Optimal investment with labour income (6/10/12)

(i) Let's suppose that agent's wealth dynamics are

$$dw_t = rw_t\,dt + \theta_t\left(\sigma\,dW_t + (\mu - r)\,dt\right) + (aL_t - c_t)\,dt$$

where $a>0$ is the wage rate, and L_t is the rate of working. The objective of the agent is to achieve

$$V(w) \equiv \sup E\left[\int_0^\infty e^{-\rho t}\,U(c_t, L_t)\,dt \,\Big|\, w_0 = w\right]$$

where U is C^2, concave, increasing in c, decreasing in L. This looks rather different from the usual story, but if we consider the problem

$$\max\ U(c,L) \quad \text{st} \quad c - aL = y$$

and let $\varphi(y)$ be the supremum, we would find that

$$\begin{cases} U_c = \lambda \\ U_L = -a\lambda \end{cases}$$

would characterise the solution in terms of the multiplier λ which is to be found. Once this is done, we would have dynamics

$$dw_t = rw_t\,dt + \theta_t\left(\sigma\,dW_t + (\mu - r)\,dt\right) - y_t\,dt$$

and objective

$$\sup E\left[\int_0^\infty e^{-\rho t}\,\varphi(y_t)\,dt \,\Big|\, w_0 = w\right]$$

which is quite conventional again.

(ii) If we had a story of an entrepreneur, the wealth dynamics might be

$$dw_t = rw_t + \theta_t\left(\sigma\,dW_t + (\mu(L_t) - r)\,dt\right) - c_t\,dt$$

where $\mu(\cdot)$ is increasing concave. The HJB here would be

$$0 = \sup_{c,L,\theta}\left[-\rho V + \left(rw + \theta(\mu(L) - r) - c\right)V_w + \tfrac{1}{2}\sigma^2\theta^2 V_{ww} + U(c,L)\right]$$

A bit more interesting!

A simpler version of FBH (11/10/12).

(i) Here is a simpler version of the FBH story which reveals some difficulties. There is no bank, just an agent who controls the production process through capital and labour. We have

$$\dot{K}_t = Z_t f(K_t, L_t) - C_t - \delta K_t$$

where the agent's objective is to achieve

$$V(T, k, z) = \sup E\left[\int_0^T e^{-\rho s} U(C_s, L_s)\, ds + e^{-\rho T} \varphi(K_T) \,\Big|\, K_0 = k, Z_0 = z \right]$$

By the MPOC, we shall have

$$0 = \sup_{C,L} \left[-\rho V - V_t + U(C, L) + (Z f(K,L) - C - \delta K) V_K + \mu Z V_Z + \tfrac{1}{2}\sigma^2 Z^2 V_{ZZ} \right]$$

with

$$V(0, k, z) = \varphi(k).$$

The first-order conditions we derive are of course

$$\begin{cases} U_C = V_K \\ U_L + Z V_K f_L = 0 \end{cases}$$

Now let's compare with the P-L approach, with multiplier $d\eta_t = \eta_t(a_t dW_t + b_t dt)$. We will have

$$\sup E\left[\int_0^T \left\{ e^{-\rho t} U(C_t, L_t)\, dt + \eta_t (Z_t f(K_t, L_t) - C_t - \delta K_t) dt + K_t \eta_t b_t\, dt \right\} + e^{-\rho T} \varphi(K_T) - \left[\eta_t K_t \right]_0^T \right]$$

$$= \sup E\left[\int_0^T \left\{ e^{-\rho t} U(C_t, L_t) + \eta_t (Z_t f(K_t, L_t) - C_t) + K_t \eta_t (b_t - \delta) \right\} dt + e^{-\rho T} \varphi(K_T) - \eta_T K_T + \eta_0 K_0 \right]$$

and the FOCs for C, L give us

$$(1) \qquad \begin{cases} e^{-\rho t} U_C = \eta \\ e^{-\rho t} U_L + \eta Z f_L = 0 \end{cases}$$

and FOC for K leads to

$$(2) \qquad \eta (Z f_K + b - \delta) = 0.$$

Now the story would go that if we take (K_t, Z_t, η_t) as the state vector, then we can use (1) to deduce the current values of c, L, (2) to deduce the current value of b, so we know how to evolve K, η, Z according to

$$dK = (Z f(K,L) - \delta K - c)\,dt$$

$$dZ = Z(\sigma dW + \mu dt)$$

$$d\eta = \eta\,(a dW + b dt)$$

APART FROM the fact that a is not yet known! We can't just arbitrarily set $a \equiv 0$, because we know that

$$\eta_t = e^{-\rho t} V_K(T-t, K_t, Z_t)$$

so that η certainly will have quadratic variation... and we must further have the condition

(3) $$\boxed{e^{-\rho T} \varphi'(K_T) = \eta_T}$$

to be hit at the terminal time. So understanding the martingale part of η is an inescapable part of the story, and the only place we can extract information on this is (3). It seems that we are actually forced to solve a BSDE at this point, and that is a non-trivial exercise...

