Lucas economy with leverage constraints

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Abstract

We study equilibrium in a multi-agent economy with a single productive asset. There is a financial market in which shares in the productive asset and in a single financial asset may be traded by the agents in the story. Agents have common von Neumann-Morgenstern preferences, but differ in their beliefs, as in Brown & Rogers [2]. Here, the agents may be subject to leverage constraints, and the financial asset is in non-negative net supply. Diverse beliefs give a reason why agents may hold non-zero positions in the financial asset, which, after all, does not deliver any consumption good. Without leverage constraints, agents agree on the prices of all contingent claims, but once leverage constraints are applied this consensus may be broken. Positive net supply of the financial asset can have an impact on inflation.

1 Introduction

In the classical representative-agent Lucas economy, there is a unique stock price derived from the unique state-price density; it is a great context to introduce key ideas and methods in DSGEs (dynamic stochastic general equilibrium models), but things get more interesting once we allow for agent diversity. Agent diversity is frequently introduced through different preferences, but we maintain that a more natural and interesting source of diversity comes from different *beliefs*, a claim which we shall illustrate in this paper.

In our model, there will be a single productive asset (the firm), and a single (infinitely divisible) financial asset. There will be a financial market in which shares in the firm and in the financial asset (the bond) will be traded. The key features of the model are

1. Agents have diverse beliefs;

2. There may be leverage constraints on positions held;

3. The financial asset is in non-negative net supply.

Our agents have standard von Neumann-Morgenstern preferences over consumption streams. Agent diversity is already a rationale for the existence of the financial asset, even though its possession delivers no consumption good; for example, early on, patient agents may sell some of their shares in the firm in return for more of the bond, with the intention of using the bond to buy back shares later when impatient agents have had the most of their consumption. This gives the bonds value, and in equilibrium we can find the relative prices of shares and bonds. Our agents will differ *only* in their beliefs, and in fact when we come to develop the results we will assume that *they all have a common log felicity*. This assumption allows us to develop quite explicit solutions while retaining many features of interest.

One thing we find is that *in the absence of leverage constraints, all agents agree on the prices of all assets*; they have a common state-price density. This does not mean that they take identical positions of course, because their beliefs differ; but they have to agree on prices, because if agent 1 thinks the share was worth more than agent 2 thinks it is worth, there is nothing to stop him borrowing some money to buy more of the share from agent 2 at a price that agent 2 will accept, but which agent 1 thinks is favourable to himself.

However, once we impose leverage constraints, this argument breaks down, because there now *can be* a reason that stops agent 1 from borrowing money to buy some of agent 2's shares - he may already be at his leverage limit. We will show that this can in fact happen.

A particularly interesting situation arises when short-selling is not allowed, an extreme form of leverage constraint. This can be interpreted as a cash-in-advance requirement where assets must be purchased with ready cash. In discrete time, cash-in-advance gives value to otherwise worthless money - more cash allows an agent a greater set of opportunities, so he may be willing to exchange real assets for this cash which allows him to exploit the greater opportunities arising from the relaxation of constraints on his choices. Cash-in-advance has a long history in economics. One branch of the literature takes and inventory-theoretic approach. Baumol [1] and Tobin [10] introduce models where agents trade-off exogenously imposed interest earnings and banking costs to determine equilibrium cash holdings. Romer [7] presents a discrete-time general-equilibrium version of this. Closer in line with our model are settings where cash is modelled as a transactions medium in an equilibrium economy without imposing arbitrary exogenous costs. Svensson [9] has a discrete-time cash-in-advance story where agents must decide on cash holdings before their consumption is known. Hence, they hold extra precautionary cash despite a positive interest rate. We will develop our results in a simple continuous-time infinite-horizon framework.

There has been some work in the literature on the effects of market imperfections on market equilibrium. Ross [8] examines how short-sale constraints can lead to violations of CAPM. Milne and Neave [5] investigate a similar problem in an intricate discrete-time finite-horizon equilibrium model. They show how transaction costs and trading constraints lead to market incompleteness. Similar results were obtained by Jouini and Kallal [4] using no-arbitrage arguments.

The solution we obtain is mathematically similar to Cvitanic & Karatzas' solution [3] for the Merton problem under portfolio constraints. They show that the support function of the feasible portfolio set plays a critical role in affecting the state-price density. When an agent hits his constraint, this support function simultaneously distorts the change-of-measure and discount factor in his state-price density, leading the agent to keep his portfolio within the feasible region when his unconstrained optimum would otherwise be outside. However, in Cvitanic & Karatzas' model, the stock dynamics are fixed. Hence, it is not clear from the outset how the discounted stock price can remain a martingale under different agent measures.

