Peter Ruckdeschel's question again (1/3/12)

1) The setting on p64 of WN XXXII was that with a known fixed prob^y ε the observation Y which you see is actually drawn from density $g(y)$ rather than from the informative conditional density $p(y|x)$ which tells you about the hidden variable X of interest, which has density f. The statistician will propose a kernel $k(\tilde{x}|y)$ for making a statement $\tilde{X}$ about what he thinks X is. I don't on reflection much like the objectives proposed before. Maybe better is to proceed as follows

2) The density of $\tilde{X}$ given $X = x$ will be

$$\int \left\{ (1-\varepsilon) p(y|x) + \varepsilon g(y) \right\} k(\tilde{x}|y) \, dy \equiv \varphi(\tilde{x}|x),$$

say. The distribution of $\tilde{X}$ therefore has density

$$\bar{\varphi}(\tilde{x}) \equiv \int \varphi(\tilde{x}|x) f(x) \, dx.$$

What we would like is that $\tilde{X}$ is highly informative about X, so we would look at the mutual information

$$J \equiv \iint \log\left(\frac{f(x)\varphi(\tilde{x}|x)}{f(x)\bar{\varphi}(\tilde{x})} \right) \cdot f(x)\varphi(\tilde{x}|x) \, dx \, d\tilde{x}$$

$$= \iint \log\left[\frac{\varphi(\tilde{x}|x)}{\bar{\varphi}(\tilde{x})} \right] f(x)\varphi(\tilde{x}|x) \, dx \, d\tilde{x}$$

The Statistician wants to pick k to make this big, the goal of Nature is to choose g to make it small!

3) If Nature changes g to $g + \eta$, the change in φ is

$$a(\tilde{x}) = \int \varepsilon \eta(y) k(\tilde{x}|y) \, dy$$

which is also the change in $\bar{\varphi}$. Overall, then, the change in J to leading order is

$$\iint f(x) \left[\log \frac{\varphi(\tilde{x}|x)}{\bar{\varphi}(\tilde{x})} \cdot a(\tilde{x}) + \varphi(\tilde{x}|x) \left\{ \frac{1}{\varphi(\tilde{x}|x)} - \frac{1}{\bar{\varphi}(\tilde{x})} \right\} a(\tilde{x}) \right] dx \, d\tilde{x}$$

$$= \iint f(x) a(\tilde{x}) \log\left[\frac{\varphi(\tilde{x}|x)}{\bar{\varphi}(\tilde{x})} \right] dx \, d\tilde{x}$$

So the first-order condition for optimality here is

$$\boxed{y \mapsto \iint f(x) \log\left[\frac{\varphi(\tilde{x}|x)}{\bar{\varphi}(\tilde{x})}\right] k(\tilde{x}|y)\, dx\, d\tilde{x} \quad \text{is constant}}$$

4) How does it look for the Statistician? He changes $k(\tilde{x}|y)$ to $k(\tilde{x}|y) + \nu(\tilde{x}|y)$ and thus changes $\varphi(\tilde{x}|x)$ by

$$b(\tilde{x}|x) \equiv \int \{(1-\varepsilon)p(y|x) + \varepsilon g(y)\}\, \nu(\tilde{x}|y)\, dy$$

and changes $\bar{\varphi}(\tilde{x})$ by

$$\bar{b}(\tilde{x}) \equiv \int b(\tilde{x}|x) \cdot f(x)\, dx$$

Thus the leading-order change in J is

$$\iint f(x) \left[\log\left(\frac{\varphi(\tilde{x}|x)}{\bar{\varphi}(\tilde{x})}\right) b(\tilde{x}|x) + \varphi(\tilde{x}|x) \left\{ \frac{b(\tilde{x}|x)}{\varphi(\tilde{x}|x)} - \frac{\bar{b}(\tilde{x})}{\bar{\varphi}(\tilde{x})} \right\} \right] dx\, d\tilde{x}$$

$$= \iint f(x)\, b(\tilde{x}|x) \log \frac{\varphi(\tilde{x}|x)}{\bar{\varphi}(\tilde{x})}\, dx\, d\tilde{x}$$

$$= \iiint f(x) \log\left(\frac{\varphi(\tilde{x}|x)}{\bar{\varphi}(\tilde{x})}\right) \{(1-\varepsilon)p(y|x) + \varepsilon g(y)\}\, \nu(\tilde{x}|y)\, dx\, d\tilde{x}\, dy$$

So we would need for each y

$$\boxed{\tilde{x} \longmapsto \int f(x) \log\left(\frac{\varphi(\tilde{x}|x)}{\bar{\varphi}(\tilde{x})}\right) \{(1-\varepsilon)p(y|x) + \varepsilon g(y)\}\, dx}$$

is constant

5) But does this make sense? If the statistician picked a kernel that was singular he could make infinite mutual information?! No...

Some questions of Simon Godsill (4/3/12)

Simon + Tatjana Lemke have been making use of the remarkable result of Sammorodnitsky + Taqqu that if $0 < \tau_1 < \tau_2 < \dots$ are the jump times of a standard Poisson process, and if $(W_k)_{k \geq 1}$ are IID* then

$$Z = \sum_{k \geq 1} (\tau_k)^{-1/\alpha} W_k \quad \text{is } \sout{\text{stable }(\alpha)}.$$

1) Can this result hold if the W_k took values in $\mathbb{R}^d$? It seems to me that this has to be OK, if we revisit the proof of the main result. If (τ'_k) (W'_k) constitute an independent copy of (τ_k), (W_k) then

$$Z + Z' = \sum_{j \geq 1} (\tilde{\tau}_j)^{-1/\alpha} \tilde{W}_j$$

where the $\tilde{\tau}_j$ are just the $(\tau_k) \cup (\tau'_k)$ arranged in increasing order, and these are the jump times of a Poisson process of intensity 2. Therefore

$$Z + Z' \overset{\mathcal{D}}{=} 2^{1/\alpha} Z$$

and this is the essential ingredient. However, we have as usual the problems of defining the characteristic function off some grid of points (the Hamel equation) and some conditions on the law of the W_k will be needed. But it looks like the essential argument doesn't depend on the dimension d..

2) Is there some Central Limit behaviour for the tail of the sum defining Z (now taking $W_k \equiv 1$)?

Let's write

$$Z_a = \sum_{\tau_k \geq a} \tau_k^{-1/\alpha} \equiv \sum_{\tau_k \geq a} \tau_k^{-\varepsilon} \qquad \text{for brevity}$$

Now observe that

$$Z_a = \int_a^\infty t^{-1/\alpha} \, dN_t$$

where N is the Poisson counting process. We have for any increasing Lévy process X with characteristic exponent $\psi(\cdot)$ and any deterministic function $f \geq 0$

* We certainly need to impose some conditions on the law of the W_k – finite variance would do – but I don't have S + T available to check just now.

the simple identity

$$E \exp\left(\theta \int_0^\infty f(t)\,dX_t\right) = \exp\left\{\int_0^\infty \psi(\theta f(t))\,dt\right\} \qquad (\mathrm{Re}\,\theta \le 0)$$

For a Poisson counting process we have

$$E \exp(\theta N_t) = \exp\{t(e^\theta - 1)\}$$

so $\psi(\theta) = e^\theta - 1$.

We now apply this using $f(t) = t^{-\varepsilon} I_{\{t \ge a\}}$ to discover

$$E \exp(\theta Z_a) = \exp \int_a^\infty \left(\exp\left(\tfrac{\theta}{t^\varepsilon}\right) - 1\right) dt$$

so we see that we need $\boxed{\varepsilon > 1}$ for a meaningful result. Assuming this, we get

$$E \exp \theta(Z_a - EZ_a) = \exp\left[\int_a^\infty \left\{\exp\left(\tfrac{\theta}{t^\varepsilon}\right) - 1 - \tfrac{\theta}{t^\varepsilon}\right\} dt\right].$$

Evidently this converges to 1 as $a \to \infty$, but we can ask whether it is possible to find a normalisation v_a such that

$$E \exp \theta\left\{\frac{Z_a - EZ_a}{v_a}\right\} \longrightarrow \exp\left(\tfrac{1}{2}\theta^2\right) \qquad (a \to \infty)$$

I'll suppose that $\boxed{a^\varepsilon v_a \to \infty}$ for now, and later verify that the form of v_a which we find satisfies this condition. Assuming this, we have

$$E \exp\left\{\theta \frac{Z_a - EZ_a}{v_a}\right\} = \exp\left[\int_a^\infty \left(\exp\left(\frac{\theta}{v_a t^\varepsilon}\right) - 1 - \frac{\theta}{v_a t^\varepsilon}\right) dt\right]$$

and the argument of $\exp(\cdot)$ on the RHS, that is, $\theta / v_a t^\varepsilon$, is small. Taking logs, we get to leading order

$$\int_a^\infty \frac{\theta^2}{2v_a^2} t^{-2\varepsilon}\,dt = \frac{\theta^2}{2v_a^2} \frac{a^{-(2\varepsilon - 1)}}{(2\varepsilon - 1)}$$

This tells us to take

$$v_a = \frac{a^{-\varepsilon + 1/2}}{\sqrt{2\varepsilon - 1}}$$

which we see satisfies $v_a a^{\varepsilon} \to \infty$, and gives us the Central Limit statement

$$\boxed{\frac{Z_a - EZ_a}{v_a} \xrightarrow{\mathcal{D}} N(0,1)}$$

Anything useful if we don't have $\varepsilon > 1$ (that is, $\alpha < 1$)? There's probably something we can do, but as the sums are only conditionally convergent it may be less useful

3) If

$$\boxed{m \equiv \int_0^a t^{-\varepsilon}\, dN_t, \qquad V \equiv \int_0^a t^{2\varepsilon}\, dN_t}$$

Can we get some estimates on $P(V > x \mid m)$ for big x? There's the simple Cauchy-Schwarz bound

$$\left(\int_0^a t^{-\varepsilon}\, dN_t\right)^2 \leq N_a \cdot \int_0^a t^{-2\varepsilon}\, dN_t$$

but this goes the wrong way.

If we knew that $t^{-\varepsilon} \leq 2A$ N-a.e., (which is the same as $\tau_1^{-\varepsilon} \leq 2A$) then we have $(A - t^{-\varepsilon})^2 \leq A^2$, and so

$$\int_0^a (A - t^{-\varepsilon})^2\, dN_t = A^2 N_t - 2A \int_0^a t^{-\varepsilon}\, dN_t + \int_0^a t^{-2\varepsilon}\, dN_t$$

$$\leq A^2 N_t$$

So on the event $\boxed{\tau_1 > (2A)^{-1/\varepsilon}}$ we shall have

$$\boxed{V \leq 2A\, m.}$$

Though we even have directly

$$\boxed{V = \int_0^a t^{-2\varepsilon}\, dN_t \leq \tau_1^{-\varepsilon} \int_0^a t^{-\varepsilon}\, dN_t = \tau_1^{-\varepsilon} m}$$

which comes to the same thing! Perhaps useful?

__Careful!__ $H_{-a} = \inf\{t : M_t = -a\}$ or $H_{-a} = \inf\{t : I_t \leq -a\}$? These are __not__ the same thing, because you could stop at $-a$ on the final excursion?

No you can't visit a on the final excursion unless you were there earlier!

More on (I, X, S) (5/3/12)

(1) Suppose that $(X_n)_{n\geq 0}$ is a random walk on $h\mathbb{Z}$, stopped at some stopping time, and write

$$S_n = \sup_{m\leq n} X_m, \qquad I_n = \inf_{m\leq n} X_m$$

and set $S \equiv S_\infty$, $I \equiv I_\infty$, $X \equiv X_\infty$. We shall suppose that the stopped process X is a UI martingale. Suppose we are given the joint law of (I, X, S); what can we pull out from that?

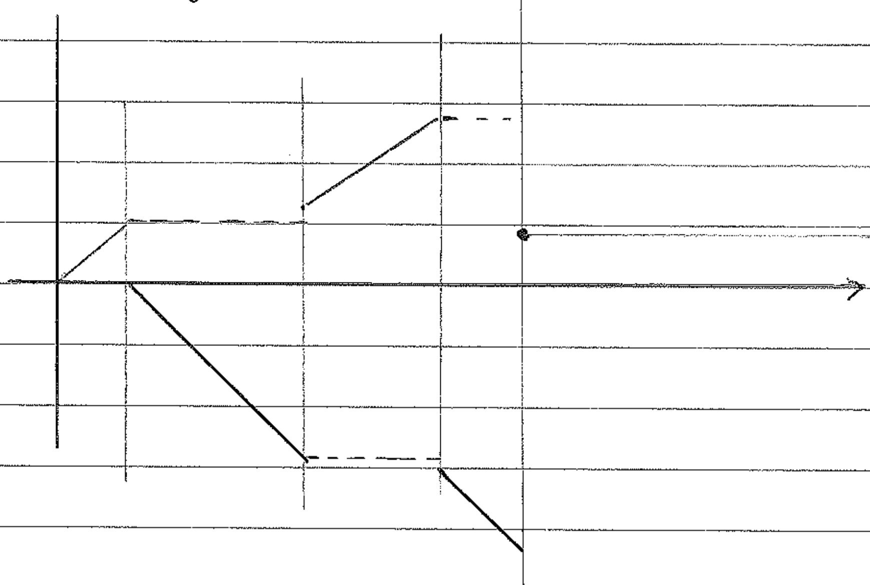
(2) Let's define

$$\tau_t \equiv \inf\{ n : S_n - I_n \geq t \}$$

with

$$M_t \equiv X(\tau_t)$$

In some sense, the martingale M is a sufficient statistic; we get paths that look like



For $b \in h\mathbb{Z}^+$, $a \in h\mathbb{Z}^+$, we have $[H_b = \inf\{t : S_t = b\}]$

$$0 = E[M(H_b \wedge H_{-a})]$$

$$= b\, P(H_b < H_{-a}) - a\, P(H_{-a} < H_b) + E[X : S < b, I > -a]$$

From the joint law, we know $E[X : S < b, I > -a]$ and $P[S < b, I > -a]$, so we can deduce

$$\boxed{P(H_b < H_{-a}),\ P(H_{-a} < H_b)}$$

Now $P[H_{-a} < H_b] = P[H_{-a} < \infty, S(H_{-a}) < b]$

So we can deduce $P[H_{-a} < \infty,\ S(H_{-a}) = b]$. Therefore we can deduce

$$\boxed{P\left[M_t = -a\right] = P\left[H_{-a} < \infty,\ S(H_{-a}) = t-a\right]}$$ ✱ (See below)

(3) What else may we deduce? If we start looking at the martingale at H_{-a} provided $H_{-a} < H_b$, and run til $H_b \wedge H_{-a-h}$, we get from OST

$$-a\,P(H_{-a} < H_b) = b\,P\left[H_{-a} < H_b < H_{-a-h}\right] - (a+h)\,P\left[H_{-a} < H_{-a-h} < H_b\right]$$
$$+ E\left[X : I = -a,\ S < b\right]$$
$$= b\,P\left[H_{-a} < H_b < H_{-a-h}\right] - (a+h)\,P\left[H_{-a-h} < H_b\right]$$
$$+ E\left[X : I = -a,\ S < b\right]$$

So from this we can deduce

$$\boxed{P\left[H_{-a} < H_b < H_{-a-h}\right]}$$

✱ Note that we have overlooked the possibility that $M_t = -a$ on the event that $\{\tau_t = \infty\}$, so we need to correct this statement to

$$\boxed{P\left[M_t = -a\right] = P\left[H_{-a} < \infty,\ S(H_{-a}) = t-a\right] + P\left[X = a,\ S - I < t\right]}$$

(4) Let μ be the joint law of (I, X, S), which must be a dist^{n} on $(h\mathbb{Z}^-) \times (h\mathbb{Z}) \times (h\mathbb{Z}^+) \equiv \mathfrak{X}$, say. Moreover, it is clearly necessary that

(C1) $$\mu\left(I \leq X \leq S\right) = 1.$$

By using the OST at $H_{-a} \wedge H_b$, we shall find that

$$\boxed{P(H_{-a} < H_b) = \frac{b}{a+b} - \frac{1}{a+b}\iiint (b-x)\, I_{\{s<b,\ i>-a\}}\ \mu(di, dx, ds)}$$

So there are further necessary conditions to be satisfied, which are not apparently implied by (C1), that is

$$\text{(C2a)} \qquad \frac{b}{a+b} - \frac{1}{a+b}\int (b-x)\, I_{\{-a<i<s<b\}}\, d\mu \geq 0$$

$$\text{(C2b)} \qquad \frac{a}{a+b} - \frac{1}{a+b}\int (x+a)\, I_{\{-a<i<s<b\}}\, d\mu \geq 0$$

We also require these expressions to be monotone increasing in b, a, respectively. From the differences we can deduce

$$P[H_b<\infty,\ I(H_b)=-a],\quad P[H_{-a}<\infty,\ S(H_{-a})=b].$$

As notation, let $T \equiv S-I$, the final value of the range, and the last time the range increases. Lets fix $a, b \in h\mathbb{Z}^+$ for now, write $t=a+b$, and denote

$$p_+ = P[H_b<\infty,\ I(H_b)=-a] = P[M_t=b,\ T\geq t]$$

$$p_- = P[H_{-a}<\infty,\ S(H_{-a})=b] = P[M_t=-a,\ T\geq t]$$

Now set $F=\{S_t=b,\ I_t=-a\}$ and notice that $F\subseteq\{T\geq t\}$. We shall have

$$b p_+ - a p_- = E[M_t\,;F] = E[M_{t+h}\,;F] \qquad (*)$$

by the OST. Now on the event F, at time $t+h$ we have either $M_{t+h}=b+h$, or $M_{t+h}=-a-h$ or $M_{t+h}\in[-a,b]$, $T=t$. We'll write

$$q_+ = P[H_{b+h}<\infty,\ I(H_{b+h})=-a]$$

$$q_- = P[H_{-a-h}<\infty,\ S(H_{-a-h})=b]$$

for the probabilities of the first two events, but notice that if the first event $\{S_{t+h}=b+h>S_t,\ I_{t+h}=-a\}$ happens then it is possible that $M_t=-a$. I've not been able to use the basic ingredients to determine the probability of $\{S_{t+h}=b+h,\ M_t=-a,\ S_t=b\}$, but I'm not sure it matters...

Now we can re-express $(*)$ to obtain a further necessary condition

$$\boxed{\text{(C3)} \qquad b p_+ - a p_- = (b+h)q_+ - (a+h)q_- + E[X\,;S=b,\ I=-a]}$$

Now notice that

(1) $$P[S=b,\ I=-a] = p_+ + p_- - (q_+ + q_-)$$

so if we set

$$\xi \equiv E[\, X \mid S=b,\ I=-a\,]$$

then the relation (C3) tells us

(2) $$\boxed{(b-\xi)p_+ - (a+\xi)p_- = (b+h-\xi)q_+ - (a+h+\xi)q_-}$$

(5) <u>Construction?</u> Suppose we're at time t, $I_t = -a$, $S = b$. Now we'd like to be able to make a construction of the next step, which we could do <u>by putting a barrier at</u> ξ <u>with prob^y</u> π. If we do that, then we'd require

(C4) $$\boxed{q_+ = p_+\left\{\pi\,\frac{b-\xi}{b+h-\xi} + (1-\pi)\,\frac{b+a+h}{b+a+2h}\right\} + p_-(1-\pi)\,\frac{h}{b+a+2h}}$$

would be the expression relating the desired probability of $\{I_{t+h} = -a,\ S_{t+h} = b+h\}$ to the probability of this event using the construction. There's a similar equation for q_-, but it turns out to add nothing new to the equations.

