Forecasting the forecasts of others (12/5/09)

(i) A paper "Forecasting the forecasts of others: implications for asset pricing" by I. Makarov + O. Rytchkov proposes a model where the value at time t of some asset is expressed as $X_t^{(1)} + X_t^{(2)}$, where the $X_t^{(i)}$ are independent AR(1) processes, assumed to have identical dynamics:

$$X_{t+1}^{(i)} = \beta X_t^{(i)} + \sigma \varepsilon_{t+1}^{(i)} \qquad \varepsilon_t^{(i)} \sim N(0, 1)$$

Agent i sees only $X^{(i)}$ together with the price process $(S_t) \equiv X_t^{(3)}$, which is determined as follows. Each period, with prob^y p, the asset is liquidated and pays $X_t^{(1)} + X_t^{(2)}$. Thus the period-$(t+1)$ return is on average

$$Q_{t+1} = p\,(X_{t+1}^{(1)} + X_{t+1}^{(2)}) + (1-p)\, X_{t+1}^{(3)} - (1+r)\, X_t^{(3)}.$$

Agent i forms his conditional expectation $E_t[Q_{t+1} \mid \mathcal{F}_t^{(i)}]$ and submits an order $\omega_i\, E_t[Q_{t+1} \mid \mathcal{F}_t^{(i)}]$. There is a supply shock $q\, \varepsilon_t^{(3)}$ at period t, again $\varepsilon^{(3)} \sim N(0,1)$, so that

$$q\, \varepsilon_t^{(3)} = -(1+r)(\omega_1+\omega_2)\, X_t^{(3)} + \sum \omega_i\, E\Big[\, p(X_{t+1}^{(1)} + X_{t+1}^{(2)}) + (1-p)\, X_{t+1}^{(3)} \,\Big|\, \mathcal{F}_t^i \Big]$$

$$\equiv -(1+r)\,\Omega\, X_t^{(3)} + \Omega \bar{E}_t\Big[\, p(X_{t+1}^{(1)} + X_{t+1}^{(2)}) + (1-p)\, X_{t+1}^{(3)} \Big]$$

Thus the price must be

$$S_t \equiv X_t^{(3)} = -\frac{q\,\varepsilon_t^{(3)}}{(1+r)\,\Omega} + \frac{1}{(1+r)}\, \bar{E}_t\Big[\, p(X_{t+1}^{(1)} + X_{t+1}^{(2)}) + (1-p)\, X_{t+1}^{(3)} \Big] \qquad (1)$$

There is thus the complicated and interesting situation that the price process is needed to calculate the agents' forecasts, but is also determined thereby.

(ii) Is it possible that X might solve an AR(1) process?

$$X_{t+1} = B\, X_t + \Sigma_0\, \varepsilon_{t+1} \quad ??$$

If we assume that $\omega_1 = \omega_2$, and restrict our attention to symmetric solutions only we would have to have

$$B = \begin{pmatrix} \beta & 0 & 0 \\ 0 & \beta & 0 \\ b_1 & b_2 & b_3 \end{pmatrix}, \quad \Sigma_0 = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ c_1 & c_2 & c_3 \end{pmatrix} \qquad \begin{matrix} b_1 = b_2 \\ c_1 = c_2 \end{matrix}$$

We therefore have a standard KF problem (with zero observational noise)

$$\begin{cases} X_{t+1} = BX_t + \varepsilon_{t+1} \\ Y_{t+1} = CX_{t+1} \end{cases}$$

where $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ for agent 1, $C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ for agent 2. If V_t is the conditional covariance of X_t given $\mathcal{Y}_t^{(1)}$, then the updating is the usual story:

$$\left(\begin{matrix} X_{t+1} \\ Y_{t+1} \end{matrix} \middle| \mathcal{Y}_t^{(1)} \right) \sim N \left(\begin{pmatrix} B\hat{X}_t \\ CB\hat{X}_t \end{pmatrix}, \begin{pmatrix} BV_tB^T + \Sigma & (BV_tB^T+\Sigma)C^T \\ C(BV_tB^T+\Sigma) & C(BV_tB+\Sigma)C^T \end{pmatrix} \right)$$

so

$$V_{t+1} = M_t - M_tC^T(CM_tC^T)^{-1}CM_t \qquad M_t \equiv BV_tB+\Sigma.$$

Clearly for agent 1, the covariance matrix is just $\begin{pmatrix} 0 & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & 0 \end{pmatrix}$. The updating for v is thus very simple, and the steady-state equation is

$$\boxed{v = \beta^2 v + \sigma^2 - \frac{(\beta v x + \sigma y)^2}{x^2 v + y^2 + c_3^2}} \qquad \begin{pmatrix} x \equiv b_1 = b_2 \\ y \equiv c_1 = c_2 \end{pmatrix} \quad (2)$$

which is a quadratic for v. After some calculations (~/WORK/SOLO/FFO/FFO1.mw) we find that the filtering for agent 2 looks like

$$\hat{X}_{t+1} - B\hat{X}_t = \begin{bmatrix} -\frac{(\beta v x+\sigma y)y}{\sigma(x^2v+y^2+c_3^2)} & \frac{\beta v x+\sigma y}{x^2v+y^2+c_3^2} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \left(Y_{t+1} - CB\hat{X}_t\right) \quad (3)$$

so $\hat{X}^{(1)}_{t+1} - \beta\hat{X}^{(1)}_t = -ky\left(X^{(2)}_{t+1} - \beta X^{(2)}_t\right) + k\sigma\left(X^{(3)}_{t+1} - x(\hat{X}^{(1)}_t + X^{(2)}_t) - b_3X^{(3)}_t\right)$

where $\boxed{k \equiv (\beta v x + \sigma y)/\sigma(x^2v+y^2+c_3^2)}$. If we write $K^{(i)}$ for agent i's gain matrix, we have that agent i forecasts tomorrow's value of X as $B\hat{X}_t$, and this allows us to find each agent's demand. Writing $\bar{X}_t \equiv X_t^{(1)} + X_t^{(2)}$ and

$$\xi_t = E\left[X_t^{(1)} \middle| \mathcal{Y}_t^{(2)}\right] + E\left[X_t^{(2)} \middle| \mathcal{Y}_t^{(1)}\right]$$

gives us

$$\boxed{\xi_{t+1} - \beta\xi_t = -ky\left(\bar{X}_{t+1} - \beta\bar{X}_t\right) + k\sigma\left\{2X^{(3)}_{t+1} - 2b_3X^{(3)}_t - x\left(\xi_t + \bar{X}_t\right)\right\}.} \quad (4)$$

$2X^{(3)}_{t+1} - 2b_3X^{(3)}_t$

(iv) Let's develop the complicated term in the demand equation:

$$\bar{E}_t\left[p\left(X^{(1)}_{t+1} + X^{(2)}_{t+1}\right) + (1-p)X^{(3)}_{t+1}\right]$$

$$= \frac{p}{2} E\left[X^{(1)}_{t+1} + X^{(2)}_{t+1} \middle| y^{(1)}_t \right] + \frac{p}{2} E\left[X^{(1)}_{t+1} + X^{(2)}_{t+1} \middle| y^{(2)}_t\right]$$

$$+ \frac{1-p}{2} E\left[X^{(3)}_{t+1} \middle| y^{(1)}_t\right] + \frac{1-p}{2} E\left[X^{(3)}_{t+1}\middle| y^{(2)}_t\right]$$

$$= \frac{p}{2}\beta(\bar{X}_t + \xi_t) + \frac{1-p}{2} E\left[x(X^{(1)}_t + X^{(2)}_t) + b_3 X^{(3)}_t \middle| y^{(1)}_t\right] + \frac{1-p}{2} E\left[x(X^{(1)}_t + X^{(2)}_t) + b_3 X^{(3)}_t \middle| y^{(2)}_t\right]$$

$$= \frac{p}{2}\beta(\bar{X}_t + \xi_t) + \frac{1-p}{2}\left\{ x(\bar{X}_t + \xi_t) + 2b_3 X^{(3)}_t \right\}$$

So the market-clearing condition is

$$\boxed{X^{(3)}_t = \frac{-q\,\varepsilon^{(3)}_t}{(1+r)\Omega} + \frac{1}{1+r}\left\{ p\beta \frac{\bar{X}_t + \xi_t}{2} + (1-p)x \frac{\bar{X}_t + \xi_t}{2} + (1-p)b_3 X^{(3)}_t \right\}} \quad (5)$$

Introduce the notation

$$\eta_t \equiv \bar{X}_t + \xi_t$$

So we get from (4), (5)

$$\boxed{\begin{aligned} \eta_{t+1} - \beta\eta_t &= (1-ky)(\bar{X}_{t+1} - \beta\bar{X}_t) + k\sigma\left\{2X^{(3)}_{t+1} - 2b_3 X^{(3)}_t - x\eta_t\right\} \\ X^{(3)}_t &= \frac{-q\,\varepsilon^{(3)}_t}{(1+r)\Omega} + \frac{1}{1+r}\left\{ \frac{p\beta + (1-p)x}{2}\eta_t + (1-p)b_3 X^{(3)}_t\right\} \end{aligned}} \quad (6)$$

Notice $\bar{X}_{t+1} - \beta\bar{X}_t = \varepsilon^{(1)}_{t+1} + \varepsilon^{(2)}_{t+1}$, so

$$\boxed{\begin{aligned} \eta_{t+1} &= (\beta - k\sigma x)\eta_t + 2k\sigma\left(X^{(3)}_{t+1} - b_3 X^{(3)}_t\right) + (1-ky)\left(\varepsilon^{(1)}_{t+1} + \varepsilon^{(2)}_{t+1}\right) \\ \left(1 - \frac{(1-p)b_3}{1+r}\right) X^{(3)}_t &= \frac{-q\,e^{(3)}_t}{(1+r)\Omega} + \frac{p\beta + (1-p)x}{2(1+r)}\eta_t \end{aligned}} \quad (7)$$

(v) Notice from the original equations that ($\bar{\varepsilon}_t \equiv \varepsilon^{(1)}_t + \varepsilon^{(2)}_t$)

$$\begin{pmatrix} \bar{X}_{t+1} \\ X^{(3)}_{t+1} \end{pmatrix} = \begin{pmatrix} \beta & 0 \\ x & b_3 \end{pmatrix}\begin{pmatrix} \bar{X}_t \\ X^{(3)}_t \end{pmatrix} + \begin{pmatrix} \sigma & 0 \\ y & c_3 \end{pmatrix}\begin{pmatrix} \bar{\varepsilon}_{t+1} \\ \varepsilon^{(3)}_{t+1}\end{pmatrix}$$

This gives

$$X^{(3)}_{t+1} = b_3 X^{(3)}_t + x \bar{X}_t + y \bar{\varepsilon}_{t+1} + c_3 \varepsilon^{(3)}_{t+1}$$

or more compactly

$$(I - b_3 L) X^{(3)} = x L \bar{X} + y \bar{\varepsilon} + c_3 \varepsilon^{(3)}$$

or again

$$\boxed{X^{(3)} = (I - b_3 L)^{-1} \{ c_3 \varepsilon^{(3)} + y \bar{\varepsilon} + x L (I - \beta L)^{-1} \sigma \bar{\varepsilon} \}} \qquad (8)$$

This gives $X^{(3)}$ in terms of $\varepsilon^{(3)}$ and $\bar{\varepsilon}$. But the first equation of (7) gives η in terms of $X^{(3)}$:

$$\boxed{(I - (\beta - k\sigma x) L) \eta = (1 - ky) \bar{\varepsilon} + 2k\sigma (I - b_3 L) X^{(3)}} \qquad (9)$$

and at the same time the second equation of (7) gives us $(q' \equiv q/\Omega)$

$$\boxed{(1 + r - b_3 (1-p)) X^{(3)} = -q' \varepsilon^{(3)} + \tfrac{1}{2} (p \beta + (1-p) x) \eta} \qquad (10)$$

Using (8), (9) gives equations for $X^{(3)}, \eta$ which allow us to express those two processes in terms of $\bar{\varepsilon}$, $\varepsilon^{(3)}$; and then (10) has to hold identically. We can put this into Maple and see what conditions we get. Expanding the numerator of (10) in powers of L, we get from $L^3|$ that

$$\boxed{\beta = k\sigma x} \qquad (11)$$

By inspecting the coefficient of L^0, + splitting the $\bar{\varepsilon}$ and $\varepsilon^{(3)}$ terms, we get further conditions:

$$y = \frac{\beta (\sigma k p + (1-p))}{k \{ -2\sigma b_3 (1-p) - 2\sigma \beta (1-p) + (1-p) \beta + 2\sigma (1+r) + \sigma k \beta p (1 - 2\sigma) \}} \qquad (12)$$

$$c_3 = \frac{q'}{(1+r) - b_3 (1-p) - \beta (1-p) - \sigma k \beta p}$$

Passing to the coefficient of L, the coefficient of $\varepsilon^{(3)}$ only vanishes if b_3 takes one of the values

$$0, \quad \frac{1+r}{1-p}, \quad \frac{\beta(1-p) - 2\sigma\beta(1-p) + 2\sigma(1+r) + \sigma\beta k p (1 - 2\sigma)}{2\sigma(1-p)} \qquad (13)$$

The last of these would mean $y = +\infty$! So only $b_3 = (1+r)/(1-p)$ is still possible, or $b_3 = 0$. The first of these gives

$$k = -\frac{1-p}{\sigma p} \qquad (14a)$$

and the second gives

$$k = \frac{1+r-\beta(1-p)}{\sigma\beta p}. \qquad (14b)$$

Either way, this is an absolute constant, expressed in terms of the given parameters of the problem. However, we have

$$k = \frac{\beta v x + \sigma y}{\sigma(v x^2 + y^2 + \varsigma_3^2)}. \qquad (15)$$

[We also have from (2) that $v = \beta^2 v + \sigma^2 - \sigma k(\beta v x + \sigma y)$, so using (11) we have $v = \sigma^2(1-ky)$.]

Neither of the values (14a), (14b) is consistent with (15) when we use the relations (11), (12), (13) for x, y, b_3....

However, if we look again at the third possibility in (13), we don't have an immediate impossibility if the numerator in expression (12) for y is also zero; this would give $k = -(1-p)/\sigma p$, and substituting this back into (13) that $b_3 = (1+r)/(1-p)$. If we go from the beginning of the calculation with this value of b_3, we find we must have

$$y = \frac{-1}{k(2\sigma-1)} = \frac{\sigma p}{(2\sigma-1)(1-p)}$$

However, the market-clearing condition then dies: in (10), we get $0 = q' \varepsilon^{(3)}$ which doesn't happen if there is any supply shock. No supply shock would say each agent can work out what η is

(vi) But can we really believe in a stationary ergodic solution for something which is terminated each period with prob p...?! We could try to say that $\bar{X}_t$ is dividend paid on day t, but then we can't really pretend that this is not observed...

Maybe we should suppose dividends δ_t on day t are IID $N(0,\sigma^2)$, and on day $t-1$ agents get signal $\delta_t + \varepsilon_t^{(1)}$ ... wasn't this very like ~~what~~ Arnaud & I did?!

$$\left(\begin{pmatrix} \mu \\ Y_{t+\tau} \\ Z_{t+\tau} \end{pmatrix} \middle| \, y_t \right) \sim N\left(\begin{pmatrix} \hat{\mu}_t \\ e^{-\lambda t}\hat{Y}_t \\ \hat{Z}_t \end{pmatrix}, \; M_\tau V_t M_\tau + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{v^2(1-e^{-2\lambda\tau})}{2\lambda} & \frac{\rho v(1-e^{-\lambda\tau})}{\lambda} \\ 0 & & \tau \end{pmatrix} \right)$$

$$\begin{pmatrix} \hat{\mu}_{t+\tau} - \hat{\mu}_t \\ \hat{Y}_{t+\tau} - \hat{Y}_t \\ \hat{Z}_{t+\tau} - \hat{Z}_t \end{pmatrix} = V_t q_{t+\tau} (q_{t+\tau}^T V_t q_{t+\tau})^{-1} (y - q_{t+\tau}^T \hat{X}_t)$$

Alternative to mark-to-market valuation (19/5/09)

1) Let's suppose that we model the log-stock price X_t of some asset as

$$X_t - X_0 = Y_t - Y_0 + \sigma Z_t + \mu t$$

where $dY_t = v\,dW_t - \lambda Y_t\,dt$, and Z is a BM, $dW dZ = \rho dt$. Suppose we have a prior $N(\hat{\mu}_0, \sigma_0^2)$ dist^n for μ, and that we only see X occasionally, at the times of trades. How does the inference of μ, and of $\sigma Z_t + \mu t$ (the 'true' value) update?

2) Suppose we have

$$(\underline{X}_0 | \mathcal{G}_0) \equiv \left(\begin{pmatrix} \mu \\ Y_0 \\ Z_0 \end{pmatrix} \middle| \mathcal{G}_0 \right) \sim N\left(\begin{pmatrix} \hat{\mu}_0 \\ \hat{Y}_0 \\ \hat{Z}_0 \end{pmatrix}, V_0 \right)$$

where we require $\hat{Y}_0 + \sigma \hat{Z}_0 = X_0$, $q_0^T V_0 q_0 = 0$, where $q_t^T = (t, 1, \sigma)$

Then

$$\left(\begin{pmatrix} \mu \\ Y_t \\ Z_t \end{pmatrix} \middle| \mathcal{G}_0 \right) \sim N\left(\begin{pmatrix} \mu_0 \\ e^{-\lambda t}\hat{Y}_0 \\ \hat{Z}_0 \end{pmatrix}, M_t V_0 M_t + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{v^2(1-e^{-2\lambda t})}{2\lambda} & \frac{\rho v(1-e^{-\lambda t})}{\lambda} \\ 0 & \frac{\rho v(1-e^{-\lambda t})}{\lambda} & t \end{pmatrix} \right)$$

where $M_t = \mathrm{diag}(1, e^{-\lambda t}, 1)$. Now at time t we observe $y = q_t^T \underline{X}_t$; what's the conditional dist^n of $\underline{X}_t$ given y?

$$\begin{pmatrix} \hat{\mu}_t - \hat{\mu}_0 \\ \hat{Y}_t - e^{-\lambda t}\hat{Y}_0 \\ \hat{Z}_t - \hat{Z}_0 \end{pmatrix} = V_t q_t (q_t^T V_t q_t)^{-1} \left(y - q_t^T \begin{pmatrix} \hat{\mu}_0 \\ e^{-\lambda t}\hat{Y}_0 \\ \hat{Z}_0 \end{pmatrix} \right)$$

where V_t is the covariance of $\underline{X}_t$ given $\mathcal{G}_0$. The covariance of $\underline{X}_t$ given the new information is just $V_t - V_t q_t (q_t^T V_t q_t)^{-1} q_t^T V_t$, which is nice, because $q_t^T V_t q_t$ is just a scalar, so we can get quite explicit solutions now. We can also compute the likelihood of the observation, which will be needed for comparing different $(v, \lambda, \sigma, \hat{\mu}_0, \rho)$

3) Looks a bit like we would do better to allow μ to change very slowly... The point is that the window of observation can have a dominant effect on the estimation

Beauty contests again. (22/6/09).

(i) It seems that the earlier beauty contest analysis wasn't quite correct. Agent j has true likelihood-ratio martingale Λ^j_t, but acts as if he used $\tilde{\Lambda}^j_t$. The optimality conditions are

$$e^{-\rho t}\tilde{\Lambda}^j_t / c^j_t = \nu_j \tilde{\zeta}_t \qquad [\rho_j = \rho \ \forall j, \text{ we assume}]$$

and we have

$$w^j_0 = E\left[\int_0^\infty \tilde{\zeta}_t c^j_t \, dt\right] / \tilde{\zeta}_0 = \tilde{\zeta}_0^{-1} E\left[\int_0^\infty \frac{1}{\nu_j} e^{-\rho t}\tilde{\Lambda}^j_t \, dt\right] = 1/(\rho \nu_j \tilde{\zeta}_0)$$

So as before the mixing weights ν_j are proportional to $1/w^j_0$, whatever LR martingale is used.

(ii) Agent j's objective from using $\tilde{\Lambda}^j$ is

$$E \int_0^\infty e^{-\rho t}\Lambda^j_t \log\left(\frac{e^{-\rho t}\tilde{\Lambda}^j_t/\nu_j}{\tilde{\zeta}_t}\right) dt$$

$$= E\int_0^\infty e^{-\rho t}\Lambda^j_t \log\left(\frac{\delta_t \ \tilde{\Lambda}^j_t/\nu_j}{\sum \tilde{\Lambda}^i_t/\nu_i}\right) dt$$

$$= K + E\int_0^\infty e^{-\rho t}\Lambda^j_t \log\left(\frac{\pi_j \tilde{\Lambda}^j_t}{\sum \pi_i \tilde{\Lambda}^i_t}\right) dt$$

where $\pi_j = w^j_0 / \sum w^i_0$. Now if we take $m \equiv \rho e^{-\rho t} dt \times P^0$ as the reference measure on $\bar{\Omega} \equiv [0,\infty)\times\Omega$ with the optional σ-field, then $f_j \equiv \Lambda^j$ is the density of Agent j's true beliefs, and what we have is

$$\int f_j \log\left(\frac{\pi_j \varphi_j}{\sum \pi_i \varphi_i}\right) dm$$

which agent j will attempt to maximise by choice of φ_j, a density.

(iii) Suppose agent j has chosen φ_j optimally, and perturbs to $\varphi_j(1+\eta)$ where $\int \varphi_j \eta \, dm = 0$. Then the first-order change in objective is

$$\int f_j \, \eta \left\{1 - \frac{\pi_j \varphi_j}{\sum \pi_i \varphi_i}\right\} dm$$

In order that this should be zero, we require

$$1 - \frac{\pi_j \varphi_j}{\bar\varphi} = \beta_j \frac{\varphi_j}{f_j} \quad \text{for some constant } \beta_j \qquad \left[\bar\varphi \equiv \sum \pi_i \varphi_i\right]$$

Multiplying by $\bar\varphi$ gives

$$\sum_{i \neq j} \pi_i \varphi_i = \beta_j \varphi_j \bar\varphi / f_j$$

$$\Rightarrow \quad \bar\varphi = \varphi_j \{ \pi_j + \beta_j \bar\varphi / f_j \}$$

$$\Rightarrow \quad \boxed{\varphi_j = \frac{\bar\varphi}{\pi_j + \beta_j \bar\varphi / f_j}} \qquad (*)$$

Multiplying by π_j and summing gives the condition

$$\boxed{1 = \sum_j \frac{\pi_j}{\pi_j + \beta_j \bar\varphi(x)/f_j(x)}}.$$

Now the RHS is clearly decreasing in $\bar\varphi(x)$, from J to 0, so (assuming not all β_j are zero) there is a unique $\bar\varphi(x)$ which achieves the equality.

