed. Keep [BRW] in mind, and hope for the best! So, write $g \in \mathcal{N}_{1,c}$ if $g \in D(Q)$ and

$$(3.1) \qquad Qg = \mu Vg + cg$$

for some complex number $\mu = \mu(g)$ with $\mathcal{R}(\mu) < 0$. Then, $\exp(-\mu\varphi_t - ct)g(X_t)$ is a martingale (right-continuous under the right hypotheses) which is bounded on $[0,\tau_u^+]$ for every $u \geq 0$. Apply the optional-sampling theorem at time τ_t^+ to obtain

$$(3.2) \qquad \tilde{P}_c^+(t)g^+ = \underset{\sim}{E}^{\cdot}[\exp(-c\tau_t^+)g^+ \circ \tilde{X}^+(t)] = e^{\mu t}g^+ \text{ on } E^+,$$

where g^+ denotes the restriction of g to E^+. Note that the fact that $c > 0$ takes care of difficulties associated with the possibility that $\tau_t^+ = \infty$.

Let $\mathcal{N}^+_{1,c} \equiv \{g^+ : g \in \mathcal{N}_{1,c}\}$. We say that $\mathcal{N}^+_{1,c}$ is <u>full</u> on E^+ if whenever ν is a complex-valued measure of finite total variation on E^+,

$$\int_{E^+} g^+(x)\nu(dx) = 0 \quad (\forall g^+ \in \mathcal{N}^+_{1,c}) \Longrightarrow \nu = 0.$$

(3.3) OBVIOUS LEMMA. Let $c > 0$. <u>Suppose that</u> $\mathcal{N}^+_{1,c}$ <u>is full on</u> E^+. <u>Then</u> $\{\tilde{P}^+_c(\cdot)\}$ <u>is uniquely determined by (3.2)</u>. <u>Moreover</u>, $\{\tilde{R}^+_c(\cdot)\}$ <u>is the unique subMarkovian resolvent on</u> E^+ <u>such that</u>

$$\forall g^+ \in \mathcal{N}^+_{1,c}, \quad 2\lambda\tilde{R}^+_c(\lambda)g^+ = g^+ \ \underline{\text{where}}\ \lambda = -\mu(g).$$

4. <u>Example</u>. Suppose that X is Brownian motion on $\mathbb{R}$, and that

(4.1) $$V(x) = 1 \ (x > 0); \quad 0 \ (x = 0); \quad -K \ (x < 0);$$

where $K > 0$. Then equation (3.1) takes the form:

(4.2) $$g = \lambda R_{\lambda+c}(g + Vg).$$

Now it is well known that for $\mathcal{R}(\alpha) > 0$,

$$\beta R_\alpha h(x) = \int_{\mathbb{R}} e^{-\beta|y-x|} h(y)dy, \qquad \beta \equiv (2\alpha)^{\frac{1}{2}}, \quad \mathcal{R}(\beta) > 0.$$

It is now easy to show that to obtain a bounded solution g of (4.2) we must choose λ <u>real</u> with $\lambda > c$, and that we then have (with the normalisation $g(0) = 1$) $g = g_{c,\lambda}$, where

$$g_{c,\lambda}(x) \equiv \cos[(2\lambda-2c)^{\frac{1}{2}}x] + \frac{(2K\lambda+2c)^{\frac{1}{2}}}{(2\lambda-2c)^{\frac{1}{2}}} \sin\ [(2\lambda-2c)^{\frac{1}{2}}x] \qquad (x \geq 0);$$

$$\equiv \exp[(2K\lambda+2c)^{\frac{1}{2}}x] \qquad (x < 0).$$

Thus,

$$\underset{\sim}{E}^x[\exp(-c\tau^+_t)g_{c,\lambda}(\tilde{X}^+_t)] = \exp(-\lambda t)g_{c,\lambda}(\tilde{X}^+_t) .$$

Let $c \downarrow\downarrow 0$ to obtain for $x \geq 0$, and with $\gamma \equiv (2\lambda)^{\frac{1}{2}} > 0$,

$$(4.3) \qquad \underset{\sim}{E}^x[\cos\gamma\tilde{X}_t^+ + K^{\frac{1}{2}}\sin\gamma\tilde{X}_t^+;\ \tau_t^+ < \infty] = \exp(-\tfrac{1}{2}\gamma^2 t)[\cos\gamma x + K^{\frac{1}{2}}\sin\gamma x].$$

Now let $\gamma \downarrow\downarrow 0$ to obtain

$$\underset{\sim}{P}^x[\tau_t^+ < \infty] = 1, \qquad \forall t.$$

Assume for the moment that

(4.4) <u>the functions</u> $\{g_\gamma^+ : \gamma > 0\}$ <u>on</u> $[0,\infty)$, <u>where</u>

$$g_\gamma^+(x) \equiv \cos\gamma x + K^{\frac{1}{2}}\sin\gamma x, \qquad x \in [0,\infty),$$

<u>are full on</u> $[0,\infty)$.