Can we choose π to make (C4) hold, $0 \le \pi \le 1$? If we consider LHS − RHS for $\pi = 0$, we get

$$q_+ - \left\{p_+\,\frac{b+a+h}{b+a+2h} + p_-\,\frac{h}{b+a+2a}\right\} \le 0$$

which we see by thinking what would happen if instead of stopping X as planned on the event F we actually let the random walk continue to $\{-a-h,\ b+h\}$. The probability of exiting at $b+h$ is $(p_+(b+a+h) + p_-h)/(b+a+2h)$, which must be at least as large as the probability of exiting there if we might have stopped before the exit time, viz. q_+.

So now we want to consider what would happen if $\pi = 1$; we would like to have

$$q_+ - p_+\,\frac{b-\xi}{b+h-\xi} \ge 0$$

because then we would know there was a value of $\pi \in [0,1]$ such that the construction would work. If this inequality were <u>not</u> true, then we'd have (using (2)) that

$$q_+(b+h-\xi) - p_+(b-\xi) = q_-(a+h+\xi) - p_-(a+\xi) < 0$$

but this is not yet conclusive.

Indeed, it seems that the conjecture must be <u>false</u>: Suppose we were to run the RW without restriction until we reach a moment when $S = K$ and $I = -K$ (if we were to reach $|X| = K+1$ before this happens, then stop there) When we have $S = K$ and $I = -K$ by symmetry we must have $\xi = 0$; but if the rule is to stop when the RW has next moved by 1 then the desired inequality can't be satisfied!

(6) So the conclusion is that

<u>we must look for a characterization of the law of</u> (I, X, S, σ)

where $\sigma = +1$ if last strict ladder epoch prior to stopping was an upper ladder epoch
$= -1$ if ———————————— lower ————

Everything that could be deduced before can still be deduced, but now we can move a bit further. Take the event $F = \{ H_b < \infty,\ I(H_b) = -a \}$ where $a+b = t$, where $P(F) = p_+$ On this event, we start to count the martingale increments til we hit $b+h$ or $-a-h$. Then OST gives us

$$\cancel{b\, p_+ = (b+h)\, P[\, H_{-a} < H_b < H_{b+h} < H_{-a-h} \,]}$$

$$- (a+h)\ P[\, H_{-a} < H_b < H_{-a-h} < H_{b+h} \,]$$

$$+ E[\, X : S = b,\ I = -a,\ \sigma = +1 \,]$$

and the final term on the RHS, and the LHS, are both known. Moreover, we have

$$p_+ = P[\, H_{-a} < H_b < H_{b+h} < H_{-a-h} \,] + P[\, H_{-a} < H_b < H_{-a-h} < H_{b+h} \,]$$

$$\cancel{+ P[\, S = b,\ I = -a,\ \sigma = +1 \,]}$$

so from these two linear equations we can deduce the two unknowns

$$P[\, H_{-a} < H_b < H_{b+h} < H_{-a-h} \,] \text{ and } P[\, H_{-a} < H_b < H_{-a-h} < H_{b+h} \,]$$

Once we have this, we can do the barrier construction

Production and investment (17/3/12)

1) The idea here is to have a fairly standard growth model

$$Y_t = Z_t f(K_t) = I_t + C_t$$

$$dK_t = (I_t - \delta K_t)\,dt$$

with objective

$$V(z,k) = \sup E\left[\int_0^\infty e^{-\rho t}\, U(C_s)\,ds \;\Big|\; Z_0 = z,\ K_0 = k\right]$$

where $dZ = Z(\sigma dW + \mu dt)$ puts a bit of randomness into the model. We'll suppose

$$U(x) = \frac{x^{1-R}}{1-R}, \qquad f(k) = A k^\alpha$$

for some $R > 1$, $A > 0$, $\alpha \in (0,1]$. This way, we see that there is the scaling relation

$$V(\lambda^{1-\alpha} z, \lambda k) = \lambda^{1-R} V(z,k)$$

so we have $V(z,k) = k^{1-R}\, V(k^{\alpha-1} z, 1) \equiv U(k)\, h(x)$, where $x \equiv k^{\alpha-1} z$.
The HJB equation for this is $(c \equiv ks)$

$$0 = \sup_c\left[-\rho V + U(c) + \mu z V_z + \tfrac12 \sigma^2 z^2 V_{zz} + (z f(k) - c - \delta k) V_k\right]$$

$$= \sup_s U(k)\left[-\rho h + s^{1-R} + \mu x h' + \tfrac12 \sigma^2 x^2 h'' + (Ax - s - \delta)\big((1-R)h - (1-\alpha)x h'\big)\right]$$

which is optimised when we take

$$\boxed{s^{-R} = h + \frac{1-\alpha}{R-1}\, x h'}$$

Thus we shall have

$$\boxed{0 = -\rho h + \mu x h' + \tfrac12 \sigma^2 x^2 h'' - \big((R-1)h + (1-\alpha)x h'\big)(Ax - \delta) + R\left\{h + \frac{1-\alpha}{R-1}\, x h'\right\}^{(R-1)/R}}$$

Some simplification can happen. Set $b \equiv (R-1)/(1-\alpha)$ and write $h(x) = g(\log x) \equiv g(w)$, to find that

$$\boxed{0 = -\rho g + (\mu - \tfrac12\sigma^2) g' + \tfrac12 \sigma^2 g'' - (1-\alpha)(bg + g')(Ae^w - \delta) + R\, b^{(1-R)/R}\, (bg + g')^{(R-1)/R}}$$

We know that $h > 0$, and h is decreasing, since $R > 1$. Numerics by the crude approach does not appear to give this...

2) We can reduce to the case $\delta = 0$ if we set

$$\tilde{K}_t \equiv e^{\delta t} K_t, \quad \tilde{I}_t \equiv e^{\delta t} I_t, \quad \tilde{C}_t \equiv e^{\delta t} C_t$$

for then

$$\frac{d\tilde{K}}{dt} = \tilde{I}, \qquad e^{\delta t} Y_t = Z_t e^{(1-\alpha)\delta t} f(\tilde{K}_t) = \tilde{I}_t + \tilde{C}_t$$

so that we modify the dynamics of Z to

$$dZ_t = Z_t \left(\sigma\, dW_t + (\mu + (1-\alpha)\delta)\, dt \right)$$

and $U(c_t) = U(e^{-\delta t} \tilde{c}_t) = e^{\delta(R-1)t} U(\tilde{c}_t)$ so we have to replace ρ by

$$\boxed{\rho' = \rho - \delta(R-1)}$$

and we must insist that $\boxed{\rho' > 0}$. So we can and shall now assume $\delta = 0$, and write ρ for ρ', so the problem is like it was originally but with zero depreciation.

3) If we take this situation, we can consider what happens when $z = 0$ (that is, $x = 0$). There will never be any output, all that can happen is that we consume the capital:

$$\max \int_0^\infty e^{-\rho t} U(c_t)\, dt \quad \text{s.t.} \quad \int_0^\infty c_t\, dt = K_0$$

so we find optimal c is to take

$$c_t = K_0\, e^{-\rho t/R} \cdot (R/\rho)^{-1}$$

and we get objective

$$U(K_0) \int_0^\infty e^{-\rho t} \left(\frac{R}{\rho}\right)^{R-1} e^{-\rho(1-R)t/R}\, dt$$

$$= U(K_0) (R/\rho)^{R-1}\, R/\rho = U(K_0) (R/\rho)^R$$

so we conclude that

$$\boxed{h(0) = (R/\rho)^R}$$

4) Write $g(w) = e^{-bw} \varphi(w)$ and the equation at the bottom of the previous page reads

$$\boxed{\begin{aligned} 0 = \left[-\rho - b(\mu - \tfrac{1}{2}\sigma^2) + \tfrac{1}{2} b^2 \sigma^2 \right] \varphi + (\mu - \tfrac{1}{2}\sigma^2 - b\sigma^2)\varphi' + \tfrac{1}{2}\sigma^2 \varphi'' - (1-\alpha)\varphi'(Ae^{w} - \delta) \\ + R\, b^{(1-R)/R}\, e^{bw/R} (\varphi')^{(R-1)/R} \end{aligned}}$$

More on preferences with limited look-ahead (21/3/12)

This is probably all somewhere in the Ekeland - Lazrak - Pirvu - Björk - Murgoci stuff, but I worked it out for the book, so it's probably worth recording somewhere in written form before typing it up.

(1) The first thing is to solve the base case for later comparison. Here we do a fixed horizon problem with intermediate consumption and terminal value:

$$\max \; E\left[\int_0^T h(s)\, U(c_s)\, ds \;+\; A\, U(w_T)\right]$$

where U is the usual CRRA, single risky asset etc. If $V(t,w)$ is the value function, then

$$0 = \sup\left[V_t + (rw + \theta(\mu - r) - c)V_w + \tfrac12 \sigma^2 \theta^2 V_{ww} \;+\; h(t)\, U \right]$$

and we know from scaling that $V(t,w) = f(t)\, U(w)$, so we get $\quad (y \equiv \theta/w,\; q \equiv c/w)$

$$0 = \sup U(w)\left[\dot f + (r + y(\mu - r) - q)(1-R) f - \frac{R(1-R)}{2}\sigma^2 y^2 f + h q^{1-R} \right]$$

so we get $y = \pi_M$, $f = h q^{-R} \Rightarrow q = (h/f)^{1/R}$, and the overall story is

$$\boxed{0 = \dot f - (R-1)(r + \kappa^2/2R) f + R\, h^{1/R} f^{1-1/R}}$$

(2) Suppose now that the agent has rolling objective

$$E_t\left[\int_t^{t+T} u(s-t, c_s)\, ds \;+\; g(w_{t+T})\right]$$

where $u(s,c) = h(s)\, U(c)$, $g(w) = A\, U(w)$. The agent in a Nash equilibrium will consume at rate $a w_t$, and invest πw_t. If we set

$$\varphi(t,w) = \tilde E\left[\int_t^T h(s)\, U(c_s)\, ds \;+\; A\, U(w_T)\right]$$

for the value of using that strategy, what we find is that

$$w_t = w_0 \exp\{ \sigma \pi W_t - \tfrac12 \sigma^2 \pi^2 t \;+\; (r + \pi(\mu - r) - a)t \}$$

so $\displaystyle E w_t^{1-R} = w_0^{1-R} \exp\left[(1-R)\{ r + \pi(\mu-r) - a - \tfrac12 \sigma^2 \pi^2 \} t \;+\; \tfrac12 \sigma^2 \pi^2 t (1-R)^2 \right]$

$$= w_0^{1-R} \exp\left[t(1-R)\{ r + \pi(\mu - r) - \tfrac12 R \sigma^2 \pi^2 - a \} \right]$$

So if we have $\pi = \pi_M$ we shall find

$$E w_t^{1-R} = w_0^{1-R} \exp\left[-t(R-1)(r + \kappa^2/2R) \;+\; a(R-1)t \right] \equiv w_0^{1-R} e^{mt}, \text{ say.}$$

Therefore

$$\boxed{\varphi(t,w) = u(w)\left[\int_t^T e^{m(s-t)} h(s)\, a^{1-R}\, ds + A e^{m(T-t)}\right]}$$

What we have to do is suppose that this was the value function and look at the HJB equation at time 0. We need to know that the presumed constant strategy is actually optimal at that time:

$$\sup_{c,\theta}\left[\dot\varphi + (rw+\theta(\mu-r)-c)\varphi_w + \tfrac{1}{2}\sigma^2\theta^2\varphi_{ww} + h(0)u(c)\right] = 0$$

Note that

$$\dot\varphi(t,w) + m\,\varphi(t,w) = -u(w)\,a^{1-R}\,h(t)$$

For brevity, let's write

$$Q = \int_0^T e^{ms} h(s)\, a^{1-R}\, ds + A e^{mT}$$

So the HJB is

$$\sup_{c,\theta}\Big[-m\,Q\,u(w) - a^{1-R}u(w)h(0) + (rw+\theta(\mu-r)-c)\frac{(1-R)}{w}Q\,u(w) - \frac{R(1-R)}{2}\sigma^2\frac{\theta^2}{w^2}Q\,u(w) + h(0)u(c)\Big] = 0$$

$$= -mQ\,u(w) - a^{1-R}u(w)h(0) + r(1-R)Q\,u(w) + Q\,u(w)\frac{\kappa^2}{2R}(1-R) + u(w)\,R\,Q^{1-1/R}h_0^{1/R}$$

where

$$c = w\,(h_0/Q)^{1/R}$$

So we shall need

$$\boxed{0 = -mQ - a^{1-R}h(0) + r(1-R)Q + \frac{\kappa^2}{2R}(1-R)Q + R\,Q^{1-1/R}h_0^{1/R}}$$

with

$$\boxed{a = (h_0/Q)^{1/R}} \qquad (*)$$

If we have ↑, then the previous equation will be satisfied. So we have to adjust a to make (*) hold.

Multitime step optimization with transaction costs (23/3/12)

1) Matt Killeya has been studying some ways to deal with portfolio choice with transaction costs. The main issue is that while it is easy to do a one-step portfolio choice story, where we change only if the immediate improvement beats the cost of portfolio change, this could lead to us being trapped in a poor portfolio which we would certainly want to change if we looked ahead 10 periods, say.

But if we had a story where we chose to move a portfolio, on the assumption that returns are IID, we would move to the MV optimal & never again change... So in order to capture what we really think is going on, we need to incorporate possibility that the means change...

2) Suppose that return Y_t on day t is $Y_t = \mu_t + \varepsilon_t$, where $\varepsilon \sim N(0, \sigma_\varepsilon^2)$ and $\mu_{t+1} - \mu_t$ are IID $N(0, \sigma^2)$. If $(\mu_t | \mathcal{Y}_t) \sim N(\hat{\mu}_t, V_t)$ then we get the usual recursive story

$$\hat{\mu}_{t+1} = \hat{\mu}_t + \frac{V_t + \sigma^2}{V_t + \sigma^2 + \sigma_\varepsilon^2}(Y_{t+1} - \hat{\mu}_t)$$

$$= \beta \hat{\mu}_t + (1-\beta) Y_{t+1}$$

in steady state, where V has converged to its limit value

$$V_\infty = \frac{\sigma\sqrt{\sigma^2 + 4\sigma_\varepsilon^2} - \sigma^2}{2}, \quad \beta = \frac{2\sigma_\varepsilon^2}{\sigma\sqrt{\sigma^2+4\sigma_\varepsilon^2} + 2\sigma_\varepsilon^2}.$$

So conditional on $\mathcal{Y}_t$, $Y_{t+1} \sim N(\hat{\mu}_t, v)$ where $v \equiv \sigma_\varepsilon^2 + V_\infty$, and therefore

$$Y_{t+1} = \hat{\mu}_t + Z_{t+1}, \qquad \hat{\mu}_{t+1} = \hat{\mu}_t + (1-\beta) Z_{t+1}$$

so $\hat{\mu}_t$ is a zero-mean Gaussian random walk.

3) The change in wealth is

$$w_{t+1} - w_t = \theta_{t+1} \cdot Y_{t+1} - \varphi(\theta_{t+1} - \theta_t)$$

where $\varphi(\cdot)$ is the (convex) transaction cost function. We could consider the objective

$$\cancel{V(m,\theta) = \sup E\left[\sum_{t\geq 0} \rho^t U(w_t) \;\middle|\; \hat{\mu}_0 = m,\ \theta_0 = \theta\right]}$$

or $$\cancel{v(m,\theta) = \sup E\left[\sum_{t\geq 1} \rho^t U(w_t - w_{t-1}) \;\middle|\; \hat{\mu}_0 = m,\ \theta_0 = \theta\right]}$$

... neither looks good...

Kalman filtering: for the record (29/3/12)

This is a straightforward calculation which needs to be done every so often, but it's hard to find a good reference – the engineers' accounts tend to be weighed down with jargon and bad notation... Let's do a nice version to refer back to.

(1) Hidden process X evolves as

$$dX = dM + AX\,dt$$

where M is a cts martingale, $dM\,dM^T = \Sigma_{XX}\,dt$, with Σ a constant matrix. The observation process Y is either given in the form

$$Y_t = CX_t + N_t$$

or

$$dY_t = CX_t\,dt + dN_t$$

where N is a continuous constant-covariance martingale. If we had the first form, then $dY = CAX\,dt + (dN + C\,dM)$, so we can reduce to the second form. So let's just standardise to

$$\boxed{\begin{array}{ll} dX = dM + AX\,dt & dM\,dM^T = \Sigma_{XX}\,dt \\ dY = dN + CX\,dt & dM\,dN^T = \Sigma_{XY}\,dt \\ & dN\,dN^T = \Sigma_{YY}\,dt \end{array}}$$

(2) Introduce the innovations martingale ν by

$$d\nu = dY - C\hat{X}\,dt$$

and express $\hat{X}$ as

$$d\hat{X} = H\,d\nu + A\hat{X}\,dt$$

for some stochastic integrand H to be identified. Now we see

$$d(XY^T) = (XX^TC^T + AXY^T + \Sigma_{XY})\,dt + \mathcal{Y}\text{-mart}$$

$$d(\hat{X}Y^T) = (\hat{X}\hat{X}^TC^T + A\hat{X}Y^T + H\Sigma_{YY})\,dt + \mathcal{Y}\text{-mart}$$

and projecting the first onto $\mathcal{Y}_t$ gives the conclusion

$$\boxed{VC^T = H\Sigma_{YY} - \Sigma_{XY}}$$

To get hold of V, we know there will be no martingale part, so

$$d(XX^T) = (XX^TA^T + AXX^T + \Sigma_{XX})\,dt + \mathcal{Y}\text{-mart}$$

$$d(\hat{X}\hat{X}^T) = (\hat{X}\hat{X}^TA^T + A\hat{X}\hat{X}^T + H\Sigma_{YY}H^T)\,dt + \mathcal{Y}\text{-mart}$$

hence

$$\boxed{\begin{aligned}\frac{dV}{dt} &= VA^T + AV + \Sigma_{XX} - H\Sigma_{YY}H^T \\ &= VA^T + AV + \Sigma_{XX} - (VC^T + \Sigma_{XY})\Sigma_{YY}^{-1}(CV + \Sigma_{YX})\end{aligned}}$$

Optimal investment for a Bayesian agent (29/3/12)

Suppose we've got a single risky asset and

$$Y_t = \frac{1}{\sigma} \log S_t$$

has dynamics

$$dY_t = dW_t + \alpha_t \, dt$$

where α_t is itself moving as

$$d\alpha_t = \varepsilon \, dW'_t$$

for some $\varepsilon > 0$, and W' independent of W. Then using the results on the previous two pages, we have in terms of the innovations BM ν that

$$\begin{cases} dY = d\nu + \hat{\alpha} \, dt \\ d\hat{\alpha} = \varepsilon \, d\nu \end{cases}$$

so we have the joint dynamics

$$\begin{cases} dS = S\left(\sigma \, dY + \tfrac{1}{2}\sigma^2 dt \right) = S\left(\sigma \, d\nu + (\sigma\hat{\alpha} + \tfrac{1}{2}\sigma^2) dt \right) \\ d\hat{\alpha} = \varepsilon \, d\nu \end{cases}$$