For proper densities, there must also be

$$1 = \int \varphi_j \, dm \; ;$$

Cross multiplying $(*)$, this gives

$$\pi_j + \int \beta_j \frac{\varphi_j \bar\varphi}{f_j} dm = 1$$

$$\boxed{\therefore \; \beta_j = (1 - \pi_j) \Big/ \int \varphi_j \bar\varphi \frac{dm}{f_j}}$$

(iv) Maybe a fixed-point theorem would work here, but would we have (in the dynamic context where the problem arose) the martingale consistency conditions ?? Seems completely unlikely!

(v) Let's try a much simpler story which hopefully illustrates the idea in a very simple setting. Suppose we have CARA agents $j = 1, \ldots, J$ and an asset which will be worth X at time 1. Agent j believes it is $N(\alpha_j, v_j)$ so he tries to optimise

$$\max E\, U(\theta(X-S_0)) = \max -\gamma_j^{-1}\exp\left(-\gamma_j\,\theta(\alpha_j - S_0) + \tfrac{1}{2}\gamma_j^2\theta^2 v_j\right)$$

leading to

$$\theta_j = \frac{\alpha_j - S_0}{\gamma_j v_j}$$

and for market clearing

$$S_0 = \sum p_j \alpha_j \qquad \text{where } p_j \propto (\gamma_j v_j)^{-1}, \ \sum p_j = 1.$$

The maximised objective is

$$-\gamma_j^{-1}\exp\left\{-\frac{(\alpha_j - S_0)^2}{2v_j}\right\}$$

Now suppose that agent j pretends to believe $X \sim N(\tilde{\alpha}_j, v_j)$; the new holdings will be

$$\tilde{\theta}_j = \frac{1}{\gamma_j v_j}(\tilde{\alpha}_j - \tilde{S}_0) \qquad \text{where } \tilde{S}_0 = \sum p_j \tilde{\alpha}_j$$

and his objective will be

$$-\gamma_j^{-1}\exp\left[-\gamma_j\tilde{\theta}_j(\alpha_j - \tilde{S}_0) + \tfrac{1}{2}\gamma_j^2\tilde{\theta}_j^2 v_j\right]$$

$$= -\gamma_j^{-1}\exp\left[-\gamma_j\tilde{\theta}_j(\alpha_j - \tilde{\alpha}_j) - \frac{(\tilde{\alpha}_j - \tilde{S}_0)^2}{2v_j}\right]$$

We suppose that the agents' choices of $\tilde{\alpha}_j$ are such as to be Pareto efficient; if noone else changes, neither do you! The aim of agent j is therefore to pick $\tilde{\alpha}_j$ so as to

$$\max\left\{(\tilde{\alpha}_j - \tilde{S}_0)(\alpha_j - \tilde{\alpha}_j) + \tfrac{1}{2}(\tilde{\alpha}_j - \tilde{S}_0)^2\right\}$$

assuming all other $\tilde{\alpha}_i$ are held fixed. Some straightforward calculus leads to

$$\tilde{\alpha}_j = (1-p_j)\alpha_j + p_j\tilde{S}_0$$

Multiply by p_j and sum on j, and you get

$$\tilde{S}_0 = \sum p_j(1-p_j)\alpha_j + \left(\sum p_j^2\right)\tilde{S}_0$$

$$\therefore \boxed{\tilde{S}_0 = \frac{\sum p_j(1-p_j)\alpha_j}{\sum p_j(1-p_j)}}$$

Agent j's gain from this is measured by

$$G_j = (\tilde{\alpha}_j - \tilde{S}_0)(\alpha_j - \tilde{\alpha}_j) + \tfrac{1}{2}(\tilde{\alpha}_j - \tilde{S}_0)^2 - \tfrac{1}{2}(\alpha_j - S_0)^2$$

From simulations, we know that it can be that $G_j < 0$ for all j. It also appears that it is impossible that $G_j > 0$ for all j. It also appears that $\sum p_j G_j < 0$, and $\sum p_j(1-p_j) G_j < 0$.

So the Keynesian beauty contest can actually result in everyone being worse off! You appear always to have someone worse off.

(vi) Suppose only agent j decides to fake his beliefs: then as before

$$\tilde{\alpha}_j = (1-p_j)\alpha_j + p_j \tilde{S}_0 \quad \text{where now } \tilde{S}_0 = S_0 + p_j(\tilde{\alpha}_j - \alpha_j)$$

Hence

$$\boxed{(1-p_j^2)\,\tilde{\alpha}_j = (1-p_j-p_j^2)\,\alpha_j + p_j S_0}$$

This clearly improves things for agent j. Notice that the weight $1-p_j-p_j^2$ on α_j can be <u>negative</u>, if p_j exceeds the golden ratio

(vii) If we take the general situation where agents $j \in F$ fake their beliefs to $\tilde{\alpha}_j$, then just as before for $j \in F$

$$\tilde{\alpha}_j = (1-p_j)\alpha_j + p_j \tilde{S}_0$$

$$= (1-p_j)\alpha_j + p_j \left\{ \sum_{i \in F} p_i \tilde{\alpha}_i + \sum_{i \notin F} p_i \alpha_i \right\}$$

Multiply by p_j and sum over $j \in F$:

$$\Big(1 - \sum_{j\in F} p_j^2\Big)\Big(\sum_{j\in F} p_j \tilde{\alpha}_j\Big) = \sum_{j \in F} p_j(1-p_j)\alpha_j + \Big(\sum_{j\in F} p_j^2\Big)\Big(\sum_{i\notin F} p_i \alpha_i\Big)$$

$$\Rightarrow \boxed{\sum_{j\in F} p_j \tilde{\alpha}_j = \frac{\sum_{j\in F} p_j(1-p_j)\alpha_j + \Big(\sum_{j\in F} p_j^2\Big)\Big(\sum_{i\notin F} p_i\alpha_i\Big)}{1-\sum_{j\in F} p_j^2}}$$

(viii) Could there ever be some proper subset F such that no agent would benefit by switching? Again, simulations make it very likely the answer is no.

Volume of insurance business again (3/7/09)

1) Returning to the story that Angus & I looked at about volume of insurance business, if firm i $(i=1,\dots,N)$ thinks the cost of insurance is c_i and they charge p_i for it, then the volume of business they do is $F_i(p)$, where we insisted

(i) $F_i(\cdot)$ is decreasing in p_i

(ii) $F_i(\cdot)$ is increasing in p_j

(iii) $k \mapsto F_i(kp)$ is decreasing

The Pareto efficiency condition for maximising profit $(p_i - c_i)\,F_i(p)$ gives

$$\frac{1}{p_i - c_i} = -\frac{F_i'(p)}{F_i(p)}$$

so we may consider F_i of the form

$$F_i(p) = \frac{\left(a + b\sum_{j\neq i} p_j\right)^{\theta}}{\left(p_i + \gamma + \lambda \sum_{j\neq i} p_j\right)^{\alpha}}$$

[previously we used $\lambda = 1$, which I believe is too restrictive] For property (i), we need $\alpha > 0$. Concerning (ii), we get

$$0 < \frac{\partial}{\partial p_j}\log F_i = \frac{b\theta}{a + b\sigma} - \frac{\alpha\lambda}{p_i + \gamma + \lambda\sigma} \qquad \left[\sigma \equiv \sum_{j\neq i} p_j\right]$$

so $\quad b\theta(p_i + \gamma + \lambda\sigma) > \alpha\lambda(a + b\sigma)$

whence

$$\boxed{b\theta\gamma \geq a\alpha\lambda, \quad \theta \geq \alpha}$$

As for (iii), we have likewise

$$0 > \frac{\partial}{\partial k}\log F_i(kp) = \frac{\theta b\sigma}{a + kb\sigma} - \frac{\alpha(p_i + \lambda\sigma)}{p_i + \gamma + \lambda\sigma k},$$

equivalently,

$$\frac{\theta b\sigma}{a + kb\sigma} \leq \frac{\alpha\lambda\sigma}{\gamma + \lambda\sigma k},$$

that is, $\theta b(\gamma + \lambda\sigma k) \leq \alpha\lambda(a + b\sigma k)$. This implies $\theta \leq \alpha$ $\therefore \theta = \alpha$, and $b\gamma \leq \lambda a$; but using $\square$ gives $b\gamma = \lambda a$. Putting this altogether, we get that within this assumed class of volume-of-business functions, the only one which has the desired properties is

$$\boxed{F_i(p) \propto \left(\frac{\gamma + \lambda \sum_{j\neq i} p_j}{p_i + \gamma + \lambda \sum_{j \neq i} p_j} \right)^{\alpha}}$$

[$\gamma = 0$!?]
(p14)

2) Assuming F_i has this form, the Pareto efficiency condition is

$$\frac{1}{p_i - c_i} = \frac{\alpha}{p_i + \gamma + \lambda \sum_{j\neq i} p_j} \qquad \forall i$$

Which is

$$\alpha p_i = \alpha c_i + \gamma + (1-\lambda) p_i + \lambda \sum p_j$$

Summing,

$$\alpha 1 \cdot p = \alpha 1 \cdot c + N\gamma + (1-\lambda) 1 \cdot p + N\lambda 1 \cdot p$$

so

$$\boxed{1 \cdot p = \frac{\alpha 1 \cdot c + N\gamma}{\alpha + \lambda - 1 - N\lambda}}$$

Assuming

$$\alpha + \lambda - 1 - N\lambda > 0$$

this all makes sense, and gives

$$\boxed{p_i = \frac{\alpha c_i + \gamma + \lambda 1 \cdot p}{\alpha + \lambda - 1}}$$

3) Suppose that N is large, $\boxed{\lambda = \beta N^{-1}}$. Then we have the average price is

$$\bar{p} \equiv N^{-1} 1 \cdot p = \frac{\alpha \bar{c} + \gamma}{\alpha - 1 - \beta + \beta N^{-1}} \approx \frac{\alpha \bar{c} + \gamma}{\alpha - 1 - \beta}$$

and the volume of business is approximately

$$\left(\frac{\gamma + \beta \bar{p}}{p_i + \gamma + \beta \bar{p}} \right)^{\alpha}$$

Suppose that the distribution of c values among the firms is given by G, with corresponding distribution H for prices

$$\boxed{p_i \doteq \frac{\alpha c_i + \gamma + \beta \bar{p}}{\alpha - 1}}$$

Then we can write down an integral for the total volume of business which is underfunded if the true break-even price is p_0.

4) An alternative large-N story we could tell would be with λ constant, $\alpha = N\nu$ for $\nu > \lambda$. This gives

$$F_i(p) \simeq \text{const}\ \exp\left[-\nu p_i / \lambda \bar{p}\right]$$

$$p_i \simeq \frac{\nu c_i + \lambda \bar{p}}{\nu} = c_i + \lambda \bar{p}/\nu$$

This seems less desirable, as it forces $p_i - c_i = \text{const}$.

5) Suppose we define

$$\mu(x) = \sum_{i=1}^{N} I_{\{p_i \le x\}}$$

Now if the actual cost of the insurance is z, then the total amount of business which is underwritten at below cost will be

$$\int_0^{\gamma} \left(\frac{\gamma + \beta\bar{p}}{x + \gamma + \beta\bar{p}}\right)^{\alpha} \mu(dx)$$

where $\bar{p} = \int x\, \mu(dx)/N$. Thus if G is the distribution of the actual cost, then the expected volume of business at risk will be

$$\int_0^{\infty} \bar{G}(x) \left(\frac{\gamma + \beta\bar{p}}{x + \gamma + \beta\bar{p}}\right)^{\alpha} \mu(dx).$$

If we assumed that the empirical implied distn of cost is what we use for G, then

$$\bar{G}(x) \equiv P(c > x) = P\left(p = \frac{\alpha c + \gamma + \beta\bar{p}}{\alpha - 1} > \frac{\alpha x + \gamma + \beta\bar{p}}{\alpha - 1}\right)$$

$$= N^{-1} \bar{\mu}\left(\frac{\alpha x + \gamma + \beta\bar{p}}{\alpha - 1}\right)$$

6) To explore this numerically, note that we always see the combination $\kappa \equiv \gamma + \beta\bar{p}$ together, and that

$$F(p) = \left(\frac{\kappa}{p + \kappa}\right)^{\alpha}$$

So we could say we want $\bar{p} = 1$, and fix a value of α, then say that if we double p_i we expect our volume of business to get multiplied by $\pi \in (0,1)$; then

$$\kappa = (2\pi^{1/\alpha} - 1)/(1 - \pi^{1/\alpha})$$

Or perhaps better is to say that if we raise price from $\bar{p}$ to $\rho\bar{p}$ we shall halve the volume of business?

7) Some simple comparative statics, on the assumption that $\bar{c}$ is held fixed. We have

$$\bar{p} = \frac{\alpha\bar{c} + \gamma}{\alpha - 1 - \beta} = \bar{c} + \frac{(1+\beta)\bar{c} + \gamma}{\alpha - 1 - \beta}$$

which decreases with α, and increases with β, γ. This as expected; if we increase α, the volume of business for fixed p's will fall, thus giving firms an incentive to drop prices. Increases of β, γ have the effect of making $F_i(p)$ less sensitive to p_i, so the firms will let p_i rise a bit.

Notice

$$p_i = \frac{\alpha c_i + \gamma + \beta\bar{p}}{\alpha - 1} = \frac{\alpha}{\alpha - 1}(c_i - \bar{c}) + \frac{\alpha\bar{c} + \gamma}{\alpha - 1 - \beta}$$

(Need to see some numerics, perhaps assuming that the population dist^n of costs is $\Gamma(a, a)$ for some $a > 0$.)

Arguably, $\boxed{\gamma = 0}$ because if the average price in the population were 0, you will get no business if you charge a positive price. This also helps cut down the number of parameters to be studied:

$$\bar{p} = \frac{\alpha\bar{c}}{\alpha - 1 - \beta}$$

Higher α means more competitive market, $\bar{p}$ closer to $\bar{c}$; raising $\beta < \alpha - 1$ means more profit opportunity

However, in favour of $\boxed{\gamma > 0}$ is the fact that with $\gamma = 0$ the map $k \mapsto F_i(kp)$ is constant

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7/1/10. It would appear to be a natural condition that $\sum_i F_i(p)$, the total volume of business, should be decreasing in each of the p_j, and numerics shows conclusively that it's <u>not</u>.

We'd also expect some saturation, that if $p_i \downarrow 0$, then $F_i(p)$ remains bdd, and if $p_j \to \infty$ $(j \neq i)$ $F_i(p)$ again remains bounded.

Diverse beliefs and filtering (8/7/09)

(i) This is an interesting little tale where we have some underlying OU process X with values in $\mathbb{R}^n$

$$dX_t = dW_t - BX_t\,dt \equiv dW_t + b_t\,dt$$

where W is a P-Brownian motion, X is a $\mathbb{P}^0$-Brownian motion, and

$$\Lambda_t = \left.\frac{dP}{d\mathbb{P}^0}\right|_{\mathcal{F}_t}$$

is the LR martingale. We'll suppose that you observe

$$Y_t = \sigma X_t + \sigma_\varepsilon \tilde{W}_t$$

where $\tilde{W}_t$ is a BM independent of W. Let $\mathcal{Y}_t \equiv \sigma(Y_s : s \le t)$ denote the observation filtration. Now if we project onto the observation filtration, using measure $\mathbb{P}$, what we see is

$$Y_t = \Sigma\, d\nu_t + \sigma\hat{b}_t\,dt$$

where $\Sigma\Sigma^T = \sigma\sigma^T + \sigma_\varepsilon\sigma_\varepsilon^T$, where ν is a $(\mathbb{P}, \mathcal{Y})$-Brownian motion.
We shall write $\hat{Z}$ for the $(\mathbb{P}, \mathcal{Y})$-optional projection of Z, ${}^oZ$ for the $(\mathbb{P}^0, \mathcal{Y})$-optional projection.

(ii) <u>Can we characterise</u> ${}^o\Lambda_t$, <u>the</u> $(\mathbb{P}^0, \mathcal{Y})$ - <u>optional projection of</u> Λ? The point of doing this would be to handle things like

$$E^0\left[\int_0^\infty \Lambda_t\, e^{-\rho t}\varphi_t\,dt\right] = E^0\left[\int_0^\infty {}^o\Lambda_t\, e^{-\rho t}\varphi_t\,dt\right]$$

where φ is some suitable $\mathcal{Y}$-adapted process. Notice that

$$dL_t / L_t = \left(-\Sigma^{-1}\sigma\hat{b}_t\right) d\nu_t$$

is a $(\mathbb{P}, \mathcal{Y})$-martingale which changes Y into $\Sigma\, dZ$, where Z is a $\mathcal{Y}$-BM in the new measure; the guess of course is that

$$ {}^o\Lambda_t = \frac{1}{L_t}.$$

What we have is that Y is a $(\mathbb{P}^0, \mathcal{Y})$-BM$(\Sigma)$, and that $Y_t - \int_0^t \sigma b_s\,ds$ is a $(\mathbb{P}, \mathcal{F})$-martingale. Therefore $Y_t - \int_0^t \sigma\hat{b}_s\,ds$ is a $(\mathbb{P}, \mathcal{Y})$-BM$(\Sigma)$, by considering the quadratic variation. But the change-of-measure martingale from $\mathbb{P}^0$ to $\mathbb{P}$ is ${}^o\Lambda_t$, and $1/{}^o\Lambda$ is the change-of-measure from $\mathbb{P}$ to $\mathbb{P}^0$. But L does that job!

A few calculations lead to

$$\boxed{\frac{d\Lambda_t}{\Lambda_t} = (\Sigma^{-1}\sigma\hat{b}_t)\cdot dY_t}$$

This just leaves us to elucidate the estimate $\hat{b}_t$.

(iii) We have

$$\begin{cases} dX = dW - BX\,dt \\ dY = \sigma dX + \sigma_\varepsilon dW = \sigma dW - \sigma BX\,dt + \sigma_\varepsilon dW \\ \qquad\qquad\qquad\qquad\quad = \Sigma d\nu - \sigma B\hat{X}\,dt \end{cases}$$

Hence

$$d(XY^T) = -X(\sigma BX)^T dt - BXY^T dt + \sigma^T dt + d(\mathbb{P}, \mathcal{Y})\text{-loc mart}$$

$$\therefore\ d(\hat{X}Y^T) = -(XX^T)^\wedge B^T\sigma^T dt - B\hat{X}Y^T dt + \sigma^T dt + d(\mathbb{P}, \mathcal{Y})\text{-loc mart}$$

Also

$$d(\hat{X}Y^T) = -\hat{X}(\sigma B\hat{X})^T dt - B\hat{X}Y^T dt + \theta\Sigma^T dt$$

if we use $d\hat{X} = \theta d\nu - B\hat{X}\,dt$. Comparing the two leads to

$$\boxed{\theta\Sigma^T = \sigma^T - v B^T\sigma^T}$$

where $v = (XX^T)^\wedge - \hat{X}\hat{X}^T$ is the conditional variance of X. By expanding XX^T, projecting, and then expanding $\hat{X}\hat{X}^T$, we deduce that

$$\frac{dv}{dt} = I - Bv - vB^T - \theta\theta^T$$

For steady-state, we therefore have to solve

$$\boxed{0 = I - Bv - vB^T - (I - vB^T)(\Sigma^{-1}\sigma)^T(\Sigma^{-1}\sigma)(I - Bv),}$$

and then express θ in terms of v. We shall have

$$\boxed{d\hat{X} = \theta d\nu - B\hat{X}\,dt = \theta\Sigma^{-1}(dY + \sigma B\hat{X}\,dt) - B\hat{X}\,dt}$$

(iv) Let's now specialize to the scalar case, $n=1$, so as to make a bit more progress. We solve for the limiting variance v by solving the quadratic

$$0 = 1 - 2Bv - (1-Bv)^2\,\sigma^2/\Sigma^2$$

and then we see

$$\boxed{d\hat{X} = a\,dY - \beta\hat{X}\,dt} \qquad \begin{cases} a = \theta/\Sigma \\ \beta = B - \theta\sigma B/\Sigma \end{cases}$$

Probably too complicated to do a diverse-beliefs story...

Note: (i) $\max_c \{ U(c) + b\,U(w-c) \}$ when $c^{-R} = b\,(w-c)^{-R}$

$\therefore \quad \dfrac{c}{w} = \dfrac{1}{1+b^{1/R}}$

$= U(w)\,\left(1+b^{1/R}\right)^R$

(ii) $\max_c \{ \Delta t\, U(c) + b\, U(w-c) \} = \Delta t \quad U(w) \left(1 + \left(b/\Delta t\right)^{1/R}\right)^R$

(iii) $\max_c \{ \Delta t\, U(c) + b\, U(w - c\Delta t) \} = U(w) \left(\Delta t + b^{1/R}\right)^R$

when $\dfrac{c}{w} = \dfrac{1}{\Delta t + b^{1/R}}$,

the interpretation being that we consume at rate $c\,dt$

Get recursion

$$\varphi_{n+1}(\pi) = \left\{ \Delta t + \left(\beta(1-R)\,A_n(\pi) \right)^{1/R} \right\}^R$$

Stochastic optimal control in HMM (25/7/09)

1) Suppose we work in discrete time, and that there is some vector of assets whose log returns Y_t in period t are drawn from density f_j when $\xi_t = j$. Here, ξ_t is a finite-state Markov chain. The state of the system at any time is (w, π), where w is the agent's wealth, and π is his posterior. Now the updating of the posterior is relatively easy:

$$\pi_{t+1}(j) = \frac{\sum_k \pi_t(k)\, p_{kj}\, f_j(Y_{t+1})}{\sum_k \sum_\ell \pi_t(k)\, p_{k\ell}\, f_\ell(Y_{t+1})}$$

So we have $\pi_{t+1} = G(\pi_t, Y_{t+1})$.