Then the transition function $\{\tilde{P}^+(\cdot)\}$ is uniquely determined by the fact that its resolvent $\{\tilde{R}^+(\cdot)\}$ satisfies:

$$2\lambda\tilde{R}^+(\lambda)g_\gamma^+ = g_\gamma^+ \qquad (\lambda = \tfrac{1}{2}\gamma^2).$$

Let us make an intelligent guess about $\{\tilde{R}^+(\cdot)\}$. <u>Let</u> $\tilde{Y}^+$ <u>be the Markov process on</u> $[0,\infty)$ <u>which behaves like Brownian motion away from</u> 0, <u>never 'exits</u> 0 <u>continuously', and jumps from</u> 0 <u>according to the Lévy measure</u>

$$(4.5) \qquad J(dx) = \text{constant.}\, x^{-(1+\alpha)}dx, \quad 0 < \alpha < 1, \quad \tan\tfrac{1}{2}\pi\alpha = K^{-\frac{1}{2}}.$$

Let $\{{}_0R^+(\cdot)\}$ be the resolvent of Brownian motion on $(0,\infty)$ killed at 0. Then the resolvent $\{\tilde{U}^+(\cdot)\}$ of Y is given by

$$\tilde{U}^+(\lambda)h^+(x) = {}_0R^+(\lambda)h^+(x) + e^{-\gamma x}J({}_0R^+(\lambda)h^+)/\lambda J({}_0R^+(\lambda)I_{(0,\infty)}),$$

where h^+ denotes an arbitrary bounded function on $[0,\infty)$, and, as always, $\gamma \equiv (2\lambda)^{\frac{1}{2}}$. It is easily checked that

$$(4.6) \qquad {}_0R^+(\lambda)g_\gamma^+(x) = (2\lambda)^{-1}[\cos\gamma x + K^{\frac{1}{2}}\sin\gamma x - e^{-\gamma x}],$$

$${}_0R^+(\lambda)I_{(0,\infty)}(x) = \lambda^{-1}[1 - e^{-\gamma x}].$$

The essential fact is that for J as at (4.5),

$$(4.7) \qquad \int_{(0,\infty)}(\cos\gamma x + K^{\frac{1}{2}}\sin\gamma x - 1)\, J(dx) = 0, \qquad \forall\gamma > 0.$$

This is known in the theory of stable processes, and is intuitively obvious because

$$\int_{(0,\infty)}(1 - e^{i\gamma x})x^{-(1+\alpha)}dx = (-i\gamma)^{\alpha}\int_{(0,\infty)}(1 - e^{-y})y^{-(1+\alpha)}dy,$$

so that

$$\arg\int_{(0,\infty)}(1 - e^{i\gamma x})x^{-(1+\alpha)}dx = \arg[(-i)^{\alpha}] = -\tfrac{1}{2}\pi\alpha,$$

and hence

$$\int_{(0,\infty)}(\sin\gamma x)x^{-(1+\alpha)}dx = (\tan\tfrac{1}{2}\pi\alpha)\int_{(0,\infty)}(1-\cos\gamma x)x^{-(1+\alpha)}dx.$$

See page 168 of Gnedenko and Kolmogorov [4] for more rigour.

Putting the pieces together, we find that

$$\forall\gamma > 0, \qquad 2\lambda\tilde{U}^{+}(\lambda)g_{\gamma}^{+} = g_{\gamma}^{+}.$$

Hence, $\tilde{X}^{+}$ has the same transition function as $\tilde{Y}^{+}$.

5. Kingman's proof that (4.7) ==> (4.5). As things have turned out, the fact that (4.7) ==> (4.5) follows from our probabilistic method - see §6. However, Kingman's proof of this fact is one of the few sensible pieces of analysis in this area at the moment - contrast §6! - and it may well point to better things.