For the standard infinite-horizon objective

$$V(w, a) = \sup E\left[\int_0^\infty e^{-\rho t} U(c_t) \, dt \;\Big|\; W_0 = w, \hat{\alpha}_0 = a \right]$$

$$= U(w) \, f(a)$$

by the usual scaling story. The HJB here is

$$0 = \sup_{c,\theta} \Big[-\rho V + U(c) + \{ rw + \theta(\sigma a + \tfrac{1}{2}\sigma^2 - r) - c \} V_w + \tfrac{1}{2}\sigma^2\theta^2 V_{ww}$$

$$+ \tfrac{1}{2}\varepsilon^2 V_{aa} + \theta\sigma\varepsilon V_{aw} \Big]$$

$x \equiv c/w, \quad q \equiv \theta/w$

$$= \sup U(w) \Big[-\rho f + x^{1-R} + (1-R)\{ r + q(\sigma a + \tfrac{1}{2}\sigma^2 - r) - x \} f - R(1-R) \frac{\sigma^2 q^2}{2} f$$

$$+ \tfrac{1}{2}\varepsilon^2 f'' + (1-R) q \sigma \varepsilon f' \Big]$$

which we can optimize in the usual way. We get

$$\boxed{x^{-R} = f, \quad \sigma^2 R q = \sigma a + \tfrac{1}{2}\sigma^2 - r + \sigma\varepsilon f'/f}$$

and the HJB becomes

$$0 = -\rho f + R f^{1-1/R} + r(1-R)f + \tfrac{1}{2}\varepsilon^2 f'' + \frac{(1-R)f}{2\sigma^2 R}\left(\sigma a + \tfrac{1}{2}\sigma^2 - r + \frac{\sigma\varepsilon f'}{f}\right)^2$$

When $\varepsilon = 0$, $\mu = \sigma a + \frac{1}{2}\sigma^2$, we find $f = \gamma_M^{-R}$ as we should, but otherwise it is different

Beating a benchmark (30/3/12)

1) In this story, with a conventional single risky asset log Brownian market, we consider the problem of a fund manager who takes initial wealth w_0 and is constrained to make $W_T \geq b\,\xi_T$, where $0 < b < 1$, and ξ is some benchmark process with the same starting value $\xi_0 = w_0$. So what he has to do is to split the time-0 wealth into the $b\xi_0$ required to generate the guarantee, and the $x_0 = (1-b)w_0$ which can be invested freely. So the problem is

$$\sup_{x_T \geq 0} E\, U(b\xi_T + x_T) \quad \text{subject to} \quad E[\zeta_T x_T] = (1-b)w_0$$

The first-order conditions from the Lagrangian formulation give us

$$U'(b\xi_T + x_T) - \lambda\zeta_T \leq 0$$

with equality where $x_T > 0$, and hence we deduce that

$$\boxed{x_T = \left(I(\lambda\zeta_T) - b\xi_T\right)^+}$$

2) Let's work through an example. Suppose $w_0 = 1$, $U'(x) = x^{-R}$, $\xi_T = \exp(\sigma W_T + (\mu - \frac{\sigma^2}{2})T)$ the stock. We have

$$I(\lambda\zeta_T) = \lambda^{-1/R} \exp\left(\frac{\kappa}{R} W_T + (r + \tfrac{1}{2}\kappa^2)T/R\right)$$

so that x_T will be positive if and only if

$$(\sigma - \tfrac{\kappa}{R})W_T < -\log(b\lambda^{1/R}) + (r + \tfrac{1}{2}\kappa^2)T/R - (\mu - \tfrac{\sigma^2}{2})T \equiv a - \tfrac{1}{R}\log\lambda$$

We can reasonably assume $\kappa/\sigma R \equiv \pi_M < 1$ here. How do we fix λ? By the budget constraint

$$1 - b = E\left[\zeta_T\left(I(\lambda\zeta_T) - b\xi_T\right)^+\right]$$

$$= E\left[\lambda^{-1/R}\zeta_T^{1-1/R} - b\zeta_T\xi_T \,:\, (\sigma - \kappa/R)W_T < a - \tfrac{1}{R}\log\lambda\right]$$

$$= \lambda^{-1/R}\int_{-\infty}^{c} \exp\left(-\frac{x^2}{2T} + (1 - \tfrac{1}{R})(-rT - \kappa x - \tfrac{1}{2}\kappa^2 T)\right)\frac{dx}{\sqrt{2\pi T}}$$

$$- b\int_{-\infty}^{c} \exp\left\{-\frac{x^2}{2T} + (\sigma - \kappa)x + \left(\mu - \frac{\sigma^2}{2} - r - \frac{\kappa^2}{2}\right)T\right\}\frac{dx}{\sqrt{2\pi T}}$$

$[c \equiv (a - \tfrac{1}{R}\log\lambda)/(\sigma - \kappa/R)]$

$$= \lambda^{-1/R} \exp\left(-\frac{R-1}{R}(r+\tfrac{1}{2}\kappa^2)T + \left(\frac{\kappa(R-1)}{R}\right)^2 \frac{T}{2}\right) \Phi\left(\frac{c + \kappa(R-1)T/R}{\sqrt{T}}\right)$$

$$- b \exp\left(\left(\mu - \frac{\sigma^2}{2} - r - \frac{\kappa^2}{2}\right)T + \tfrac{1}{2}(\sigma-\kappa)^2 T\right) \Phi\left(\frac{c' - (\sigma-\kappa)T}{\sqrt{T}}\right)$$

We could then use this to locate a value for λ.

3) How would the problem change if the manager promised that the gain in the fund would be at least $b \times$ gain in benchmark? This would need

$$w_T - w_0 \geq b(\xi_T - \xi_0),$$

Which is $\quad w_T \geq b\xi_T + (1-b)w_0 \equiv \underline{w}$, say. As before, we would find that

$$x_T = (I(\lambda\xi_T) - \underline{w})^+$$

So the condition for $x_T > 0$ will be ($w_0 = 1$ for simplicity)

$$\lambda^{-1/R} \exp\left[\frac{\kappa}{R} W_T + (r + \tfrac{1}{2}\kappa^2)\frac{T}{R}\right] > b \exp\left[\sigma W_T + (\mu - \tfrac{1}{2}\sigma^2)T\right] + (1-b)$$

This time there is a <u>finite interval</u> of W_T - values in which $x_T^* > 0$. For any λ, we would have to find this interval $[\underline{a}, \bar{a}]$ numerically, then integrate over it, and adjust λ to match budget constr.

Parisian options by excursion thy (20/4/12)

I was looking at this when Schröder visited this week, so it may be worth recording some of the things.

We have a BM B_t and we kill it at

$$\tau \equiv \inf \{ t : \quad B_s < 0 \quad \text{for all} \quad t - D \leq s \leq t \}$$

the first time the path has stayed continuously below 0 for duration $D > 0$. A parisian option pays $\varphi(B_T)$ at time T if $T < \tau$, zero else, where φ is a test function.

The obvious thing is to suppose $T \sim \exp(\lambda)$ independently of B, and try to calculate

$$E[\varphi(B_T) : T < \tau]$$

by excursion methods. Various useful formulae are recorded in VI.50 of R&W. The excursion entrance law has density

$$n_t(x) = n(\{f : \zeta(f) > t,\ f_t \in dx\}) / dx$$

$$= |x| \exp(-x^2/2t) / \sqrt{2\pi t^3}$$

and hence

$$n(\zeta > t) = 2/\sqrt{2\pi t}$$

and the excursion measure of marked excursions is $\sqrt{2\lambda}$. We observe the point process of excursions until the first one which contains a mark or spends more than D below 0.

$$n(f > 0,\ f \text{ is marked}) = \tfrac{1}{2}\sqrt{2\lambda}$$

$$n(f < 0,\ \tau < \zeta \wedge T) = \frac{1}{\sqrt{2\pi D}} \cdot e^{-\lambda D}$$

$$n(f < 0,\ T < \tau \wedge \zeta) = \int_0^D \lambda e^{-\lambda t} \left(\int_{-\infty}^0 \frac{|x|\, e^{-x^2/2t}}{\sqrt{2\pi t^3}}\, dx \right) dt$$

$$= \int_0^D \lambda e^{-\lambda t} \frac{dt}{\sqrt{2\pi t}}.$$

Adding these three gives the denominator:

$$Q \equiv \boxed{\frac{e^{-\lambda D}}{\sqrt{2\pi D}} + \sqrt{2\lambda}\, \Phi(\sqrt{2\lambda D})}$$

after some calculations (of course, $B_0 = 0$ is a maintained hypothesis)

Likewise, the numerator will be

$$\int_0^\infty \lambda e^{-\lambda t} \left(\int_0^\infty \frac{x e^{-x^2/2t}}{\sqrt{2\pi t^3}} \varphi(x)\, dx \right) dt + \int_0^D \lambda e^{-\lambda t} \left(\int_{-\infty}^0 \frac{|x| e^{-x^2/2t}}{\sqrt{2\pi t^3}} \varphi(x)\, dx \right) dt$$

$$= \int_0^\infty \varphi(x)\, \lambda e^{-x\sqrt{2\lambda}}\, dx + \int_0^D \lambda e^{-\lambda t} \int_{-\infty}^0 \frac{|x| e^{-x^2/2t}}{\sqrt{2\pi t^3}} \varphi(x)\, dx\, dt$$

[pp 20-21 of WN XIX (5/2/01) does this already ... !]

More realistic tax story (30/4/12)

(1) The simple tax story from WN XVI, p11, doesn't treat the business of tax credits properly. If at time 0 we have wealth w_0 and tax credit z_0, and we then trade to wealth w_{h-} at the end of the year before we have to pay tax, then the tax to be paid will be $(\Delta w \equiv w_{h-} - w_0)$

$$(\tau\Delta w - z_0)^+$$

leaving tax credit $(z_0 - \tau\Delta w)^+$, and wealth $w_h = w_{h-} - (\tau\Delta w - z_0)^+$. Suppose the objective is

$$\sup E\Big[\sum_{n\geq 0} \beta^n U(w_{nh}) \,\Big|\, w_0 = w,\ z_0 = z\Big] \equiv V(w,z)$$

with $\beta = e^{-\rho h}$. For intermediate form of the problem, set

$$V_N(w,z) = \sup E\Big[\sum_{n=0}^{N} \beta^n U(w_{nh}) \,\Big|\, w_0 = w,\ z_0 = z\Big]$$

and assume that $U'(x) = x^{-R}$. Then scaling gives $V_N(\lambda w, \lambda z) = \lambda^{1-R} V_N(w,z)$ so we conclude that

$$V_N(w,z) = U(w)\, f_N(z/w)$$

for some functions $f_N(\cdot)$, $f_0 = 1$. The Bellman equation is

$$V_N(w,z) = \sup\Big[U(w) + \beta\, E\big(V_{N-1}(w_h, z_h)\big) \,\Big|\, w_0 = w,\ z_0 = z\Big]$$

so

$$U(w) f_N(z/w) = \sup U(w)\Big[1 + \beta\, E\Big\{f_{N-1}(z_h/w_h)\Big(\frac{w_h}{w_0}\Big)^{1-R}\Big\} \,\Big|\, w_0 = w,\ z_0 = z\Big]$$

Dividing by $U(w)$, and standardizing to $w_0 = 1$, we get

$$f_N(z) = \sup/\inf\Big[1 + \beta\, E\Big(f_{N-1}(z_h/w_h)\, w_h^{1-R}\Big) \,\Big|\, z_0 = z\Big]$$

[sup/inf = sup if $0<R<1$; = inf if $R>1$] Note that $z_h > 0 \Rightarrow w_h = w_{h-}$. We use all the tax credit iff $w_{h-} \geq w_0 + z_0/\tau$, so we have

$$w_h = \min\{w_{h-},\ (1-\tau)w_{h-} + \tau w_0 + z\}$$

so inside the expectation is

$$f_{N-1}\left(\frac{(z - \tau\Delta w)^+}{w_{h-}}\right)\cdot\Big(\max\{w_{h-},\ (1-\tau)w_{h-} + \tau w_0 + z\}\Big)^{1-R}$$

(2) Things are not so bad.

Proposition f is concave increasing.

Proof Let's consider two possible start values (w_0', z_0') and (w_0'', z_0''), where $w_0' = 1 = w_0''$, and take some $\alpha \in (0,1)$. Suppose that $h=1$ for ease of exposition, and that the (optimal) gain generated in period is Δ_n', Δ_n'' proportionally, so that $w_{(n+1)-}' = (1+\Delta_n) w_n'$, and we get

$$\begin{cases} w_{n+1}' = (1+\Delta_n')w_n' - (\tau \Delta_n' w_n' - z_n')^+ \\ w_{n+1}'' = (1+\Delta_n'')w_n'' - (\tau \Delta_n'' w_n'' - z_n'')^+ \end{cases}$$

Notice that

$$w_{n+1}' + z_{n+1}' = w_n' + z_n' + (1-\tau)\Delta_n' w_n'$$

so that

$$w_{n+1}' + z_{n+1}' = \sum_{j=0}^{n} (1-\tau)\Delta_j' w_j'$$

Now suppose we define $\bar{w}_n \equiv \alpha w_n' + (1-\alpha) w_n''$, $\bar{z}_n \equiv \alpha z_n' + (1-\alpha) z_n''$, $\bar{\Delta}_n = \dfrac{\alpha w_n' \Delta_n' + (1-\alpha) w_n'' \Delta_n''}{\bar{w}_n}$

If we start from tax credit $\bar{z}_0 \equiv z_0$ and use the gains $\bar{\Delta}_n$, then $(\bar{w}_n, \bar{z}_n)$ are not the wealths and tax credits generated; these will be some other values (w_n, z_n). We certainly have

$$\left.\begin{aligned} \bar{w}_{n+1} + \bar{z}_{n+1} &= \sum_{j=0}^{n} (1-\tau)\bar{\Delta}_j \bar{w}_j \\ \text{and} \quad w_{n+1} + z_{n+1} &= \sum_{j=0}^{n} (1-\tau)\bar{\Delta}_j w_j \end{aligned}\right\} \quad (*)$$

I claim that for all n, $w_n \geq \bar{w}_n$ and $\bar{z}_n \geq z_n$. This is clearly true for $n=0$. We have

$$\begin{aligned} \bar{z}_{n+1} &= \alpha(z_n' - \tau\Delta_n' w_n')^+ + (1-\alpha)(z_n'' - \tau\Delta_n'' w_n'')^+ \\ &\geq (\bar{z}_n - \tau \bar{\Delta}_n \bar{w}_n)^+ \\ &\geq (\bar{z}_n - \tau\bar{\Delta}_n w_n)^+ \quad \text{if inductive hypothesis true} \\ &\geq (z_n - \tau\bar{\Delta}_n w_n)^+ \quad \text{"} \\ &= z_{n+1} \end{aligned}$$

Now from $(*)$ we learn that

$$w_{n+1} = \sum_0^n (1-\tau)\bar{\Delta}_j w_j - z_{n+1} \geq \sum_0^n (1-\tau)\bar{\Delta}_j \bar{w}_j - z_{n+1} = \bar{w}_{n+1} + \bar{z}_{n+1} - z_{n+1} \geq \bar{w}_{n+1}$$

as required. Thus the wealths w_n generated dominate the convex combinations $\bar{w}_n$, so the value is concave.

FBH driven by jumps (3/5/12)

(1) The original FBH story did not allow for bank failure because paths were continuous and if the bank was in danger it could alter interest rates to avoid failure. But let's now change the story by assuming $Z \equiv 1$, but the losses of capital come as actual jumps. Let's suppose that capital distress comes as $K_- \, dN$, where N is increasing, random jumps of size < 1, coming with intensity $\psi(q_t)$ as before. We now have to understand what happens when there is such an event; let's write $\lambda \equiv S_-/(S_- + D_-)$ for the proportion of the firm funded by equity. It seems to me the resolution goes as follows

(i) The distressed capital $K_- dN$ is placed in the hands of an administrator. The book value of this is $F \equiv p_- K_- dN$, and so the value of equity is marked down to $S_- - \lambda F$, value of debt marked down to $D_- - (1-\lambda) F$, no other changes;

(ii) The administrator sells the distressed capital, raising

$$P \equiv (1-\gamma) \tilde{p} \, K_- dN$$

from the sale, where $\tilde{p}$ is the price achieved for the capital (which could be different from p_-, p, ...) and γ is the proportion that has to be scrapped.