The structure of this article is as follows. In Section 2 we present the framework of the DSGE model, including the diverse-beliefs characterization, the leverage constraints, and the preferences of agents. By suitable choice of numeraire, we may (and shall) suppose that the financial asset has constant value. Section 3 contains the main part of the analysis. As is usual, we first elucidate the solutions to individual agents' optimization problems using the Pontryagin-Lagrange approach to identify dual-optimal multiplier processes. Since we assume common logarithmic felicity for all agents, the solution is remarkably complete, and we are able to make the agents' demands quite explicit. This then allows us to pin down the equilibrium stock price process and the equilibrium inflation using market clearing. Finally in Section 4 we present some numerics.

This study presents examples of some rather surprising phenomena. We find an equilibrium in which agents do not necessarily agree on the prices of all contingent claims, but where they think a traded asset is overvalued, they are prevented by their leverage constraints from exploiting what they see as a mispricing. Nonetheless, we are working in a market which is in a sense complete, because the filtration is a Brownian filtration and therefore we have the stochastic integral representation property. Every contingent claim can be replicated, but not necessarily by a trading strategy which respects the leverage constraints.

Throughout, we will be quite relaxed about technical conditions, such as the difference between local martingales and martingales, the question of duality gap, and transversality. All these are issues that have to be dealt with properly to achieve proof; but our view is that first we should find out what the results will be if we sidestep these issues, and then if we care to invest the effort, to determine conditions to ensure the points we have assumed.

2 General set-up.

Throughout, we work in a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ satisfying the usual conditions, and supporting a Wiener process $(W_t)_{t\geq 0}$. We will assume that processes are continuous Itô processes; for simplicity, we shall assume where needed that *local martingales are in fact martingales*, which would of course need to be verified for any particular situation.

There will be a single productive asset generating a flow $(\tilde{\delta}_t)$ of consumption good, satisfying

$$d\tilde{\delta}_t = \tilde{\sigma}_t \, dW_t + \tilde{\mu}_t \, dt. \tag{1}$$

There are J agents in the economy, and agent j aims to maximize his objective[1]

$$\mathcal{U}_j(\tilde{c}) \equiv E^j \left[\int_0^\infty U_j(t, \tilde{c}_t) \, dt \right] \tag{2}$$

over possible consumption streams $(\tilde{c}_t)$. Here, P^j is the probability expressing agent j's beliefs about the world; we assume that all P^j are absolutely continuous with respect to P, with likelihood-ratio martingales

$$\left. \frac{dP^j}{dP} \right|_{\mathcal{F}_t} = \Lambda_t^j, \tag{3}$$

where Λ^j solves the SDE

$$d\Lambda_t^j = \lambda_t^j \, \Lambda_t^j \, dW_t, \qquad \Lambda_0^j = 1. \tag{4}$$

In terms of Λ^j, the objective (2) can of course be expressed as

$$\mathcal{U}_j(c) = E \left[\int_0^\infty \Lambda_t^j \, U_j(t, \tilde{c}_t) \, dt \right]. \tag{5}$$

Whenever we are discussing the optimization problem of a single agent, for notational simplicity we omit the label for that agent.

Now we specify what the possible consumption streams are for an agent. There will be a financial market with two assets in it, a single infinitely-divisible share (S_t), ownership of which entitles the holder to the entire stream $\tilde{\delta}$ of consumption good; and a financial asset (B_t) which we may think of as a bank account or bond. Ownership of B delivers no consumption good; the process B is a continuous *finite-variation Itô process.* We shall assume throughout that both S and B are strictly positive, and that $B_0 = 1$. There is a net supply $A_0 \geq 0$ of the bond[2].

Financial assets are denominated in units of cash, and there is a finite-variation price-level process $(p'_t)_{t \geq 0}$ which converts consumption-good values to cash values, so that

$$\delta_t \equiv p'_t \tilde{\delta}_t \tag{6}$$

is the cash value of the stream of consumption good. We shall use tildes to denote processes denominated in units of consumption good.

[1]The utility functions U_j are all assumed C^2, strictly concave, and satisfy the Inada conditions.

[2]We might think that the net supply of the bond could be allowed to vary with time, but as we shall see, within the assumptions we are making this cannot happen. Nevertheless, modelling some mechanism for (say) monetary policy choices would be an interesting extension of this study.

The agent is going to choose a triple (n, φ, c) of processes linked by

$$
\begin{aligned}
w_t &= n_t S_t + \varphi_t B_t, &(7)\\
dw_t &= n_t(dS_t + \delta_t\,dt) + \varphi_t\,dB_t - c_t\,dt, &(8)\\
w_t &\geq 0. &(9)
\end{aligned}
$$

This is the familiar specification of an admissible self-financing wealth process; n_t is the number of shares held at time t, φ_t is the number of bonds held at time t, (12) expresses the fact that the wealth at time t is the sum of these, (13) expresses the self-financing property that the change in wealth is the gains from trade, and (14) is the admissibility condition that wealth should never go negative. Of course, the initial wealth w_0 of the agent is given. We shall denote by $\mathcal{A}(w_0)$ the set of all such achievable triples (n, φ, c) that can be sustained from initial wealth w_0.