Suppose that (4.7) holds for some non-negative measure J on $(0,\infty)$. For complex γ with $\mathcal{I}(\gamma) \geq 0$, define

$$f(\gamma) \equiv \int_{(0,\infty)}(1 - e^{i\gamma x})J(dx).$$

Then f is analytic in $\{\mathcal{I}(\gamma) > 0\}$, and continuous on $\{\mathcal{I}(\gamma) \geq 0\}$. Moreover, $\mathcal{R}(f(\gamma)) \geq 0$ (with equality at $\gamma = 0$ and perhaps at multiples of a purely real θ). Now, f is real and positive on the upper imaginary axis $\{\mathcal{R}(\gamma) = 0,\ \mathcal{I}(\gamma) > 0\}$; and, since (4.7) holds, $(1 - K^{\frac{1}{2}}i)f$ is imaginary on the right half $\{\mathcal{R}(\gamma) > 0,\ \mathcal{I}(\gamma) = 0\}$ of the real axis. Hence, in the first quadrant the harmonic function φ, where

$$\varphi(\gamma) \equiv \arg f(\gamma) = (\log f(\gamma)),$$

stays bounded between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$, and has boundary values as shown in Figure 1.

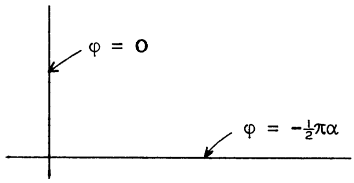

Figure 1

Hence, $\varphi(\gamma) = \alpha \arg(-i\gamma)$. Thus $\log f(\gamma)$ is determined up to an additive constant, and

$$f(\gamma) = \text{constant}.(-i\gamma)^{\alpha}.$$

In particular, for real $\theta > 0$,

$$\int_{(0,\infty)} (1 - e^{-\theta x}) J(dx) = \theta^{\alpha},$$

and so J is determined.

6. **Proof of (4.4).** One of the main difficulties which we have encountered in this work is that of proving that various classes of functions are full.

We have so far failed to adapt Kingman's method to prove (4.4); and we do need (4.4) to show that $\tilde{X}^+$ cannot exit 0 continuously. Note that since (4.4) implies that $\tilde{X}^+$ and $\tilde{Y}^+$ have the same transition function, it follows that (4.4) implies Kingman's result that (4.7) ⟹ (4.5). However, Kingman's method of proof proves to be useful in the study of analogues of (4.7).

We now prove (4.4) by a bizarre probabilistic method. We know from (4.3) that

$$\int_{(0,\infty)} \tilde{P}_t^+(0,dx)(\cos\gamma x + K^{\frac{1}{2}}\sin\gamma x) = \exp(-\tfrac{1}{2}\gamma^2 t),$$

so that

$$(6.1) \qquad \int_{(0,\infty)} \tilde{P}_t^+(0,d\gamma)(\cos\gamma x + K^{\frac{1}{2}}\sin\gamma x) = \exp(-\tfrac{1}{2}tx^2).$$

Suppose now that ν is a signed (or, more generally, complex-valued) measure on $(0,\infty)$ of finite total variation such that

$$(6.2) \qquad \int_{(0,\infty)} (\cos\gamma x + K^{\frac{1}{2}}\sin\gamma x)\nu(dx) = 0, \qquad \forall\gamma > 0.$$

Integrate (6.2) with respect to the measure $\tilde{P}_t^+(0,d\gamma)$ over $\gamma \in (0,\infty)$ to obtain

$$\int_{(0,\infty)} \exp(-\tfrac{1}{2}tx^2)\nu(dx) = 0, \qquad \forall t > 0.$$

Hence (on putting $t = s^{-1}$ and multiplying by $(2\pi s^3)^{-\frac{1}{2}}$),

$$\int_{(0,\infty)} (2\pi s^3)^{-\frac{1}{2}} \exp(-x^2/2s)\nu(dx) = 0.$$

Multiply by $\exp(-\frac{1}{2}\theta^2 s)$, where $\theta > 0$, and integrate over s in $(0,\infty)$ to obtain

$$\int_{(0,\infty)} e^{-\theta x}\nu(dx) = 0, \qquad \forall\theta > 0.$$

Hence $\nu = 0$. You can easily check that the various appeals to Fubini's theorem are justified.