The required money P comes λP from households/equity holders, by withdrawals from x, and $(1-\lambda) P$ from new loans. So we now have

$$\text{debt} = D_- - (1-\lambda) F + (1-\lambda) P = D_- - (1-\lambda)(F - P)$$

$$\text{equity} = S_- - \lambda F + \lambda P = S_- - \lambda (F - P)$$

$$\text{retained deposits} = x_- - P$$

and the administrator holds P in cash

(iii) The administrator splits the loss $F - P$ according to the equity first rule

$$\text{loss to } S = \lambda F \wedge (F - P)$$

$$\text{loss to } B = (F - P - \lambda F)^+$$

Therefore S will get back

$$P_S = \lambda F - (\lambda F \wedge (F - P)) = (P - (1-\lambda) F)^+$$

B gets $\quad P_B = (1-\lambda) F - ((1-\lambda) F - P)^+ = P \wedge ((1-\lambda) F)$

and $P_S + P_B = P$. So the equityholders put P_S back into their bank account, so that retained deposits now stand at

$$x_- - P + P_S = x_- - P_B$$

and the bank sits on its P_B, so bank equity is $Q_- + P_B$

(iii) The total value of deposits is now

$$x_- - P_B + D_- - (1-\lambda)(F-P) = \Delta_- - P_B - (1-\lambda)(F-P)$$

Now the depositors have taken λP out and put P_S in, so if there were no losses, there would be

$$\Delta_- - \lambda P + P_S = \Delta_- + (1-\lambda)P - P_B$$

which is more than what debt + retained deposits actually amounts to: so the loss to depositors at this moment stands at $(1-\lambda)F$. We'll assume that bank equity makes good a fraction $\xi \in [0,1]$ of this loss, so that we end up with return deposits

$$= x_- - P_B + \xi(1-\lambda)F$$

and bank equity

$$= Q_- + P_B - \xi(1-\lambda)F$$

(iv) Finally, let's look at the capital account. There is now capital $K_- - \gamma K_- dN$ and if the price level is p we shall have

$$p K_-(1-\gamma dN) = D + S$$

$$= D_- + S_- - (F-P)$$

$$= p_- K_- - K_- dN \left[p_- - (1-\gamma)\tilde{p} \right]$$

whence

$$p(1-\gamma dN) = p_-(1-\gamma dN) - (1-\gamma)(p_- - \tilde{p})\, dN$$

so

$$\Delta p\,(1-\gamma dN) = (1-\gamma)(\tilde{p} - p_-)\, dN.$$

Now suppose the price level $\tilde{p}$ achieved at the forced sale was $\nu \Delta p + p_-$: Then

$$\nu = (1-\gamma dN)/(1-\gamma)dN$$

Which is huge if dN is small!! This appears to me to be absurd, so I would assume that

$$\boxed{p_- = \tilde{p} = p}$$

(v) With these assumptions, we can summarize how things stand after the defaults:

$$K = K_- - \gamma K_- dN$$
$$D = D_- - (1-\lambda)\gamma F \qquad [F \equiv pK_- dN]$$
$$S_- = S_- - \lambda\gamma F$$
$$x = x_- - P_B + \xi(1-\lambda)F = x_- - (1-(\gamma\vee\lambda) - \xi(1-\lambda))F$$
$$Q = Q_- + P_B - \xi(1-\lambda)F$$

This all makes sense, but is different from what we got for the earlier analysis!
Notice that $D_- = (1-\lambda)\, p_- K_- = (1-\lambda)\, pK_-$ so

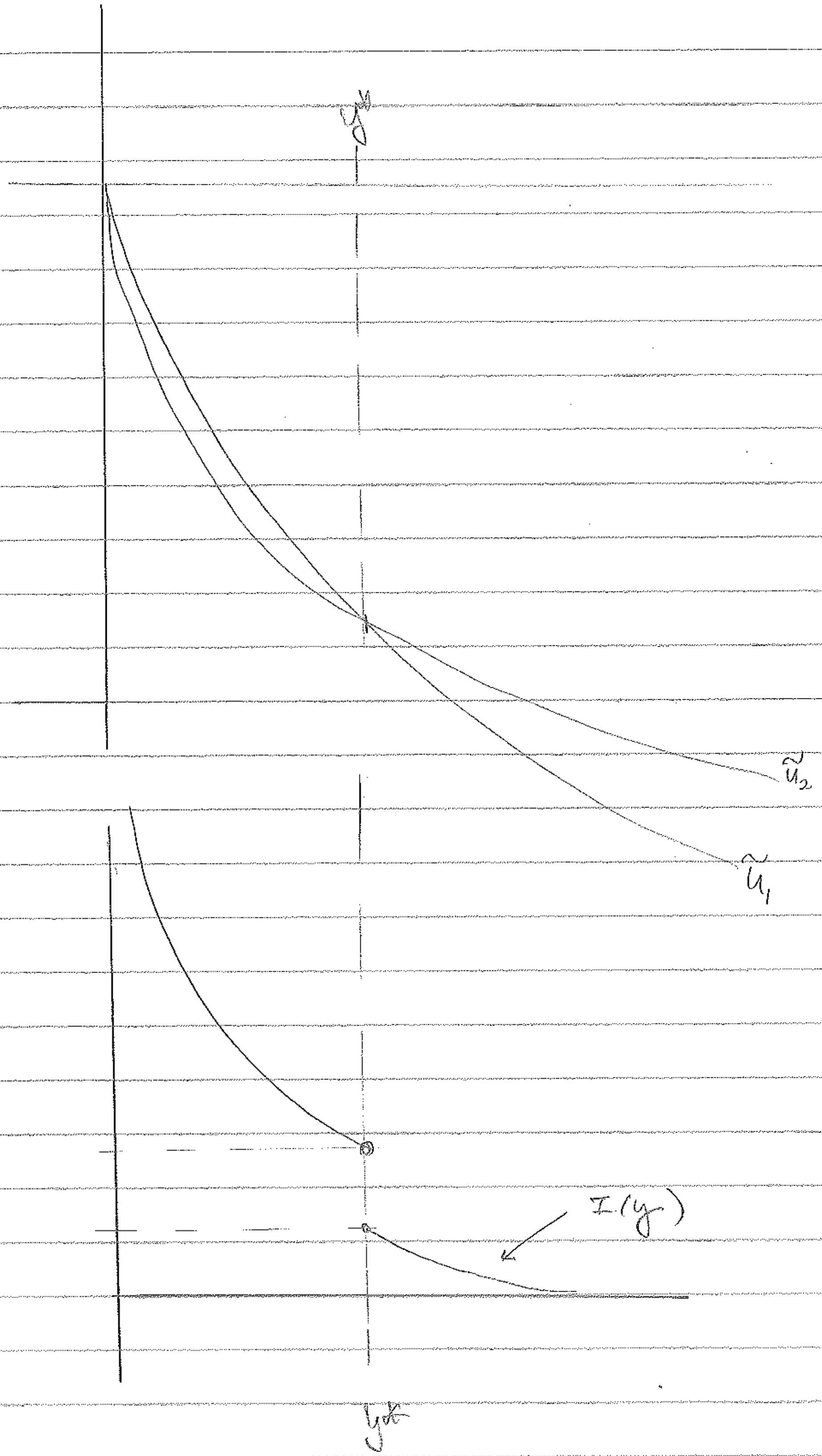
$$\cancel{D = (1-\lambda)(pK_- - \gamma F) = (1-\lambda)Kp}$$

If the jump causes one of the inequalities to break, then we choose dA so as to restore them, with the result that K falls by γdA, and $\tilde{D}$ falls by $(1-\gamma)\, dA$. One observation is that as D, S both drop by the same proportion as pK when there's a default, we won't find leverage violated then, though reserving or capital adequacy constraints might be.

The notational choices here look a bit suboptimal; let's change $\lambda' \equiv D/(S+D) \equiv 1-\lambda$, $\varepsilon \equiv 1-\xi$ which is the fractional loss imposed on depositors. Then

$$K = K_- - \gamma K_- dN$$
$$D = D_- - (1-\lambda)\gamma\, pK_- dN$$
$$S = S_- - \lambda\, \gamma p K_- dN$$
$$x = x_- - \{1-(\gamma\vee\lambda) - (1-\varepsilon)(1-\lambda)\}\, pK_- dN$$
$$= x_- - \{\varepsilon(1-\lambda) - (\gamma-\lambda)^+\}\, pK_- dN$$

$$Q = Q_- + \{\varepsilon(1-\lambda) - (\gamma-\lambda)^+\}\, pK_- dN$$


y^*
$\tilde{u}_2$
$\tilde{u}_1$
$I(y)$
y^*

Simplified hedge fund tale (4/5/12)

Moritz has been looking at a simpler version of the hedge fund manager's problem where there is no change in AUM, and the manager gets

$$X = \alpha(w_T - w_0)^+ + \beta w_T$$

at time 1 for some $\alpha, \beta > 0$. As usual, he wants to

$$\max E\, U(X)$$

with CRRA U, and we'll suppose the standard complete single-asset market. We have $U(X) = u(w_1)$ where

$$u(w) = u_1(w) \vee u_2(w)$$

with

$$u_1(w_T) = U((\alpha+\beta)w_T - \alpha w_0), \quad u_2(w_T) = U(\beta w_T)$$

where

$$\tilde{u}_1(y) = \tilde{U}\left(\frac{y}{\alpha+\beta}\right) - \frac{\alpha y w_0}{\alpha+\beta}, \quad \tilde{u}_2(y) = \tilde{U}(y/\beta)$$

so we have of course that

$$\tilde{u}(y) = \tilde{u}_1(y) \vee \tilde{u}_2(y)$$

At what value of y does the definition switch? If $U'(x) \equiv x^{-R}$, $R' \equiv 1/R$, we look to solve $\tilde{u}_1(y) = \tilde{u}_2(y)$, that is

$$\frac{1}{R'-1}\left(\frac{y}{\alpha+\beta}\right)^{1-R'} - \frac{\alpha y w_0}{\alpha+\beta} = \frac{1}{R'-1}\left(y/\beta\right)^{1-R'}$$

that is

$$\left(\frac{y}{\alpha+\beta}\right)^{-R'} - \frac{\alpha(R'-1)w_0}{\alpha+\beta} = \left(y/\beta\right)^{-R'}$$

so

$$y^{-R'}\left\{(\alpha+\beta)^{R'} - \beta^{R'}\right\} = \frac{\alpha(R'-1)w_0}{\alpha+\beta}$$

which determines the critical value y_* at which the two dual utilities cross.

As usual, the optimal terminal wealth is

$$w_T^* = I(\lambda \zeta_T)$$

for some $\lambda > 0$ chosen to match bcs. Let's take it a bit further. We have that

$\zeta_t = \exp(-\kappa W_T - aT)$ where $a \equiv r + \frac{1}{2}\kappa^2$, so $\lambda\zeta_T > y_*$ iff

$$\kappa W_T + aT < \log(\lambda/y_*)$$

$$\text{iff } W_T < \frac{1}{\kappa}\left\{\log(\lambda/y_*) - aT\right\} \equiv q, \text{ say}$$

In this region, $I(y) = -\tilde{u}_2'(y) = \beta^{-1}(y/\beta)^{-R'}$, whereas for smaller y we shall have

$$I(y) = -\tilde{u}_1'(y) = \frac{1}{\alpha+\beta}\left(y/(\alpha+\beta)\right)^{-R'} - \frac{\alpha W_0}{\alpha+\beta}$$

The budget constraint to fix λ is therefore

$$W_0 = E\left[\zeta_T\, I(\lambda\zeta_T)\right]$$

$$= E\left[\lambda^{-R'}\left(\frac{\zeta_T}{\beta}\right)^{1-R'} ; \lambda\zeta_T > y_*\right]$$

$$+ E\left[\lambda^{-R'}\left(\frac{\zeta_T}{\alpha+\beta}\right)^{1-R'} - \frac{\alpha W_0}{\alpha+\beta} ; \lambda\zeta_T < y_*\right]$$

We could in principle work out the value when we're at (t, W_t) from this, but it's quite a slog...

Wiener-Hopf again: an observation (22/5/12) [See also WN XXII p31]

(1) Coming back to the attempt to use WH factorization of Markov chains to attack the two-sided exit problem for Lévy processes, we can consider a Markov chain (irreducible) on the finite set $E_- \cup \{0\} \cup E_+ \equiv I$, and have $v(i) = -1$ ($i \in E_-$), $v(0)=0$, $v(i) = +1$ on E_+. The two parts of the statespace only communicate via 0, so the Q-matrix is

$$Q = \begin{array}{c} \left(\begin{array}{c|c|c} Q_- & q_- & 0 \\ \hline u_- & -q_0 - c & u_+ \\ \hline 0 & q_+ & Q_+ \end{array} \right) \end{array} \begin{array}{l} E_- \\ 0 \\ E_+ \end{array}$$

where $c \geq 0$ is a killing rate in 0, and $u_- 1 + u_+ 1 = q_0$. If we time-change out the time in 0, the Q-matrix becomes

$$\bar{Q} = \left(\begin{array}{c|c} Q_- + \dfrac{q_- u_-}{q_0 + c} & \dfrac{q_- u_+}{q_0 + c} \\ \hline \dfrac{q_+ u_-}{q_0 + c} & Q_+ + \dfrac{q_+ u_+}{q_0 + c} \end{array} \right) \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Now set $\varphi_t \equiv \int_0^t v(X_s)\,ds$, $A_t \equiv \int_0^t I_{\{X_s = 0\}}\,ds$, $\tau_t = \inf\{u : A_u > t\}$ and consider:

$$\exp\{i\theta\varphi_t - \alpha A_t\}\, h(X_t) \quad \text{is a martingale}$$

iff

$$Qh + (i\theta v - \alpha I_{\{0\}})h = 0$$

so to find α, h to solve this we get the system of linear equations:

$$\begin{pmatrix} Q_- - i\theta & q_- & 0 \\ u_- & -q_0 - c - \alpha & u_+ \\ 0 & q_+ & Q_+ + i\theta \end{pmatrix} \begin{pmatrix} h_- \\ h_0 \\ h_+ \end{pmatrix} = 0$$

So $h_- = -(Q_- - i\theta)^{-1} q_- h_0$, $h_+ = -(Q_+ + i\theta)^{-1} q_+ h_0$ and

$$u_- h_- - (q_0 + c + \alpha) h_0 + u_+ h_+ = 0.$$

Setting $c = 0$, we learn that this can only hold for non-zero h when α is the

Lévy exponent ; rearranging gives

$$\alpha = \boxed{\psi(i\theta) = -q_0 - u_-(Q_- - i\theta)^{-1}q_- - u_+(Q_+ + i\theta)^{-1}q_+ .}$$

2) In trying to find the WH factorization of the Markov chain, we have to look for (λ, f) such that

$$\exp(\lambda\varphi_t - cA_t)\, f(X_t) \quad \text{is a martingale,}$$

that is

$$Qf + (\lambda v - c I_{\{0\}})f = 0$$

which (replacing $i\theta$ by λ) is exactly the equation system we saw before. So what this says is that

$$\boxed{\lambda \text{ is an eigen-value iff } \quad c = \psi(\lambda)}$$

3) How would it look for the two-sided exit problem for a Lévy process exiting $[-a,b]$?
I want to find $E \exp(sZ_T)$ for $Re(s)=0$, T the first exit time of the Lévy process. If the additive functional φ exits at level b while $X_t = j \in E^+$, then we know that

$$E\left[\exp(sH_0) \mid X_0 = j\right] = \left((-s-Q^+)^{-1}q_+\right)_j$$

so for the Markov chain formulation we need to calculate ($\tau \equiv \inf\{t : \varphi_t \notin [-a,b]\}$)

$$E\left[\exp(s\varphi_\tau - cA_\tau)\, g(X_\tau)\right]$$

where

$$\begin{aligned} g(j) &= \left((-s-Q^+)^{-1}q_+\right)_j && \text{for } j \in E^+ \\ &= \left((s-Q^-)^{-1}q_-\right)_j && \text{for } j \in E^- \end{aligned}$$

Alternatively, if we set

$$f(j) = \begin{cases} e^{bs}\left((-s-Q^+)^{-1}q_+\right)_j & \text{for } j \in E^+ \\ e^{-as}\left((s-Q^-)^{-1}q_-\right)_j & \text{for } j \in E^- \end{cases}$$

then we are looking for $E\left[\exp(-cA_\tau) f(X_\tau)\right]$. Here's how. Set

$$h(x,j) = E\left[e^{-cA_\tau} f(X_\tau) \mid \varphi_0 = x, X_0 = j\right]$$

and write $h_\pm$ for restrictions of h to $E_\pm$, $h = [h_- ; h_+]$. What we have is

~~$h(\varphi_t, X_t)$ is a martingale, so by Ito we get~~

$$0 = Qh + V h_x \quad \Rightarrow \quad V^{-1}Qh + h_x = 0 \Rightarrow h(x,\cdot) = \exp(-xV^{-1}Q)\, w$$

for some w to be found. But we notice that if $S = \begin{pmatrix} I & \Pi^+ \\ \Pi^- & I \end{pmatrix}$ then

$$\exp(xV^{-1}Q) = S \begin{pmatrix} e^{-xG^-} & \cdot \\ \cdot & e^{xG^+} \end{pmatrix} S^{-1}$$

so we shall have

$$h(x,\cdot) = S \begin{pmatrix} e^{xG^-} & \cdot \\ \cdot & e^{-xG^+} \end{pmatrix} S^{-1} w$$

and

$$\begin{cases} h_- = \left(e^{xG^-}, \ \Pi^+ e^{-xG^+} \right) S^{-1} w \\ h_+ = \left(\Pi^- e^{xG^-}, \ e^{-xG^+} \right) S^{-1} w \end{cases}$$

Using the BCs at $x = -a, b$ gives

$$\left.\begin{aligned} f_+ &= \left(\Pi^- e^{bG^-}, \ e^{-bG^+} \right) S^{-1} w \\ f_- &= \left(e^{-aG^-}, \ \Pi^+ e^{aG^+} \right) S^{-1} w \end{aligned}\right\} \Rightarrow f = \begin{bmatrix} e^{-aG^-} & \Pi^+ e^{aG^+} \\ \Pi^- e^{bG^-} & e^{-bG^+} \end{bmatrix} S^{-1} w$$

This then gives us

$$h(0,\cdot) = w = S \begin{pmatrix} e^{-aG^-} & \Pi^+ e^{aG^+} \\ \Pi^- e^{bG^-} & e^{-bG^+} \end{pmatrix}^{-1} f$$

and we now need to mix over the initial law $(u_-, u_+)/(q_0 + c)$ for the final answer for $a, \theta > 0$,

$$\int_0^\infty \exp\left(-\frac{a^2}{2t} - \frac{\theta^2 t}{2}\right)\frac{dt}{\sqrt{2\pi t}} = \frac{e^{-a\theta}}{\theta}$$

Evolution of firm size (25/5/12)

After hearing Yannis' talk in Minneapolis, I come back to thinking about the stochastic portfolio stuff, and wonder whether the firm size story can be approached differently.

(1) Suppose you have a firm which starts off at size 1, and whose size goes like a GBM, $X_t = \exp(\sigma W_t + (\mu - \frac{1}{2}\sigma^2)t) \equiv \exp(\sigma W_t + ct)$ where we'll insist $c<0$; all firms eventually die. If for the moment we hold μ, σ fixed, and we generate new firms with these characteristics at rate α, what is the invariant / limiting distribution of log price? It would have a density

$$\alpha \int_0^\infty \exp\left(-(x-ct)^2/2\sigma^2 t\right) \frac{dt}{\sqrt{2\pi t \sigma^2}} = \begin{cases} \dfrac{\exp(2xc/\sigma^2)}{|c|} & (x>0) \\[2ex] \dfrac{1}{|c|} & (x<0) \end{cases}$$

according to Maple ... and me. This can be written $\exp\{2(x \wedge 0)/\sigma^2\}/|c|$ if we so wish ...

(2) If you now suppose that firms are being born with unit size according to the measure $m(d\mu, d\sigma)$, then if you look among the population of firms of log-size x, we see the distribution of (μ, σ) values proportional to the density

$$\frac{\exp\{2x(\mu - \frac{1}{2}\sigma^2)/\sigma^2\}}{|\mu - \frac{1}{2}\sigma^2|} \; m(d\mu, d\sigma)$$

for $x>0$

$$= \frac{\exp\{-2x|c|/\sigma^2\}}{|c|} \; m(d\mu, d\sigma)$$

and this is clearly decreasing with c.

(3) The restriction to $c<0$ is a bit unnatural from a modelling point of view. Let's suppose that there is independent killing of firms at rate $\varepsilon>0$, so that now the limiting dist^n of firms has a density

$$\int_0^\infty \exp\left(-\frac{(x-ct)^2}{2\sigma^2 t} - \varepsilon t\right) \frac{dt}{\sqrt{2\pi\sigma^2 t}} = \frac{\exp\left\{\dfrac{xc}{\sigma^2} - \dfrac{|x|}{\sigma^2}\sqrt{c^2+2\varepsilon\sigma^2}\right\}}{\sqrt{c^2+2\varepsilon\sigma^2}}$$

$$K_{\frac{1}{2}}(x) = K_{-\frac{1}{2}}(x) = e^{-x}\sqrt{\frac{\pi}{2x}}$$

Notice that

$$\frac{K_{-\frac{1}{2}}\left(\alpha\sqrt{\delta^2+x^2}\right)}{\left(\sqrt{\delta^2+x^2}/\alpha\right)^{\frac{1}{2}}} = e^{-\alpha\sqrt{\delta^2+x^2}}\sqrt{\frac{\pi}{2}} \cdot \frac{1}{\sqrt{\delta^2+x^2}}$$

We have for a generalised hyperbolic with parameters $\lambda \in \mathbb{R}$, $\alpha, \beta, \delta, \mu \in \mathbb{R}$, with $\gamma \equiv \sqrt{\alpha^2-\beta^2}$ the density is

$$\frac{(\gamma/\delta)^{\lambda}}{\sqrt{2\pi}\,K_{\lambda}(\delta\gamma)}\, e^{\beta(x-\mu)}\, \frac{K_{\lambda-\frac{1}{2}}\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)}{\left(\sqrt{\delta^2+(x-\mu)^2}/\alpha\right)^{\frac{1}{2}-\lambda}}$$

Take $\mu=0$, $\lambda=0$, and we're looking at a generalized hyperbolic law.