We are going to suppose that the agents are leverage-constrained; for some $K,\ L \in [0, 1)$ we demand that for all t

$$
\begin{aligned}
-\varphi_t B_t &\leq K n_t S_t, &(10)\\
-n_t S_t &\leq L \varphi_t B_t. &(11)
\end{aligned}
$$

A special case of this would be when $L = 0$, which is a short-sales constraint on the agent's holdings of the stock and bond. If we take $K = L = 1$, we are back to the admissibility constraint (14), but this is only a limiting case of the situation we consider, since we assume that both K and L are strictly less than 1.

We observe that if we define $\bar{w}_t = w_t/B_t$, $\bar{S}_t = S_t/B_t$, $\bar{\delta}_t = \delta_t/B_t$, and $\bar{c}_t = c_t/B_t$, then the equations (7), (8), (9) become

$$
\begin{aligned}
\bar{w}_t &= n_t \bar{S}_t + \varphi_t, &(12)\\
d\bar{w}_t &= n_t(d\bar{S}_t + \bar{\delta}_t\,dt) - \bar{c}_t\,dt, &(13)\\
\bar{w}_t &\geq 0. &(14)
\end{aligned}
$$

The consumption stream $\bar{c}$ measure in units of B corresponds to a consumption stream

$$\tilde{c}_t = B_t \bar{c}_t / p'_t. \tag{15}$$

The conclusion from this is that *we may take $B \equiv 1$ in the equations* (12),(13), *so long as we understand that the price level process should be replaced by the finite-variation process*

$$p_t \equiv p'_t / B_t. \tag{16}$$

To explain a bit more fully, if we have that $B_t = \exp(\int_0^t r_s\, ds)$ and $p'_t = \exp(\int_0^t \alpha_s\, ds)$, then the price level process adjusted for riskless interest will be

$$p_t = \exp(\int_0^t (\alpha_s - r_s)\, ds). \tag{17}$$

We shall therefore proceed under the assumption that B is identically 1 in the dynamics (12), (13), *and refer to this asset as the bond.* The process δ appearing in (13) is related to $\tilde{\delta}$ by

$$\delta_t = p_t \tilde{\delta}_t \tag{18}$$

so that the drift μ_t of δ is related to the given drift $\tilde{\mu}_t$ of $\tilde{\delta}$ by

$$\mu_t = \tilde{\mu}_t + (\alpha_t - r_t). \tag{19}$$

We will derive μ from equilibrium, which will tell us the equilibrium values for $(\alpha - r)$, in effect, inflation once riskless interest has been accounted for. It is not possible to separate out the values of α, r individually.

An important point to note is that $\delta_t = p_t \tilde{\delta}_t$ is the product of the given output process $\tilde{\delta}$ and a finite-variation process, so the volatility σ of δ is in fact the given $\tilde{\sigma}$; *the volatility of δ is known, and is equal to $\tilde{\sigma}$.*

We will develop the individual agent's optimal solution at a fairly general level, but very quickly we shall make the restrictive assumption

$$U_j(t, c) = U(t, c) \equiv e^{-\rho t} \log c \qquad \forall j. \tag{20}$$

As is readily verified, the Fenchel dual of U is

$$\tilde{U}(t, y) = e^{-\rho t} \big[-1 - \rho t - \log y \big]. \tag{21}$$

By assuming log utility, we are able to solve remarkably explicitly, and indeed, when there is no leverage constraint, this situation is completely solved in Section 4 of [2].

To summarize then, the given components of the story are the consumption stream $\tilde{\delta}$, the objectives $\mathcal{U}_j$ of the agents, and their beliefs P^j. We are going to find equilibrium solutions, that is, a price process S and a price level process p such that if each agent optimizes his objective over achievable triples (n, φ, c), then the markets clear; the consumption stream is exactly consumed, the total holdings of shares at all times equal the unit supply, and the total holdings of the bonds at all times equals the available supply. The first step in finding an equilibrium is of course to consider the optimization problem of an individual agent, which we shall solve by the Pontryagin-Lagrange method. Again for ease of exposition, we shall assume that there is *no duality gap*, which is an assumption that often requires a lot of work to justify, but this is technical backfilling of results which we think are already intriguing.

3 Diverse agents equilibrium.

To derive the equilibrium, we use the Pontryagin-Lagrange approach, introducing a strictly positive Lagrangian semimartingale ζ and non-negative Lagrangian processes ξ and η to take care of the leverage constraints (10), (11). The set $\mathcal{A}(w_0)$ of admissible triples (n, φ, c) must now satisfy (10), (11) in addition to (12), (13), (14).