2) Suppose the agents are CRRA, trying to max $E\left[\sum_{t\geq 0} \beta^t\, U(c_t)\right]$. The value function $V(w,\pi) = U(w)\,\varphi(\pi)$, and the aim is to solve for φ. We have

$$V(w,\pi) = \sup_{c,q}\left[U(c) + \beta\, E^\pi\left\{ U\left((w-c)q\cdot e^Y + r(w-c)(1-q\cdot 1)\right)\varphi(G(\pi,Y))\right\}\right]$$

$$= \sup_c \left[U(c) + \beta (w-c)^{1-R} \underbrace{\sup_q E^\pi\left\{ U(q\cdot e^Y + r(1-q\cdot 1))\,\varphi(G(\pi,Y))\right\}}_{\equiv A(\pi), \text{ say.}}\right]$$

Optimization over c is easy

$$\frac{w-c}{c} = \left(\beta(1-R)A(\pi)\right)^{1/R} \quad\Rightarrow\quad \boxed{\frac{c}{w} = \frac{1}{1+\left(\beta(1-R)A(\pi)\right)^{1/R}}}$$

and substituting back into Bellman eq^{ns} gives

$$V(w,\pi) = U(w)\varphi(\pi) = U(w)\left(1+\left(\beta A(\pi)(1-R)\right)^{1/R}\right)^R$$

$$\therefore\quad \boxed{\varphi(\pi)^{1/R} = 1 + \left(\beta(1-R)A(\pi)\right)^{1/R}}$$

3) Policy improvement won't work so well here, because if we want to determine the value of a particular policy $\theta_n(\pi)$, the equations (once optimized over c) are not linear in $\varphi_n(\pi)$

$\therefore$ Only hope is value improvement.

Following this along, we can use $\varphi_0(\pi) \equiv 1$, corresponding to consumption at just two periods. Then

$$V_{n+1}(w,\pi) = U(w)\,\varphi_{n+1}(\pi)$$
$$= \sup_c \Big[U(c) + \beta \sup_\theta U(w-c)\, E^\pi\big\{ (\theta e^Y + (1-\theta) r)^{1-R} \varphi_n(G(\pi,Y)) \big\} \Big]$$
$$= \sup_c \Big[U(c) + U(w-c) \underbrace{\beta E^\pi\big\{ (\theta^*_{n+1}(\pi) e^Y + (1-\theta^*_{n+1}(\pi)) r)^{1-R} \varphi_n(G(\pi,Y)) \big\}}_{\equiv A_n(\pi)} \Big]$$

and hence $\varphi_{n+1}(\pi)^{1/R} = 1 + A_n(\pi)^{1/R}$. Should be OK recursively; this is value improvement, not policy improvement.

4) If we consider this as a discretisation of a continuous-time problem, using time step Δt, then the objective would better be

$$E\Big[\sum_{n\ge 0} \beta^n U(c_n) \cdot \Delta t \Big]$$

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = B(\alpha,\beta)$$

Bayesian analysis of a two-state HMM regression model (31/7/09)

1) Christina Erlwein looks at a story where you see observations

$$Y_t = \theta(\xi_t) F_t + \varepsilon_t$$

where the F_t is an observed factor, $\xi_t \in \{0,1\}$ is the hidden Markov chain, and $\varepsilon_t \sim N(0, \tau(\xi_t)^{-1})$. The idea is to find out about the unknowns θ_0, θ_1, τ_0, τ_1 and p_{01}, p_{10}, the Markovian transition probabilities.

Presumably the whole story would generalise to vector-valued Y, F, and even to more than 2 states, but this is already hard enough for now.

2) I'm going to try doing a Bayesian analysis on the assumption that we see $(Y_i, F_i, \xi_i)_{i=1}^t$. This would give a posterior for the (unobserved) path of ξ; if we do it right, we can integrate out the parameters, find posterior prob^{ty} of all paths, and then just keep the most likely ones. This is approximate, but prevents exponential explosion of the stored data

In more detail then, suppose that for the prior at time 0 we have

$$p_{01} \sim B(\alpha_0^0, \beta_0^0) \qquad p_{10} \sim B(\alpha_0^1, \beta_0^1)$$

$$\tau_i \sim \Gamma(a_0^i, b_0^i), \qquad \theta(i) \sim N(0, (K_0^i \tau_i)^{-1}), \quad P(\xi_0 = 0) = 1.$$

Suppose we fix a path $\xi_1, \dots, \xi_t$. Then the likelihood of the given observations is

$$\propto B(p_{01} | \alpha_0^0, \beta_0^0) B(p_{10} | \alpha_0^1, \beta_0^1) \Gamma(\tau_0 | a_0^0, b_0^0) \Gamma(\tau_1 | a_0^1, b_0^1) . \prod_{j=1}^t p_{\xi_{j-1}\xi_j}$$

$$\exp\Big\{ -\tfrac{1}{2} \sum_{i=1}^t (Y_i - \theta(\xi_i) F_i)^2 \tau(\xi_i) + \tfrac{1}{2} \sum_{i=1}^t \log \tau(\xi_i) \Big\} \exp\Big\{ -\tfrac{1}{2} \sum_{j=0}^1 \theta_j^2 K_0^j \tau_j \Big\} \sqrt{\tau_0 \tau_1}$$

Let's write $N_{ij}(t)$ for the number of transitions from state i to state j by time t. Then the terms involving the transition probabilities become

$$B(p_{01} | \alpha_0^0, \beta_0^0) B(p_{10} | \alpha_0^1, \beta_0^1) \prod_{ij=0}^1 p_{ij}^{N_{ij}(t)}$$

which is a Beta, up to scaling. Integrating gives

$$\frac{B(\alpha_0^0 + N_{01}(t), \beta_0^0 + N_{00}(t))}{B(\alpha_0^0, \beta_0^0)} \cdot \frac{B(\alpha_0^1 + N_{10}(t), \beta_0^1 + N_{11}(t))}{B(\alpha_0^1, \beta_0^1)}$$

The rest of the Bayesian updating splits into two similar bits — let's just focus on the $\xi=0$ part of the story. We have a likelihood (suppressing the label $\xi=0$ when convenient)

$$\propto \tau^{a_0-1} e^{-b_0\tau} \exp\Big[-\frac{\tau}{2} \sum_{\xi_i=0} (Y_i - \theta F_i)^2 + \sum_{\xi_i=0} \tfrac{1}{2}\log\tau - \tfrac{1}{2} K_0 \tau \theta^2 \Big] \sqrt{\tau 2\pi}$$

$$= \tau^{a_0-1} e^{-b_0\tau} \exp\Big[-\frac{\tau}{2} K_t \theta^2 + \theta\tau \sum_{\xi_i=0} Y_i F_i - \frac{\tau}{2} \sum_{\xi_i=0} Y_i^2 + \frac{m}{2}\log\tau \Big] \sqrt{\tau 2\pi}$$

where $K_t = K_0 + \sum_{\xi_i=0} Y_i^2$, $m = |\{i : \xi_i = 0, i \le t\}|$

$$= \tau^{a_0-1} \exp\Big[-\frac{\tau}{2} K_t (\theta - \hat\theta_t)^2 + \frac{\tau}{2} K_t \hat\theta_t^2 - \frac{\tau}{2} \sum_{\xi_i=0} Y_i^2 - b_0\tau + \frac{m}{2}\log\tau \Big] \sqrt{\tau 2\pi}$$

where $K_t \hat\theta_t = \sum'_{\xi_i=0} Y_i F_i$

$$= \tau^{a_0 + m/2 - 1} \exp\Big[-\frac{\tau}{2} K_t (\theta - \hat\theta_t)^2 - \tau\Big(b_0 + \tfrac{1}{2}\sum'_{\xi_i=0} Y_i^2 - \tfrac{1}{2} K_t \hat\theta_t^2 \Big) \Big] \sqrt{\tau 2\pi}$$

Integrate out (τ, θ) in this gives

$$K_t^{-1/2} \Big(b_0 + \tfrac{1}{2} \sum'_{\xi_i=0} Y_i^2 - \tfrac{1}{2} K_t \hat\theta_t^2 \Big)^{-a_0 - m/2} \Gamma(a_0 + \tfrac{m}{2})$$

and rescaling by the time-0 constants gives

$$\sqrt{\frac{K_0}{K_t}} \left(\frac{b_0}{b_0 + \frac{1}{2}\sum_{\xi_i=0} Y_i^2 - \frac{1}{2} K_t \hat\theta_t^2} \right)^{a_0 + m/2} \frac{\Gamma(a_0 + m/2)}{\Gamma(a_0)}.$$

So for each of the paths we care about, we need to keep a record of

$$\sum_{\xi_j=0,1} Y_j^2, \quad \sum_{\xi_j=0,1} Y_j F_j, \quad N_{ij}(t)$$

which should permit us to do the updating.

Optimal investment with random lifetime (2/8/09)

(This must be the thing Mark Davis + Michel Vellekoop did?)

Suppose you live to random time τ, which is totally inaccessible, with deterministic hazard rate $h(\cdot)$. You wish to invest and consume admissibly so as to max $E\left[\int_0^\tau \varphi(t)\, U(c_t)\,dt\right]$, where φ is a deterministic function of time which you choose. How to do it?

1) Define

$$V(t,w) \equiv \sup E_t\left[\int_t^\tau \varphi(s) U(c_s)\,ds \;\Big|\; \tau > t,\ w_t = w\right]$$

Let's suppose CRRA U, so that $V(t,w) = f(t)\,U(w)$. For this, the HJB is

$$0 = \sup\left[\varphi(t) U(c) + V_t + (rw + \theta(\mu - r) - c) V_w + \tfrac{1}{2}\sigma^2\theta^2 V_{ww} - h(t) V\right]$$

$$= \sup\Big[\varphi(t) U(c) + U(w) f' + f(1-R) U(w)\left\{r + z(\mu - r) - \tfrac{c}{w}\right\} - R(1-R) U(w) \tfrac{\sigma^2}{2} z^2 f - h(t) U(w) f(t)\Big]$$

$$= \sup U(w)\left[\varphi(t) x^{1-R} + f' + f(1-R)\{r + z(\mu - r) - x\} - \tfrac{\sigma^2 z^2}{2} R(1-R) f - hf\right]$$

whence $\sigma^2 R z = (\mu - r)$, $x^{-R}\varphi = f$ so we have that

$$\boxed{\theta = \frac{\mu - r}{\sigma^2 R} w \equiv \pi_M w, \qquad \frac{c}{w} = \left(\frac{\varphi(t)}{f(t)}\right)^{1/R}}$$

Thus the investment is exactly what it was for the standard problem, but the consumption rate varies with time in some interesting way. HJB says

$$0 = f' + r(1-R) f + f(1-R)\frac{\kappa^2}{2R} + R\varphi^{1/R} f^{1-1/R} - hf$$

$$= f' - \left(h + (R-1)(r + \kappa^2/2R)\right) f + R\varphi^{1/R} f^{1-1/R}$$

If we set $b \equiv (R-1)(r + \kappa^2/2R)$ and define $\psi(t) = \exp\left(-bt - \int_0^t h(s)\,ds\right) = e^{-bt}\bar{F}(t)$ we get an ODE for $g(t) \equiv f(t)\,\psi(t)$:

$$\boxed{g'(t) + \tilde{\varphi}(t)\, g(t)^{1-1/R} = 0} \qquad \text{where } \tilde{\varphi}(t) = (\varphi(t)\psi(t))^{1/R} R$$

Hence $$\boxed{\frac{d}{dt}\left[g(t)^{1/R}\right] = \frac{1}{R} g' g^{1/R - 1} = -\frac{\tilde{\varphi}}{R}}$$

To solve this, you need to put in some BCs. I'd suggest we assume $\tau \le T_1$ a.s. (though this isn't really needed) and $\boxed{\varphi(t) = 0 \text{ for } t \ge T_0}$ $(T_0 < T_1)$

There may be something here. We can get an expression for the covariance of $(Y_{t_1}, \ldots, Y_{t_n})$ in terms of the parameters. For a given prior over $\Theta = (\alpha, \sigma, \lambda, \varepsilon)$ we could now do a numerical MLE. We could then pretend the parameters are the truth and do some filtering.

Even if this didn't work sufficiently simply to be a M2M alternative it could be a useful signalling methodology...

Alternative to mark-to-Market again (3/8/09)

1) Returning to the story on p6, it seems to me better to offer the dynamics for the observed price Y:

$$Y_t = Y_0 + \int_0^t \varepsilon W_s \, ds + Z_t - Z_0,$$

where $dZ_t = \sigma dW_t' - \lambda Z_t dt$ is an independent OU process. This way we have a drift, $\varepsilon W_t dt$, in the price, which can vary a bit. We can then stack the processes W_t, $\xi_t \equiv \int_0^t W_s ds$, Z_t into state vector $X_t \equiv (Z_t, \xi_t, W_t)^T$ evolving as

$$dX_t = \begin{pmatrix} 0 & \sigma \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} dW \\ dW' \end{pmatrix} + \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix} X_t \, dt \equiv \mathcal{Z} \begin{pmatrix} dW \\ dW' \end{pmatrix} + B X_t dt$$

We have

$$\exp(tB) = \begin{pmatrix} e^{-\lambda t} & 0 & 0 \\ 0 & 1 & \varepsilon t \\ 0 & 0 & 1 \end{pmatrix},$$

and after some calculations

$$X_t = e^{tB} X_0 + \eta, \qquad \eta \sim N\left(0, \begin{pmatrix} \sigma^2(1-e^{-2\lambda t})/2\lambda & 0 & 0 \\ 0 & \varepsilon^2 t^3/3 & \varepsilon t^2/2 \\ 0 & \varepsilon t^2/2 & t \end{pmatrix}\right)$$

2) But the standard BS model for the log of an asset price does not fit into this! Maybe better is to combine the two and do

$$Y_t = Y_0 + \int_0^t \varepsilon W_s \, ds + Z_t - Z_0 + a\beta_t$$

for some third independent BM. Stacking the state vector as $X_t = (\beta_t, Z_t, \xi_t, W_t)^T$ we get

$$X_t = \begin{pmatrix} X_0^1 \\ e^{-\lambda t} X_0^2 \\ X_0^3 + \varepsilon t X_0^4 \\ X_0^4 \end{pmatrix} + N\left(0, \begin{pmatrix} a^2 t & 0 & 0 & 0 \\ 0 & \frac{\sigma^2}{2\lambda}(1-e^{-2\lambda t}) & 0 & 0 \\ 0 & 0 & \varepsilon^2 t^3/3 & \varepsilon t^2/2 \\ 0 & 0 & \varepsilon t^2/2 & t \end{pmatrix}\right)$$

– but can we estimate the parameters too? We have $(a, \sigma, \lambda, \varepsilon)$; we could get an expression for the likelihood, but only numerics will be available

Maybe more satisfying (but not easier) is to tell a story where

$$d Y_t = \sigma_1 \, dW_t^1 + \lambda (Z_t - Y_t)\, dt$$

$$d Z_t = \sigma_2 \, dW_t^2 + \mu_t \, dt - \varepsilon Z_t \, dt$$

$$d \mu_t = \sigma_3 \, dW_t^3$$

where the signal we care about is Z, observation process is Y. Idea would be that the underlying Z is basically a BM with drift, with perhaps some mean reversion (though taking $\varepsilon = 0$ would also be interesting). The drift μ can itself also evolve slowly.

We have

$$dX \equiv d\begin{pmatrix} Y \\ Z \\ \mu \end{pmatrix} = \sigma \, dW + BX \, dt, \qquad B = \begin{pmatrix} -\lambda & \lambda & 0 \\ 0 & -\varepsilon & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

After some calculations, arrive at

$$\exp(tB) = \frac{1}{\lambda - \varepsilon} \begin{pmatrix} (\lambda-\varepsilon) e^{-\lambda t} & \lambda (e^{-\varepsilon t} - e^{-\lambda t}) & e^{-\lambda t} + \frac{\lambda - \cancel{\varepsilon} - \lambda e^{-\varepsilon t}}{\varepsilon} \\ 0 & (\lambda-\varepsilon) e^{-\varepsilon t} & \frac{\lambda - \varepsilon}{\varepsilon}(1 - e^{-\varepsilon t}) \\ 0 & 0 & \lambda - \varepsilon \end{pmatrix}$$

OK, but this is getting ever more cumbersome. For the transitions of X, I think it will be best to do computer algebra + numerics.

Optimal investment with HMM again (29/8/09)

(1) Suppose wealth equation is

$$dw_t = rw_t\,dt + \theta_t\,\sigma\left(dW_t + \frac{\mu(\xi_t)-r}{\sigma}\,dt\right) - c_t\,dt$$

$$= rw_t\,dt + \theta_t\,\sigma\left(dW_t + h(\xi_t)\,dt\right) - c_t\,dt$$

$$= rw_t\,dt + \theta_t\,\sigma\,dY_t - c_t\,dt$$

where ξ_t is an unobserved Markov chain with intensity matrix Q. If we define the innovations BM N_t by

$$dY_t = dN_t + \hat{h}_t\,dt$$

and let $\pi_t(x) \equiv P(\xi_t = x \mid \mathcal{Y}_t)$ then the filtering equation (RW, VI.11.1) is

$$d\pi_t(x) = \pi_t(x)(h(x) - \hat{h}_t)\,dN_t + (Q^T\pi_t)(x)\,dt$$

(2) Let's now specialise the case of a two-state chain, $I = \{1, 2\}$, $p_t \equiv \pi_t(1) \equiv 1-\pi_t(2)$ so that $\hat{h}_t = p_t h_1 + (1-p_t)h_2$ and

$$\left[Q = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}\right]$$

$$\begin{cases} dp_t = p_t(h_1 - \hat{h}_t)\,dN_t + (-\alpha p_t + \beta(1-p_t))\,dt \\ dw_t = rw_t\,dt + \theta_t\sigma(dN_t + \hat{h}_t\,dt) - c_t\,dt \end{cases}$$

If we let $V(w,p) = U(w)\,f(p)$ be the value-f^{n} for CRRA investor, HJB gives

$$0 = \sup\Big[U(c) - \rho V + \{rw + \theta\sigma(ph_1 + (1-p)h_2) - c\}V_w + (\beta - (\alpha+\beta)p)V_p$$
$$+ \tfrac{1}{2}\sigma^2\theta^2 V_{ww} + \sigma\theta\,p(1-p)(h_1-h_2)V_{wp} + \tfrac{1}{2}p^2(1-p)^2(h_1-h_2)^2 V_{pp}\Big]$$

$$\left.\begin{aligned} \therefore\ c^{-R} &= V_w \\ \sigma\theta V_{ww} + p(1-p)(h_1-h_2)V_{wp} + (ph_1 + (1-p)h_2)V_w &= 0 \end{aligned}\right\}$$

are the optimality conditions. After some algebra we get

$$\boxed{\begin{aligned} 0 = R f^{1-1/R} - \rho f + r(1-R)f + (\beta - (\alpha+\beta)p)f' \\ + (1-R)\left((ph_1 + (1-p)h_2)f + (1-p)(h_1-h_2)f'\right)^2/2Rf \\ + \tfrac{1}{2}p^2(1-p)^2(h_1-h_2)^2 f'' \end{aligned}}$$

NB

The problem with this analysis is that it's open-loop, not closed-loop. The perturbation story for agent i assumes that all other agents have committed to their extraction path $x_j(t)$. However, we wouldn't have this under feedback; if i alters his path, then the other agents will adjust in response!

Games with exhaustible resources (4/9/09)

(i) There's a preprint with this title by Harris, Howison + Sircar. The idea is that you have N energy producers, and producer i produces at rate $q_i(t)$. The total amount produced by time t by producer i is $z_i(t) = \int_0^t q_i(s)ds$. The idea is that the cost of production is $b_i(z)$, increasing; as you deplete cheap sources, you have to turn to more expensive ones. In their story, they choose $b_i(z) = I_{\{z \geq a_i\}}$, but this is probably unnecessarily restrictive.

If total rate of production at time t is

$$Q(t) \equiv \sum_{i=1}^{N} q_i(t)$$

then the price you get for your energy is $P(Q(t))$, where $P : [0,\infty) \to \mathbb{R}$ is C^2 strictly decreasing, ultimately negative (even we have $P' < 0$).

If we write $Q_{-i}(t) = \sum_{j \neq i} q_j(t)$ then the value to producer i from extracting according to q_i is

$$\int_0^\infty e^{-rt} \; q_i(t) \Big(P(q_i(t) + Q_{-i}(t)) - b_i(z_i(t)) \Big) dt$$

How is this to be optimized?

(ii) For the time being let's drop the subscript i, and write $F(t,x) = P(x + Q_{-i}(t))$. It's clear that you would never produce if it were unprofitable. Suppose that the optimal q^* has been found, and we consider perturbing this to $q^* + \eta$ (where $\eta_t \geq 0$ if $q^*_t = 0$). The first-order change in the objective is

$$\int_0^\infty e^{-rt} \Big\{ \eta_t \Big(F(t,q^*_t) - b(z_t) + q^*_t F'(t,q^*_t) \Big) - q^*_t b'(z^*_t) \int_0^t \eta_s ds \Big\} dt$$

$$= \int_0^\infty e^{-rt} \eta_t \Big(F(t,q^*_t) - b(z_t) + q^*_t F'(t,q^*_t) - \int_t^\infty e^{-r(s-t)} q^*_s b'(z^*_s) ds \Big) dt$$

≤ 0

Hence we deduce q^* should satisfy

$$\boxed{\begin{aligned} F(t,q_t) - b(z_t) + q_t F'(t,q_t) - \int_t^\infty e^{-r(s-t)} q_s b'(z_s) ds &\leq 0 \\ -q_t \big(F(t,q_t) - b(z_t)\big) &\leq 0 \end{aligned}}$$

and at least one always holds with equality.

Of course, we're looking for Pareto-efficient q_i. Notice that if we take the first variational inequality with equality, we get

$$e^{-rt}\Big(F(t,q) - b(z) + q F'(t,q)\Big) - \int_t^\infty e^{-rs} q\, b(z)\, ds \Big) = 0$$

so differentiating wrt t gives us

$$\boxed{-r\Big(F - b(z) + q F'\Big) + \dot{q}\big(2F' + q F''\big) + \dot{F} + q\dot{F}' = 0} \qquad (*)$$

Now the condition $\bar{\rho} \equiv \sup_Q \{ -Q\, P''(Q)/P'(Q) \} < 2$ appears in the paper of HHS as a condition that makes things work nicely. If we had that condition, then

$$F''(t,q) = P''(q + Q_{-i}) < \frac{-2P'(q+Q_{-i})}{q + Q_{-i}} \leq \frac{-2P'(q+Q_{-i})}{q}$$

$$= \frac{-2F'(t,q)}{q}$$

so that the coefficient of $\dot{q} \equiv \ddot{z}$ in the boxed equation is certainly negative. This may be significant.