Notice that this is

$$\frac{\exp\left\{ \theta x - |x| \sqrt{\theta^2+2\tilde{\varepsilon}} \right\}}{\sigma^2 \sqrt{\theta^2+2\tilde{\varepsilon}}} \qquad (\tilde{\varepsilon} \equiv \varepsilon/\sigma^2)$$

where $\theta \equiv c/\sigma^2$. So if we roll the factor $\frac{1}{\sigma^2}$ into the measure m, what really matters is the distribution of θ under the resulting measure. Thus the distribution of the firm log-sizes in steady state will have a density proportional to

$$\int_{-\infty}^{\infty} \frac{\exp\left(\theta x - |x| \sqrt{\theta^2+2\varepsilon}\right)}{\sqrt{\theta^2+2\varepsilon}} F(d\theta)$$

for some distribution function F.

(4) There's the classic plot of log size against log rank from Fernholz' book which shows quite convincingly that

$$\log \text{size} \approx A - B \log \text{rank}$$

If we denote the distribution of firm size by G, then the approximate size of the rank-j observation will be $G^{-1}\left(\frac{N-j}{N}\right)$, so the relationship suggested by the Fernholz plot would then give

$$\log G^{-1}\left(\frac{N-j}{N}\right) \approx A - B \log(j/N)$$

$$\Rightarrow \quad G^{-1}(u) \approx \text{const.}\,(1-u)^{-B}$$

which would suggest $G(y) \approx 1-(y/A)^{-1/B}$, so the density $G'(y) \sim y^{-1-1/B}$.
The heuristic argument would lead to the conclusion that the tail of the log-size distribution is exponential approximately

Crossing the spread (29/5/12)

(1) This is a challenging question from Shamir & Wampin. Suppose we have time T in which to acquire a quantity Q of some stock. Let the number of our quotes at the bid at time t be z_t, the number of quotes from others be x_t, and let y_t denote the number of quotes at the ask. Let q_t be the remaining quantity we have to buy at time t. New quotes come in at rate λ, and disappear at rate $f(s)$, where s is the number of quotes sitting the other side of the spread, where f is increasing. The idea here is that if there were many quotes at the bid, then sellers would realise that many people wanted to buy, and would pull out their orders and reinsert them higher up. A random story would be the ideal, but it'll be hard enough just as a deterministic problem:

$$\dot{y} = \lambda - f(x+z)$$

$$\dot{x} = \lambda - \frac{x}{x+z} f(y)$$

[$f(s) = \mu + \varepsilon s$ would be a simple choice]

$$\dot{q} = -\frac{z}{x+z} f(y) - \frac{dA}{dt}$$

where A_t denotes the number of agressor trades done by time t. If you reach $y=0$ before T, then the market has moved away from you, and you have to buy at the next pip up, costing B per unfilled order. If you reach T before y hits 0, you have to cross the spread at cost $b < B$ per unfilled order, and each agressor trade costs you b; so the overall cost is

$$b\int_0^{T\wedge\tau} dA_s + b\, q_T\, I_{\{T<\tau\}} + B\, q_\tau\, I_{\{\tau<T\}}$$

where τ is the time y hits zero.

(2) Trying to solve for value function is not good - 1 time, 3 space variables... Pontryagin also looks rather hopeless, as you have three multiplier functions. Maybe the best one can hope to do is to generate randomly selected decreasing paths for q on some discrete-time grid, solve the first-order Euler scheme (if $|\Delta q| > f(y)\Delta t$ then we need to do agressor trades) and evaluate the outcomes... then pick the best. Or we could try optimizing over the Δq_i, though the discontinuous functional for the objective makes it v. unlikely this would work.

Equilibria in complete markets (3/6/12)

(i) Speaking to Hayne Leland, he has a paper with He on diffusion asset prices in a complete market equilibrium, assuming a constant riskless rate. The question makes sense in a complete market without the diffusion assumptions, so let's see what we can do.

(ii) Suppose the probability world is a one-dimensional Brownian probability space, with $(W_t)_{t\geq 0}$ the generating BM. Take a strictly positive $\mathcal{F}_T$-measurable Λ_T, $E\Lambda_T = 1$, to be interpreted as the state-price density (with no real loss of generality, set $r=0$), with representation of $\Lambda_t \equiv E(\Lambda_T | \mathcal{F}_t)$ as

$$d\Lambda_t = \Lambda_t(-\kappa_t)\,dW_t$$

where κ is the market price of risk process. What we should have for the asset price process S_t is that $\Lambda_t S_t$ is a martingale, and $S_T = I(\Lambda_T)$ for some positive decreasing function I, the inverse marginal utility. We have

$$dS_t = S_t(\sigma_t\,dW_t + \mu_t\,dt)$$

for some previsible processes σ, μ to be identified. Now we have

$$\Lambda_t S_t = E[\Lambda_T S_T | \mathcal{F}_t] = E[\Lambda_T I(\Lambda_T) | \mathcal{F}_t] = S_0 + \int_0^t H_u\,dW_u$$

for some previsible H, by the Brownian integral representation theorem. Therefore we know

$$d(\Lambda_t S_t) = \Lambda_t S_t(\sigma_t - \kappa_t)\,dW_t = H_t\,dW_t$$

so once we've found H we can express

$$\sigma = \kappa + H/\Lambda S$$

and $\mu = \kappa\sigma$, so we've got the dynamics of S. To summarize then, if we give ourselves:

(a) inverse marginal utility $I(\cdot)$

(b) $\Lambda_T \in L^1(\mathcal{F}_T)$, $\Lambda_T > 0$, $E\Lambda_T = 1$

(c) $r \equiv 0$

we are able to find the market price of risk process and the evolution of the market asset in equilibrium assuming we can solve for the Brownian integral representation.

(iii) However, this seems a slightly unnatural choice of starting point. From an economic standpoint, it would be more natural to suppose we are given U, and the terminal wealth S_T and work from that.

(a) Define $\Lambda_T = U'(S_T) / EU'(S_T)$ for the risk-neutral measure

$$\Lambda_t \equiv E[\Lambda_T | \mathcal{F}_t]$$

(b) Now the equilibrium price process ought to be S_t,

$$\Lambda_t S_t = E_t(\Lambda_T S_T) = E_t[S_T U'(S_T)]$$

In the semimartingale setting, if we can identify the stochastic integrands

$$d\Lambda_t = -\Lambda_t \kappa_t \, dW_t, \qquad d(\Lambda_t S_t) = \Lambda_t S_t \tilde{H}_t \, dW_t$$

then we know as before that $\tilde{H}_t = (\sigma_t - \kappa_t)$, so we can deduce μ_t, σ_t.

If we assumed that S was a diffusion

$$dS_t = S_t \{ \sigma(t, S_t) \, dW_t + \mu(t, S_t) \, dt \}$$

then $\Lambda_t = h(t, S_t)$ for some h such that $\mathcal{L} h = 0$, $h(T, \cdot) = U'(\cdot)$ and then

$$d(\Lambda S) = d(hS)$$

$$\doteq \left[S h_t + \mu(S h_s + h) + \tfrac{1}{2}\sigma^2 (S h_{ss} + 2 h_s) \right] dt$$

$$= (\mu h + \sigma^2 h_s) \, dt = 0$$

We notice

$$d\Lambda = -\kappa \Lambda \, dW = \sigma S h_s \, dW \quad \Rightarrow \quad \boxed{\kappa = -\sigma S h_s / h}$$

and so if we match notation with He + Leland, what they call f is our $-S h_s / h$.

I've checked that their PDE (7) does indeed follow from $\frac{1}{2}\sigma^2 S^2 h_{ss} + \mu S h_s + h_t = 0$, along with the condition that

$$-S h_s / h = \mu / \sigma^2$$

(for checking, it seems best to write $h(t, S) = \exp(v(t, x))$ where $x \equiv \log S$; this would make $f \equiv -S h_s / h = -v_x$)

However, the whole thing is rather freaky: supposing that terminal wealth is the terminal value of a diffusion is OK, but then insisting on the wealth process being that diffusion all along the path is just weird...

LAF in natural scale (6/6/12)

We've been looking at trying to approximate transition densities of diffusions by various transformations, so, for example, if we work with a regular one-dimensional diffusion X solving

$$dX_t = \sigma(X_t)\,dW_t + b(X_t)\,dt$$

then the scale function is $s'(x) = \exp\left(-\int^x \frac{2b}{\sigma^2}(v)\,dv\right)$, and the diffusion $Y_t = s(X_t)$ is a diffusion in natural scale solving the SDE

$$dY_t = (\sigma s')(X_t)\,dW_t \equiv h(Y_t)\,dW_t$$

say.

Now if we look at the action functional for Y, it is

$$\int_0^T \tfrac{1}{2}\left|\dot{y}_t / h(y_t)\right|^2 dt \equiv \int_0^T \psi(y_t, q_t)\,dt \qquad [q_t = \dot{y}_t]$$

The ODE to be satisfied along the least-action path is

$$0 = D_y\psi - q\,D_{yq}\psi - \dot{q}\,D_{qq}\psi$$

$$= -\frac{q^2 h'(y)}{h(y)^3} + \frac{2q^2 h'(y)}{h(y)^3} - \frac{\dot{q}}{h(y)^2}$$

implying $\quad h(y)\,\ddot{y} = \dot{y}^2\,h'(y)$

whence

$$\frac{d}{dt}(\log \dot{y}) = \frac{d}{dt}\left(\log h(y)\right) \quad \Rightarrow \quad \boxed{\dot{y}_t = A\,h(y_t)}$$

If we write $y_t = s(x_t)$ we learn that x_t should satisfy

$$\boxed{\dot{x}_t = A\,\sigma(x_t)}$$

for some constant A and initial condition x_0 to be chosen to match the BCs at $0, T$.

Wiener-Hopf again (13/6/12)

(i) As part of the project to get hold of the WH factors of a Lévy process using the old BRW story of factorizing the MC, let's look at the other end of it;

$$\frac{c}{c-\psi(i\theta)} = E\, e^{i\theta \bar{Z}(\tau)} \; E\, e^{i\theta \underline{Z}(\tau)}$$

where the Spitzer-Rogozin identity identifies the R/L WH factors:

$$E\, e^{i\theta \bar{Z}(\tau)} = \exp \int_0^\infty \frac{e^{-ct}}{t} \left(\int_0^\infty (e^{sx}-1) P(X_t \in dx) \right) dt \qquad (s = i\theta)$$

How can this be expressed in terms of $\psi(\cdot)$??

(ii) Let's focus on the inner integral, or something very close to it,

$$I_t(z) \equiv \int_0^\infty e^{-zx} (e^{sx}-1)\, P(X_t \in dx)$$

where $\mathrm{Re}(z) > 0$. We can evaluate this by Plancherel's identity as follows. If we set $g(x) \equiv e^{-zx}(e^{sx}-1)$ then the Fourier transform of g is

$$\hat{g}(\theta) = \int_0^\infty e^{i\theta x - zx} (e^{sx}-1)\, dx = \frac{1}{z-s-i\theta} - \frac{1}{z-i\theta}$$

Therefore

$$I_t(z) = \int \hat{g}(\theta)\, e^{t\psi(-i\theta)} \frac{d\theta}{2\pi}$$

$$= \int \left[\frac{1}{z+i\theta-s} - \frac{1}{z+i\theta} \right] e^{t\psi(i\theta)} \frac{d\theta}{2\pi}$$

From this we conclude that

$$\int_0^\infty e^{-ct} I_t(z)\, dt = \int \left\{ \frac{1}{z+i\theta-s} - \frac{1}{z+i\theta} \right\} \frac{1}{c-\psi(i\theta)} \frac{d\theta}{2\pi}$$

and if we now integrate w.r.t. c from some value out to some large N, we have

$$\int_0^\infty \frac{e^{-ct} - e^{-Nt}}{t} I_t(z)\, dt = \int \left\{ \frac{1}{z+i\theta-s} - \frac{1}{z+i\theta} \right\} \log \left\{ \frac{N-\psi(i\theta)}{c-\psi(i\theta)} \right\} \frac{d\theta}{2\pi}$$

The next observation is that for any z_1, z_2 with positive real part,

$$\int \left(\frac{1}{z_1+i\theta} - \frac{1}{z_2+i\theta} \right) \frac{d\theta}{2\pi} = 0$$

as can be checked, so we can find that letting $N \to \infty$ gives us

$$(I - uv)^{-1} = I + \frac{uv}{1 - vu}$$

$$(M - uv)^{-1} = M^{-1}\left(I + \frac{uv\,M^{-1}}{1 - v \cdot M^{-1} u}\right)$$

$$\int_0^\infty \frac{e^{-ct}}{t} I_t(z)\, dt = \int \left(\frac{1}{z+i\theta-s} - \frac{1}{z+i\theta} \right) \log\left\{ \frac{c}{c-\psi(i\theta)} \right\} \frac{d\theta}{2\pi}$$

Notice that for any $\theta \in \mathbb{R}$, $Re\left(c-\psi(i\theta)\right) = c + \frac{\sigma^2\theta^2}{2} + \int (1-\cos\theta x)\,\mu(dx) \geq c$.

We would now need to let $z \downarrow 0$, $z \in (0,\infty)$, to get the log of the right WH factor. This is going to involve a Cauchy principal value, not so very tractable...

(iii) How would it be using the matrix WH approach? I think we need to slightly modify the approach used on pp 31-33 by setting $v(0) = \varepsilon > 0$, which doesn't change the validity of the approximation method, but does avoid the two time changes. We have

$$V^{-1}Q = \left(\begin{array}{c|cc} -Q_- & -q_- & 0 \\ \hline \varepsilon^{-1}u_- & -\frac{q_0+c}{\varepsilon} & \varepsilon^{-1}u_+ \\ 0 & q_+ & Q_+ \end{array} \right) = \begin{pmatrix} -A & -B \\ C & D \end{pmatrix}$$

For notational simplicity, set $\alpha \equiv \varepsilon^{-1}$, and notice for $\lambda > 0$ we have

$$u_+ (\lambda - Q_+)^{-1} q_+ = q_0 \, h_+(\lambda)$$

where $h_+(\lambda)$ is the Laplace transform of the duration of excursions into $E_+ \equiv \{1,\dots,n\}$ for the chain; equivalently, the Laplace transform of the upward jumps. Notice that $h_+(0) = u_+1/q_0$ which is the prob'y an excursion away from 0 went into E_+.

Therefore

$$(\lambda-D)^{-1} = \begin{bmatrix} \varepsilon/(\varepsilon\lambda + q_0 + c - q_0 h_+(\lambda)), & \dfrac{u_+(\lambda-Q_+)^{-1}}{\varepsilon\lambda + q_0 + c - q_0 h_+(\lambda)} \\[2ex] \dfrac{\varepsilon(\lambda-Q_+)^{-1} q_+}{\varepsilon\lambda + q_0 + c - q_0 h_+(\lambda)}, & (\lambda-Q_+)^{-1}\left\{ I + \dfrac{q_+(\lambda-Q_+)^{-1}u_+}{1-u_+(\lambda-Q_+)^{-1}q_+} \right\} \end{bmatrix}$$

We therefore have

$$(\lambda - V^{-1}Q) = \begin{bmatrix} (\lambda + A + B(\lambda-D)^{-1}C)^{-1} & (\lambda+A+B(\lambda-D)^{-1}C)^{-1}B(\lambda-D)^{-1} \\[2ex] +(\lambda-D+C(\lambda+A)^{-1}B)^{-1}C(\lambda+A)^{-1} & (\lambda-D+C(\lambda+A)^{-1}B)^{-1} \end{bmatrix}$$

Notice

$$B(\lambda-D)^{-1}C = q_- \, e_0^T(\lambda-D)^{-1}e_0 \, \alpha u_- = \frac{q_- u_-}{\varepsilon\lambda + q_0 + c - q_0 h_+(\lambda)}$$

$$C(\lambda+A)^{-1}B = \alpha e_0 u_-(\lambda+A)^{-1}q_- e_0^T = -\alpha q_0 h_-(-\lambda)\, e_0 e_0^T$$

where we define $q_0 h_-(s) \equiv u_-(s-Q_-)^{-1}q_-$. Hence

$$\left(\lambda-D+C(\lambda+A)^{-1}B\right)^{-1} = \left(\lambda-D-\alpha q_0 h_-(-\lambda)e_0e_0^T\right)^{-1}$$

$$= (\lambda-D)^{-1}\left\{ I + \frac{\alpha q_0 h_-(-\lambda)\, e_0 e_0^T(\lambda-D)^{-1}}{1-\alpha q_0 h_-(-\lambda)\, e_0^T(\lambda-D)^{-1}e_0} \right\}$$

$$= (\lambda-D)^{-1}\left\{ I + \frac{q_0 h_-(-\lambda)}{\varepsilon\lambda+q_0+c-q_0h_+(-\lambda)-q_0h_-(-\lambda)}\, e_0\,(1, \alpha u_+(\lambda-Q_+)^{-1}) \right\}$$

and

$$\left(\lambda+A+B(\lambda-D)^{-1}C\right)^{-1} = \left(\lambda+A+\alpha q_- e_0^T(\lambda-D)^{-1}e_0 u_-\right)^{-1}$$

$$= \left(\lambda+A+\frac{q_-u_-}{\varepsilon\lambda+q_0+c-q_0h_+(-\lambda)}\right)^{-1}$$

$$= (\lambda+A)^{-1}\left\{ I - \frac{q_-u_-(\lambda+A)^{-1}}{\varepsilon\lambda+q_0+c-q_0h_+(-\lambda)+u_-(\lambda+A)^{-1}q_-} \right\}$$

$$= (\lambda+A)^{-1}\left\{ I - \frac{q_-u_-(\lambda+A)^{-1}}{\varepsilon\lambda+q_0+c-q_0h_+(-\lambda)-q_0h_-(-\lambda)} \right\}$$

This could be developed further, but let's just see where it's all heading. We have

$$E e^{-\lambda\bar{Z}(\tau)} = \int_0^\infty ce^{-ct}E(e^{-\lambda\bar{Z}_t})\,dt$$

$$= \int_0^\infty ce^{-ct}P(\xi>\bar{Z}_t)\,dt \qquad \xi\sim\exp(\lambda)$$

$$= 1-\int_0^\infty ce^{-ct}P(\xi<\bar{Z}_t)\,dt$$

$$= 1-P(\varphi \text{ reaches } \xi \text{ before killing})$$

$$= 1-e_0^T\int_0^\infty \lambda e^{-\lambda x}\exp(xG_+)1\,dx$$

$$= 1-\lambda e_0^T(\lambda-G_+)^{-1}1$$

Now from the matrix WH factorization we have

$$S^{-1}(\lambda - V^{-1}Q)^{-1}S = \begin{pmatrix} (\lambda+G_-)^{-1} & \cdot \\ \cdot & (\lambda-G_+)^{-1} \end{pmatrix}$$

So we will understand the right WH factor of the Lévy process if we understand

$$e_0^T \; S^{-1}(\lambda - V^{-1}Q)^{-1} S 1 = e_0^T \; S^{-1}(\lambda - V^{-1}Q)^{-1} S \, I_{E_+} = e_0^T (\lambda - G_+)^{-1} 1$$

The $(\lambda - V^{-1}Q)^{-1}$ we understand reasonably well, but what do we know of $S1$, SI_{E_+}, $e_0^T S^{-1}$??
This looks more problematic as it appears to need knowledge of π_+, π_-.