Proposition 1. *A triple (ζ, ξ, η) is in the dual-feasible set $\mathcal{D}$ if ζ is a strictly positive continuous semimartingale and ξ, η are non-negative adapted processes such that*

$$d(\zeta S) + \zeta S(K\eta + \xi)\, dt + \zeta\delta\, dt \qquad \text{is a (local) martingale;} \tag{22}$$

$$d\zeta + \zeta(\eta + L\xi)\, dt \qquad \text{is a (local) martingale.} \tag{23}$$

Assuming the transversality property

$$\zeta_t w_t \to 0 \quad \text{a.s., in } L^1 \quad (t \to \infty), \tag{24}$$

the duality relationship

$$\sup_{(n,\varphi,c)\in\mathcal{A}(w_0)} \mathcal{U}(c) \le \inf_{(\zeta,\xi,\eta)\in\mathcal{D}} E\Big[\int_0^\infty \Lambda\tilde{U}(t, p/\zeta)\, dt + w_0\zeta_0 \Big] \tag{25}$$

holds.

Proof. If ζ is a strictly positive continuous semimartingale, and ξ, η are non-negative adapted processes, we find that

$$\begin{aligned} \sup_{(n,\varphi,c)\in\mathcal{A}(w_0)} \mathcal{U}(c) \quad &\le \quad \sup_{(n,\varphi,c)\in\mathcal{A}(w_0)} E\Big[\int_0^\infty \big\{ \Lambda U(t, c/p)\, dt + \zeta dw + w d\zeta + dw\, d\zeta + \\ &\qquad + (KnS + \varphi)\zeta\eta\, dt + (nS + L\varphi)\zeta\xi\, dt \big\} + w_0\zeta_0 \Big]. \end{aligned} \tag{26}$$

To understand why this is, notice that from the integration-by-parts formula, the terms $\zeta dw + w d\zeta + dw\, d\zeta$ are the differential of ζw, and because of (24) we have

$$w_0\zeta_0 = -[w_t\zeta_t]_0^\infty.$$

Thus the inclusion of these two terms on the right-hand side makes no change to the left-hand side of (26). To understand the effect of the terms involving ξ and η, notice that if $(n, \varphi, c) \in \mathcal{A}(w_0)$, then these terms contribute something non-negative to the right-hand side, and so we have if anything increased the objective, whence the stated inequality.

We now use (12) and (13) to rework the right-hand side of (26), collecting terms in n and in φ:

$$\begin{aligned} \sup_{(n,\varphi,c)\in\mathcal{A}(w_0)} \mathcal{U}(c) \quad &\le \quad \sup_{(n,\varphi,c)\in\mathcal{A}(w_0)} E\Big[\int_0^\infty \big\{ \Lambda U(t, c/p)\, dt - \zeta c\, dt \\ &\qquad + n\{\zeta(dS + \delta\, dt) + S\, d\zeta + dS\, d\zeta + \zeta S(K\eta + \xi)\, dt\} + \\ &\qquad + \varphi\{\, d\zeta + \zeta(\eta + L\xi)\, dt\} \big\} + w_0\zeta_0 \Big] \\ &= \quad \sup_{(n,\varphi,c)\in\mathcal{A}(w_0)} E\Big[\int_0^\infty \big\{ \Lambda\tilde{U}(t, p\zeta/\Lambda)\, dt + \\ &\qquad + n\{d(\zeta\, S) + \zeta\delta\, dt + \zeta S(K\eta + \xi)\, dt\} + \\ &\qquad + \varphi\{d\zeta + \zeta(\eta + L\xi)\, dt\} \big\} + w_0\zeta_0 \Big]. \end{aligned} \tag{27}$$

Because n and φ are arbitrary previsible integrands, we deduce the dual-feasibility conditions (22), (23), and (25) follows immediately. $\square$

If the inequality (25) is strict, then we say there is a *duality gap*. If there is a duality gap, there is little we can do. Verifying that there is no duality gap usually requires some careful analysis exploiting the explicit features of the situation under consideration. We shall avoid this here, by simply *assuming that there is no duality gap*, that is, (25) holds with equality. This will take us directly to some interesting results without a long and dreary technical detour.

3.1 Finding the dual-optimal multipliers.

We now suppose that ζ and S have dynamics

$$\begin{aligned} d\zeta_t &= -\zeta_t\,(a_t\,dW_t + b_t\,dt), && (28)\\ dS_t &= S_t(v_t\,dW_t + m_t\,dt). && (29) \end{aligned}$$

Proposition 2. *Under our assumptions of no duality gap, and the transversality relation* (24)*, the dual-optimal choices for* a, b *are*

$$\begin{aligned} \bar{a} &= \frac{M + z_p^+ - z_m^-}{v}, && (30)\\ \bar{b} &= \frac{Lz_p^+}{1-L} + \frac{z_m^-}{1-K}, && (31) \end{aligned}$$

where

$$\begin{aligned} z_p &= -M - \lambda v - \frac{Lv^2}{1-L}, && (32)\\ z_m &= -M - \lambda v + \frac{v^2}{1-K}, && (33) \end{aligned}$$

and $M \equiv m + \delta/S$*. Moreover,* $v > 0$ *and*

$$\zeta_0 = 1/\rho w_0. \qquad (34)$$

Proof. By considering the drifts of (22), (23) we deduce that the dual-feasibility conditions are