(iii) To solve this, we could suppose that z solves the nonlinear 2nd order ODE $(*)$ while $F > b(z)$, but $\dot{z} \equiv q = 0$ when $F \leq b(z)$. We could then do an ODE in N dimensions, with arbitrary initial condition $q(0) \equiv \dot{z}(0)$. Having solved this, we could then vary $q(0)$ until we match the condition

$$F(0, q_0) - b_i(0) + q_i(0) F'(0, q_0) = \int_0^\infty e^{-rs} q_i(s)\, b_i(z_i(s))\, ds$$

or again

$$F(Q_0) - b_i(0) + q_i(0) F'(Q_0) = \int_0^\infty e^{-rs} q_i(s)\, b_i(z_i(s))\, ds$$

Depending on how well the ODE solver copes with this discontinuous coefficient, we should be able to deal with this

$$V_w = \delta^{-R} g', \qquad V_\delta = \delta^{-R}\left[(1-R)g - z\,g'(z)\right]$$

$$V_{ww} = \delta^{-1-R} g'', \quad V_{w\delta} = -\delta^{-1-R}\left(Rg' + z g''\right),$$

$$V_{\delta\delta} = \delta^{-1-R}\left\{-R(1-R)g + 2Rz g' + z^2 g''\right\}$$

Investment/consumption problem with random endowment (5/9/09)

Suppose that an agent receives a log-Brownian endowment stream

$$d\delta_t = \delta_t (v\, dW_t + b\, dt)$$

and then invests in a correlated log-Brownian market:

$$dw_t = r w_t\, dt + \theta_t (\sigma dW'_t + (\mu - r) dt) - c_t\, dt + \delta_t\, dt, \qquad dW dW' = \rho\, dt.$$

to achieve

$$V(w,\delta) = \sup E\left[\int_0^\infty e^{-\beta t} U(c_t) dt \,\Big|\, w_0 = w, \delta_0 = \delta \right] \qquad U'(x) = x^{-R}.$$

The HJB is

$$0 = \sup\Big[U(c) - \beta V + (rw + \theta(\mu - r) - c + \delta) V_w + b\delta V_\delta + \tfrac{1}{2}\sigma^2\theta^2 V_{ww} + \rho\sigma\theta v\, V_{w\delta} + \tfrac{1}{2}\delta^2 v^2 V_{\delta\delta} \Big]$$

and we get some scaling: $V(w,\delta) = \delta^{1-R} g(w/\delta)$. After some calculations, we get $(z \equiv w/\delta)$

$$0 = \tilde{U}(V_w) - \beta V + (rw + \delta) V_w + b\delta V_\delta - \frac{\left((\mu - r) g' - \sigma\rho v (R g' + z g'') \right)^2}{2\sigma^2 g''} + \tfrac{1}{2} v^2 \left(z^2 g'' + 2Rzg' - R(1-R) g \right)$$

$$= \delta^{1-R}\Big[\tilde{U}(g') - \beta g + (rz + 1) g' + b\left((1-R) g - z g' \right) - \frac{\left((\mu - r) g' - \sigma\rho v (R g' + z g'') \right)^2}{2\sigma^2 g''} + \tfrac{1}{2} v^2 \left(z^2 g'' + 2Rzg' - R(1-R) g \right) \Big]$$

This should be OK to solve by the usual methods.

Optimal $l = q^{R/(\alpha - \gamma R)} \, (g')^{1/\alpha - \gamma R}$

Optimal $\Delta = q^{\alpha/(\alpha - \gamma R)} \, (g')^{\gamma/(\alpha - \gamma R)}$

Production economy with irreversible capital investment (5/9/09)

Let's suppose we have a homogeneous-of-degree-1 production function $f(K,L)$ and investment in financial assets. Wealth equation is

$$\begin{cases} dw_t = rw_t\,dt + \theta_t(\sigma dW_t + (\mu-r)dt) - c_t\,dt + f(K_t,L_t)dt - dJ_t \\ dK_t = -\delta K_t\,dt + dJ_t \end{cases}$$

with objective

$$V(w,K) = \sup E\left[\int_0^\infty e^{-\beta t} U(c_t,L_t)\,dt \;\middle|\; w_0 = w,\ K_0 = k\right]$$

and $U(c,L) = c^{1-R}L^{\alpha}/(1-R)$, J increasing. This is a bit rough for the felicity, but it does help us to do a scaling story:

$$V(\lambda w, \lambda K) = \lambda^{1-R+\alpha}\, V(w,K)$$

So $V(w,K) = K^{1-R+\alpha}\, g(x)$, $x \equiv w/K$. Now we go for HJB:

$$0 = \sup_{\theta,c,L}\left[U(c,L) - \beta V + (rw + \theta(\mu-r) - c + f(K,L))V_w + \tfrac{1}{2}\sigma^2\theta^2 V_{ww} - \delta K V_K\right]$$

with side condition $V_w \geq V_K$, equal when $dJ > 0$. Write $s = c/K$, $l = L/K$, and we get

$$0 = \sup_{s,l,y}\Big[U(s,l) - \beta g + \{rx + y(\mu-r) - s + f(1,l)\}g' + \tfrac{1}{2}\sigma^2 y^2 g'' - \delta\left((1-R+\alpha)g - xg'\right)\Big],$$

$$g' \geq (1-R+\alpha)g - xg'.$$

To make things a bit more explicit, take f to be Cobb-Douglas: $f(K,L) = A\,K^{1-\gamma}L^{\gamma}$, and after lengthy calculations we get the max'd value of $U(s,l) + g'(f(1,l)-s)$ is

$$(g')^{(\alpha+\gamma(1-R))/(\alpha-\gamma R)}\; q^{\alpha/(\alpha-\gamma R)}\; \frac{\gamma R - \alpha}{\gamma(1-R)}$$

where $q \equiv -\gamma A(1-R)/\alpha$. Optimization over y is simple, so looks like we could handle this with usual tricks.

Games with exhaustible resources again (10/9/09)

(i) To understand better the problem studied on pp 25-26, firstly consider the static problem where agent i has production cost a_i and aims to maximise (over quantity Q_i produced) his profit

$$Q_i\left(P(Q)-a_i\right) \qquad\qquad Q=\sum_j Q_j$$

The Nash equilibrium condition is

$$P(Q)-a_i+Q_iP'(Q)=0 \qquad \text{if } Q_i>0$$

so that $Q_i=-(P(Q)-a_i)^+/P'(Q)$, where we now see that Q must be set so that

$$QP'(Q)+\sum_j(P(Q)-a_j)^+=0. \tag{1}$$

The NE is therefore given by

$$\boxed{q_i^*(a)=-(P(Q^*)-a_i)^+/P'(Q^*)} \tag{2}$$

where Q^* solves (1).

(ii) Now we look again at the continuous time problem. Perturbing optimal $\bar{q}$ to $\bar{q}+\eta$ gives first-order change for agent i equal to

$$\int_0^\infty e^{-rt}\eta_t\left\{\bar{q}_i(t)P'(Q_t)+P(Q_t)-b_i(\bar{z}_i(t))-\int_t^\infty e^{-r(s-t)}\bar{q}_i(s)\,b_i'(\bar{z}_i(s))\,ds\right\}dt$$

$$\equiv\int_0^\infty e^{-rt}\eta_t\left\{\bar{q}_i(t)P'(Q_t)+P(Q_t)-\bar{a}_i(t)\right\}dt, \tag{3}$$

say, where we define

$$\boxed{\bar{a}_i(t)\equiv b_i(\bar{z}_i(t))+\xi_i(t)\equiv b_i(\bar{z}_i(t))+\int_t^\infty e^{-r(s-t)}\bar{q}_i(s)\,b_i'(\bar{z}_i(s))\,ds.} \tag{4}$$

We shall have to have that $\{\ \}$ in the integral in (3) must be zero when $\bar{q}_i(t)>0$, since perturbation η is possible in either direction. Thus for a NE we will have to have

$$\boxed{\bar{q}(t)=q^*(\bar{a}(t))} \tag{5}$$

Notice also that

$$(6)\begin{cases} \dfrac{d\xi_i}{dt} = r\xi_i - \bar{q}_i(t)\, b_i'(\bar{z}_i(t)) & (\bar{q}_t \equiv q^*(b(\bar{z}_t)+\xi_t)) \\[2ex] \dfrac{d\bar{z}_i}{dt} = \bar{q}_i(t) \equiv q^*_i(b(\bar{z}_t)+\xi_t) \end{cases}$$

so that $(\xi, \bar{z})$ solves a non-linear first-order ODE. The boundary conditions are $\bar{z}_0 = 0$, and $\xi_\infty = 0$, since ultimately the quantities z extracted get so large that we switch to alternatives, and $b' = 0$ there.

(iii) Numerical solution of (6) is complicated because if you start with some arbitrary values of ξ_0, you can't ensure that ξ stays positive, and then you get problems with calculating $\bar{q} = q^*(b(\bar{z}) + \xi)$. It seems therefore best to reverse time in the ODE.

To do this, it seems helpful to impose a bit more structure – this is not necessary, but just helps to fix ideas. Suppose agent i has initial reserves V_i, $V_1 \geq V_2 \geq \dots$ and that extraction when there remains x in the oilfield costs $\varphi(x)$. Suppose the alternatives cost c to produce; then the fields will be depleted to level $V_c = \varphi^{-1}(c)$ (φ is of course decreasing).

Thus

$$b_i(z) = \varphi(V_i - z) \wedge c.$$

If t^*_i is time of exhaustion of field i, it seems natural to suppose that t^*_1 is the largest (can easily deal with a situation where this is not true, of course). So introduce the variable $\tau \equiv t^*_1 - t$. This changes the ODE (6) to $d\xi_i/d\tau = -d\xi_i/dt$, $d\bar{z}_i/d\tau = -d\bar{z}_i/dt$, and now at $\tau = 0$ we have initial condition $\xi_i = 0\ \forall i$, $\bar{z}_1 = V_1 - V_c$, and $\bar{z}_j > V_j - V_c$ for $j > 1$.

We have to search over initial values of $\bar{z}$ so that as τ increases, (*) all the $\bar{z}_i$ hit zero at the same time. The trick here would be to modify the ODE to

$$\frac{d\bar{z}_i}{d\tau} = -1 \qquad \text{if } \bar{z}_i < 0$$

and then we can check that they all hit 0 together by seeing whether they are all together at some big τ-value.

(*) but this is not correct; some producers might not start at time 0.

(iv) Maybe it's better to work with the reserves $x_i(t)$ (so a negative value means you've stop extracting and switched to alternatives). The cost of production $\varphi(x)$ is decreasing with x, x_c is the critical level of reserves where $\varphi(x_c) = c$.
Then

$$\begin{cases} \dfrac{d\xi_i}{dt} = r\xi_i + \bar{q}_i \; \varphi'(\bar{x}_i) \\ \dfrac{d\bar{x}_i}{dt} = -\bar{q}_i = -q_i^*(a_t) \qquad , \quad a_t \equiv \xi_t + \varphi(\bar{x}_t) \end{cases}$$

So in reverse time τ, we get

$$\begin{cases} \dfrac{d\xi_i}{d\tau} = -r\xi_i - \bar{q}_i \; \varphi'(\bar{x}_i) \\ \dfrac{d\bar{x}_i}{d\tau} = \bar{q}_i = q_i^*(\xi_t + \varphi(\bar{x}_t)) \end{cases}$$

with conditions $\xi_0 = 0$, and $\bar{x}(0)$ to be determined. The evolution keeps going until

$$\tau^* \equiv \sup\{t : \bar{x}_i(t) < V_i \text{ for some } i\}$$

and the success criterion is that $x_i(\tau^*) = V_i$ for all i. [It makes sense to work in terms of reserves in excess of x_c?]

(V) For numerical stability, it may be best to use $w \equiv \bar{x}_1$ as the independent variable, which increases (in reversed time) from V_2 to V_1. Writing $\tilde{x}$, $\tilde{\xi}$ for the corresponding functions $\bar{x}$, ξ expressed as functions of w, this would give

$$\begin{cases} \dfrac{d\tilde{\xi}_i}{dw} = \left(-r\tilde{\xi}_i - q_i^*(\tilde{\xi} + \varphi(\tilde{x}))\right) / q_1^*(\tilde{\xi} + \varphi(\tilde{x})) \\ \dfrac{d\tilde{x}_i}{dw} = q_i^*(\tilde{\xi} + \varphi(\tilde{x})) / q_1^*(\tilde{\xi} + \varphi(\tilde{x})) \end{cases}$$

With the structural assumptions imposed at (iii), it's obvious that the ordering of $x_i(t)$ remains unchanged with time, and that the gaps between them decrease as t increases.

In Danielsson et al, they have in effect $S_t = \varphi(q_t) Z_t$, where q_t is the dollar value of the agents' investment in the stock.

Some thoughts on price impact (22/9/09)

(1) This is a simple story which I've written up in some notes for the OZ book, but want to develop here some way. Suppose that the market price of a stock S_t satisfies

$$S_t = \varphi(\nu_t) Z_t$$

where Z is a standard log brownian motion, $dZ = Z(a\,dW + b\,dt)$, and ν_t is the number of units of the asset being held by some large investor at time t. We'll suppose that ν is a continuous semimartingale. If x_t denotes the cash holdings at time t of the agent, then

$$\tilde{w}_t = x_t + \Phi(\nu_t) Z_t$$

is his liquidation wealth at time t, where $\Phi(x) = \int_0^x \varphi(y)\,dy$. It isn't hard to show that

$$\boxed{d\tilde{w}_t = (r\tilde{w}_t - c_t)\,dt + \Phi_t\,(dZ_t - rZ_t\,dt)}$$

where $\Phi_t \equiv \Phi(\nu_t)$. If we define the mark-to-market value $w_t = x_t + \nu_t S_t$ $= x_t + \nu_t \varphi(\nu_t) Z_t$ at time t, then always $w \geq \tilde{w}$, and we have

$$\boxed{dw_t = (rw_t - c_t)\,dt + \nu_t(dS_t - rS_t\,dt) + \tfrac{1}{2}\varphi'(\nu_t) Z_t\, d\langle \nu \rangle_t .}$$

(2) What if we look at the process $y_t \equiv w_t / Z_t$? We find after some calculation that

$$\boxed{\begin{aligned} dy_t = {} & (a\nu\varphi(\nu) + \alpha\,\nu\varphi'(\nu) - ay)\,dW + \Big\{ \tfrac{1}{2}\alpha^2 \nu \varphi'' + (\beta\nu + \tfrac{1}{2}\alpha^2)\varphi' \\ & + (b - r - a^2)\nu\varphi + y(r - b + a^2) \Big\}\,dt, \end{aligned}}$$

where $\boxed{d\nu = \alpha\,dW + \beta\,dt.}$

Now if we follow Danielsson-Shin-Zigrand in supposing that at all times the large agent invests up to some VaR constraint, we would have

$$\nu_t \sigma_t S_t = \lambda w_t, \qquad [\text{assuming } \nu \geq 0]$$

for some constant λ, where $dS_t = S_t(\sigma_t\,dW_t + \mu_t\,dt)$. This says equivalently

$$\boxed{\nu\varphi(\nu)\sigma = \lambda y.}$$

Now we know that $\boxed{\sigma = a + \alpha\varphi'/\varphi}$ from $S = \varphi(\nu) Z$

Divide by r, let $r \downarrow 0$, to learn

$$\lambda \{ g'(0) + \varphi(0) \} = a\varphi(0)$$

$$\boxed{\therefore\ g'(0) = \frac{(a-\lambda)\,\varphi(0)}{\lambda}}$$

So suppose we now seek a solution of the form $y = f(\nu)$. This would lead to

$$\sigma = \frac{\lambda f(\nu)}{\nu\varphi(\nu)} = a + \frac{\alpha\varphi'}{\varphi}$$

and

$$dy = f'(\nu)\,\alpha\, dW + f\nu$$
$$= f'(\nu)\left\{\frac{\lambda f(\nu)}{\nu\varphi(\nu)} - a\right\}\frac{\varphi(\nu)}{\varphi'(\nu)}\,dW + f\nu$$

so when we compare with the earlier expression for dy, we learn that

$$\frac{f'(\nu)\varphi(\nu)}{\varphi'(\nu)}\left\{\frac{\lambda f(\nu)}{\nu\varphi(\nu)} - a\right\} = a\nu\varphi(\nu) + \nu\varphi'(\nu)\left\{\frac{\lambda f(\nu)}{\nu\varphi(\nu)} - a\right\}\cdot\frac{\varphi(\nu)}{\varphi'(\nu)} - a f(\nu)$$

This leads to a differential equation for f

$$\boxed{(f' - \nu\varphi')\left(\frac{\lambda f}{\nu\varphi} - a\right)\frac{\varphi}{\varphi'} = a(\nu\varphi - f)}$$

Maple seems not to be able to solve it though...

(3) Introduce
$$\xi = \frac{X_t}{Z_t} = y_t - \nu_t\,\varphi(\nu_t)$$

which we find satisfies

(*) $$d\xi = -(a\xi + \alpha\varphi(\nu))\,dW + \left((r + a^2 - b)\xi - \beta\varphi(\nu) - \tfrac{1}{2}\alpha^2\varphi'(\nu)\right)dt$$

Under leverage / VaR story, $y = \lambda^{-1}\sigma\nu\varphi(\nu)$, so $\xi = \lambda^{-1}\nu\varphi(\nu)\{\sigma - \lambda\}$, and we have again

(**) $$\sigma = a + \alpha\varphi'(\nu)/\varphi(\nu) = \frac{\lambda y}{\nu\varphi(\nu)} = \frac{\lambda(\xi + \nu\varphi(\nu))}{\nu\varphi(\nu)}$$

Suppose $\xi = g(\nu)$: then we would have (comparing the c/e of dW in (*) and the Itô expansion)

$$\alpha g'(\nu) = -a g(\nu) - \alpha\varphi(\nu) \Rightarrow \alpha = \alpha(\nu) = -\frac{a g(\nu)}{g'(\nu) + \varphi(\nu)}$$

Using (**), we deduce

$$\boxed{\lambda\{g(\nu) + \nu\varphi(\nu)\} = \nu\left\{a\varphi(\nu) - \frac{a g(\nu)\varphi'(\nu)}{g'(\nu) + \varphi(\nu)}\right\}}$$

This is an ODE for g... any chance of solving? Numerically of course... use $g(0) = 0$?

Try $\boxed{g = \varphi h}$ as a substitution, so we have the ODE

$$\lambda(h+\nu) = a\nu\left\{1 - \frac{h\varphi'}{\varphi + \varphi h' + \varphi' h}\right\} = a\nu \, \frac{\varphi + \varphi h'}{\varphi + \varphi h' + \varphi' h}$$

Thus if we set $\boxed{H \equiv h + \nu}$ we have

$$\lambda H = \frac{a\nu\varphi\, H'}{\varphi H' + \varphi'(H-\nu)} \quad \therefore \quad \lambda H \varphi H' + \lambda H(H-\nu)\varphi' = a\nu\varphi H'$$

$$\left[H'(0) = a/\lambda \text{ if } g(0)=0\right]$$

$$\therefore \quad \varphi H'(a\nu - \lambda H) = \lambda H(H-\nu)\varphi'$$

$$\Rightarrow \quad \frac{H'(a\nu - \lambda H)}{H(H-\nu)} = \frac{\lambda\varphi'}{\varphi}$$

$$\Rightarrow \quad H'\left\{-\frac{a}{H} + \frac{a-\lambda}{H-\nu}\right\} = \frac{\lambda\varphi'}{\varphi}$$

Maple can't do this one... Might try choosing H and then recovering φ ??

Not clear we can have φ monotone very easily...

We can try for a series solution for H, if we assume $\varphi(x) = e^{\varepsilon x}$. We have

$$H(x) = \frac{ax}{\lambda} + \varepsilon\left(1 - \frac{a}{\lambda}\right)x^2 + \varepsilon^2\left(1 - \frac{\lambda}{a}\right)x^3 + \varepsilon^3\frac{(a-2\lambda)^2}{\lambda a^2}(\lambda - a)x^4$$

$$+ \frac{\varepsilon^4(\lambda-a)}{\lambda a^3}\left(-23\lambda a^2 + 4a^3 + 41a\lambda^2 - 23\lambda^3\right)x^5$$

$$+ \frac{\varepsilon^5(\lambda-a)}{\lambda a^4}\left(395a^2\lambda^2 - 156\lambda a^3 - 426a\lambda^3 + 22a^4 + 166\lambda^4\right)x^6 + O(x^7)$$

(4) Seems like we need to distinguish two cases:

<u>Case $\lambda > a$</u> : Notice $H(\nu) = \{g(\nu) + \nu\varphi(\nu)\}/\varphi(\nu) = W/Z\varphi(\nu)$, so we shall expect $H > 0$ for $\nu > 0$. If at some time we were to get $H > \nu$, then we'd have $H' < 0$, and so H would get pulled back below ν fairly soon. If we ever got $H < a\nu/\lambda$, then H would be decreasing and would never get back above $a\nu/\lambda$. So looks like we should have $a\nu/\lambda < H < \nu$, and indeed if we solve out from ν = small, the initial value of H seems to have negligible impact on the trajectory.

<u>Case $\lambda < a$</u> : If $H > a\nu/\lambda > \nu$, then $H' < 0$. Likewise, if $H < \nu$ we get $H' < 0$. So seems like the natural home for H is in $(\nu, a\nu/\lambda)$, where $H' > 0$. What seems to happen is that $H \downarrow 0$ $(\nu\uparrow)$ ... ??! This one is potentially delicate, because if H gets too close to $a\nu/\lambda$, there is a massive upward push + the

solution gets pushed up to $a\nu/\lambda$ + can't be solved beyond that. Otherwise, if H were ever to get down below ν it never gets above ν again. So it seems like we must have H growing linearly, and since

$$H' = \frac{\lambda\varphi'}{\varphi} \frac{H(H-\nu)}{a\nu - \lambda H}$$

we get H' tending to a constant, so the only way this can be happening is if $H-\nu$ remains $O(1)$. What seems likely is that $h(\nu) \equiv H(\nu) - \nu$ has a limit as $\nu \to \infty$, so we'll see as $\nu \to \infty$ $\quad (\varphi(\nu) = \exp \epsilon\nu)$

$$1 = \frac{\lambda\varepsilon}{a-\lambda} h_\infty \qquad \therefore h_\infty = \frac{a-\lambda}{\lambda\varepsilon}.$$

The ODE for h is

$$h' = \frac{a\nu - \lambda(1+\epsilon h)(h+\nu)}{\lambda(h+\nu) - a\nu}$$

which might best be solved backwards from $\nu = \infty$ (thinking of the possible explosion in the ODE for H).