Numerical solution methods for HJB? (17/6/12)

(i) A fairly general form in which the HJB equation appears is as

$$(1) \qquad \sup_\theta \Phi(x, Df, \theta) = 0 \qquad \forall x$$

where we write Df for the vector of f and all its derivatives up to order 2 at the point x. Let's suppose that $\Phi(x, Df, \cdot)$ is always strictly concave with a unique maximizer, and let's also suppose that $\Phi(x, \cdot, \theta)$ is linear. In that case, if we write

$$\bar{\Phi}(x, Df) = \sup_\theta \Phi(x, Df, \theta)$$

then evidently $\bar{\Phi}(x, \cdot)$ is convex, being the supremum of a family of linear functions.

(ii) If we now try to solve numerically, setting down a discrete set $x_1, \dots, x_N$ of x-values where we want to find $f(x_i) \equiv z_i$, we do a finite-difference approximation to the derivatives of f and so find the discretized form of (1)

$$\sup_\theta \Phi(x_i, z, \theta) = 0 \quad \text{for all } i$$

where now we formally include the values of the function at all points, even though for each i only a few of these values feature in the equation. Once again we may take sup over θ and, writing $\psi_i(z) \equiv \sup_\theta \Phi(x_i, z, \theta)$ we have

$$(2) \qquad \psi_i(z) = 0 \qquad \forall i$$

where each ψ_i is convex, and, we assume, smooth. Now this is a system of N non-linear equations in N unknowns. Such a system does not have to have any solution, or a unique solution (for example, if $N=2$, $\psi_1(z) = \frac{1}{2}|z|^2 - 1$, and $\psi_2(z) = \frac{1}{2}|z-a|^2 - 1$, the zero sets are two circles, which can intersect in 0, 1, or 2 places). However, in the situations we consider, there ought to be a solution, since we are supposing the HJB equation is well posed. Uniqueness will in general be an issue because in the case of no freedom to control we would just be solving a second-order linear PDE which typically has multiple solutions and we will need to impose boundary conditions to nail things down.

(iii) Solving (2) can be done by Newton's method. If we have an approximation $z^{(n)}$ to the root, we can try moving to $z^{(n)} + \eta$ where

$$\psi_i(z^{(n)} + \eta) \approx \psi_i(z^{(n)}) + D\psi_i(z^{(n)}) \cdot \eta$$

which would tell us to do

$$\eta = -\left(D\psi(z^{(n)})\right)^{-1}\psi(z^{(n)}).$$

Since each of the functions ψ_i is convex, and Newton's method approximates the functions by the tangents, then looks for a zero of those linear approximations, it follows that $\psi_i(z^{(n+1)}) \geq 0 \;\forall i$, since ψ_i lies everywhere above each of its tangents.

The Newton method is known to converge extremely rapidly once you get close enough to a root, but can struggle to get close enough to any root. In this tale, we could always do a reduced Newton method, where we just use a step $\beta\eta$ for some $\beta \in (0,1)$, or we could stick in some steps of policy improvement if it helped.

Numerics for utilities bounded below (20/6/12)

We have the HJB equation

$$0 = \sup_{\theta, c} \Big[-\rho V + U(c) + \{ rw + \theta(\mu - r) - c \} V' + \tfrac{1}{2}\sigma^2\theta^2 V'' - G(w,\theta)(V+\kappa) \Big]$$

$$\text{with} \qquad G(w,\theta) = I_{\{w<0\}} \left(\tfrac{1}{2}\sigma^2 \nu^2 \theta^2 + b w^2 \right)$$

$$= -\rho V + \tilde{U}(V') + rwV' - \tfrac{1}{2}\kappa^2 \frac{V'^2}{V'' - I_{\{w<0\}}\nu^2(V+\kappa)} - I_{\{w<0\}} b w^2 (V+\kappa)$$

$$\equiv \Phi(w, DV).$$

Now let us write $A \equiv V'' - I_{\{w<0\}} \nu^2 (V+\kappa)$ and consider the first-order change in Φ when we perturb V to $V+\eta$. We see

$$\Delta\Phi = -\rho\eta + \tilde{U}'(V')\eta' + rw\eta' - \frac{\kappa^2 V'\eta'}{A} + \tfrac{1}{2}\kappa^2 \frac{V'^2(\eta'' - \nu^2 I_{\{w<0\}}\eta)}{A^2} - I_{\{w<0\}} b w^2 \eta$$

$$= -\left(\rho + bw^2 I_{\{w<0\}} + \frac{\kappa^2 V'^2 \nu^2}{2A^2} \right)\eta + \left(\tilde{U}'(V') + rw - \frac{\kappa^2 V'}{A} \right)\eta' + \frac{\kappa^2 V'^2}{2A^2}\eta''$$

and the Newton method leads us to the following equation for η:

$$\Phi(w, DV) + \Delta\Phi = 0$$

Unfortunately, this really doesn't seem to work so well... the old policy improvement play runs better!

Remarks on a couple of papers of Hayne Leland (22/6/12)

1) There's a nice preprint 'Options and Expectations' around the theme 'Who should buy options?' The framework is a single-asset complete log Brownian/binomial market, so as usual in equilibrium the 'market' agent would hold the market portfolio, so

$$\zeta_t = U'(w_t^M)$$

where U is the assumed common utility of all agents. Some agents would want to buy/sell options - why? The explanation in this paper is that agents have different beliefs, so agent j with LR martingale Λ_t^j would want

$$\Lambda_T^j \, U'(w_T^j) \propto \zeta_T$$

and some examples are given of how that works out.

2) The paper 'On dynamic investment strategies' with J. C. Cox (JEDC 24, 2000, 1859-1880) again considers a complete Black-Scholes market with one risky asset, all coefficients constant. The questions addressed are

(i) Suppose we propose portfolio θ and consumption rate c which are functions of time and the current price of the asset only; what do they look like?

(ii) Suppose the portfolio is a function of time and current wealth only; what is possible? (We want path independent wealth process)

(iii) When do we have in (ii) that we're seeing the optimal wealth for some agent?

The techniques used are to cast the problem in terms of a binomial tree model, then pass to limits.

Let's simplify by setting $r \equiv 0$ (this loses no generality but makes the expressions simpler) and let's put ourselves in a multi-dimensional setting, which is notationally slicker, and should reveal when results won't generalise.

The returns processes are

$$dX_t^j = \sigma_{jk}\, dW_t^k = \sigma_{jk}\, d\tilde{W}_t^k + \mu_j\, dt \quad , \quad (j=1,\dots,n)$$

(W is a BM in the risk-neutral probability), and any function of the asset prices and time will be a function of X and time also.

(i) Suppose we hold $\theta_i(t, X_t)$ in asset i, $(i=0,1,\dots,n)$ at time t, where asset 0 is cash. Suppose we consume at rate $c(t, X_t)$; when do we have that this is a self-financing trading strategy?

We shall have

$$(1)\quad \begin{cases} W_t = \sum_{i=0}^{n} \theta_i(t, X_t) \equiv \varphi(t, X_t), \text{ say} \\ dW_t = \sum_{i=1}^{n} \theta_i(t, X_t)\, dX_t^i - C(t, X_t)\, dt \end{cases}$$

If $\mathcal{L} \equiv \frac{1}{2} a_{jk} D_j D_k + \frac{\partial}{\partial t}$ is the generator of X, we shall have the conditions (which are N+S)

$$\begin{cases} (2a) & \mathcal{L}\varphi + C = 0 \\ (2b) & \theta_j = D_j \varphi \qquad (j = 1, \dots, n) \end{cases}$$

Since $\mathcal{L}$ and D_j commute, for $j = 1, \dots, n$ we get

$$(3)\qquad \mathcal{L}\theta_j = \mathcal{L} D_j \varphi = D_j \mathcal{L}\varphi = -D_j C$$

[The situation $n = 1$ is the one considered in Cox + Leland. Our (3) is their (1), our (2b) is their (3), and our (2a) is their (1) + (2)]

So put simply, if you want to generate a wealth process $W_t = \varphi(t, X_t)$, you have to use portfolios $\theta_j = D_j \varphi$, and then withdraw cash at rate $-\mathcal{L}\varphi$ ... simple!

(ii) The second question supposes that again $W_t = \varphi(t, X_t)$, but this time we have portfolio process $\theta_i(t, W_t)$. Therefore

$$(4)\quad \begin{cases} W_t = \sum_{i=0}^{n} \theta_i(t, W_t) = \varphi(t, X_t) \\ dW_t = \sum_{i=1}^{n} \theta_i(t, W_t)\, \sigma_{ik}\, dW^k - C(t, W_t)\, dt. \end{cases}$$

Hence we have as before

$$\begin{cases} (5a) & \mathcal{L}\varphi + C = 0 \\ (5b) & \theta_j = D_j \varphi. \end{cases}$$

Note
$$D_{jk}\varphi = D_k \theta_j = \frac{\partial}{\partial x_k} \theta_j = \theta_j'(t, w) \frac{\partial w}{\partial x_k} = \theta_j'(t, w) \frac{\partial \varphi}{\partial x_k} = \theta_k \theta_j'$$

Now as $D_{jk}\varphi = D_{kj}\varphi$, this implies that $\dfrac{\theta_j'}{\theta_j} = \dfrac{\theta_k'}{\theta_k} \quad \forall j, k = 1, \dots, n$

and so

$$\boxed{\theta_j(t,w) = \alpha_j(t)\, f(t,w)} \qquad \text{for } j = 1,\dots,n$$

for some functions $\alpha_j(\cdot)$ of t alone, and a common f. Now apply D_i to (5a) and learn

$$\begin{aligned}
0 &= D_i \mathcal{L}\varphi + D_i C \\
&= \mathcal{L} D_i\varphi + D_i C \\
&= \mathcal{L}\,\theta_i + D_i C \\
&= \dot\theta_i + \tfrac{1}{2} a_{jk} D_j(D_k \theta_i) + D_i C \\
&= \dot\theta_i + \tfrac{1}{2} a_{jk} D_j(\theta_i' \theta_k) + D_i C \\
&= \dot\theta_i + \tfrac{1}{2} a_{jk}\, \theta_j(\theta_i''\theta_k + \theta_i'\theta_k') + C'\theta_i \\
&= \dot\alpha_i f + \alpha_i \dot f + \frac{\alpha_i}{2}(\alpha_j a_{jk}\alpha_k)(ff'' + f'^2) + \alpha_i f C'
\end{aligned}$$

This tells us that $\dot\alpha_i(t)/\alpha_i(t)$ must be the same for all i so that apart from a multiplicative constant, the α_j are all the same. Absorbing the dependence on t into $f(t,w)$ we may wlog assume $\alpha_j(t) = \alpha_j(0)$ for all t, that is, the α_i are constants. By renormalising, we may also assume that

$$\sum_{j=1}^{n} \alpha_j(0) = 1.$$

Notice that we can conclude: $\nabla\varphi(t,X) \propto \alpha$, so it follows that

$$\boxed{\varphi(t,X) = g(t,\alpha\cdot X)}$$

which reduces everything to the one-dimensional situation. We have (using (5b))

$$\mathcal{L}\varphi = \dot\varphi + \tfrac{1}{2}\sigma^2 \frac{\partial}{\partial x}\theta_1(t,w) = \dot\varphi + \tfrac{1}{2}\sigma^2\theta_1' \frac{\partial w}{\partial x} = \dot\varphi + \tfrac{1}{2}\sigma^2 \theta_1' \frac{\partial\varphi}{\partial x}$$

$$(6) \qquad = \dot\varphi + \tfrac{1}{2}\sigma^2\theta_1\theta_1'$$

and then differentiating (5a) gives us

(7) $\quad 0 = D_1(\mathcal{L}\varphi) + D_1 c = \mathcal{L}(D_1\varphi) + D_1 c = \mathcal{L}\,\theta_1 + D_1 c$

Now $\quad D_1 c = \frac{\partial}{\partial x_1} c(t, \varphi(t,x_1)) = c'(t,w) \,.\, \theta_1(t,w)$

and
$$\mathcal{L}\,\theta_1 = \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial x_1^2}\,\theta_1(t,\varphi(t,x_1)) + \frac{\partial}{\partial t}\,\theta_1(t,\varphi(t,x_1))$$
$$= \tfrac{1}{2}\sigma^2 \frac{\partial}{\partial x_1}\Big(\theta_1'(t,\varphi(t,x_1)) \,.\, \theta_1(t,\varphi(t,x_1))\Big) + \dot\theta_1 + \theta_1'\,\dot\varphi$$
$$= \tfrac{1}{2}\sigma^2\,\theta_1\,(\theta_1''\theta_1 + (\theta_1')^2) + \dot\theta_1 + \theta_1'\,\dot\varphi$$

So from (7) we conclude that
$$\boxed{0 = \tfrac{1}{2}\sigma^2\,\theta_1\,(\theta_1''\theta_1 + (\theta_1')^2) + \dot\theta_1 + \theta_1'\,\dot\varphi + \theta_1 c'}$$

C+L have (eq^n (22) in the present notation)
$$0 = \tfrac{1}{2}\sigma^2\,\theta_1^2\,\theta_1'' - c\,\theta_1' + \dot\theta_1 + c'\theta_1$$

So the two are in agreement iff
$$-c = \tfrac{1}{2}\sigma^2\,\theta_1\theta_1' + \dot\varphi$$

However, the RHS is $\mathcal{L}\varphi$, according to (6), so this is just a restatement of (5a).

(iii) If an agent with utility u optimally invests, his terminal wealth is $w_T^* = I(\zeta_T)$, ζ being the state-price density process. His wealth at time t satisfies
$$\zeta_t w_t^* = E_t(\zeta_T w_T^*) = E_t(\zeta_T\, I(\zeta_T)) = h(t, X_t)$$

for some function h, and $\zeta_t = \exp(-\kappa\tilde W_t - \tfrac{1}{2}\kappa^2 t)$, $dX = \sigma(d\tilde W + \kappa\, dt)$.
We shall have that w_t^* is increasing in X_t if $\kappa > 0$, decreasing if $\kappa < 0$.
Since $D\varphi = \theta_1$, the final result Proposition 3 of C+L follows.

A question from Pawel Z. (25/6/12)

Suppose you receive an income stream δ_t of consumption good $d\delta = \delta(\sigma dW + \mu dt)$ but that the good has a limited storability. So you put the good into store, where there is quantity X_t in store at time t:

$$dX_t = (\delta_t - \lambda x_t - q_t)\,dt$$

where $\lambda > 0$ is decay, q_t is rate of consumption withdrawal. The aim is to obtain

$$V(x,\delta) = \sup E\left[\int_0^\infty e^{-\rho t} U(q_t)\,dt \,\middle|\, x_0 = x,\ \delta_0 = \delta\right]$$

There's a scaling $V(ax, a\delta) = a^{1-R} V(x,\delta)$ which implies $V(x,\delta) = x^{1-R} v(\delta/x)$.
Setting $s \equiv \delta/x$ for the scaled variable, we have HJB

$$0 = \sup_c \left[U(c) - \rho V + (\delta - \lambda x - c) V_x + \mu \delta V_\delta + \tfrac{1}{2}\sigma^2 \delta^2 V_{\delta\delta} \right] \qquad (q \equiv c/x)$$

$$= \sup_q x^{1-R}\left[U(q) - \rho v(s) + (s - \lambda - q)\left\{(1-R)v(s) - s v'(s)\right\} + \mu s v'(s) + \tfrac{1}{2}\sigma^2 s^2 v''(s) \right]$$

Hence we conclude

$$0 = \tilde{U}\big((1-R)v - sv'\big) - \rho v + (s-\lambda)\big((1-R)v - sv'\big) + \mu s v' + \tfrac{1}{2}\sigma^2 s^2 v''$$

Boundary conditions? As $\delta \downarrow 0$, we end up with a deterministic problem with no inflow of consumption good. If we let $f(x) \equiv V(x,0) = x^{1-R} v(0)$, we get the HJB

$$0 = \sup_c \left[U(c) - \rho f(x) - (\lambda x + c) f' \right] = \sup_q x^{1-R}\left[U(q) - \rho v(0) - (\lambda + q)(1-R) v(0) \right]$$

$$\text{so} \quad U'(q) = v(0)(1-R) \quad \Rightarrow \quad 0 = \tilde{U}\big((1-R)v(0)\big) - \rho v(0) - \lambda(1-R)v(0)$$

which determines $v(0)$.

Evolution of firm size again (29/6/12)

We've got the density of firms of log-size x earlier (p35) to be

$$\frac{\exp\{\theta x - |x|\sqrt{\theta^2+2\tilde{\varepsilon}}\}}{\sigma^2\sqrt{\theta^2+2\tilde{\varepsilon}}}$$

if we write $\theta = c/\sigma$, $\tilde{\varepsilon} = \varepsilon/\sigma^2$. What we want to do is give ourselves a prior for (θ, σ) and then deduce the distribution of (θ, σ) given log size $= x$. To keep it all reasonably tractable, let's assume $\tilde{\varepsilon}$ is constant (which would say killing rate of firm $\propto \sigma^2$, which is not obviously ridiculous), and for simplicity write ε wherever we had $\tilde{\varepsilon}$.

If we give ourselves a prior density of the form (for some $A > 0$)

$$\exp\left(-A\sqrt{\theta^2+2\varepsilon}\right) g(\sigma)$$

the posterior given observation x would come out as

$$\frac{\exp\{\theta x - (A+|x|)\sqrt{\theta^2+2\varepsilon}\}\, g(\sigma)}{\sigma^2\sqrt{\theta^2+2\varepsilon}}$$

so this is a GH distribution (holding σ constant) with $\lambda = \mu = 0$, $\alpha = A + |x|$, $\beta = x$, $\delta = \sqrt{2\varepsilon}$, so $\gamma^2 = A^2 + 2A|x|$.

So in the posterior, we have σ and $\theta \equiv c/\sigma$ are independent, and θ has a GH dist^n, with

$$E(\theta|x) = x\sqrt{2\varepsilon}\; K_1(\gamma\sqrt{2\varepsilon})/K_0(\gamma\sqrt{2\varepsilon}).$$

We could use $g(\sigma) \propto \sqrt{\sigma}\exp\left(-\tfrac{1}{2}(\sigma + \tfrac{1}{\sigma})\right)$

so that the posterior for σ will be a first passage time density

$$\sigma^{-2} g(\sigma) \propto \exp\left(-\tfrac{1}{2}(\sigma + 1/\sigma)\right)/\sqrt{2\pi\sigma^3}$$

$$= \exp\left(-\tfrac{1}{2\sigma}(\sigma \pm 1)^2 \pm 1\right)/\sqrt{2\pi\sigma^3}$$

This has mean 1, so it fits together quite neatly.

Various plots I've looked at are rather too neatly symmetrical, and I'm concerned that we seem to have that roughly log size goes up linearly with expected θ... could other behaviours be accommodated? What stylized facts need to be explained?