$$\begin{aligned} 0 &= M - av - b + K\eta + \xi, && (35)\\ 0 &= -b + \eta + L\xi, && (36) \end{aligned}$$

where

$$M \equiv m + \delta/S. \qquad (37)$$

We can now solve the linear equations (35) and (36) to learn that

$$\begin{aligned} a &= \frac{M+z}{v}\,, && (38)\\ b &= \frac{z^-}{1-K} + \frac{Lz^+}{1-L}, && (39) \end{aligned}$$

where

$$z \equiv (1-L)\xi - (1-K)\eta. \tag{40}$$

Recall that the dual variables ξ, η are required to be non-negative, so z can take any real value. The positive and negative parts of z tell us ξ and η.

Under the assumption of no duality gap,

$$\sup_{(n,\varphi,c)\in\mathcal{A}(w_0)} \mathcal{U}(c) = \inf_{(\zeta,\xi,\eta)\in\mathcal{D}} E\left[\int_0^\infty \Lambda_t \tilde{U}(t, p_t\zeta_t/\Lambda_t)\,dt + w_0\zeta_0\right]. \tag{41}$$

From the assumed logarithmic form (20) of the felicity, the expression to be minimized over ζ is

$$E\left[\int_0^\infty -\Lambda_t e^{-\rho t}\log\zeta_t\;dt + w_0\zeta_0\right] = E\left[\int_0^\infty e^{-\rho t}\{\tfrac{1}{2}a_t^2 + \lambda_t a_t + b_t\}\;dt + w_0\zeta_0 - \rho^{-1}\log\zeta_0\right] \tag{42}$$

from (28) and the Cameron-Martin-Girsanov effect of change of measure as change of drift. The equation (34) follows from this by simple calculus. Now if we write $Q \equiv \frac{1}{2}a^2 + \lambda a + b$, we have from (38) and (39) that

$$\begin{aligned}\frac{dQ}{dz} &= \frac{\lambda + a}{v} + \frac{L}{1-L}I_{\{z>0\}} - \frac{1}{1-K}I_{\{z<0\}}\\ &= \frac{\lambda}{v} + \frac{M}{v^2} + \frac{z}{v^2} + \frac{L}{1-L}I_{\{z>0\}} - \frac{1}{1-K}I_{\{z<0\}}.\end{aligned} \tag{43}$$

Inspection of (43) shows that the gradient of Q is increasing in z, with an upward jump at $z = 0$.

Thus, defining z_p, z_m as at (32), (33), we notice that $z_m > z_p$; that if $z_p > 0$ then $z = z_p$ minimizes Q; that if $z_m < 0$ then $z = z_m$ minimizes Q; and otherwise the minimizing choice of z is 0. Notice also that $z_p > 0$ if and only if $\xi > 0$, which implies that the constraint (11) is tight; the agent is as short of the stock as he can be. Since this is what would be expected for an agent whose value of λ is low, we deduce that we will always have

$$v > 0. \tag{44}$$

□

Altogether then, the minimizing values of a, b are

$$\begin{aligned}\bar{a} &= \frac{M + z_p^+ - z_m^-}{v},\\ \bar{b} &= \frac{Lz_p^+}{1-L} + \frac{z_m^-}{1-K},\end{aligned}$$

and this (together with (34)) identifies *the agent's dual-optimal choice of* ζ. The agent takes the dynamics of S as given, therefore v and M are known to the agent, as is his likelihood-ratio drift λ, so $\bar{a}$ and $\bar{b}$ are known to the agent, and therefore this recipe is completely explicit. Importantly, since agents have different λ, they do *not* in general have the same state-price density process ζ; agents who are not leverage-constrained (that is, $z_p < 0 < z_m$) have common values of $\bar{a}$, $\bar{b}$. A consequence of this is that the agents do not agree on the prices of contingent claims at all times; however, when there *is* disagreement, an agent who thinks that an asset is valued too cheaply is not able to exploit this because he is leverage-constrained, and may not borrow more cash to buy more of the share he thinks is undervalued.

3.2 Agent demands.

The optimization over c which gave us (27) tells us that

$$c_t = \frac{e^{-\rho t}\Lambda_t}{\zeta_t}, \tag{45}$$

so we have the key relation

$$\zeta_t c_t = e^{-\rho t}\Lambda_t. \tag{46}$$

Although agents may not always all agree that market assets are correctly priced, each agent believes that his future consumption is correctly priced. To see why, we reprise the analysis that took us to (27), making use of (22), (23):

$$\begin{aligned}
d(\zeta w) &= \zeta\, dw + w\, d\zeta + dw\, d\zeta \\
&= \zeta n(dS + \delta\, dt) - \zeta c\, dt + (nS + \varphi)d\zeta + ndS\, d\zeta + \\
&= n\, d(\zeta S) + \zeta\delta\, dt + \varphi\, d\zeta - \zeta c\, dt \\
&\doteq -n\zeta S(K\eta + \xi)dt - \varphi\zeta(\eta + L\xi)dt - \zeta c\, dt \\
&= -\zeta\eta(KnS + \varphi)dt - \zeta\xi(nS + L\varphi)dt - \zeta c\, dt \\
&= -\zeta c\, dt,
\end{aligned}$$

where $\doteq$ signifies that the two sides differ by a (local) martingale. The final step comes from complementary slackness. Therefore

$$M_t \equiv \zeta_t w_t + \int_0^t \zeta_s c_s\, ds \quad \text{is a non-negative (local) martingale.}$$