(This looks feasible, but we may need to take $s \equiv 1/\nu$ as the independent variable)

Further observations on price impact (4/10/09)

(1) A more general framework for price impact would be where we suppose

$$S_t = f(t, \nu_t, Z_t)$$

where ν_t is the number of units of asset held at time t, and f is sufficiently smooth. As usual, Z is Brownian motion. If we set

$$F(t, x, Z) \equiv \int_0^x f(t, \nu, Z)\, d\nu$$

and define the liquidation value $\underline{w}_t$ of an agent's portfolio as $\underline{w}_t \equiv x_t + F(t, \nu_t, Z_t)$ where x_t is the cash balance

$$dx_t = (rx_t - c_t)dt - \{F(t, \nu_t, Z_t) - F(t, \nu_{t-}, Z_t)\}$$

assuming simple ν, then we should have

$$d\underline{w}_t = (rx_t - c_t)dt + \{F_t dt + F_Z dZ + \tfrac{1}{2} F_{ZZ} dt\}$$

This makes intuitive sense: the change in wealth from the risky asset account is only coming from the variation of Z and t, not from changes in ν.

(2) The mark-to-market wealth $w_t = x_t + \nu_t f(t, \nu_t, Z_t)$ evolves differently. I get that

$$\boxed{dw_t = (rw_t - c_t)dt + \nu_t(dS_t - rS_t dt) + \tfrac{1}{2} f_\nu \, d\langle \nu \rangle.}$$

The story I tell has

$$S_t = \varphi(\nu_t) \exp\{aZ_t + (b - \tfrac{1}{2}a^2)t\},$$

and the story from DSZ has the expression

$$S_t = \exp\{\nu_t S_t + rt + \gamma Z_t\}$$

in effect, though this is a bit dodgy in that for some values of ν, Z there might be no solution...

(3) What we imagine here is that we shall have $\nu_t = g(w_t, Z_t)$. But suppose we had the V@R condition

$$\nu_t S_t = k \bar{w}_t / \sigma_t, \qquad \bar{w}_t = \int_{-\infty}^t \lambda e^{\lambda(s-t)} w_s \, ds \quad ?$$

Or the more basic condition $\nu_t S_t = k w_t / \sigma_t$? Taking this basic story first, and using $\nu_t = g(t, w_t, Z_t)$, $S_t = f(t, \nu_t, Z_t)$, we get that

(Maple: ../SOLO/PriceImpact/PI_3.mw)

$$\boxed{\sigma = \frac{f_v g_z + f_z}{f\{1 - \nu f_v g_w\}}}$$

and g must solve

$$\boxed{kw = g\{ f_v (kw g_w + g_z) + f_z \}}$$

If we try writing $\gamma(s) = g(t, e^{ks}, s)$, then $\dot{\gamma} = ke^{ks} g_w + g_z$, so we find the simpler ODE

$$k e^{ks} = \gamma(s) \{ f_v(t, \gamma(s), s) \dot{\gamma} + f_z(t, \gamma(s), s) \}$$

which would give $\gamma = g$ along some trajectory. Equally, we could have

$$\gamma(s) = g(t, b e^{ks}, s)$$

for some $b \neq 1$, and recover the same ODE for γ.

$$kbe^{ks} = \gamma(s) \{ f_v(t, \gamma, s) \dot{\gamma} + f_z \} = \gamma(s) \frac{d}{ds} f(t, \gamma(s), s)$$

How tractable this may be will depend on the exact form of f.

Games with exhaustible resources (14/10/09)

1). Despite the earlier problems, it may still be possible to solve the problem of Harris et al by reduction to the correct ODE. What we have to find is some vector field V which is the NE solution in the following sense. Agent j receives an objective value of

$$\int_0^\infty \Phi_j(t, x_t, p_t)\,dt,$$

where $p_t = \dot{x}_t$, and x is an integral curve of V. Agent j may perturb the vector field V to $V+\eta_j$, where η_j is constrained to lie in the 1-dimensional subspace that agent j controls. The vector field V has to have the property that for all j the possible perturbations available to agent j do not improve agent j's objective.

We have the flow map F satisfying

$$x_t \equiv F(t,x) = x + \int_0^t V(x_s)\,ds$$

so that

$$DF(t,x) = I + \int_0^t DV(F(s,x_s))\,DF(s,x_s)\,ds$$

solves a linear ODE.

2) If V is perturbed to $V+\eta$, then the integral curve x is perturbed to $x+\delta x$, where to first order

$$(x+\delta x)^{\cdot} = V(x+\delta x) + \eta(x+\delta x)$$

$$\doteq V(x) + DV(x)\,\delta x + \eta(x)$$

so that

$$(\delta x)^{\cdot} = DV(x)\,\delta x + \eta(x) \equiv DV(x)(\delta x + \varepsilon) \qquad (1)$$

again solves a linear ODE, where we write $\eta \equiv DV(x)\,\varepsilon$. Now let's bring δx back to the tangent space at x_0: define ξ by

$$DF(t,x)\,\xi_t = (\delta x)_t$$

so that when we differentiate we shall have

$$(\delta x)^{\cdot}_t = DV(x_t)(\delta x)_t + DF(t,x)\,\dot{\xi}_t$$

$$= DV(x)\left\{(\delta x)_t + \varepsilon_t\right\} \qquad \text{(see (1))}$$

so that

$$DF(t,x)\,\dot{\xi}_t = DV(x_t)\,\varepsilon_t = \eta(x_t).$$

3) Now consider the change in agent j's objective. To leading order this is

$$\int_0^\infty \left\{ D_x \Phi_j(t, x_t, p_t)\,(\delta x)_t + D_p \Phi_j(t, x_t, p_t)\,(\dot{\delta x_t}) \right\} dt$$

$$= \int_0^\infty \left\{ D_x \Phi_j(t, x_t, p_t) - \frac{d}{dt} D_p \Phi_j(t, x_t, p_t) \right\} (\delta x)_t \, dt$$

$$= \int_0^\infty \left\{ D_x \Phi_j(t, x_t, p_t) - \frac{d}{dt} D_p \Phi_j(t, x_t, p_t) \right\} DF(t, x_t)\, \xi_t \, dt$$

$$= \int_0^\infty \left\{ D_x \Phi_j(t, x_t, p_t) - \frac{d}{dt} D_p \Phi_j(t, x_t, p_t) \right\} DF(t, x_t) \left(\int_0^t DF(s, x_s)^{-1} \eta(x_s)\, ds \right) dt$$

$$= \int_0^\infty \int_s^\infty \left(D_x \Phi_j(t, x_t, p_t) - \frac{d}{dt} D_p \Phi_j(t, x_t, p_t) \right) DF(t, x_t)\, DF(s, x_s)^{-1} \Big) dt \; \eta(x_s)\, ds$$

However, it seems to me actually <u>hopeless</u>: even though we might try to combine these conditions, one for each j, and evaluate DF along some integral curve, the problem is the evolving DF along the curve requires knowledge of DV along the curve, and all we are going to get from such an analysis would be knowledge of V along the curve...

Guarantees problem in discrete time (15/10/09)

(1) Suppose we set up a binomial tree with $u = 1-d$, and $r=0$, so that RN prob of 'up' is $1/2$. Let's suppose objective prob π of up will be $> 1/2$. Suppose we set at time t a lower bound L_t which will apply to consumption at all later times, and the objective is

$L \left< \begin{matrix} u \\ d \end{matrix} \right.$

$$\sup E\left[\sum_{k=0}^{T} \alpha_k U(c_k) + \sum_{k=0}^{T-1} \beta_k U(L_k) \,\Big|\, w_0 = w,\ L_0 = L \right]$$

where $\alpha_k \geq 0$, $\beta_k \geq 0$ is some given sequence of constants. We insist that the L_t form an increasing process

(2) Suppose U is CRRA: $U'(x) = x^{-R}$, and set

$$V_t(w,L) = \sup E\left[\sum_{k=t}^{T} \alpha_k U(c_k) + \sum_{k=t}^{T-1} \beta_k U(L_k) \,\Big|\, w_t = w,\ L_t = L \right].$$

Usual scaling says $V_t(\lambda w, \lambda L) = \lambda^{1-R} V_t(w, L)$, so we may set

$$V_t(w,L) \equiv L^{1-R} v_t(z) \equiv L^{1-R} v_t(w/L).$$

We have $V_T(w,L) = \alpha_T U(w)$ if $w \geq L$. For feasibility, we must have

$$w_t \geq (T-t+1) L_t .$$

3) DP equations give

$$V_t(w,L) = L^{1-R} v_t(w/L)$$

$$= \sup_{\substack{c \geq L \\ L' \geq L}} \Big[\alpha_t U(c) + \beta_t U(L') + \pi V_{t+1}((w-c)u, L') + (1-\pi) V_{t+1}((w-c)d, L') \Big]$$

$L' = L\ell$

$c = Lx$

$$= L^{1-R} \sup_{\substack{x \geq 1 \\ \ell \geq 1}} \left[\alpha_t U(x) + \beta_t U(\ell) + \pi \ell^{1-R} v_{t+1}\left(\frac{(z-x)u}{\ell} \right) + (1-\pi) \ell^{1-R} v_{t+1}\left(\frac{(z-x)d}{\ell} \right) \right]$$

Need $(z-x)d \geq (T-t)\ell$ for feasibility

This optimization over two variables, x, ℓ, may be simplified if we write

$$\frac{(z-x)}{\ell} = a \quad \therefore \quad x = z - a\ell,$$ and with a fixed, we have

$$\alpha_t\, U(z-a\ell) + \beta_t\, U(\ell) + \ell^{1-R}\left\{\pi\, v_{t+1}(au) + (1-\pi)\, v_{t+1}(ad)\right\}$$

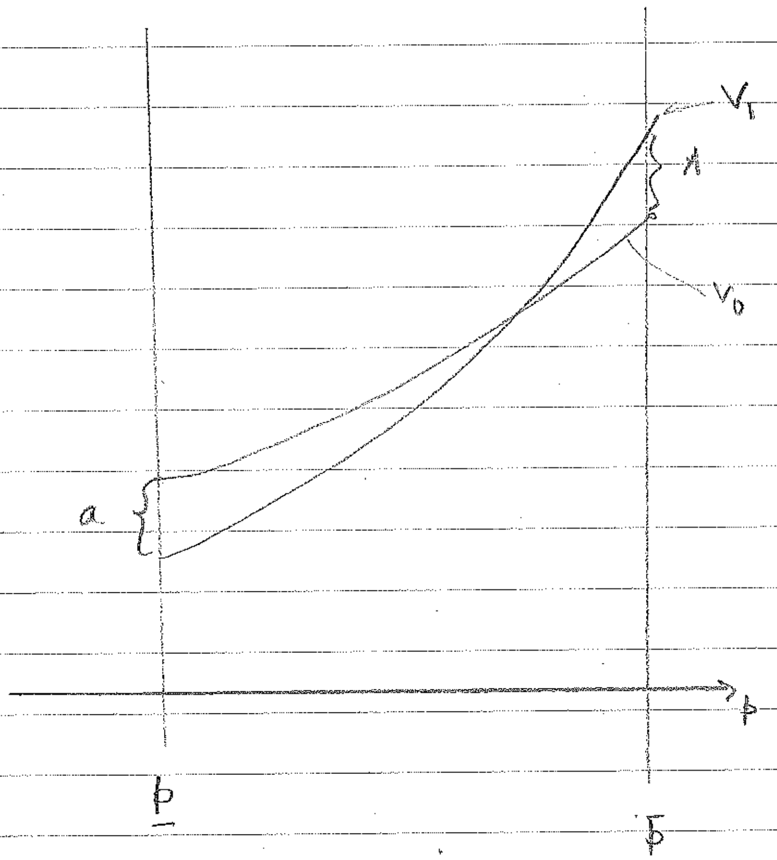
to optimize. Calculus gives (differentiating w.r.t. ℓ)

$$\alpha_t\, a\left(\frac{z}{\ell} - a\right)^{-R} = \beta_t + (1-R)\left\{\pi\, v_{t+1}(au) + (1-\pi)\, v_{t+1}(ad)\right\}$$

which leads to

$$\frac{z}{\ell} - a = \left(\frac{\alpha_t a}{\beta_t + (1-R)\left\{\pi\, v_{t+1}(au) + (1-\pi)\, v_{t+1}(ad)\right\}}\right)^{1/R}$$

provided of course that this value of ℓ is ≥ 1: else we use $\ell = 1$.



V_1
A
V_0
a
p
$\underline{p}$
$\bar{p}$

Mothballing an asset (5/11/09)

(1) Suppose you have some factory which can produce a good at unit rate for cost c_1 if in production, and costs c_0 (per unit time) if it has been mothballed. The price p_t of the good produced fluctuates, following

$$dp_t = p_t(\sigma dW_t + \mu dt).$$

Assume $r=0$ (everything is referred to discounted values) and we aim to

$$\max E\Big[\int_0^\infty e^{-\rho t}\big((p_t - c_1)I_{\{\xi_t=1\}} - c_0 I_{\{\xi_t=0\}}\big)dt - \sum_{t\geq 0} e^{-\rho t} I_{\{\Delta\xi_t=1\}} A$$
$$- \sum_{t\geq 0} e^{-\rho t} I_{\{\Delta\xi_t=-1\}} a\Big]$$

where $\xi_t = 1$ if factory is producing at time t, $\xi_t=0$ if not.

(2) Write $V(p,\xi)$ for the value function so that

$$e^{-\rho t}V(p_t,\xi_t) + \int_0^t e^{-\rho s}\big((p_s - c_1)I_{\{\xi_s=1\}} - c_0 I_{\{\xi_s=0\}}\big)ds$$
$$- \sum_{0<s\leq t} e^{-\rho s}\big(A I_{\{\Delta\xi_s=1\}} + a I_{\{\Delta\xi_s=-1\}}\big)$$

Hence we have while we're not intervening that

$$-\rho V + \mu p V' + \tfrac{1}{2}\sigma^2 p^2 V'' - c_\xi + p I_{\{\xi=1\}} = 0$$

If $-\alpha < 0 < \beta$ solve $Q(t) = \frac{\sigma^2}{2}t(t-1) + \mu t - \rho = 0$, then the value function looks like

$$\begin{cases} V_1 = B_1 p^\beta + K_1 p^{-\alpha} - \dfrac{1}{Q(1)}p - \dfrac{c_1}{\rho} \\ V_0 = B_0 p^\beta + K_0 p^{-\alpha} - c_0/\rho \end{cases}$$

As $p\to 0$, you would never want the project, so $K_0=0$. Also, as $p\to\infty$, growth can't be faster than p (if we assume $\rho>\mu$ for well posed problem) Hence $B_1=0$, and we have

$$V_1 = K_1 p^{-\alpha} + \frac{p}{\rho-\mu} - \frac{c_1}{\rho}, \quad V_0 = B_0 p^\beta - c_0/\rho$$

Assume $c_1 > c_0$. We need to select change-over points $\underline{p}$, $\bar{p}$ such that

$$\left.\begin{array}{l} \text{at } \underline{p},\ V_1 \text{ pastes to } V_0 - a \text{ in } C^1 \text{ fashion} \\ \text{at } \bar{p},\ V_0 \text{ pastes to } V_1 - A \text{ in } C^1 \text{ fashion} \end{array}\right\}.$$

Why should older people come out of stocks? (5/11/09)

1) Suppose you are able to invest in some risky stocks,

$$dS_t^i = S_t^i \left(\sigma_{ij} \, dW_t^j + \mu^i dt \right)$$

or a riskless bank account bearing interest at rate r. There's a rule of thumb in the investment management industry that when you are age A you should put $(100-A)\%$ of your wealth into stocks - yet this is not what the Merton story would give you...

2) The evolution of wealth of an individual is

$$dw_t = r w_t \, dt + \theta_t \cdot \left(\sigma \, dW_t + (\mu - r) \, dt \right) + (\varepsilon_t - c_t) \, dt$$

where ε_t is the earnings stream of the individual, and it is this which needs to be taken into account. Suppose we evaluate the NPV of all future earnings at time t:

$$y_t = \zeta_t^{-1} E\left[\int_t^{\infty} \zeta_u \, \varepsilon_u \, du \,\middle|\, \mathcal{F}_t \right]$$

so that the effective wealth of the investor is not w_t but $\bar{w}_t \equiv w_t + y_t$.

If we now follow the Merton rule, we should see

$$\begin{cases} c_t^* = \gamma_M \bar{w}_t \\ \theta_t^* = \pi_M \bar{w}_t \equiv R^{-1} (\sigma \sigma^T)^{-1} (\mu - r) \, \bar{w}_t \end{cases}$$

and so

$$\frac{\theta_t^*}{w_t} = \pi_M \left(w_t + y_t \right) / w_t = \pi_M \left\{ 1 + \frac{y_t}{w_t} \right\}$$

This somehow explains what's happening, because we expect $y_t \downarrow$ as $t \uparrow$, so in terms of your actual wealth w, the proportion you put into stocks would decline... all a bit trivial really... !

Increasing risk aversion as you get older would have a similar effect.

$$E\left[e^{-r t_1} : X_{t_1} = b\right] = \frac{e^{\alpha a} - e^{-\beta a}}{e^{\alpha a + \beta b} - e^{-\alpha b - \beta a}}$$

$$E\left[e^{-r t_1} : X_{t_1} = a\right] = \frac{e^{\beta b} - e^{-\alpha b}}{e^{\alpha a + \beta b} - e^{-\alpha b - \beta a}}$$

Infrequent portfolio revision (17/11/09)

(i) Suppose we have $X_t = \sigma W_t + \mu t$ and we want to invest in X, taking profit whenever $X_{t_k} - X_{t_{k-1}} \in \{-a, b\}$ for some $a, b > 0$. Each time we take profit, there is a portfolio revision which we assume costs c. Then ~~if~~ φ is the expected discounted long-term gain,

$$\varphi = E\left\{ \sum_{k \geq 1} e^{-r t_k} (X(t_k) - c) \right\}$$

we shall have

$$\varphi = E\left[e^{-r t_1} (X(t_1) - c) \right] + E\, e^{-r t_1} \varphi$$

so

$$\varphi = \frac{E\left[e^{-r t_1} (X(t_1) - c) \right]}{1 - E\left(e^{-r t_1} \right)}$$

where $t_1 = \inf\{t > 0 : X_t \in \{-a, b\}\}$. After some calculations, we have

$$\varphi = - \frac{(b-c)(e^{\alpha a} - e^{-\beta a}) + (a+c)(e^{-\alpha b} - e^{\beta b})}{e^{\beta b} - e^{-\beta a} + e^{\alpha a} - e^{-\alpha b} + e^{-\beta a - \alpha b} - e^{\beta b + \alpha a}}$$

It seems this is impossible to optimise over a, b in closed form. Here, of course, $-\alpha < 0 < \beta$ are the roots of $\frac{1}{2}\sigma^2 x^2 + \mu x - r = 0$.

(ii) How would this be if we didn't know μ with precision? We would of course optimize after mixing over μ. What appears to happen is that if the drift is upward, then it's best to have a very big. If we took $a = \infty$, then the optimization over b would concern

$$\max_b \left(\frac{e^{\beta b} - 1}{b - c} \right)^{-1}$$

if there was just one alternative for μ (but maximising the averaged version of this else). If we maximize $(b-c)/(\beta b + (\beta b)^2/2)$ (using expansion of e^x), we find

$$b = c + \sqrt{c^2 + 2c/\beta}$$

More generally, if we max $(b-c)/(a_1 b + a_2 b^2)$ we get

$$\boxed{b = c + \sqrt{c^2 + c a_1/a_2}.}$$

If we fix ~~some~~ exit levels a,b, what's the mean time to hitting the exit set?
Giving Maple the routine calculations to do leads to a mean exit time of ($k = 2\mu/\sigma^2$)

$$\frac{b(e^{ka}-1) - a(1-e^{-kb})}{\mu(e^{ka}-e^{-kb})}$$

Deterministic epidemics (24/11/09)

We have the usual

$$\dot S = -\alpha IS$$
$$\dot I = \alpha IS - \beta I$$

and suppose there is an associated cost of the epidemic $\gamma \int_0^T I_s\,ds$, where T is the time the epidemic finishes. Write $\tilde S(v) = S(\varphi(v))$, $\tilde I(v) = I(\varphi(v))$ so that

$\tilde S'(v) = \dot S(\varphi(v))\,\varphi'(v) = \varphi'(v)\,(-\alpha\,(IS)(\varphi(v)))$ so we demand that

$$\varphi'(v) = \frac{1}{(IS)(\varphi(v))}$$

so that φ is inverse to $\int_0^{\cdot} I_t S_t\,dt$, and we get

$$\begin{cases} \tilde S' = -\alpha \\ \tilde I' = \alpha - \beta/\tilde S \end{cases}$$

The objective transforms

$$\gamma \int_0^T I(s)ds = \gamma \int_0^{\tilde T} I(\varphi(v))\,\frac{dv}{(IS)(\varphi(v))} = \gamma \int_0^{\tilde T} \frac{dv}{\tilde S_v}$$

and we can solve the coupled ODE:

$$\tilde S_v = S_0 - \alpha v, \qquad \tilde I_v = I_0 + \alpha v + \frac{\beta}{\alpha}\log\left(\frac{S_0 - \alpha v}{S_0}\right)$$

Thus the loss is $\gamma \int_0^{\tilde T} \frac{dv}{\tilde S_v} = \frac{\gamma}{\beta}\int_0^{\tilde T}(\alpha - \tilde I')\,dv = \frac{\gamma}{\beta}\left(\alpha \tilde T + I_0\right)$

Suppose we want to end the epidemic by time $\tilde T = \tau$; then this requires that

$$\frac{S_0}{S_0 - \alpha\tau} = \exp\left(\alpha(I_0 + \alpha\tau)/\beta\right) \Rightarrow \boxed{S_0 = \frac{\alpha\tau\, e^{\alpha(I_0+\alpha\tau)/\beta}}{e^{\alpha(I_0+\alpha\tau)/\beta} - 1}}$$

So if you started off with S_{0-} susceptibles, and by vaccination (at unit cost) could reduce this immediately to some other lower value, if your aim was to finish the epidemic at τ the overall cost of this would be

$$\frac{\gamma}{\beta}(\alpha\tau + I_0) + (S_{0-} - S_0)$$
$$= \frac{\gamma}{\beta}(\alpha\tau + I_0) + S_{0-} - \frac{\alpha\tau\, e^{\alpha(I_0+\alpha\tau)/\beta}}{e^{\alpha(I_0+\alpha\tau)/\beta} - 1}$$

Interesting if $\dfrac{e^{\alpha I_0/\beta}}{e^{\alpha I_0/\beta} - 1} > \dfrac{\gamma}{\beta} > 1$!