Law of (I, X, S) again (13/7/12)

(i) Let's come back to the situation where $(\xi_t)_{t \in h\mathbb{Z}^+}$ is a simple symmetric random walk on $h\mathbb{Z}$, which we stop at some stopping time T to create a martingale

$$X_t \equiv \xi_{t \wedge T}$$

With $S_t = \sup\{\xi_u : u \le t\}$, $I_t \equiv \inf\{\xi_u : u \le t\}$, $g_t^+ \equiv \sup\{u \le t : S_u > S_{u-h}\}$, $g_t^- = \sup\{u \le t : I_u < I_{u-h}\}$, and defining

$$\sigma_t = +1 \quad \text{if } g_t^+ > g_t^- ; \quad = -1 \text{ else}$$

We are concerned with N+S conditions on the law of (I, X, S, σ) at time T for there to be a stopping time T achieving that law. So if we set $\mathfrak{X} = -h\mathbb{Z}^+ \times h\mathbb{Z} \times h\mathbb{Z}^+ \times \{-1, +1\}$ we say that a measure m on $\mathfrak{X}$ is consistent if there exists some stopping time T such that

$$\mathcal{L}(I_T, X_T, S_T, \sigma_T) = m$$

The question is, can we find N+S conditions on a measure m defined on $\mathfrak{X}$ to be consistent?

(ii) Necessary conditions. The measure m only deals with the joint dist^n of $(I_T, X_T, S_T, \sigma_T)$ – there are no sample paths or first passage times for m. But if m is consistent, we have an identification of the law of m with the law of these functionals of the stopped RW, so there will be probabilities/distributions associated with (for example) the hitting times

$$H_x \equiv \inf\{t : X_t = x\}$$

Let's use $\bar{m}(\cdot)$ for such things, as a way of emphasizing that we are dealing with numbers which have a proper probabilistic interpretation if m is consistent, but otherwise are simply algebraic expressions in terms of m.

In what follows, a, b are generic elements of $h\mathbb{Z}^+$. To start with, we look at the OST at $H_a \wedge H_b$:

$$(1) \begin{cases} 1 = \bar{m}(H_{-a} < H_b) + \bar{m}(H_b < H_{-a}) + m(S < b, I > -a) \\ 0 = -a\, \bar{m}(H_{-a} < H_b) + b\, \bar{m}(H_b < H_{-a}) + m(X : S < b, I > -a) \end{cases}$$

These allow us to deduce expressions:

$$\left.\begin{aligned} \bar m(H_b < H_{-a}) &= \frac{a - m(a+X : S<b,\ I>-a)}{a+b} &\equiv \varphi(b,-a) \\ \bar m(H_{-a} < H_b) &= \frac{b - m(b-X : S<b,\ I>-a)}{a+b} &\equiv \varphi(-a,b) \end{aligned}\right\}$$

Say. Thus φ is a function defined on $(h\mathbb{Z}^+)^2 \setminus \{(0,0)\}$ in terms of m. We shall certainly need

$$\boxed{\begin{array}{llll} \text{(NC 1a)} & \varphi(b,-a) \uparrow m(S \geq b) & \text{as } a \uparrow \infty \\ \text{(NC 1b)} & \varphi(-a,b) \uparrow m(I \leq -a) & \text{as } b \uparrow \infty \end{array}}$$

If these conditions hold, by looking at the first equation of (1), we see that $\varphi(b,-a) \downarrow 0 \quad (b \uparrow \infty)$, $\varphi(-a,b) \downarrow 0 \quad (a \uparrow \infty)$. Notice that if NC1 holds, we may take differences to learn that

$$\bar m(H_b < \infty,\ I(H_b) = -a) = \varphi(b,-a-h) - \varphi(b,-a) \equiv \psi_+(-a,b) \quad \geq 0$$

$$\bar m(H_{-a} < \infty,\ S(H_{-a}) = b) = \varphi(-a,b+h) - \varphi(-a,b) \equiv \psi_-(-a,b) \quad \geq 0$$

If we set $\tau_t = \inf\{u : S_u - I_u = t\}$, $\tilde X_t = X(\tau_t)$, then we have

$$\begin{cases} \psi_+(-a,b) = \bar m(\tilde X_{a+b} = b) \\ \psi_-(-a,b) = \bar m(\tilde X_{a+b} = -a) \end{cases}$$

(CAREFUL: it could be that $a+b > S-I$ ...)

(iii) Let's next fix a, b in $h\mathbb{Z}^+$, not both zero, set $t = a+b$, and define

$$p_{++} = \bar m(\tilde X_t = b,\ \tilde X_{t+h} = b+h), \quad p_{--} = \bar m(\tilde X_t = -a,\ \tilde X_{t+h} = -a-h)$$

$$p_{+-} = \bar m(\tilde X_t = b,\ \tilde X_{t+h} = -a-h), \quad p_{-+} = \bar m(\tilde X_t = -a,\ \tilde X_{t+h} = b+h)$$

$$p_{+0} = \bar m(\tilde X_t = b,\ \tilde X_{t+h} \in [a,b]), \quad p_{-0} = \bar m(\tilde X_t = -a,\ \tilde X_{t+h} \in [a,b])$$

Notice that $p_{\pm 0}$ will correspond to paths which get stopped at time $t+h$. Now using the OST at time τ_{t+h} from time τ_t, we will get

$$\begin{cases} \psi_+(-a,b) = p_{++} + p_{+-} + p_{+0} \\ b\,\psi_+(-a,b) = (b+h)\,p_{++} - (a+h)\,p_{+-} + m\big(X : S=b, I=-a, \sigma=+1\big) \\ \psi_-(-a,b) = p_{--} + p_{-+} + p_{-0} \\ -a\,\psi_-(-a,b) = -(a+h)\,p_{--} + (b+h)\,p_{-+} + m(X : S=b, I=-a, \sigma=+1) \end{cases}$$

Notice that $p_{+0} = m(S=b, I=-a, \sigma=+1)$, $p_{-0} = m(S=b, I=-a, \sigma=-1)$, both known directly from m.

Let's define $v_\pm = m(X : S=b, I=-a, \sigma=\pm 1)/m(S=b, I=-a, \sigma=\pm 1)$ so that we now find

$$\left.\begin{aligned} \psi_+ &= p_{++} + p_{+-} + p_{+0} \\ b\,\psi_+ &= (b+h)\,p_{++} - (a+h)\,p_{+-} + p_{+0}\,v_+ \end{aligned}\right\}$$

implying that

$$p_{++} = \frac{(a+b+h)\,\psi_+ - (a+h+v_+)\,p_{+0}}{a+b+2h}$$

$$p_{+-} = \frac{h\,\psi_+ - (b+h-v_+)\,p_{+0}}{a+b+2h}$$

$$p_{--} = \frac{(a+b+h)\,\psi_- - (b+h-v_-)\,p_{-0}}{a+b+2h}$$

$$p_{-+} = \frac{h\,\psi_- - (a+h+v_-)\,p_{-0}}{a+b+2h}$$

Some consequences: we must have (in order that $p_{+-}, p_{-+} \geq 0$) necessary conditions:

$$\boxed{\begin{aligned} &(\mathrm{NC2a}) \quad \frac{p_{+0}}{\psi_+} \leq \frac{h}{b+h-v_+} \\ &(\mathrm{NC2b}) \quad \frac{p_{-0}}{\psi_-} \leq \frac{h}{a+h+v_-} \end{aligned}}$$

There is a further condition, because $\tilde{m}(\tilde{X}_{a+b+h} = b+h) = p_{++} + p_{-+}$

so there has to be the consistency conditions

$$\boxed{\begin{aligned} \psi_+(-a, b+h) &= p_{++} + p_{-+} \\ \psi_-(-a, b+h) &= p_{+-} + p_{--} \end{aligned}}$$

Proposition. The consistency conditions are universally valid

Proof Multiplying by $(a+b+2h)$, the task is to show that (the other relation is symmetric)

$$(a+b+2h)\{p_{++} + p_{-+}\} = (a+b+2h)\psi_+(-a, b+h)$$

$$\equiv (a+b+2h)[\varphi(b+h, -a-h) - \varphi(b+h, -a)].$$

Let's firstly develop the left-hand side.

$$LHS = (a+b+h)\psi_+(-a,b) - (a+h+v_+)p_{+0} + h\psi_-(-a,b) - (a+h+v_-)p_{-0}$$

$$= (a+b+h)\{\varphi(b,-a-h) - \varphi(b,-a)\} + h\{\varphi(-a,b+h) - \varphi(-a,b)\}$$
$$- (a+h)m(S=b, I=-a) - m(X: S=b, I=-a)$$

$$= a+h - m(a+h+X: S<b, I>-a-h) - \{a - m(a+X: S<b, I>-a)\} - h(\varphi(b,-a)+\varphi(-a,b))$$
$$+ h\varphi(-a,b+h) - m(a+h+X: S=b, I=-a)$$

$$= h - m(a+h+X: S<b, I>-a-h) + m(a+X: S<b, I>-a) - h\{1 - m(S<b, I>-a)\}$$
$$+ h\varphi(-a,b+h) - m(a+h+X: S=b, I=-a)$$

$$= -m(a+h+X: S<b, I>-a-h) + m(a+h+X: S<b, I>-a) - m(a+h+X: S=b, I=-a)$$
$$+ h\varphi(-a,b+h)$$

So if we set $A_1 = \{S<b, I>-a\}$, $A_2 = \{S<b, I>-a-h\} \supseteq A_1$, and $A_3 = \{S=b, I=-a\}$, disjoint from A_1, then $(A_2 \cup A_3)\setminus A_1 = \{S \le b, I=-a\}$ $= \{S < b+h, I=-a\}$. So altogether

$$\boxed{LHS = -m(a+h+X: S<b+h, I=-a) + h\varphi(-a,b+h).}$$

Now $RHS = a+h - m(a+h+X: S<b+h, I>-a-h) - \{a - m(a+X: S<b+h, I>-a)\}$
$$- h\varphi(b+h,-a)$$

$$= h - m(a+h+X: S<b+h, I>-a-h) + m(a+X+h: S<b+h, I>-a)$$
$$- hm(S<b+h, I>-a) - h\varphi(b+h,-a)$$

We have RHS = LHS iff

$$\varphi(-a, b+h) + \varphi(b+h, -a) = 1 - m(S < b+h, I > -a)$$

which is identically true. $\square$

Distribution of firm sizes again (28/8/12)

(1) We have seen that if log firm size evolves as a BM with volatility σ and drift $c = \theta\sigma^2$, and firms get killed at rate $\varepsilon > 0$, then density of log firm size in steady state (assuming firms are born at unit rate) is

$$\frac{\exp\left\{\theta x - |x|\sqrt{\theta^2 + 2\tilde{\varepsilon}}\right\}}{\sigma^2\sqrt{\theta^2+2\tilde{\varepsilon}}} \equiv \varphi(x\,|\,\varepsilon, \theta, \sigma)$$

where $\tilde{\varepsilon} \equiv \varepsilon/\sigma^2$, and if we integrate this over $x \in \mathbb{R}$ we get total mass $1/\varepsilon$.

(2) A Generalised Hyperbolic distribution with parameters $(\lambda, \alpha, \beta, \delta, \mu)$ has density

$$x \mapsto \frac{(\gamma/\delta)^\lambda}{\sqrt{2\pi}\;K_\lambda(\delta\gamma)}\, e^{\beta(x-\mu)}\, \frac{K_{\lambda-\frac12}\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)}{\left(\sqrt{\delta^2+(x-\mu)^2}\,/\alpha\right)^{\frac12-\lambda}} \qquad (x \in \mathbb{R})$$

$$\equiv GH(x\,|\,\lambda, \alpha, \beta, \delta, \mu)$$

where $\gamma = \sqrt{\alpha^2-\beta^2}$, so we need $|\beta| \leq |\alpha|$. Observing that

$$K_{\frac12}(x) = K_{-\frac12}(x) = e^{-x}\sqrt{\pi/2x}$$

we see that

$$\varphi(x\,|\,\varepsilon, \theta, \sigma) = e^{\theta x}\sqrt{\frac{2}{\pi}}\;\frac{K_{-\frac12}\left(|x|\sqrt{\theta^2+2\tilde{\varepsilon}}\right)\,|x|^{\frac12}}{\sigma^2\left(\sqrt{\theta^2+2\tilde{\varepsilon}}\right)^{\frac12}}.$$

(3) Suppose we take a prior density over (θ, σ) of the form $\exp(-A\sqrt{\theta^2+2\tilde{\varepsilon}})$. $g(\sigma)\,d\theta\,d\sigma$. Then if we integrate out the θ-variable we have

$$\int_{-\infty}^{\infty} \exp\left(\theta x - (A+|x|)\sqrt{\theta^2+2\tilde{\varepsilon}}\right)\frac{d\theta}{\sqrt{\theta^2+2\tilde{\varepsilon}}}\cdot\frac{g(\sigma)}{\sigma^2}\,d\sigma$$

$$= \int_{-\infty}^{\infty} GH\left(\theta\,|\,0, A+|x|, x, \sqrt{2\tilde{\varepsilon}}, 0\right) 2\sqrt{\pi}\,K_0\left(\sqrt{2\tilde{\varepsilon}}\sqrt{A^2+2A|x|}\right) d\theta \cdot \frac{g(\sigma)}{\sigma^2}\,d\sigma$$

$$= \sqrt{2\pi}\,K_0\left(\sqrt{2\tilde{\varepsilon}}\sqrt{A^2+2A|x|}\right)\frac{g(\sigma)}{\sigma^2}\,d\sigma$$

Suppose we use

$$g(\sigma) = q\,\sigma^{-\nu}$$

for some $\nu > 1$, normalising constant q. Then if we do the integration

$$\frac{1}{\tilde{\varepsilon}} = \iiint \varphi(x|\varepsilon,\theta,\sigma)\exp\left(-A\sqrt{\theta^2+2\tilde{\varepsilon}}\right) q\,\sigma^{-\nu}\,dx\,d\theta\,d\sigma$$

$$= \iint \sqrt{2\pi}\; K_0\left(\sqrt{2\varepsilon/\sigma^2}\sqrt{A^2+2A|x|}\right) q\,\sigma^{-\nu}\,\frac{d\sigma}{\sigma^2} \qquad \left(\tfrac{1}{\sigma} \equiv y\right)$$

$$= \int_{-\infty}^{\infty} dx \int_0^{\infty} \sqrt{2\pi}\; K_0\left(y\sqrt{2\varepsilon}\sqrt{A^2+2A|x|}\right) q\,y^{\nu}\,dy$$

$$= \int_{-\infty}^{\infty} \left\{2\varepsilon(A^2+2A|x|)\right\}^{-(\nu+1)/2} \sqrt{2\pi}\,q\left(\int_0^{\infty} K_0(s)\,s^{\nu}\,ds\right) dx$$

$$= 2^{\nu-1}\,\Gamma\left(\frac{\nu+1}{2}\right)^2 \cdot \sqrt{2\pi}\,q \int_{-\infty}^{\infty} \left\{2\varepsilon(A^2+2A|x|)\right\}^{-(\nu+1)/2} dx$$

$$= 2^{\nu}\,\Gamma\left(\frac{\nu+1}{2}\right)^2 \sqrt{2\pi}\,q\,(2\varepsilon)^{-(\nu+1)/2} A^{-(\nu+1)} \cdot A \cdot \int_0^{\infty} (1+2x)^{-(\nu+1)/2}\,dx$$

$$= (2/A)^{\nu}\,\Gamma\left(\frac{\nu+1}{2}\right)^2 \sqrt{2\pi}\,q\,(2\varepsilon)^{-(\nu+1)/2} \cdot \frac{1}{2} \cdot \frac{2}{\nu-1}$$

This determines what the normalisation q should be.

(4) We now have the joint density of (x,θ,σ):

$$\frac{\exp\left[\theta x - (A+|x|)\sqrt{\theta^2+2\tilde{\varepsilon}}\right]}{\sqrt{\theta^2+2\tilde{\varepsilon}}}\, q\,\sigma^{-\nu-2} \equiv h(x,\theta,\sigma)$$

What is most interesting is what would be the conditional distribution of (θ,σ) given a value of x. It seems to be somewhat easier to ask about the Sharpe ratio $\kappa \equiv c/\sigma \equiv \theta\sigma$ and σ. The joint density of (x,κ,σ) is easily seen to be

$$\frac{1}{\sigma}\,h(x,\kappa/\sigma,\sigma) = \frac{\exp\left[\left(\kappa x - (A+|x|)\sqrt{\kappa^2+2\varepsilon}\right)/\sigma\right]}{\sqrt{\kappa^2+2\tilde{\varepsilon}}}\, q\,\sigma^{-\nu-2}$$

If we condition on x and integrate out σ, what we end up with is

$$\propto (\kappa^2+2\varepsilon)^{-\frac{1}{2}} \left[(A+|x|)\sqrt{\kappa^2+2\varepsilon} - \kappa x \right]^{-\nu-1} \qquad (*)$$

The joint density for κ and $\tau \equiv 1/\sigma$ is seen to be

$$\propto \frac{\tau^\nu}{\sqrt{\kappa^2+2\varepsilon}} \exp\left\{ -\tau \left((A+|x|)\sqrt{\kappa^2+2\varepsilon} - \kappa x \right) \right\}.$$

So we have a neat interpretation of the conditional joint law of (κ, τ) given x; first we choose κ according to the density $(*)$, and then given κ, τ will have a $\Gamma(\nu+1, \beta)$ distribution, where

$$\beta = (A+|x|)\sqrt{\kappa^2+2\varepsilon} - \kappa x.$$

Use of expert managers (12/9/12)

(1) Here's a little story coming out of the earlier study with Tom + Mattheus in 2008 on business lines. The basic tale is that there are N assets with dynamics

$$dS_t^i / S_t^i = \sigma_{ij}\, dW_t^j + \mu_t^i\, dt \qquad i = 1, \dots, N$$

$$d\mu_t^i = v_{ij}\, d\tilde{W}_t^j + b_i(\bar{\mu}_i - \mu_t^i)\, dt.$$

There is a boss who can invest himself in the assets, or who can employ managers to do it for him. Manager i can see the process μ_t^i, but the principal cannot. Manager i charges a proportional fee δ_i. Let's suppose the boss wants to maximise the expected long-run growth rate. We'll discuss later what the objective of the managers is.