Assuming that M is in fact a martingale, and that (24) holds, we deduce that

$$\zeta_t w_t = E_t\left[\int_t^\infty \zeta_u c_u\, du\right]. \tag{47}$$

Using (46) gives us that

$$\zeta_t w_t = \rho^{-1} e^{-\rho t}\Lambda_t,$$

whence

$$c_t = \rho w_t = \frac{e^{-\rho t}\Lambda_t}{\zeta_t}. \tag{48}$$

This allows us to find the agent's demand n_t for the stock, and φ_t for the bond. Indeed, from (48), (28), and (4) we have

$$\frac{dw_t}{w_t} = -\rho dt + \lambda_t \, dW_t + a_t \, dW_t + (\, b_t + a_t^2 + a_t \lambda_t \,) \, dt, \tag{49}$$

which we compare with (13) to discover that

$$\begin{aligned} nSv &= (a+\lambda)w, & (50) \\ nSM &= w(\, b + a^2 + a\lambda \,). & (51) \end{aligned}$$

3.3 Market clearing.

The analysis of this section has so far focused on a single agent, so to lighten notation we have omitted the label of the agent, but now we need to reintroduce that label to discuss market clearing. Market clearing of the consumption good market requires that

$$\sum_j c_t^j = \delta_t = \rho \sum_j w_t^j, \tag{52}$$

using (48). From (12) we have also that

$$\sum_j w_t^j = \sum_j \{\, n_t^j S_t + \varphi_t^j \,\} = S_t + A_0 = \delta_t / \rho, \tag{53}$$

using (52). Similarly, from (13) we conclude that

$$\sum_j dw^j = dS,$$

which explains why we have insisted that the supply of bonds is constant - the analysis shows that it has to be, and indeed there would be no economic reason for the supply to change without introducing some further element into the modelling.

By taking the Itô expansion of the last two terms in (53), and matching the terms in dW_t and in dt, we learn that

$$\begin{aligned} v_t S_t &= \sigma_t \delta_t / \rho, & (54) \\ m_t S_t &= \mu_t \delta_t / \rho. & (55) \end{aligned}$$

However, if we take (50), (51) and sum over all the agents, we obtain alternative expressions:

$$\begin{aligned} v_t S_t &= \sum_j w_t^j (a_t^j + \lambda_t^j), & (56) \\ M_t S_t &= \sum_j w_t^j (b_t^j + a_t^j (a_t^j + \lambda_t^j)), & (57) \end{aligned}$$

where the variables a_t^j, b_t^j are expressed in terms of M_t and v_t by (30) and (31).

4 Simulation.

In principle, market clearing should determine the equilibrium price process S, and the price-level process p, but the equations to be solved are sufficiently complicated that closed-form solution is not generally possible. So in this section we make a computational study of the equilibria which arise.

We suppose that the process $\tilde{\delta}$ and its associated volatility $\tilde{\sigma}$ and drift $\tilde{\mu}$ are given and known, as are the processes λ^j in the definition of the beliefs P^j of the agents. As we remarked earlier, since $\delta_t = p_t\tilde{\delta}_t$ and p is a finite-variation process, the volatility $\sigma_t = \tilde{\sigma}_t$ is therefore known, but the drift of δ is not. The total supply of bonds A_0 is given. The initial values w_0^j of the agents' wealths are given and satisfy the clearing condition (53), and the initial value S_0 is given, and is consistent with (53). In view of (34), we have that $\zeta_0^j = 1/\rho w_0^j$ for all j.

The simulation is going use a simple first-order Euler scheme to evolve the processes S, μ, w^j, Λ^j, ζ^j, while at the same time evolving the processes δ, σ, and λ^j in accordance with the given recipes. So we suppose that we have reached time t and have calculated S_t, w_t^j, Λ_t^j, ζ_t^j; the goal now is to make a small step forward to time $t' = t+h > t$. The increment ΔW of the driving Brownian motion is simulated, and is used to create the increments of the various processes.

To generate the change in S we shall need (see (29)) v_t and m_t. We get v_t from (54), we will explain shortly where we get m_t. To generate the change in ζ^j, we need (see (28)) a_t^j and b_t^j, which are derived from (30), (31). This is not immediately available, because we do not know M. However, upon inspection of (32), (33), (30) we see that each a^j is a continuous non-decreasing piecewise-linear function of M, so we adjust M until the clearing condition (56) is satisfied, and this gives us $M = m + \delta/S$, whence m follows. Having found M, we now know all the a^j and b^j, so we can make the next step of the ζ^j. We can make the next steps of Λ^j because the processes λ^j were given to us. In order to step δ forward, we need to know μ_t, which we derive from (55).