Multiplier-wealth correspondence (26/11/09)

(i) When we solve for equilibrium prices in a diverse-beliefs model with a central planner, we find the first-order conditions

$$U_j'(t, c_t^j)\,\Lambda_t^j = \nu_j \zeta_t$$

so that market clearing gives

$$\delta = \sum c^j = \sum I_j(t, \nu_j \zeta / \Lambda^j)$$

We are therefore interested to understand the map $(\delta, \nu) \mapsto \zeta$ as defined by this implicit equation. Here are some small remarks.

(ii) If we simplify by removing t from the notation, and have

$$\delta = \sum I_j(\nu_j \zeta)$$

then if we change ν to ν', where $\nu_j' = \nu_j$ for $j \neq k$, $\nu_k' > \nu_k$, then what can we say about ζ'? Clearly $\zeta' < \zeta$. Now let's look at

$$\sum I_j\left(\nu_j' \frac{\nu_k}{\nu_k'} \zeta\right) = \sum_{j \neq k} I_j(\nu_j', \nu_k \zeta / \nu_k') + I_k(\nu_k \zeta)$$

$$= \sum_{j \neq k} I_j(\nu_j \nu_k \zeta / \nu_k') + \delta - \sum_{j \neq k} I_j(\nu_j \zeta)$$

$$= \delta + \sum_{j \neq k} \left\{ I_j\left(\nu_j \frac{\nu_k}{\nu_k'} \zeta\right) - I(\nu_j \zeta) \right\}$$

$$> \delta$$

if we assume strict monotonicity of the I_j (else we have $\geq$). Hence we get the bounds

$$\frac{\nu_k}{\nu_k'} \zeta < \zeta' < \zeta.$$

Applying this repeatedly

$$\left| \log \zeta(\nu) - \log \zeta(\nu') \right| \leq \sum \left| \log \nu_j - \log \nu_j' \right|$$

which gives that

$$\boxed{\text{the map } y \mapsto \log \zeta(e^y) \text{ is Lipschitz.}}$$

Moreover, the Lipschitz constant is 1, so this is even uniform in t!

Super-replication questions (7/12/09)

(i) Bruno Dupire has been investigating robust derivatives, such as the two-time-point Asian

$$\left(\tfrac{1}{2}\left(S(T_1)+S(T_2)\right)-K\right)^+$$

and this seems to me to be a candidate for numerica

if you discretize the distribution of $X \equiv e^{-rT_1}S_1$ and $Y = \ldots$ finite grid, you seek a joint pmf $F_{ij} = P(X=x_i, Y=y_j)$ which sat... constraints

$$p_i = \sum_j F_{ij} \quad \forall i, \quad q_j = \sum_i F_{ij} \quad \forall j, \quad \sum_j F_{ij}(y_j - x_i) = 0 \quad \forall i$$

which are the requirements that the marginals are correct, plus the martingale property. Finding the law F to satisfy these and maximise the price is the LP

$$\max_{F\geq 0} \sum_{ij} F_{ij} h_{ij} + \sum_i \lambda_i \left(p_i - \sum_j F_{ij}\right) + \sum \eta_j \left(q_j - \sum_i F_{ij}\right) \sum_i \gamma_i \sum_j F_{ij}(y_j - x_i)$$

$$= \max_{F\geq 0} \sum_{i,j} F_{ij}\left(h_{ij} - \lambda_i - \eta_j \cdot \gamma_i (y_j - x_i)\right) + \lambda\cdot p + \eta \cdot q$$

where $h_{ij} = \left(\tfrac{1}{2}(e^{-rT_1}x_i + e^{-rT_2}y_j) - K\right)^+$. The dual feasibility condition is that

$$h_{ij} \leq \lambda_i + \eta_j - \gamma_i(y_j - x_i)$$

which is to say that we take a position of the form $f_1(S_1) + f_2(S_2) + g(S_1)(S_2 - S_1)$ which dominates the Asian payoff. When I try this numerically, I find you get very large counterbalancing positions in S_1 and S_2, which only happens because the discretized S_1, S_2 have compact support ...

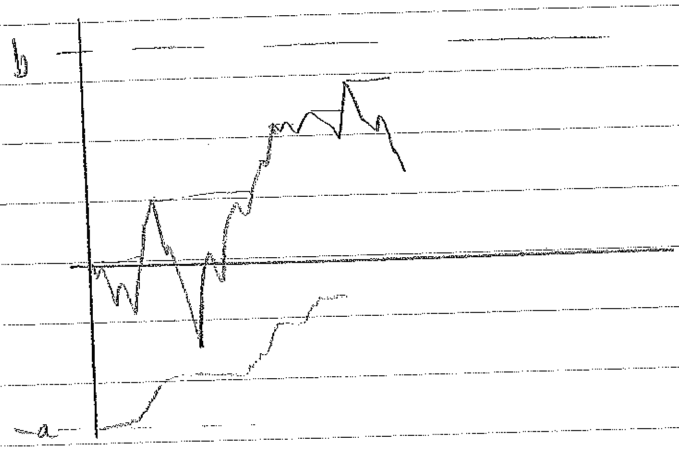
(ii) Assuming we have a non-negative payoff, we can write the super-replication as

$$f_1(S_1) + f_2(S_2) + g(S_1)S_2 \geq \text{payoff} \geq 0$$

Dividing by S_2, we see

$$\liminf_{S_2\to\infty} \frac{1}{S_2} f_2(S_2) \geq \sup_{S_1}\left(-g(S_1)\right)$$

If we can be sure that f_2 grows at worst linearly, we know g is bounded below, so wlog $g \geq 0$; ~~notice that if we have any other super-replication~~ $\cancel{\tilde{f}_1(S_1) + \tilde{f}_2(S_2) + \tilde{g}(S_1)S_2}$, ~~then we can take the min and reduce cost. One super-replication is~~ $\cancel{\tfrac{1}{2}(S_1-K)^+ + \tfrac{1}{2}(S_2-K)^+}$, ~~so for this example, we do have~~ $\cancel{g \geq 0}$. ~~If~~ $\cancel{f_2}$ ~~is bdd below, we could also ask~~ $\cancel{f_2 \geq 0}$ ~~by adding a constant to~~ $\cancel{f_1}$


b
a

Trading out of a position with a rising stop (9/12/09)

(i) Suppose $X_t = \sigma W_t + \mu t$, $\bar{X}_t = \sup_{u \le t} X_u$,

$$T \equiv \inf\{t : X_t \le \bar{X}_t - a \ \text{ or } \ X_t > b\}$$

for positive constants a, b. I want to find

$$E\left[e^{-\lambda T - \theta X_T}\right]$$

for $\lambda, \theta > 0$.

Let $-\alpha < 0 < \beta$ be roots of $\frac{1}{2}\sigma^2 t^2 + \mu t - \lambda = 0$. Now we are looking at excursions down from $\bar{X}$; there is marking at rate λ, and we want $E\left[e^{-\theta X_T} : T < \text{first mark}\right]$

Set

$A = \{$excursions which are λ-marked before reaching 0 or $-a\}$

$B = \{$excursions which get to $-a$ with no mark$\}$.

To calculate the excursion measure of these sets, we need to find

$$E^x\left[1 - e^{-\lambda H_0 \wedge H_{-a}}\right] = \frac{1 - e^{-\beta a}}{e^{\alpha a} - e^{-\beta a}}\left(1 - e^{-\alpha x}\right) + \frac{e^{\alpha a} - 1}{e^{\alpha a} - e^{-\beta a}}\left(1 - e^{\beta x}\right)$$

and

$$E^x\left[e^{-\lambda H_{-a}} : H_{-a} < H_0\right] = \frac{e^{-\alpha x} - e^{\beta x}}{e^{\alpha a} - e^{-\beta a}}$$

Thus

$$n(A) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} E^{-\varepsilon}\left[1 - e^{-\lambda(H_0 \wedge H_{-a})}\right] = \frac{\beta e^{\alpha a} + \alpha e^{-\beta a} - (\alpha + \beta)}{e^{\alpha a} - e^{-\beta a}}$$

$$n(B) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} E^{-\varepsilon}\left[e^{-\lambda H_{-a}} : H_{-a} < H_0\right] = \frac{\alpha + \beta}{e^{\alpha a} - e^{-\beta a}}$$

Hence

$$\boxed{\nu \equiv n(A) + n(B) = \frac{\alpha + \beta e^{(\alpha+\beta)a}}{e^{(\alpha+\beta)a} - 1}}$$

is the rate of excursions which stop the point process. Hence

$$E\left[e^{-\theta X_T} ; T < \text{first mark}\right] = e^{-\nu b - \theta b} + \int_0^b \nu e^{-\nu y} \frac{n(B)}{\nu} e^{-\theta(y-a)}\, dy$$

$$= e^{-(\nu+\theta)b} + \frac{n(B) e^{\theta a}}{\nu + \theta}\left(1 - e^{-(\nu+\theta)b}\right)$$

(ii) This is basically everything we need. For example, by setting $\theta = 0$ we get

$$E[e^{-\lambda T}] = \frac{(\alpha+\beta)(1-e^{-\nu b}) + \nu e^{-\nu b}(e^{\alpha a} - e^{-\beta a})}{\nu(e^{\alpha a} - e^{-\beta a})}.$$

Again, by differentiating wrto theta, we deduce that

$$E[X_T e^{-\lambda T}] = \frac{(\alpha+\beta)e^{-\nu b}(\nu(a-b)-1) + \nu^2 b e^{-\nu b}(e^{\alpha a} - e^{-\beta a}) + (\alpha+\beta)(1-a\nu)}{\nu^2(e^{\alpha a} - e^{-\beta a})}$$

If we send $a \to \infty$, we get

$$E[X_T e^{-\lambda T}] = b e^{-\nu b}$$

which checks out correctly!

Sending b to infinity gives

$$E[e^{-\lambda T}] = \frac{\alpha+\beta}{\nu(e^{\alpha a} - e^{-\beta a})}$$

$$E[X_T e^{-\lambda T}] = \frac{(\alpha+\beta)(1-a\nu)}{\nu^2(e^{\alpha a} - e^{-\beta a})}$$

(iii) What's the mean time til exit? If we differentiate $E[e^{-\lambda T}]$ with respect to λ, and let $\lambda \downarrow 0$, we find after putting it all into Maple that

$$E[T] = -\frac{2a^2(\xi-1)^3(1+c-\xi)[1-e^{-kc}]}{\sigma^2 c^2 (\xi-1)^3}$$

where $c = 2\mu a/\sigma^2$, $k = b/a(e^c-1)$, $\xi = e^c$,

$$= \frac{\sigma^2}{2\mu^2}(e^c-1-c)(1-e^{-kc})$$

Option pricing under diverse beliefs (10/12/09)

Suppose we go back to the diverse beliefs setting, with log agents, and under the reference measure P^0 we have

$$d\delta_t = \delta_t \, \sigma \, dX_t$$

where X is a BM under P^0. Let's suppose also that agents think the drift in X is constant;

$$d\Lambda^j_t = \Lambda^j_t \, \alpha_j \, dX_t, \quad \text{so} \quad \Lambda^j_t = \exp(\alpha_j X_t - \tfrac{1}{2}\alpha_j^2 t)$$

We have

$$\zeta_t \delta_t = \sum e^{-\rho_j t} \Lambda^j_t / \nu_j,$$

$$S_t = \delta_t \frac{\sum e^{-\rho_j t} \Lambda^j_t / \nu_j \rho_j}{\sum e^{-\rho_j t} \Lambda^j_t / \nu_j}$$

What's the price of a European put option? We need to calculate

$$E^0\left[\zeta_T \, Y\right]$$

for $Y = (K - S_T)^+$, and the neat way to do this is to set $\frac{d\tilde{P}}{dP_0} = \frac{\delta_0}{\delta_T} e^{-\sigma^2 T}$, so we get

$$\delta_0^{-1} e^{\sigma^2 T} \tilde{E}\left[\zeta_T \delta_T \, Y\right] = \delta_0^{-1} e^{\sigma^2 T} \sum_j \frac{e^{-\rho_j T}}{\nu_j} \tilde{E}\left[\Lambda^j_T \, Y\right].$$

Under $\tilde{P}$, $X_t = \tilde{X}_t - \sigma t$, where $\tilde{X}$ is a $\tilde{P}$-BM, so $\Lambda^j_T = e^{\alpha_j(\tilde{X}_T - \sigma T) - \frac{1}{2}\alpha_j^2 T}$ $= e^{-\sigma\alpha_j T} \exp(\alpha_j \tilde{X}_T - \frac{1}{2}\alpha_j^2 T)$, so if we set $d\tilde{P}^j = e^{\alpha_j \tilde{X}_T - \frac{1}{2}\alpha_j^2 T} d\tilde{P}$, we get that the option price is

$$\frac{1}{\delta_0 \zeta_0} e^{\sigma^2 T} \sum_j \nu_j^{-1} e^{-\rho_j T - \sigma\alpha_j T} \tilde{E}^j\left[Y\right]$$

where under $\tilde{P}^j$ we shall have $X_t = \tilde{X}^j_t + (\alpha_j - \sigma)t$, with $\tilde{X}^j$ a BM ($\tilde{P}^j$)

A point to take care over: this is a dividend-paying stock, so there's no put-call parity.

Maybe we try $\rho_j = \varepsilon \gamma_j$, and let $\varepsilon \downarrow 0$. This would have the effect of pushing δ_0 to 0 while we keep $S_0 = 1$.

Why moving-average crossovers? (17/12/09)

Suppose we have bivariate AR(1) process X_t;

$$X_{t+1} = AX_t + \varepsilon_t$$

with observation process

$$Y_t = CX_t + \eta_t \qquad C \equiv (0,1).$$

What's the KF evolution of $\hat{x}$? The usual kinds of calculations give

$$\begin{cases} \hat{X}_{t+1} = A\hat{X}_t + \dfrac{M_t C^T}{\Sigma_\eta + CM_tC^T}(Y_{t+1} - CA\hat{X}_t) \\ V_{t+1} = M_t - \dfrac{M_t C^T C M_t}{\Sigma_\eta + CM_tC} \\ M_t = \Sigma_\varepsilon + AV_tA^T \end{cases}$$

Suppose we've identified $M_\infty \equiv M$, the limiting form. Then

$$\hat{X}_{t+1} = \left(I - \frac{\binom{M_{12}}{M_{22}}(0\ 1)}{\Sigma_\eta + M_{22}} \right) A\hat{X}_t + \frac{\binom{M_{12}}{M_{22}}}{\Sigma_\eta + M_{22}} Y_{t+1}$$

so if $\mu \equiv \binom{M_{12}}{M_{22}} / (\Sigma_\eta + M_{22})$, we shall see

$$\begin{aligned} \hat{X}_{t+1} &= (I - \mu(0,1))A\hat{X}_t + \mu Y_{t+1} \\ &= B\hat{X}_t + \mu Y_{t+1} \\ &= \sum_{j\geq 0} B^j \mu Y_{t+1-j} \end{aligned}$$

Diagonalising B will now give us some idea about the underlying in terms of EWMAs of the observations.

For example

$$A = \begin{pmatrix} \theta & 0 \\ \beta & (1-\beta) \end{pmatrix} \qquad \beta \in (0,1), \quad \theta \in (0,1]$$

would give us convergence to the OU/RW $X^{(1)}$. Or again

$$A = \begin{pmatrix} \beta & 0 \\ 1 & 1 \end{pmatrix} \qquad \beta \in (0,1)$$

models $X^{(2)}_{t+1} = X^{(2)}_t + Z_t + \varepsilon_t$ where $Z_t = X^{(1)}_t$ is an AR(1) process.

Something intriguing emerges from numerical examples: we always find

$$\hat{Z}^1_t = a \sum_{j\geq 0} \left(\frac{\lambda_1^j}{1-\lambda_1} - \frac{\lambda_2^j}{1-\lambda_2} \right) Y_{t-j}$$

where λ_1, λ_2 are the e-values of B. This is perhaps not so surprising, in that if we added a constant to all of the observations Y_t, then this is exactly the observation sequence we would see based on the process $\tilde{X}_t = (X^{(1)}_t, a + X^{(2)}_t)$. The inference we draw about $X^{(1)}$ would be unaffected!

We can prove this is true by considering

$$e_1^T \sum_{j\geq 0} B^j \mu = e_1^T (I - B)^{-1} \mu$$

$$= \frac{1}{\mu_2 \beta - \mu_1 - \mu_2} \; e_1^T \begin{pmatrix} -\mu_2 & \mu_1 \\ \mu_2 - 1 & -1+\beta-\mu_1 \end{pmatrix} \mu$$

$$= 0$$

Thus we know that $e_1^T \sum_{j\geq 0} B^j \mu \, Y_{t-j}$ must be a linear combination of two EWMAs of past Y values, with exponential factors the two e-values of B; the weights on those two EWMAs have just been shown to be equal and opposite by considering the special case of constant Y

Volume of business once more (7/1/10)

(a) Returning the story where you have firms $1, \dots, N$ charging prices $p_1, \dots, p_N$ for the same product, we have supposed in the past that the volume of business $F_i(p)$ which firm i gets should satisfy

(i) $F_i(p)$ decreases with p_i

(ii) $F_i(p)$ increases with p_j $(j \neq i)$

(iii) $k \mapsto F_i(kp)$ decreases

However, I'm not sure these conditions are just what we need. I think we should certainly have

(iii)′ $\sum_j F_j(p)$ should be decreasing in each p_i

and for the examples studied earlier it is <u>not</u>. Moreover, (iii) is not necessarily what you want; if customers would not pay more than 100 for something, but be relatively insensitive to price below 100, then if we had $p = (50, 80)$ the first firm would get (a bit over) half the business – but if $k = 1.5$, $kp = (75, 120)$ and the first firm would get all the business.

So let's go with (i), (ii), (iii)′ – what examples could we get of this?

(b) Suppose customer n picks a firm at random, and looks at the price p_i charged by this firm. With probability $\bar{G}(p_i)$ $(\bar{G} \downarrow)$ he accepts, else he chooses another firm at random, and accepts that price with prob^y $\bar{G}(p_j)$.

Thus the volume of business done by firm i is

$$\frac{A}{N} \left\{ \bar{G}(p_i) + \sum_{j \neq i} \frac{1}{N-1} G(p_j)\, \bar{G}(p_i) \right\}$$

$$= \frac{A}{N} \bar{G}(p_i) \left\{ 1 + \sum_{j \neq i} \frac{1}{N-1} G(p_j) \right\}$$

This clearly satisfies (i), (ii), and we also have (iii)′, because if firm k raises p_k to p_k', then any customer who accepts with price p_k' will also accept with price p_k. However, the price firm i charges is chosen to maximise $(p - c_i)\bar{G}(p)$, so is not affected by what the other firms charge.

(c) Another somewhat better story is that customer n has a reservation price X_n with dist^n G, and he accepts the first price p_i iff $p_i \le X_n$, else he picks another p_j at random, and accepts that

So firm i gets business

$$\frac{A}{N}\left\{ \bar{G}(p_i) + \frac{1}{N-1}\sum_j{}' I_{\{p_i < p_j\}} (\bar{G}(p_i) - \bar{G}(p_j)) \right\}$$

$$= \frac{A}{N}\left\{ \bar{G}(p_i) + \frac{1}{N-1}\sum_j (\bar{G}(p_i) - \bar{G}(p_j))^+ \right\}$$

$$\equiv F_i(p)$$

The conditions for Pareto efficiency will be

$$\frac{1}{p_i - c_i} = -\frac{F_i'(p)}{F_i(p)} = \frac{g(p_i)(1+\sum I_{\{p_j > p_i\}}/(N-1))}{\bar{G}(p_i) + \frac{1}{N-1}(\sum_j \bar{G}(p_i) - \bar{G}(p_j))^+)}$$

This will likely be a mess to work with though...

... but in fact it seems to be quite simple to handle numerically!

(d) We could tell some story where customers are Bayesian, and of different types. They accept a price if it is less than their prior mean price, else they go to another firm, but with updated prior from seeing the rejected price. The analysis here would be a mess; a special case would be no learning (bigoted views) which would be just like (c), so we can't expect the analytical solution to be any more tractable.

(e) (15/1/10) Here's a better tale. Individuals have reservation price with density g. At each step, they pick a firm at random + inspect their price. If this is less than their res. price, they accept. If not, with prob^y β they try again, else give up. If $\bar{\mu}(t)$ is proportion of firms charging more than t, the Vol of business done by a firm with price p is

$$\frac{A}{N}\left\{\int_p^\infty g(t)\,dt + \int_p^\infty g(t)\,\bar{\mu}(t)\beta\,dt + \int_p^\infty g(t)\,\beta^2\bar{\mu}(t)^2\,dt + \cdots\right\}$$

$$= \frac{A}{N}\int_p^\infty \frac{g(t)\,dt}{1-\beta\bar{\mu}(t)}$$

Thus for a firm (j, say) with cost c_j, the profit as a f^n of $p = p_j$ is

$$(p - c_j)\int_p^\infty \frac{g(t)\,dt}{1-\beta\bar{\mu}(t)}$$

with FOC

$$0 = -\int_p^\infty \frac{g(t)\,dt}{1-\beta\bar{\mu}(t)} + \frac{(p-c_j)\,g(p)}{1-\beta\bar{\mu}(p)}.$$

<u>Autocovariance of Markov-modulated sequences (14/1/10)</u>

(i) Suppose $(\xi_t)_{t\in\mathbb{Z}}$ is a stationary ergodic irreducible finite-state Markov chain, and that we see some sequence $(X_t)_{t\in\mathbb{Z}}$ which is, conditional on ξ, independent, and

$$(X_t|\xi) \sim F(\cdot|\xi_t)$$

Write π for the stationary dist^n of ξ, μ_j for $\int x\, F(dx|j)$, and $\bar\mu = \sum_j \pi_j \mu_j$ for the overall mean of X. Write

$$\tilde\mu_j = \mu_j - \bar\mu$$

What is the autocovariance of (X_t)?