(2) Suppose that the boss chooses to do proportional investment $\theta_t^i = p_i w_t$ for $i = 1, \dots, J$, where p_i are fixed constants, and then lets the managers do their stuff for the other assets, so $\theta_t^j = \pi_t^j w_t$ for $j = J+1, \dots, N$, and the managers invest optimally. Then

$$dw_t = w_t \left\{ r\,dt + (p, \pi_t)\big(\sigma dW + (\mu - q)\,dt\big) \right\} \qquad q = r + \begin{pmatrix} 0 \\ \delta \end{pmatrix}$$

and if we use horizon T, the value $V(t, w, \mu) = \log w + h(T-t, \mu)$ should satisfy

$$0 = \sup_\pi \Big[\dot{V} + (r + (p,\pi)(\mu - q))\, wV_w + \tfrac{1}{2}(p,\pi)a(p,\pi)^T\, W^2 V_{WW}$$
$$+ (B(\bar{\mu} - \mu), D_\mu V) + \tfrac{1}{2}\alpha_{ij} D_{\mu_i} D_{\mu_j} V \Big]$$

$$(\alpha \equiv v v^T)$$

$$= \sup_\pi \Big[-\dot{h} + (r + (p,\pi)(\mu - q)) - \tfrac{1}{2}(p,\pi)a(p,\pi)^T + (B(\bar{\mu} - \mu), D_\mu h) + \tfrac{1}{2}\alpha_{ij} D_{\mu_i} D_{\mu_j} h \Big]$$

Let's partition $\mu - q = \begin{pmatrix} \xi^b \\ \xi^m \end{pmatrix}$, $a = \begin{pmatrix} a^{bb} & a^{bm} \\ a^{mb} & a^{mm} \end{pmatrix}$, so that

$$\pi^* = (a^{mm})^{-1}(\xi^m - a^{mb} p)$$

and

$$\boxed{\begin{aligned} 0 = -\dot{h} + r + p\cdot\xi^b - \tfrac{1}{2} p\, a^{bb} p + \tfrac{1}{2}(\xi^m - a^{mb}p)\cdot(a^{mm})^{-1}(\xi^m - a^{mb}p) \\ + (B(\bar{\mu} - \mu), D_\mu h) + \tfrac{1}{2}\alpha_{ij} D_{\mu_i} D_{\mu_j} h \end{aligned}} \qquad (*)$$

Now let's notice $h(0,\cdot) = 0$, and set $H(s,\mu) \equiv \int_0^\infty e^{-st} h(t,\mu)\,dt$. Now multiply $(*)$ by e^{-st} and integrate dt, to discover that

$$0 = -sH + s^{-1}\Big(r + p\cdot\xi^b - \tfrac{1}{2} p\cdot a^{bb} p + \tfrac{1}{2}(\xi^m - a^{mb}p)\cdot(a^{mm})^{-1}(\xi^m - a^{mb}p) \Big) + \mathcal{L}H$$

where $\mathcal{L}$ is the generator of the OU process. Hence

$$H = s^{-1} R_s \varphi$$

where $\varphi = r + p\cdot\xi^b - \frac{1}{2} p\cdot a^{bb} p + \frac{1}{2}(\xi^m - a^{mb}p)(a^{mm})^{-1}(\xi^m - a^{mb}p)$.
We think that h will grow linearly with t, and we find the rate by multiplying H by s and letting $s \to 0$. The long-run growth rate is

$$E^\infty \varphi(\mu)$$

where P^∞ is the limiting dist^n of μ, viz. $N(\bar\mu, \Sigma)$, where $\Sigma_{ij} = \alpha_{ij}/(b_i + b_j)$.
Using this,

$$\boxed{\begin{aligned} E^\infty \varphi(\mu) = r &+ p\cdot(\bar\mu^b - r) - \tfrac{1}{2} p\cdot a^{bb} p + \tfrac{1}{2}\operatorname{tr}\big((a^{mm})^{-1}\Sigma^{mm}\big) \\ &+ \tfrac{1}{2}(\bar\mu^m - \delta^m - r - a^{mb}p)\cdot(a^{mm})^{-1}(\bar\mu^m - \delta^m - r - a^{mb}p) \end{aligned}}$$

Optimizing over p will give us

$$\boxed{p^* = (a^{-1})^{bb}(\bar\mu^b - r) + (a^{-1})^{bm}(\bar\mu^m - \delta^m - r)}$$

and the optimized value is $(y = \bar\mu - q)$

$$\boxed{\begin{aligned} r + \tfrac{1}{2}\operatorname{tr}\big((a^{mm})^{-1}\Sigma^{mm}\big) &+ \tfrac{1}{2}(\bar\mu^m - \delta^m - r)\cdot(a^{mm})^{-1}(\bar\mu^m - \delta^m - r) \\ &+ \tfrac{1}{2}\, y^T \begin{pmatrix} I \\ -(a^{mm})^{-1}a^{mb} \end{pmatrix} (a^{-1})^{bb} \big(I, \ -a^{bm}(a^{mm})^{-1}\big)\, y. \end{aligned}}$$

(3) Assuming the managers set their fees low enough that they get hired, then the value to the boss will be

$$r + \tfrac{1}{2}\operatorname{tr}(a^{-1}\Sigma) + \tfrac{1}{2}(\bar\mu - \delta - r)\, a^{-1} (\bar\mu - \delta - r)$$

and the optimal investment proportions will be $a^{-1}(\mu - r - \delta)$, which averages out at $a^{-1}(\bar\mu - r - \delta)$. The j^{th} agent would have objective

$$\delta_j \, (a^{-1}(\bar\mu - r - \delta))_j$$

so the derivative wrt δ_j will be

$$(a^{-1}(\bar\mu - r - \delta))_j - \delta_j \, (a^{-1})_{jj} = 0$$

at optimality. A Pareto efficient choice of the δ's would require all these to be zero.

.. or is it? This seems rather bogus because if δ_j remains fixed, the best growth rate that manager j can achieve will come by maximising the growth rate for the boss ... hard to tell a good story here?

OR We could say that the δ arrived at should satisfy

$$\boxed{r + \tfrac{1}{2}\operatorname{tr}(a^{-1}\Sigma) + \tfrac{1}{2}(\bar\mu - \delta - r)\, a^{-1} (\bar\mu - \delta - r) \geq r + \tfrac{1}{2}(\bar\mu - r)\, a^{-1} (\bar\mu - r)}$$

as a constraint to ensure that all the managers are used. If that happens, then we could argue about the individual agents' shares, but all this is irrelevant compared with the exponential growth rate ...

[Papak. particularly did not like my comments! He said did I believe EU explains everything, what about the Allais paradox (which I need to dig into, particularly the experimental basis for the claims) But there is a serious point in all this. The Cumulative Prospect "Theory" says that agents misunderstand big/small probabilities and instead of ranking distributions F according to

$$\int U(x)\, d\, F(x)$$

with U concave incr, we should rank according to

$$\int \varphi(x)\, d(\psi(F(x)))$$

where φ is incr, but not necessarily concave, and ψ is the prob'y distortion. The first extension makes sense, but the second does not: suppose we had some <u>bivariate</u> data, and we attempt to rank by

$$\int \varphi(x_1, x_2)\, d(\psi(F(x_1, x_2)))$$

- then $\psi \circ F$ is <u>not in general a distribution function</u>. Moreover, if we just rotate the data to $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ our conclusions shouldn't be altered, but $\psi(F(x_1, x_2))$ will be!

Thoughts from a paper of Andrikogiannopoulou & Papakonstantinou (12/9/12)

This paper looked at some data from an online gambling site, + tried to tell a behavioural finance story for how + why people gamble, what influences them and so on. This led me to wonder what would happen if we consider an agent whose objective is

$$E \sum_{n\geq 0} \beta^n \varphi(w_n)$$

where w_n is the wealth at time n, $\beta \in (0,1)$ fixed, and φ is increasing, but non-negative, defined only on $\mathbb{R}^+$, initially convex before coming concave. Each period, the agent gets ε of new income from working, and may choose to stake up to his total wealth on a gamble. He may choose from distributions $F_1, \dots, F_N$ which are the returns delivered from a unit stake on N gambles. If the value function is $V(w)$, then*

$$V(w) = \varphi(w) + \beta \sup_{\substack{1\leq j\leq N \\ 0\leq \theta \leq w+\varepsilon}} \int_0^\infty V(w+\varepsilon - \theta + \theta x)\, F_j(dx)$$

and this could be found at least approximately by value improvement, or indeed by policy improvement.

We could concoct interesting families of F as mixtures of a $\Gamma(1)$ and another Γ, so densities of the form

$$p\,\lambda e^{-\lambda x} + (1-p)\,(\beta x)^{\alpha-1} e^{-\beta x}\,\beta/\Gamma(\alpha)$$

which have mean

$$p/\lambda + (1-p)\,\alpha/\beta$$

Variance $\quad 2p/\lambda^2 + (1-p)\,\alpha(\alpha+1)/\beta^2 - \left(p/\lambda + (1-p)\alpha/\beta\right)^2$

third moment

$$p\,\frac{6}{\lambda^3} + (1-p)\,\frac{\alpha(\alpha+1)(\alpha+2)}{\beta^3}$$

* Numerically, may be better to write as $V(w) = \varphi(w) + \beta \sup_{\substack{1\leq j\leq N \\ 0\leq t\leq 1}} \int_0^\infty V\big((w+\varepsilon)(1-t+tx)\big)\, F_j(dx)$

The hedging story for (I, X, S, σ) (17/9/12)

(i) What Moritz & I have been able to establish is that a probability measure m on $\mathfrak{X} = -h\mathbb{Z}^+ \times h\mathbb{Z} \times h\mathbb{Z}^+ \times \{-1, 1\}$ is the joint law of I, X, S, σ for a symmetric simple random walk on $h\mathbb{Z}$ stopped at some a.s. finite stopping time if and only if for all $a, b \in h\mathbb{Z}^+$ not both zero

$$\begin{cases} m(b+h-X : S=b, I=-a, \sigma=+1) \leq h\,\psi_+(-a,b) & (1) \\ m(a+h+X : S=b, I=-a, \sigma=-1) \leq h\,\psi_-(-a,b) \end{cases}$$

Here, $\psi_+(-a,b) = \varphi(b,-a-h) - \varphi(b,-a)$, $\psi_-(-a,b) = \varphi(-a,b+h) - \varphi(-a,b)$ and

$$\begin{cases} \varphi(b,-a) = \dfrac{a - m(a+X : S<b, I>-a)}{a+b} \\ \varphi(-a,b) = \dfrac{b - m(b-X : S<b, I>-a)}{a+b} \end{cases}$$

(ii) Let's now assume that the stopped martingale is uniformly integrable (which we could achieve by stopping once the martingale exits $[-Nh, Nh]$ with no impact on the measure m on $([-Nh, 0] \times [-Nh, Nh] \times [0, Nh] \times \{-1, 1\}) \cap \mathfrak{X}$.)

Let's then notice that

$$(a+b)\,\varphi(b,-a) = m(a+X : S \geq b \text{ or } I \leq -a)$$

$$= m(a+X : S \geq b, I > -a)$$

since $m(a+X : I \leq -a) = 0$ (*)

$$= (a+b)\, m(S \geq b, I > -a) - m(b-X : S \geq b, I > -a)$$

So the inequality (1) can be stated as

$$\boxed{\begin{aligned} 0 \leq h\, m(S \geq b, I = -a) &- \frac{h}{a+b+h} m(b-X : S \geq b, I > -a-h) + \frac{h}{a+b} m(b-X : S \geq b, I > -a) \\ &- m(b+h-X : S=b, I=-a, \sigma=+1) \end{aligned}}$$

(*) This has to hold for m, as it's joint law for a martingale – hard to see from the inequalities WHY, though!

Now if we were to be trying to find some extremal price for some derivative $\Phi(I, X, S, \sigma)$, subject to the law of X being given by market prices, we would have a constrained maximization

$$\max_m \int \Phi\, dm + \sum_K \eta_K \left\{ C(K) - \int (X-K)^+ dm \right\}$$

$$+ \sum_{\substack{a,b\geq 0 \\ s\in\{+1,-1\}}} \lambda^s_{ab} \left\{ h\, m(S\geq b, I=-a) - \frac{h}{a+b+h} m(b-X;\, S\geq b, I>-a-h) \right.$$

$$+ \frac{h}{a+b} m(b-X;\, S\geq b, I>-a)$$

$$\left. - m(b+h-X : S=b, I=-a, \sigma=+1) - \zeta^s_{ab} \right\}$$

where the ζ^s_{ab} are slack variables, ≥ 0. The Lagrangian then takes the form

$$\max_m \int \Big\{ \Phi - \sum_K \eta_K (X-K)^+ + \sum \lambda^s_{ab} \Big(h\, I_{\{S\geq b, I=-a\}} - \frac{h}{a+b+h} (b-X) I_{\{S\geq b, I>-a-h\}}$$

$$+ \frac{h}{a+b} (b-X) I_{\{S\geq b, I>-a\}} - (b+h-X) I_{\{S=b, I=-a, \sigma=+1\}} \Big) \Big\} dm$$

$$+ \text{ the rest}$$

Now since $dm \geq 0$, we shall have to have that

$$\Phi \leq \sum \eta_K (X-K)^+ - \sum \lambda^s_{ab} (\cdots)$$

constitutes the superhedge, with equality where the extremal m puts mass. The first term in the hedge, $\sum \eta_K (X-K)^+$, is easy to understand; it is simply a time-0 position in calls. But the subsequent term,

$$h\, I_{\{S\geq b, I=-a\}} - \frac{h}{a+b+h} (b-X) I_{\{S\geq b, I>-a-h\}} + \frac{h}{a+b} (b-X) I_{\{S\geq b, I>-a\}}$$

$$- (b+h-X) I_{\{S=b, I=-a, \sigma=+1\}}$$

is harder to see, because it is not clear how it would be constructed in an adaptive fashion. There are four terms in the expression; for the middle two, what we shall do is to replicate the payoff (almost) by buying forward $\frac{h}{a+b+h} I_{\{I(H_b) > -a-h\}} - \frac{h}{a+b} I_{\{I(H_b) > -a\}}$

units of the underlying at the time H_b, and closing out the $\frac{h}{a+b+h}$ long units at H_{-a-h}, closing out the $-h/_{a+b}$ units at H_{-a}.

By doing this, so long as X stays strictly above $-a$, we generate exactly the correct random variable;

if X falls to $-a-h$, closing out the first generates $-h$, closing out the second generates $+h$, so net zero;

so it is only if the asset falls to global minimum $-a$ that the position is worth something different from the two terms we are trying to match: the position is worth

$$\frac{h}{a+b+h}(X-b) - h \quad \text{on } \{S \geq b, \ I(H_b) > -a, \ I = -a\}$$

Whereas what we are trying to match on that same event would be worth

$$\frac{h}{a+b+h}(X-b)$$

We'll now describe a hedge for the other two pieces that will nearly (but not quite) match the claim we want, which is

$$h I_{\{S \geq b, \ I = -a\}} - (b+h-X) I_{\{S=b, \ I=-a, \ \sigma=+1\}}$$

cancels the discrepancy from the other hedge

$$= h I_{\{H_b<\infty, \ I(H_b)=-a=I\}} + h I_{\{H_b<\infty, \ I(H_b)>-a=I\}}$$

$$+ (X-b) I_{\{S=b, \ I=-a, \ \sigma=+1\}} - h I_{\{S=b, \ I=-a, \ \sigma=+1\}}$$

The hedge we use for this is to buy forward $I_{\{H_b<\infty, \ I(H_b)=-a\}}$ units of the asset at time H_b, and to then close out the position at the first exit from $[-a, b]$. If the first exit takes place at $b+h$, then the position will be worth h, exactly as it needs to be. If no exit takes place, the hedge is worth $(X-b) I_{\{S=b, I=-a, \sigma=+1\}}$. If exit happens at $-a-h$, then what we get is our hedge is worth $-(a+b+h)$ and the thing we tried to match was worth 0.

So if we use the approximate hedge, we collect a term $+\lambda^+_{ab}(a+b+h)$ on the event $\{H_{-a} < H_b < H_{-a-h} < H_{b+h}\}$, so we make this profit from the specified hedge. This profit would be zero if λ^*_{ab} was zero; or if λ^*_{ab} was positive, but the event $\{H_{-a} < H_b < H_{-a-h} < H_{b+h}\}$ had zero probability.

If $\lambda^*_{ab} > 0$, then the slack variable z^*_{ab} will be zero, and for the optimal in the inequality corresponding to λ^*_{ab} will have to hold with equality. Looking back through the proof, what this implies is that $\tilde{p}_{+-} = 0$, so indeed the event $\{H_{-a} < H_b < H_{-a-h} < H_{b+h}\}$ never happens!

McKean's winding number problem again (3/10/12)

(i) Let's come back to the old problem where (X_t) is a BM started at $x<0$, $Y_t = \int_0^t X_s\,ds$, and τ is $\inf\{t>0 : Y_t = 0\}$. Then McKean's result is

$$P^x[X_\tau \in dz]/dz \propto \frac{z^{3/2}\sqrt{|x|}}{|x|^3 + z^3}$$

The old L-McK-R-W approach did an expansion in eigenfunctions, but maybe what we want is to find some suitably 'nice' functions $f(x,y;\theta)$ so that

$$E^x\left[f(X_\tau, 0;\theta)\right] \quad \text{is available in simple form,}$$

so that we can deduce the form of the law. Now a natural class of functions for $f(x,0;\theta)$ would be $f(x,0;\theta) = x^\theta$ for $0 \le \theta < \frac{1}{2}$, for we get

$$\int_0^\infty \frac{z^{3/2}\sqrt{|x|}}{|x|^3+z^3}\cdot z^\theta\,dz = \int_0^\infty |x|^\theta \frac{t^{3/2+\theta}\,dt}{1+t^3} \qquad t^3 = s^2$$

$$= |x|^\theta \int_0^\infty \frac{2}{3}\,\frac{s^{2/3+2\theta/3}}{1+s^2}\,ds$$

$$= |x|^\theta\cdot\frac{2}{3}\cdot\frac{\pi}{2}\cdot\sec\left(\frac{\pi}{3}(1+\theta)\right)$$

(ii) Suppose we set for $x\in\mathbb{R}$, $y\le 0$, and $\theta\in[0,\frac{1}{2})$ fixed

$$f(x,y;\theta) = E^{x,y}\left[X_\tau^\theta\right]$$

and do some scaling. If $(X_t)_{t\ge0}$ is BM started at x, then (cX_{t/c^2}) is BM started at cx, and

$$\int_0^t \tilde{X}_s\,ds = \int_0^t cX(s/c^2)\,ds = c^3\int_0^{t/c^2} X_u\,du$$

$$= c^3\left(Y_{t/c^2} - Y_0\right)$$

So $\tilde{Y}_t \equiv c^3 Y(t/c^2) = c^3 Y_0 + \int_0^t \tilde{X}_s\,ds$, so $(\tilde{X}_t, \tilde{Y}_t)$ has the same law as (X,Y) when started from (cx, c^3y). Therefore

$$E^{(cx,c^3y)}\left[|X_\tau|^\theta\right] = E^{(x,y)}\left[\tilde{X}_{\tilde{\tau}}^\theta\right] = c^\theta E^{(x,y)}\left[X_\tau^\theta\right] = c^\theta f(x,y;\theta)$$

giving the scaling relationship

$$\boxed{f(cx, c^3y; \theta) = c^\theta f(x,y;\theta)}$$

Taking $c = y^{-1/3}$ gives us

$$f(x,y;\theta) = y^{\theta/3} g(xy^{-1/3};\theta)$$

for some function g of a single argument $\xi = xy^{-1/3}$ which we need to discover

The PDE

$$\tfrac{1}{2} f_{xx} + x f_y = 0$$

for f now becomes

$$\boxed{\tfrac{1}{2} g''(\xi) + \xi \left\{ \tfrac{\theta}{3} g(\xi) - \tfrac{\xi}{3} g'(\xi) \right\} = 0}$$

with the condition that for $x > 0$, $\lim_{y \downarrow 0} f(x,y;\theta) = x^\theta \lim_{y \downarrow 0} (xy^{-1/3})^{-\theta} g(xy^{-1/3};\theta)$

$$= x^\theta \lim_{s \to \infty} s^{-\theta} g(s;\theta) = x^\theta$$

so there has to be a BC at infinity for g.

Asking Maple to solve this leads to Whittaker functions