4.1 Constant $\tilde{\sigma}$, $\tilde{\mu}$, λ^j.

As a first example, we can consider a situation where the volatility and drift of the output process $\tilde{\delta}$ are constant, and each agent has a fixed and constant opinion on what the drift in $\tilde{\delta}$ is. We present in Figures 1 and 2 some plots of the solution for a typical example. We notice from (53) that if A_0 is zero then S is a multiple of δ, and therefore has the same volatility and drift. Looking back at (19), we see that the plot of inflation $\alpha - r$ would then just be a shift of the plot of $\mu = m$, so the second and third plots in Figure 1 would look the same. While they have similar shape, they are not the same, because of the parameter choice $S_0/A_0 = 0.7$. A consequence of such a relatively large value of A_0 is that δ cannot become too small - see (53).

There is no reason why $\tilde{\delta}$ cannot get arbitrarily close to zero, so the only way that could happen is if inflation became arbitrarily large.

Observe (see Figure 2)that the most optimistic agent (in red) does at times take on the largest possible amount of stock, subject to his leverage constraint; likewise, the most pessimistic agent (in blue) does at times take on as little of the stock as his leverage constraint allows.

4.2 Bayesian agents.

An interesting variant of the dynamics (1) of $\tilde{\delta}$ arises when we suppose that the drift $\tilde{\mu}$ is not constant but is instead an OU process centred at $\bar{\mu}$, so that

$$d\tilde{\mu} = \sigma_\mu \, dW' + \beta(\bar{\mu} - \tilde{\mu}) \, dt \tag{58}$$

for some Brownian motion W' independent[3] of W, and $\beta > 0$. Working in the observation filtration of the process $\tilde{\delta}$, the standard Kalman filtering story (see, for example, Chapter VI.9 in [6]) tells us that

$$\begin{aligned} d\tilde{\delta}/\tilde{\delta} &= \tilde{\sigma} \, d\hat{W} + \hat{\mu} \, dt, && (59) \\ d\hat{\mu} &= \kappa \, d\hat{W} + \beta(\bar{\mu} - \hat{\mu}) \, dt, && (60) \end{aligned}$$

where

$$\kappa = \sqrt{\beta^2 \tilde{\sigma}^2 + \sigma_\mu^2} - \beta\tilde{\sigma} \tag{61}$$

and $\hat{W}$ is the innovations Brownian motion. Since the only change from the constant case in Section 4.1 is in the drift of $\tilde{\delta}$, the dynamics of δ (which is the output stream denominated in units of the numeraire B) evolves exactly as before, and when we come to calculate the effect on inflation (19), the drift $\tilde{\mu}$ is no longer constant, but is calculated from the evolution (60), using the simulated increments of δ.

An example of this is shown in Figure 3, when $\sigma_\mu = 0.9$, $\bar{\mu} = 0.05$, and $\beta = 1$. Notice that the top two panels coincide with the top two panels of Figure 1, but the third panel, the inflation plot, is different, as would be expected.

[3]Constant correlation is not hard to handle, but adds little to the story.

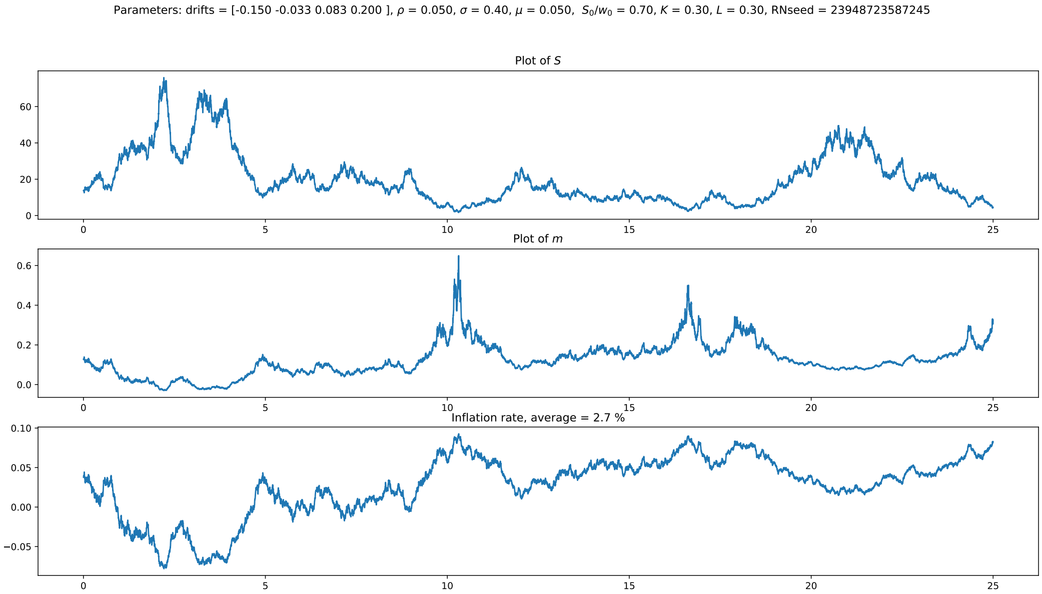

Figure 1: Plot of S, m, and inflation for constant growth rate $\tilde{\mu} = 0.05$.