7. Now, of course, there is much more to study in connection with the above example. In particular, the question mentioned earlier about how killing X at rate c induces a killing of $\tilde{X}^+$, is rather interesting. It is clear that $\tilde{X}^+$ is killed according to a <u>discontinuous</u> multiplicative functional which takes into account the jumps of $\tilde{X}^+$ from 0. But we are not going to become involved with the analytic complexities of that problem now.

Instead, we end with an example of a very different type.

8. <u>Example</u>. Let $\{B_t\,;\, t \geq 0\}$ be a Brownian motion on $\mathbb{R}$, starting at 0, with drift $\mu > 0$, so that the law of $\{B_t - \mu t\,;\, t \geq 0\}$ is Wiener measure. Define :

$$V_t \equiv M_t - B_t \,, \qquad \varphi_t \equiv 2M_t - B_t = V_t + M_t \,,$$

$$\tau_t^+ \equiv \inf\{s\,;\, \varphi_s > t\}, \qquad \tilde{V}_t^+ \equiv V(\tau_t^+) \,.$$

Now V is a time-homogeneous strong Markov process, and M is local time at 0 for V. Thus φ is a fluctuating continuous additive functional for V. Obviously, $P[\tau_t^+ < \infty] = 1$.

The results of Rogers and Pitman [7] make it plain that the transition semigroup $\{\tilde{P}_t^+\}$ of $\tilde{V}^+$ is given by the following formulae:

$$\tilde{P}_t^+(0,dy) = 2\mu\, e^{-2\mu y}(1 - e^{-2\mu t})^{-1}\, dy \quad \text{on} \quad [0,t]\,, \tag{8.1.i}$$

and, for $x > 0$,

$$\tilde{P}_t^+(x,\{x+t\}) = e^{-2\mu t}(1 - e^{-2\mu x})(1 - e^{-2\mu(x+t)})^{-1}\,, \tag{8.1.ii}$$

$$\tilde{P}_t^+(x,dy) = 2\mu e^{-2\mu y}(1 - e^{-2\mu t})(1 - e^{-2\mu(x+t)})^{-2}\, dy \quad \text{on} \quad [0,x+t), \tag{8.1.iii}$$

$$\tilde{P}_t^+(x,(x+t,\infty)) = 0\,. \tag{8.1.iv}$$

Here is a martingale proof in the spirit of the remainder of this paper. Begin by observing that for $\theta \geq 0$,

$$\exp(\theta\varphi_t)\ g_\theta(V_t) \text{ is a martingale,} \tag{8.2}$$

where

$$g_\theta(x) \equiv \theta e^{(2\mu-\theta)x} - (2\mu+\theta)e^{-\theta x} .$$

Indeed,

$$e^{\theta\varphi} g_\theta(V) = \theta e^{(2\mu+\theta)M - 2\mu B} - (2\mu+\theta)e^{\theta M} ,$$

so that

$$d[e^{\theta\varphi} g_\theta(V)] = (2\mu+\theta)\theta e^{\theta M}[e^{2\mu(M-B)} - 1]dM$$
$$- 2\mu\theta\, e^{(2\mu+\theta)M - 2\mu B}(dB - \mu dt).$$

But whenever M increases, $M = B$, so that

$$[e^{2\mu(M-B)} - 1]dM = 0 .$$

This observation was used by Azema and Yor [1] to find similar families of martingales of Brownian motion. Now (8.2) follows, since $\exp(\theta\varphi_t)\ g_\theta(V_t) \in L^1$ for each $t \geq 0$. By the fact that, for $u > 0$, V and φ are bounded on $[0,\tau_u)$, we deduce from the optional sampling theorem that

$$\tilde{P}_t^+ g_\theta(x) = e^{-\theta t} g_\theta(x). \tag{8.3}$$

We need only prove now that

(8.4) $\{g_\theta ; \theta \geq 0\}$ <u>is full on every interval of the form $[0,K]$</u>, and that (8.3) holds if $\{\tilde{P}_t^+\}$ is defined by (8.1). These parts are left as exercises for the reader.

We can get results for $\mu = 0$ by letting $\mu \downarrow\downarrow 0$, to obtain

$$\tilde{P}_t^+(0,dy) = t^{-1}\,dy \qquad \text{on } [0,t] ,$$

$$\tilde{P}_t^+(x,\{x+t\}) = \frac{x}{x+t} ,$$

$$\tilde{P}_t^+(x,dy) = \frac{t\,dy}{(x+t)^2} \qquad \text{on } [0,\ x+t).$$

This is a strikingly simple semigroup!

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