(ii) We have

$$E\, X_0^2 = \sum \pi_j \int x^2 F(dx|j)$$

$$= \sum \pi_j \left\{ \int (x-\mu_j)^2 F(dx|j) + \mu_j^2 \right\}$$

so if v_j denotes the variance of X_0 given $\xi_0 = j$, we see

$$\mathrm{var}(X_0) = \sum_j \pi_j v_j + \left(\sum \pi_j \mu_j^2 - \bar\mu^2\right).$$

(iii) For $t \in \mathbb{N}$,

$$E(X_0 X_t) = \sum \pi_i p_{ij}^{(t)} \mu_i \mu_j$$

$$= \sum \pi_i p_{ij}^{(t)} (\tilde\mu_i + \bar\mu)(\tilde\mu_j + \bar\mu)$$

$$= \sum \pi_i p_{ij}^{(t)} \tilde\mu_i \tilde\mu_j + \bar\mu^2$$

since $\sum \pi_j \tilde\mu_j = 0$ and π is invariant. Thus

$$\boxed{\mathrm{cov}(X_0, X_t) = \sum \pi_i p_{ij}^{(t)} \tilde\mu_i \tilde\mu_j .}$$

(iv) Notice that in the spectral representation of $P^t = M\Lambda^t M^{-1}$, the unit e-value contributes nothing to the covariance $\mathrm{cov}(X_0, X_t)$, which therefore decays geometrically. Notice though that if $\tilde\mu_j \simeq 0$ (as would be the case if all μ_j were very close) then the cov at lag $t \geq 1$ is <u>very small</u>; so we could in such a setup have no autocorrelation of returns, but significant autocorrelation of <u>absolute</u> returns.

Volume of business again (15/1/10)

1) Suppose there are N firms, and just one customer, whose reservation price has density g. If the prices charged by the firms are $p_1 < p_2 < \dots < p_N$, what's the probability that firm k eventually gets the customer? It will be

$$\frac{1}{N} \int_{p_k}^{\infty} g(t) \left\{ 1 + \beta \bar{\mu}(t) + \beta^2 \bar{\mu}(t)^2 + \dots \right\} dt$$

$$= \frac{1}{N} \int_{p_k}^{\infty} g(t) \frac{dt}{1 - \beta \bar{\mu}(t)}$$

$$= \frac{1}{N} \sum_{m=k}^{N} \frac{\bar{G}(p_m) - \bar{G}(p_{m+1})}{1 - \beta (N-m)/N} = \sum_{m=k}^{N} \frac{\bar{G}(p_m) - \bar{G}(p_{m+1})}{N(1-\beta) + \beta m}$$

which we see is bounded above by ($\beta = 1$) $\sum_{m=k}^{N} \left(\bar{G}(p_m) - \bar{G}(p_{m+1}) \right) / m$ which is also bounded above by $k^{-1} \bar{G}(p_k)$, which has a natural interpretation. So this is OK.

The FOC for firm k's price p says

$$\frac{(p-c) g(p)}{1 - \beta(N-k)/N} = \frac{\bar{G}(p) - \bar{G}(p_{k+1})}{1 - \beta(N-k)/N} + \sum_{m>k} \frac{\bar{G}(p_m) - \bar{G}(p_{m+1})}{1 - \beta(N-m)/N}.$$

This can be solved recursively quite simply ... now what can we do with the story?

2) There's a paper of Carlson + McAfee (JPE $\underset{\sim}{91}$ 480–493, 1983) which comes up with something mathematically similar (though the modelling tale is a bit different). They get a closed-form solution for uniform distn of reservation price; they allow the unit cost to depend on volume of business (which I believe should be OK for us too); and then they raise questions of market entrants. This last could be quite interesting, because there might be an intrinsic ceiling for the number of market participants ...

Accounting tale: autocovariances etc (27/1/10)

For the story of trying to value assets, there are various models. Simplifying somewhat, we had

$$(A)\quad \begin{cases} dZ_t = \mu\,dt + \sigma_2\,dW_t \\ dY_t = \lambda(Z_t - Y_t)\,dt + \sigma_3\,d\tilde W_t \end{cases}$$

$$(B)\quad \begin{cases} dZ_t = \mu\,dt + \sigma_2\,dW_t \\ d(Y_t - Z_t) = \lambda(Z_t - Y_t)\,dt + \sigma_3\,d\tilde W_t \end{cases}$$

where $W, \tilde W$ are independent BMs.

Case A. We have that $\xi_t \equiv Z_t - Y_t$ solves

$$d\xi_t = \mu\,dt - \lambda\xi_t\,dt + \sigma_2\,dW_t - \sigma_3\,d\tilde W_t$$

so this is a stationary OU process, with mean μ/λ, and variance $(\sigma_2^2 + \sigma_3^2)/2\lambda$ in steady state. From this we can deduce steady-state moments of the increments of Y: for $0 \le s \le t$

$$\boxed{\begin{aligned} E(Y_t - Y_0) &= \mu t \\ \operatorname{cov}(Y_t - Y_0, Y_s - Y_0) &= \frac{\sigma_3^2}{2\lambda}\left\{1 - e^{-\lambda t} - e^{-\lambda s} + e^{\lambda(s-t)}\right\} \\ &\quad + \frac{\sigma_2^2}{2\lambda}\left\{2\lambda s - 1 - e^{-\lambda(t-s)} + e^{-\lambda t} + e^{-\lambda s}\right\} \end{aligned}}$$

after some calculations.

Case B. This one is quite a bit easier: for $0 \le s \le t$

$$\boxed{\begin{aligned} E(Y_t - Y_0) &= \mu t \\ \operatorname{cov}(Y_t - Y_0, Y_s - Y_0) &= \sigma_2^2 s + \frac{\sigma_3^2}{2\lambda} e^{-\lambda(t-s)}\left(1 - e^{-2\lambda s}\right) \end{aligned}}$$

This one looks a lot easier to work with! For example, if we have $s_1 < s_2 \le t_1 < t_2$ then it's easy to calculate

$$\boxed{\begin{aligned} \operatorname{Cov}(Y_{s_2} - Y_{s_1}, Y_{t_2} - Y_{t_1}) &= \operatorname{cov}(\xi_{s_1} - \xi_{s_2}, \xi_{t_2} - \xi_{t_1}) \\ &= \frac{\sigma_3^2}{2\lambda} e^{-\lambda(t_1 - s_2)}\left(1 - e^{-\lambda(t_2 - t_1)}\right)\left(1 - e^{-\lambda(s_2 - s_1)}\right) \end{aligned}}$$

However, estimation of this proves difficult, especially σ_2 which will be a smaller order perturbation of the drift. So let's just insist $\sigma_2 = 0$ and restrict to Case B:

$$\left.\begin{aligned} dX_t &= \mu\, dt \\ dZ_t &= -\lambda Z_t\, dt + \sigma\, dW_t \end{aligned}\right\} \quad Y_t = X_t + Z_t .$$

Suppose we have seen (some) of Y up to time t, and have formed estimate

$$\left(\begin{pmatrix} \mu \\ Z_t \end{pmatrix} \Big| \mathcal{Y}_t \right) \sim N\left(\begin{pmatrix} \hat\mu \\ \hat Z_t \end{pmatrix}, \begin{pmatrix} V_{\mu\mu} & V_{\mu Z} \\ V_{Z\mu} & V_{ZZ} \end{pmatrix} \right)$$

We now see $\Delta Y \equiv Y_{t+h} - Y_t = h\mu - (1-e^{-\lambda h}) Z_t + \xi$, where $\xi \sim N(0, v_h)$, and $v_h \equiv \sigma^2(1-e^{-2\lambda h})/2\lambda$. We have

$$\begin{pmatrix} \mu \\ Z_{t+h} \\ \Delta Y \end{pmatrix} \Big| \mathcal{Y}_t \sim N\left(\begin{pmatrix} \hat\mu_t \\ \hat Z_t e^{-\lambda h} \\ h\hat\mu - \lambda_h \hat Z_t \end{pmatrix}, \left(\begin{array}{cc|c} V_{\mu\mu} & e^{-\lambda h} V_{\mu Z} & hV_{\mu\mu} - \lambda_h V_{\mu Z} \\ e^{-\lambda h} V_{\mu Z} & e^{-2\lambda h} V_{ZZ} + v_h & v_h - \lambda_h e^{-\lambda h} V_{ZZ} + he^{-\lambda h} V_{\mu Z} \\ \hline & & h^2 V_{\mu\mu} - 2\lambda_h h V_{\mu Z} + \lambda_h^2 V_{ZZ} + v_h \end{array}\right) \right)$$

where $\lambda_h = 1 - e^{-\lambda h}$. Write $V_Y = v_h + h^2 V_{\mu\mu} - 2h\lambda_h V_{\mu Z} + \lambda_h^2 V_{ZZ}$, so that the mean updates as

$$\begin{pmatrix} \hat\mu_{t+h} - \hat\mu_t \\ \hat Z_{t+h} - e^{-\lambda h} \hat Z_t \end{pmatrix} = \frac{1}{V_Y} \begin{pmatrix} hV_{\mu\mu} - \lambda_h V_{\mu Z} \\ v_{YZ} \end{pmatrix} \left(\Delta Y - h\hat\mu - \lambda_h \hat Z_t \right)$$

where $v_{YZ} = v_h - \lambda_h e^{-\lambda h} V_{ZZ} + he^{-\lambda h} V_{\mu Z}$. The updated covariance is just

$$\begin{pmatrix} V_{\mu\mu} & e^{-\lambda h} V_{\mu Z} \\ e^{-\lambda h} V_{Z\mu} & v_h + e^{-2\lambda h} V_{ZZ} \end{pmatrix} - \frac{1}{V_Y} \begin{pmatrix} hV_{\mu\mu} - \lambda_h V_{\mu Z} \\ v_{YZ} \end{pmatrix} \left(hV_{\mu\mu} - \lambda_h V_{\mu Z} ,\ v_{YZ} \right).$$

Market selection (4/2/10)

(1) There are some interesting papers, by Blume + Easley, and by Kogan et al, where we look at a complete market (central planner) equilibrium. The equilibrium satisfies

$$e^{-\rho_j t} u_j'(c_t^j) \Lambda_t^j = \nu_j \zeta_t \qquad (j = 1, \dots, J)$$

and Kogan et al distinguish the survival of agent j – conventionally defined as

$$P\left[\limsup \left(c_t^j / \sum c_t^i \right) > 0 \right] > 0$$

from the price impact of agent j, which they define as the negation of

for any $s>0$, $\displaystyle\lim_{t\to\infty} \frac{\zeta_{t+s}}{\zeta_t} \Big/ \frac{\zeta^*_{t+s}}{\zeta^*_t} = 1$

where ζ_t^* is SPD (for some $\nu \gg 0$) you get when both agents have same beliefs (so they only discuss two agents ...) But it seems to me that you should also consider the notion of extinction in the asset market:

$$w_t^j / \sum w_t^i \to 0 \quad \text{a.s.}$$

along with the notion of extinction in the consumption market:

$$c_t^j / \sum c_t^i \to 0 \quad \text{a.s.}$$

(2) I reckon that you can get a situation where an agent goes extinct in the consumption market but not in the asset market when agents have same beliefs, same ρ_j but possibly different utilities. We can make this happen by supposing that there is a strictly positive continuous process x_t which goes to 0 a.s., and then set

$$c_t^1 = x_t, \qquad c_t^2 = \sqrt{x_t}$$

define ζ_t by $e^{-\rho t} u_1'(c_t^1) = \zeta_t$, and choose the two utilities in such a way that

$$x u_1'(x) = \sqrt{x}\, u_2'(\sqrt{x})$$

(So we could fix $\varepsilon \in (\tfrac{1}{2}, 1)$, and set $u_1'(x) = x^{-\varepsilon}$, whence $u_2'(z) = z^{1-2\varepsilon}$.) Then clearly $c_t^1 / c_t^2 = \sqrt{x_t} \to 0$ a.s., but at all times the agents have the same wealth!

(3) Here is an example where an agent is extinguished in the consumption market, but not in the asset market, with

- same u and same discount for both agents
- $c_t^2 \equiv 1$, $\Lambda_t^2 \equiv 1$
- u is CRRA

So what we have is that (abbreviating Λ_t^1 to Λ_t)

$$\frac{\Lambda_t u'(c_t^1)}{u'(1)} = \text{const}$$

so that $c_t^1 = I(1/\Lambda_t)$, and we just have to build a positive martingale Λ which tends to zero almost surely, and has the desired properties. The martingale will drift downwards until the time τ of a single upward jump (which may not happen) After the jump, the martingale evolves as $d\Lambda = \Lambda dW$, after the next integer time

For this problem, we have $\zeta_t \propto \Lambda_t^2 u'(c_t^2) e^{-\rho t} \propto e^{-\rho t}$.

For the construction, let's take $a_n = 2^{-n}$, $n = 0, 1, \ldots$ and suppose that while $a_{n+1} < \Lambda_t < a_n$ and before the jump has happened,

$$\dot{\Lambda}_t = -2^{-n-1} \equiv -b_n$$

so that (if there is no jump) it takes time 1 to cross from a_n down to a_{n+1}.

We will have intensity ν_n of the upward jump while $\Lambda_t \in (a_{n+1}, a_n)$, jumping up by ξ_n, where of course

$$\nu_n \xi_n = b_n$$

Suppose $u'(x) = x^{-R}$. We have that agent 2's wealth at all times is ρ^{-1}.

At time n, agent 1's wealth is at least (assuming $\tau > n$)

$$E_n\left[\int_n^{n+1} e^{-\rho(s-n)} I(1/\Lambda_s)\, ds\right]$$

$$= E_n\left[\int_n^{n+1} e^{-\rho(s-n)} \Lambda_s^{1/R}\, ds\right]$$

$$\geq E_n\left[\int_n^{n+1} e^{-\rho(s-n)} \Lambda_s^{1/R}\, ds \;;\; \tau \leq n+1\right]$$

$$\geq E_n\left[\int_\tau^{n+1} e^{-\rho(s-n)} (a_{n+1} + \xi_n)^{1/R}\, ds \;:\; \tau \leq n+1\right]$$

$$= (a_{n+1} + \xi_n)^{1/R} \int_0^1 \nu_n e^{-\nu_n t} \left(\int_t^1 e^{-\rho s}\, ds\right) dt$$

$$= (a_{n+1}+\xi_n)^{1/R}\rho^{-1}\int_0^1 \nu_n e^{-\nu_n t}\,(e^{-\rho t}-e^{-\rho})\,dt$$

$$= (a_{n+1}+\xi_n)^{1/R}\rho^{-1}\Big[\frac{\nu_n}{\nu_n+\rho}\,(1-e^{-(\rho+\nu_n)}) - e^{-\rho} + e^{-\rho-\nu_n}\Big]$$

$$= (a_{n+1}+\xi_n)^{1/R}\rho^{-1} e^{-(\rho+\nu_n)}\Big[\frac{\nu_n}{\rho+\nu_n}\,e^{\rho+\nu_n} + \frac{\rho}{\rho+\nu_n} - e^{+\nu_n}\Big]$$

$$= (a_{n+1}+\xi_n)^{1/R}\rho^{-1} e^{-\rho-\nu_n}\Big[\frac{\nu_n}{\rho+\nu_n}\,e^{\nu_n}(e^{\rho}-1) + \frac{\rho}{\rho+\nu_n}(-e^{\nu_n}+1)\Big]$$

$$= \frac{(a_{n+1}+\xi_n)^{1/R}}{\rho(\rho+\nu_n)}\, e^{-\rho}\Big[\nu_n(e^{\rho}-1) - \rho(1-e^{-\nu_n})\Big]$$

$$\sim \frac{(a_{n+1}+\xi_n)^{1/R}}{\rho(\rho+\nu_n)}\, e^{-\rho}\,(e^{\rho}-1-\rho)\,\nu_n \qquad \text{since } \nu_n \to 0.$$

So if we take $\xi_n = 2^n$, $R=\frac{1}{2}$, $\nu_n = b_n/\xi_n = \frac{1}{2}\,2^{-2n}$, we get that this remains $O(1)$, indeed, converges to

$$e^{-\rho}(e^{\rho}-1-\rho)/2\rho$$

Thus on the event that there is no jump, $w_t^1/w_t^2 \geq$ const.

But once the jump happens, we can follow the GBM down to 1, and then repeat the recipe!

(4) This can it seems be modified to create an example where (amazingly!)

$$\boxed{\frac{c_t^1}{c_t^2} \xrightarrow{a.s.} 0, \qquad \frac{w_t^1}{w_t^2} \xrightarrow{a.s.} \infty}$$

As before, we'll have $u'(x) = x^{-R}$, $\Lambda_t^2 \equiv 1$, $c_t^2 \equiv 1$, $\zeta_t = e^{-\rho t}$, and we just have to construct the LR martingale Λ_t for agent 1; the consumption process is then $c_t = I(1/\Lambda_t) = \Lambda_t^{1/R}$, and the wealth at time t is

$$w_t^1 = E_t\Big[\int_t^\infty e^{-\rho(s-t)}\,\Lambda_s^{1/R}\,ds\Big].$$

We shall require that $\boxed{\rho > 1}$ which is just an irrelevant scaling condition in effect.
Fix levels $a_n = 2^{-n}$, $n \geq 0$, and values $\xi_n = A(n+1)$, $n \geq 0$, where

$$A = \frac{2R^2}{2\rho R^2+R-1} > \frac{1}{\rho}$$, and we fix $\boxed{R = 1/2}$

(the condition $\rho > 1$ is to ensure that the denominator in A is positive)

The likelihood-ratio martingale evolves as

$$d\Lambda_t = \Lambda_t \, dW_t$$

in $(a_0, \infty) \equiv (1, \infty)$, and in $(0, 1]$ will evolve by jumps; it will sit at level a_n for unit time, then with probability p_n will jump to value $\xi_n > 1$ with probability $(1-p_n)$ to value a_{n+1}, where the (p_n) and (ξ_n) are to be found. The LR martingale will be a Markov process, and we shall have

$$z_n = E\left[\int_0^\infty e^{-\rho s} \Lambda_s^{1/R} \, ds \;\middle|\; \Lambda_0 = a_n \right]$$

If we start at a_n, and just consider what happens at the first jump, we shall have

$$z_n = \frac{1-e^{-\rho}}{\rho} \cdot a_n^{1/R} + e^{-\rho}\{ p_n \, h(\xi_n) + (1-p_n) z_{n+1} \}$$

and the martingale condition

$$a_n = p_n \xi_n + (1-p_n) a_{n+1}.$$

Here, $h(\xi) = E^\xi \left[\int_0^\infty e^{-\rho s} \Lambda_s^{1/R} ds \right] = E^\xi \left[\int_0^{H_1} e^{-\rho s} \Lambda_s^{1/R} ds + e^{-\rho H_1} z_0 \right]$. Some routine calculations show that with the special choice $z_0 = A$, we have

$$h(\xi) = A \xi^{1/R} = A \xi^2 \qquad \text{for } \xi \geq 1.$$

Suppose we abbreviate $B \equiv (1-e^{-\rho})/\rho A < 1 - e^{-\rho} < 1$. Then the two conditions which p_n, ξ_n must satisfy are

$$\begin{cases} n+1 = B a_n^2 + e^{-\rho}\{ p_n \xi_n^2 + (1-p_n)(n+2) \} \\ a_n = p_n \xi_n + (1-p_n) a_{n+1} \end{cases}$$

The second gives $\xi_n = (a_n - (1-p_n) a_{n+1}) / p_n$ which is substituted into the first to give the equation

$$e^{\rho}(n+1 - Ba_n^2) = \frac{(a_n - (1-p_n) a_{n+1})^2}{p_n} + (1-p_n)(n+2)$$

Now the LHS is positive, and the RHS is $+\infty$ when $p_n = 0$, and a_n^2 when $p_n = 1$. So for there to exist a root p_n we need

$$n+1 - Ba_n^2 > a_n^2$$

equivalently, $n+1 > (B+e^{\rho})\, a_n^2$. This is hardest to satisfy when $n=0$, when the required inequality says $1 > B + e^{\rho}$ – but we already saw that this holds.

Now it's clear that the p_n must $\to 0$ for the equation to hold, and indeed we shall have

$$p_n \sim \{(e^{\rho}-1)n\}^{-1} (a_n - a_{n+1})^2 \sim \frac{2^{-2(n+1)}}{n}$$

So the p_n get small geometrically fast and the ξ_n get big like $n\, 2^n$. Since $\sum p_n < \infty$, there is, starting from a_0, a positive probability that there will <u>never</u> be an upward jump, and $\Lambda_t \to 0$. But if there <u>is</u> an upward jump, Λ will diffuse back down to a_0 and the story starts again. Eventually, Λ_t will head off to 0, and so $c_t^1 \to 0$ a.s. However, eventually the martingale Λ_t passes sequentially through the values $\xi_0, \xi_1, \ldots \to \infty$ and so $w_t^1 \to \infty$, even though $w_t^2 \equiv 1$!

Notice that we could approximate this discontinuous LR martingale with a continuous one and get the same qualitative conclusions.

(5) This can be related to KRWW definition of no price impact. In this example, $\delta_t = 1 + I(1/\Lambda_t)$, so if we put two agents into the market with the same beliefs as agent 2, the FOC would say

$$c_t^j = I(e^{\rho t}\zeta_t^0 / \nu_j) = (e^{\rho t}\zeta_t^0)^{-1/R}\, \nu_j^{1/R}$$

Hence by market clearing $\delta_t = (e^{\rho t}\zeta_t^0)^{-1/R}(\nu_1^{1/R} + \nu_2^{1/R})$ and we see that

$$\zeta_t^0 = e^{-\rho t}\, \delta_t^{-R}\, (\nu_1^{1/R} + \nu_2^{1/R})^R.$$

Thus we see $$\frac{\zeta_{t+s}^0}{\zeta_t^0} = e^{-\rho s}\left(\frac{\delta_{t+s}}{\delta_t}\right)^{-R}$$

But we know $\Lambda_t \to 0$, so $I(1/\Lambda_t) \to 0$. Hence $(\delta_{t+s}/\delta_t) \to 1$ as $t \to \infty$ for every $s \geq 0$, and thus there is no price impact according to the def^n of KRWW.