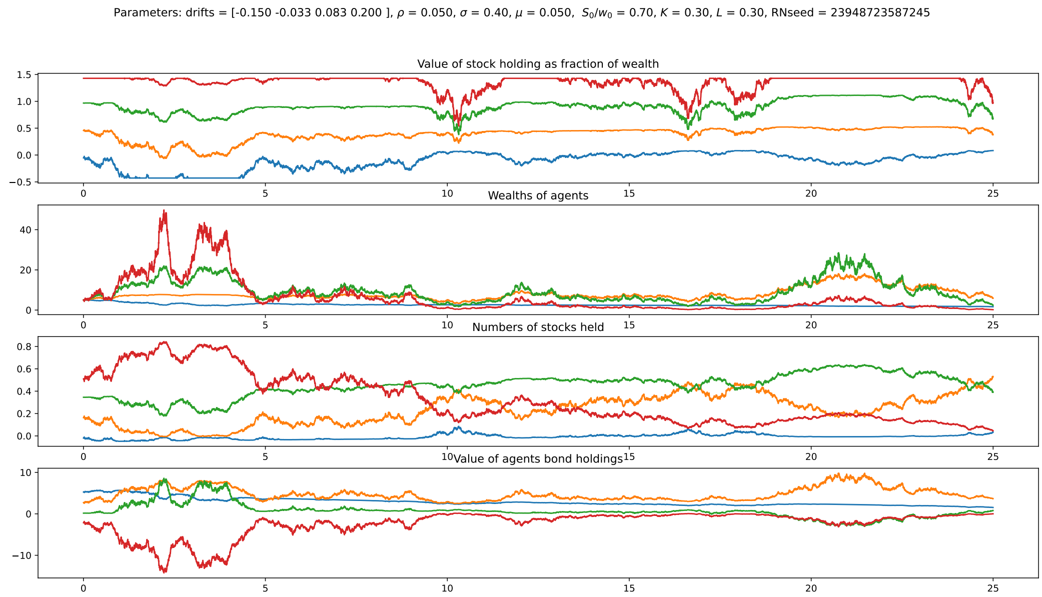

Figure 2: Agents' holdings and wealth for constant growth rate $\tilde{\mu} = 0.05$.

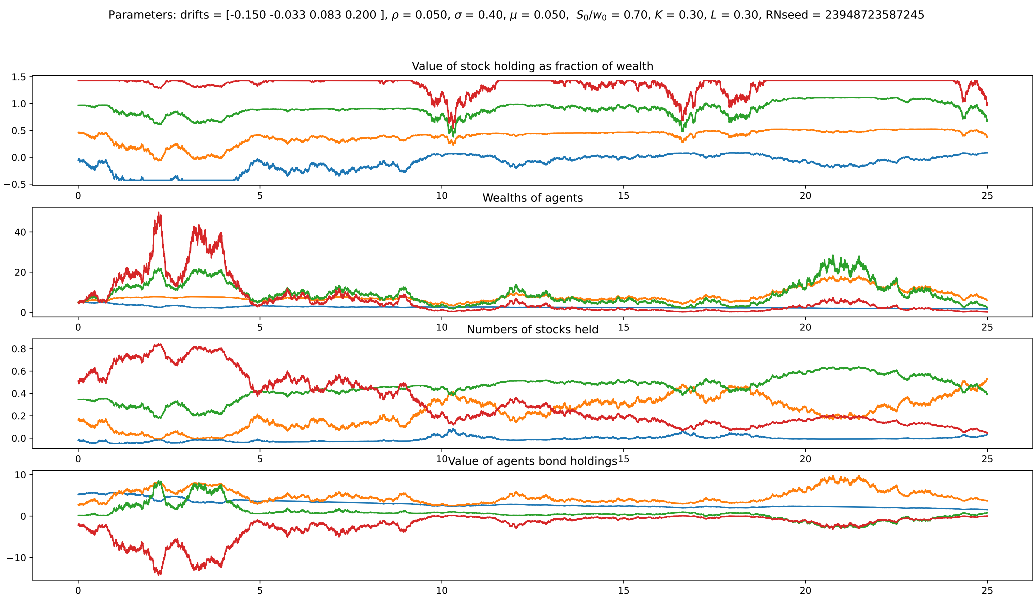

Figure 3: Agents' holdings and wealth for OU growth rate.

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