(6) However, this example can it seems be reworked to show that the notion of

No price impact from KRWW is not correct. What we have constructed is a strictly positive martingale $\Lambda_t \to 0$ a.s., and with the property that

$$E_t\left[\int_t^\infty e^{-\rho(s-t)} \Lambda_s^2 \, ds \right] \to \infty.$$

In the reference story, $\Lambda^1 = \Lambda^2 \equiv 1$, $u(x) = \log x$ [Yes! even with the best possible utility, things can go wrong] and the dividend process δ is some positive adapted process which doesn't need more identification for the moment.

$$c^j_t = \frac{1}{\nu_j e^{\rho t} \zeta^0_t} \Rightarrow \zeta^0_t = e^{\rho t}(\alpha_1 + \alpha_2)/\delta_t, \quad \alpha_j \equiv \nu_j^{-1}$$

For the comparison,

$$c^1_t = I(\nu_1 e^{\rho t} \zeta_t / \Lambda_t) = \Lambda_t e^{-\rho t} \beta_1 / \zeta_t$$

so $\zeta_t = e^{\rho t}\{\beta_1 \Lambda_t + \beta_2\}/\delta_t$ for some positive constants β_1, β_2. Now if we ensure the multipliers chosen so that $\alpha_1 + \alpha_2 = \beta_2$, then clearly

$$\zeta_t / \zeta^0_t \longrightarrow 1 \quad \text{a.s.}$$

and there is no price impact in the definition of KRWW. However, if we want to price the cashflow $\delta_t \Lambda_t$, in the reference situation we get

$$\pi^0_t \equiv \frac{1}{\zeta^0_t} E_t \int_t^\infty \zeta^0_s \delta_s \Lambda_s \, ds = \frac{\delta_t}{\alpha_1 + \alpha_2} E_t \int_t^\infty e^{-\rho(s-t)} (\alpha_1 + \alpha_2) \Lambda^1_s \, ds$$

$$= \delta_t \Lambda^1_t \rho^{-1}$$

whereas in the diverse beliefs situation, we have

$$\pi^1_t = \frac{1}{\zeta_t} E_t \int_t^\infty \zeta_s \delta_s \Lambda_s \, ds = \frac{\delta_t}{\beta_1 \Lambda_t + \beta_2} E_t \int_t^\infty e^{-\rho(s-t)} (\beta_1 \Lambda_s + \beta_2) \Lambda_s \, ds$$

$$= \frac{\delta_t}{\beta_1 \Lambda_t + \beta_2} \left\{ \beta_2 \rho^{-1} \Lambda_t + \beta_2 \, E_t \int_t^\infty e^{-\rho(s-t)} \Lambda_s^2 \, ds \right\}$$

Thus

$$\frac{\pi^1_t}{\pi^0_t} = \frac{\beta_2 \Lambda_t + \rho \, E_t\left[\int_t^\infty e^{-\rho(s-t)} \Lambda_s^2 \, ds\right]}{\Lambda_t (\beta_1 \Lambda_t + \beta_2)} \longrightarrow \infty.$$

Market selection + survival (5/3/10)

(1) Maybe the correct criterion for price impact should be that if we compute the SPD ζ when agent j has beliefs Λ^j, and the SPD ζ^0 when he has Λ^1 (say) then

$$L^1-\lim_{t\to\infty} E_t\left[\int_t^\infty \left|\frac{\zeta^0_s}{\zeta^0_t}-\frac{\zeta_s}{\zeta_t}\right|ds\right]=0$$

This would be implied by

$$E\int_t^\infty \left|\frac{\zeta^0_s}{\zeta^0_t}-\frac{\zeta_s}{\zeta_t}\right|ds \to 0$$

This would say that for any bounded cashflow process, the two pricings would agree asymptotically.

Or maybe we would demand a.s. convergence to 0 ??

(2) Suppose we want to have a market (that is, ρ, u) such that whenever the Λ^j satisfy

$$\frac{\Lambda^j_t}{\Lambda^1_t} < \varepsilon \qquad \forall t \quad \forall i,j$$

then for some constant $\eta>0$ we get

$$c^j_t/D_t \geq \eta \qquad \forall t, \ \forall j$$

where $D_t=\Sigma c^j_t$ is the total output, and we want this to hold whatever the initial wealths; then we have

$$c^j = I(\nu, x/\lambda^j)$$

so we want

$$\frac{I(\nu_2 x/\lambda_2)}{I(\nu_1 x/\lambda_1)} \leq \eta < \infty \quad \text{uniformly in } x \text{ if } \lambda_1 \geq \varepsilon\lambda_2$$

so want

$$\frac{I(\nu_2\,\varepsilon x/\lambda_1)}{I(\nu_1 x/\lambda_1)} \leq \eta < \infty \quad \text{uniformly in } x, \lambda_1 \text{ for } \varepsilon, \nu_1, \nu_2 \text{ fixed}$$

This is implied by

(A) $x\mapsto \log I(e^x)$ is uniformly continuous, which in turn is implied by

(B) the cfe of relative risk aversion $R(x) = -xU''(x)/U'(x)$ is bdd away from 0

<u>Optimal stopping of BM with learning about drift (10/3/10)</u>

(1) Suppose we have a process

$$X_t = W_t + \mu t$$

where μ is a random variable with distribution F, and suppose we desire to solve the optimal stopping problem

$$\max_{\tau \geq t} E^{(t,x)}\left[e^{-\rho(\tau - t)} g(X_\tau) \right] \equiv v(t,x)$$

where the stopping function g is given. Notice that the dynamics of X are the dynamics of a BM h-transformed by

$$h(t,x) = \int \exp\left(x\mu - \tfrac{1}{2}\mu^2 t\right) F(d\mu)$$

So we have that the generator of X is $\mathcal{G} f = \frac{1}{h} \mathcal{G}_0 (hf)$, and $\mathcal{G}_0 = \frac{1}{2} D^2$ (plus $\partial/\partial t$, of course). We shall have

$$e^{-\rho t} v(t, X_t) \quad \text{is a martingale while in continue region}$$

so

$$-\rho v + \mathcal{G} v + \frac{\partial v}{\partial t} \equiv \frac{1}{h}\left[\mathcal{G}_0 (hv) - \rho h v + h \frac{\partial v}{\partial t} \right] = 0$$

in continue region, with $v \geq g$ everywhere.

(2) In general, this is only going to be soluble by numerical scheme, but if

$$F(d\mu) = p\, \delta_{\mu_0}(d\mu) + (1-p)\, \delta_{-\mu_0}(d\mu)$$

we get

$$h(t,x) \propto e^{-\mu_0^2 t/2} \cosh(\mu_0 x + a) \equiv e^{-\mu_0^2 t/2} h_0(x)$$

and $\mathcal{G} = \frac{1}{2} D^2 + \mu_0 \tanh(\mu_0 x + a) D$ is <u>time-independent</u>, and hence v is time independent:

$$-\rho v + \mathcal{G} v = 0 \quad \text{in continue region}$$

Therefore

$$-\rho \tilde{v} + \mathcal{G}_0 \tilde{v} = 0 \quad \text{in continue region}, \qquad \tilde{v} \equiv h_0 v$$

and

$$\tilde{v} \geq \tilde{g} \equiv h_0 g$$

<u>This is a standard one-dimensional optimal stopping problem.</u>

Contagion effects again (14/3/10)

This does a few calculations in preparation for fitting the model Angus & I worked on. The GDB of countries $1, \dots, N$ are stored in $a = (a_i)_{i=1}^N$ and we think that if $y_t^i \equiv \delta_t^i - a^i$ then

$$y_{t+1}^i - y_t^i = -b y_t^i + \theta \sum_{j \neq i} y_t^j + a_i \varepsilon_{t+1}^i$$

Hence $\delta_{t+1} = \delta_t - B\delta_t + \mu + \operatorname{diag}(a)\varepsilon_{t+1}$. $\quad (\mu = Ba)$

Thus the matrix $A \equiv I - B$ has the form

$$A = \alpha Q + \nu P \qquad P = \tfrac{1}{N} 1 1^T, \quad Q = I - P$$

Let's write V for the covariance matrix of the noises: $V = \sigma_0^2 \operatorname{diag}(a_i^2)$. Then

$$W_n = \frac{1-\alpha^{2n}}{1-\alpha^2} QVQ + \frac{1-(\alpha\nu)^n}{1-\alpha\nu}\left(QVP + PVQ\right) + \frac{1-\nu^{2n}}{1-\nu^2} PVP$$

Hence

$$\bar{W}_n \equiv W_\infty - W_\infty A^n - A^n W_\infty + W_\infty$$

$$= \frac{2-2\alpha^n}{1-\alpha^2} QVQ + \frac{2-\nu^n-\alpha^n}{1-\alpha\nu}(PVQ + QVP) + \frac{2-2\nu^n}{1-\nu^2} PVP,$$

$$W_\infty 1 = \frac{1}{1-\alpha\nu} Qv + \frac{1}{1-\nu^2} Pv, \qquad (v \equiv V1)$$

$$A^k W_\infty 1 = \frac{\alpha^k}{1-\alpha\nu} Qv + \frac{\nu^k}{1-\nu^2} Pv,$$

$$p_n \equiv \Gamma 1 - A^n \Gamma 1 = (1-\nu^n)\Gamma 1,$$

$$1 \cdot W_n 1 = \frac{1-\nu^{2n}}{1-\nu^2} 1 \cdot V1 = \frac{1-\nu^{2n}}{1-\nu^2} 1 \cdot v.$$

If we set

$$\left.\begin{aligned} b_{k,n,m} &\equiv a - \Gamma W_n 1 + A^n W_\infty p_n + A^n W_\infty A^k p_m \\ \tilde{b}_{k,n,m} &\equiv a - \Gamma W_n 1 + A^n W_\infty p_n + A^{n+k} W_\infty p_m \end{aligned}\right\}$$

then

$$b_{k,n,m} - a = \left[-W_n + A^n W_\infty (1-\nu^n) + \nu^k(1-\nu^m) A^n W_\infty\right] \Gamma 1$$

$$= \Gamma\left\{ -\frac{1-(\alpha\nu)^n}{1-\alpha\nu} Qv - \frac{1-\nu^{2n}}{1-\nu^2} Pv + \left(1-\nu^n+\nu^k - \nu^{k+m}\right)\left(\frac{\alpha^n}{1-\alpha\nu} Qv + \frac{\nu^n}{1-\nu^2} Pv\right)\right\}$$

$$= \Gamma \left\{ \frac{\alpha^n \nu^k (1-\nu^m) - 1 + \alpha^n}{1-\alpha\nu} Q v + \frac{\nu^{n+k}(1-\nu^m) - 1 + \nu^n}{1-\nu^2} P v \right\},$$

$$\tilde{b}_{R_{n,m}} - a = \Gamma \left\{ \frac{\alpha^{n+k}(1-\nu^m) - 1 + \alpha^n}{1-\alpha\nu} Q v + \frac{\nu^{n+k}(1-\nu^m) - 1 + \nu^n}{1-\nu^2} P v \right\}$$

similarly. Finally,

$$A^n W_\infty A^s = \frac{\alpha^{n+s}}{1-\alpha^2} QVQ + \frac{1}{1-\alpha\nu}\left(\nu^n \alpha^s PVQ + \alpha^n \nu^s QVP \right) + \frac{\nu^{n+s}}{1-\nu^2} PVP$$

Robust optimal portfolios (19/3/10)

Raman Uppal presented a MV optimization story where you have ambiguity

$$\max_{\theta} \min_{\alpha \in I} \{ \theta \cdot \alpha - \tfrac{1}{2} \gamma \theta \cdot V \theta \}$$

where I is some box (confidence set for growth rate), V is a covariance matrix.

Seems to me that a more natural thing to do would be a Bayesian analysis, where you think the return X is $N(\mu, V)$ where we have a $N(\mu_0, V_\mu)$ prior for μ and suppose that $V \sim v(I + W_{n,p})v$, where v is PDS symmetric, W is Wishart. Maybe for more flexibility, $V \sim V_0 + v W_{n,p} v$, and W_{np} is represented as ZZ^T, where Z is $n \times p$ IID $N(0,1)$.

So the objective of the risk-averse Bayesian investor becomes

$$-E \exp(-\gamma \theta \cdot X) = -E \exp\left(-\gamma \theta \cdot \mu + \tfrac{1}{2} \gamma^2 \theta \cdot V \theta\right)$$

$$= -E \exp\left(-\gamma \theta \cdot \mu_0 + \tfrac{1}{2} \gamma^2 \theta \cdot V_\mu \theta + \tfrac{1}{2} \gamma^2 \theta \cdot V_0 \theta + \tfrac{1}{2} \gamma^2 \theta \cdot v Z Z^T v \theta\right)$$

$$= -\exp\{-\gamma \theta \cdot \mu_0 + \tfrac{1}{2} \gamma^2 \theta \cdot (V_\mu + V_0) \theta\} \, E \exp\left(\tfrac{1}{2} \gamma^2 |Z^T v \theta|^2\right)$$

so we just need to deal with the final expectation. Write $a = v\theta$, and think of stacking the columns of Z into an np-vector ξ. Then

$$|Z^T a|^2 = \xi \cdot Q \xi$$

where

$$Q = \begin{pmatrix} a a^T & 0 & 0 & \cdots \\ 0 & a a^T & 0 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

Some routine calculations give the final expectation to be

$$E \exp \tfrac{1}{2} \gamma^2 |Z^T v \theta|^2 = (1 - \gamma^2 |a|^2)^{-p/2}.$$

Thus the log of the objective, to be minimized, will be

$$-\gamma \theta \cdot \mu_0 + \tfrac{1}{2} \gamma^2 \theta \cdot (V_\mu + V_0) \theta - \tfrac{p}{2} \log\left(1 - \gamma^2 |v\theta|^2\right)$$

The FOC becomes

$$0 = -\mu_0 + \gamma (V_\mu + V_0) \theta - \frac{p \gamma v^2 \theta}{1 - \gamma^2 |v\theta|^2}$$

Hence

$$\boxed{\theta = \gamma^{-1} (V_\mu + V_0 - \lambda v^2)^{-1} \mu_0, \qquad \lambda \equiv \frac{p \gamma}{1 - \gamma^2 |v\theta|^2}}$$

Selling out of a position (19/3/10)

Suppose you have A units of a stock which you want to sell by time T. Opportunities to sell come as a Poisson proc rate λ. If you get to T with x still unsold, you incur a penalty $\frac{1}{2}\gamma x^2$, with residual value $x S_T$. Suppose $S_t = \sigma W_t + \mu t$.
Let's write

$$V(\tau, x, S) = \text{value if we have } x \text{ remaining, time } \tau \text{ to go, spot } S_t = S$$

Thus the HJB is

$$0 = -V_\tau + \tfrac{1}{2}\sigma^2 V_{SS} + \mu V_S + \lambda \sup_{0 \le y \le x} \left[V(\tau, y, S) - V(\tau, x, S) + (x-y)S \right]$$

Let's conjecture that $V(\tau, x, S) = v(\tau, x) + xS$, with $v(0,x) = -\frac{1}{2}\gamma x^2$.
The HJB says

$$0 = -v_\tau + \mu x + \lambda \sup_{0 \le y \le x} \left[v(\tau, y) - v(\tau, x) \right]$$

Now we guess $v(\tau, x) = \frac{1}{2} a_\tau x^2 + b_\tau x + c_\tau$, and find the optimising y is

$$y^* = \left(0 \vee \left(-b_\tau / a_\tau \right) \right) \wedge x$$

then we find

$$0 = -\left(\tfrac{1}{2} a' x^2 + b' x + c' \right) + \mu x + \lambda \left\{ -\frac{b^2}{2a} - \tfrac{1}{2} a x^2 - bx \right\}$$

$$= -\tfrac{1}{2} x^2 (+a' + \lambda a) + x(-b' + \mu - \lambda b) + (-c' - \lambda b^2 / 2a)$$

Thus

$$a(\tau) = -\gamma e^{-\lambda \tau},$$

$$b(\tau) = \frac{\mu}{\lambda}(1 - e^{-\lambda \tau}),$$

$$c(\tau) = \frac{\mu^2}{\lambda^2 \gamma} \{ \sinh \lambda \tau - \lambda \tau \}$$

So we optimally sell to $\quad \frac{\mu}{\gamma \lambda}(e^{\lambda \tau} - 1) \quad$ when we get the chance

Uncertain drift (29/3/10)

(1) ~~Suppose~~ We observe a process Y which evolves as

$$dY_t = \sigma_Y \, dW_t + \mu_t \, dt, \qquad \text{where} \quad d\mu_t = \sigma_\mu \, dW'_t$$

and W', W are correlated ρ. How does the filtering story look for the unobserved μ? Conventional KF story: in steady state

$$d\hat{\mu}_t = \frac{\sigma_\mu}{\sigma_Y} \left(dY_t - \hat{\mu}_t \, dt \right)$$

with limiting variance $v_\infty = \sigma_Y \sigma_\mu (1-\rho)$.

(2) Suppose agents $1, \dots, J$ each do the above filtering, each thinking σ_μ is a particular value, ρ is a particular value. Wlog take $\sigma_Y \equiv 1$. Then we get

$$d\hat{\mu}^j_t = a_j \left(dY_t - \hat{\mu}^j_t \, dt \right)$$

where $a_j = \sigma^j_\mu$ for short. Thus $\hat{\mu}^j_t = e^{-a_j t} \hat{\mu}^j_0 + \int_0^t a_j e^{a_j(s-t)} \, dY_s$

$$= e^{-a_j t} (\hat{\mu}_0 + a_j (Y_t - Y_0)) + \int_0^t a_j^2 \exp(a_j (s-t)) (Y_t - Y_s) \, ds$$

Notice that the correlation ρ_j doesn't enter into the filtering for μ, but it does affect the asymptotic variance, that is, the level of confidence we ascribe to the estimate.

(3) If we now tell a diverse-beliefs story, with Y a BM under the reference measure then

$$d\Lambda^j_t = \Lambda^j_t \hat{\mu}^j_t \, dY_t$$

since we regard $\hat{\mu}$ as a functional of Y. We can now find the DSGE for CARA agents, with the SPD

$$\log \zeta_t = -\Gamma Y_t - \beta t + \sum p_j \log \Lambda^j_t \qquad p_j \propto \gamma_j^{-1}, \ \sum p_j = 1.$$

Thus $X_t \equiv [Y_t; \hat{\mu}^1_t; \dots; \hat{\mu}^J_t]$ is Markovian, and we expect that for suitably integrable f we should be able to find φ such that

$$E \left[\int_t^\infty f(X_s) \, \zeta_s \, ds \right] = \zeta_t \, \varphi(X_t)$$

but the PDE for φ doesn't admit any nice solutions, not that I can see.

7) Jim Gatheral has a price impact story where if you trade from position $0 = x_0$ to $a = x_T$ at rate $\dot{x}_t$, the cost you incur from price impact is (in the mean)

$$\int_0^T \dot{x}_t \left(\int_0^t g(t-s) f(\dot{x}_s) ds \right) dt$$

for some increasing f which is concave near zero, convex ultimately, and the function g is decreasing positive. If you insist that there's no price manipulation possible, this places restrictions on g, f: for example, $g(t) = e^{-\lambda t}$ can only happen for $f(x) \propto x$. How do the optimal paths x look??

Is the model really plausible? If you considered two trading patterns for which $\int_0^\varepsilon \dot{x}_s ds = \int_0^\varepsilon \dot{y}_s ds = 1$, but x is short + sharp, would you really expect the profile of their impact over time to be much different? Different by a constant factor? Because that's what the Gatheral model would imply

8) If we compared agents who used different amounts of data in estimating asset growth, how would this affect the equilibrium? Maybe best proxy is to EWMA of history for estimates? Corresponds to two agents doing KF, but with different constants for variance of noise?

Interesting questions.

1) I got to comment on a paper of Christoffersen + Musto about high-water marks and hedge fund compensation schemes. This referred to an earlier paper of Goetzmann Ingersoll + Ross. But are high-water mark incentives really such a good idea? Suppose the assets in the fund at time t, S_t, evolve as

$$dS_t = S_t(\sigma dW_t + \mu_t dt) - dI_t - dC_t$$

where I_t is the total paid out to the agent, C_t the total withdrawn by principal (could be decreasing if principal puts money in). Suppose principal's objective is

$$E\left[\int_0^\tau e^{-\rho t}\, dC_t + e^{-\rho\tau}(1-\varepsilon)S_\tau I_{\{S_\tau = B\}} + e^{-\rho\tau}\pi S_\tau I_{\{S_\tau = b\}}\right]$$

and agent's is

$$E\left[\int_0^\tau e^{-\rho t} U(i_t)\,dt - \int_0^\tau e^{-\rho t} f(\mu_t)\,dt + \varepsilon e^{-\rho\tau} S_\tau I_{\{S_\tau = B\}}\right]$$

where τ is first-exit from $[b, B]$. Interpretation is that the agent chooses a default level b at which to close the fund; you only recover πS_τ at that point. The upper value B is a graceful exit where the manager's initial proportion ε of input wealth is returned to him, + the principal has the rest. It seems pretty clear that the optimal ~~contract~~ here should not involve a HWM at all... what would it look like?

2) Generally, in a contracting problem there is a system where two agents can control, each with his own objective, principal being first-mover, in that he states binding contract. How to solve these by 'policy improvement'? Numerics?

3) Suppose agents could devote effort to networking, which would improve their ability to trade. How would you model that? How could you find people becoming trading specialists without imposing that?

4) Ralf asks: if you take exponential utility of consumption, it's easy to calculate the solution to the HJB eq^n, but what problem does this solve??

5) Another one from Ralf, about approximating the price of a basket $\frac{1}{n}\sum S_i$. We have of course

$$\frac{1}{n}\sum_1^n S_i = \frac{1}{n}\sum(S_i + c) - c \geq \left(\prod_1^n (S_i + c)\right)^{1/n} - c$$

and it seems that for some c this can be a very good bound...

6) Can we make some story about <u>prominence</u> of information, and how this influences things?