(ii) What if we put the randomness onto K directly,

$$dK = (f(K,L) - \delta K - c)\,dt + \sigma K dW \quad ?$$

This time, the value is $V(t, K)$ and the HJB will be

$$0 = \sup\left[-\rho V - V_t + U(c,L) + \{f(K,L) - \delta K - c\}V_K + \tfrac{1}{2}\sigma^2 K^2 V_{KK}\right]$$

and we get the optimality conditions

$$U_c = V_K, \quad U_L + V_K f_L = 0.$$

How does P-L approach look now? We'd $d\eta = \eta\,(a dW + b dt)$, and

$$\max E\Big[\int_0^T \{e^{-\rho t} U(c,L) + \eta(f(K,L) - \delta K - c) + K\eta b + \eta a \sigma K\}\,dt$$
$$+ \varphi(K_T)\,e^{-\rho T} - \eta_T K_T + \eta_0 K_0\Big]$$

which gives us various first-order conditions:

$$\begin{cases} e^{-\rho t} U_c = \eta, \quad e^{\rho t} U_L + \eta f_L = 0 \\ f_K + b + \sigma a - \delta = 0 \\ e^{-\rho T} \varphi'(K_T) = \eta_T \end{cases}$$

and we still have the difficult BSDE terminal condition. We might avoid this by taking $\varphi \equiv 0$, which would still get us to $K_T = 0$ as a condition, but the problem is that we can only identify $b + \sigma a$, not a, b separately.

(iii) How if we allow the agent to split K between production and riskless return

$$dK = -\delta K - c + \{ f(\theta K, L)\, dt - \theta \sigma K\, dW \} + r(1-\theta) K\, dt \,?$$

At optimality, we should find $\theta = 1$.

The PL approach once again can be applied:

$$\max E \int_0^T \{ e^{-\rho t} U(c, L) + \eta(-\delta K - c + f(\theta K, L) + r(1-\theta) K)\, dt + \eta b K dt + a \eta \sigma \theta K dt \}$$
$$+ e^{-\rho T} \varphi(K_T) - \eta_T K_T + \eta_0 K_0$$

So we deduce

$$\begin{cases} e^{-\rho t} U_c = \eta \\ e^{\rho t} U_L + \eta f_L(\theta K, L) = 0 \\ K f_K(\theta K, L) - rK + a \sigma \eta K = 0 \qquad \text{(differentiate wrto } \theta) \\ \theta f_K + r(1-\theta) + \eta(b + \sigma \theta a) = 0 \\ e^{-\rho T} \varphi'(K_T) = \eta_T \end{cases}$$

No better ... there is still indeterminacy ..

<u>So what has been achieved ...?</u>

(iv) Return to the finite horizon example

$$dK = (Z f(K,L) - \delta K - C)\,dt$$

with objective

$$\Phi(K_0, Z_0) \equiv E\left[\int_0^T e^{-\rho t}\, U(C_t, L_t)\, dt\right]$$

where let's suppose that U is bounded above, $U(0,L) \equiv 0$, and once K hits 0 it remains there, with zero consumption thereafter.

If we cast the problem into P-L form we get $d\eta = \eta\,(a\,dW + b\,dt)$ and

$$\max E\left[\int_0^T \left\{ e^{-\rho t} U(C_t, L_t)\,dt + \eta_t \left(Z_t f(K_t, L_t) - \delta K_t - C_t\right) dt + K_t \eta_t b_t\, dt \right\} - \eta_T K_T + \eta_0 K_0 \right]$$

If we dig out the FOC's we get

$$\begin{cases} e^{-\rho t} U_C = \eta \\ e^{-\rho t} U_L + \eta Z f_L = 0 \\ Z f_K - \delta + b = 0 \end{cases}$$

which together determine C_t, L_t, b_t from K_t, Z_t, η_t. This allows us to write down the evolution for K, η (we need to make some choice for a; taking $a \equiv 0$ would be a natural place to start).

So let's give ourselves η_0 and using the FOCs generate a path K^*, η^* with corresponding C^*, L^*. This path may hit zero capital and get stuck there, that's allowed.

Now for a generic feasible K, L, C, we shall have

$$E \int_0^T e^{-\rho t} U(C_t, L_t)\, dt$$

$$= E\left[\int_0^T \left\{ e^{-\rho t} U(C_t, L_t) + \eta_t^* \left(f(K_t, L_t) - \delta K_t - C_t\right) + \eta_t^* b_t^* K_t \right\} dt - \eta_T^* K_T + \eta_0 K_0 \right]$$

$$\leq E \int_0^T \left\{ e^{-\rho t} U(C_t^*, L_t^*) + \eta_t^* \left(f(K_t^*, L_t^*) - \delta K_t^* - C_t^* \right) + \eta_t^* b_t^* K_t^* \right\} dt + \eta_0 K_0$$

$$= E\left[\int_0^T e^{-\rho t}\, U(C_t^*, L_t^*)\, dt + \eta_T^* K_T^* \right]$$

Thus we have achieved an <u>upper bound</u> for the objective

$$\Phi(K_0, Z_0) \leq E\left[\int_0^T e^{-\rho t} U(C_t^*, L_t^*)\,dt + \eta_T^* K_T^*\right]$$

as well as a <u>lower bound</u>

$$\Phi(K_0, Z_0) \geq E\left[\int_0^T e^{-\rho t} U(C_t^*, L_t^*)\,dt\right]$$

since (K^*, C^*, L^*) was feasible. The next stage is to vary η_0 (and also the choice of a) to bring these bounds close. If we are able to bring the bounds close, we not only have found a good approximation for the value, but we also have a reasonably good candidate for the optimal policy.

(V) Pawel points out a <u>snag</u> here, that <u>if wealth does hit zero</u> the FOCs don't hold thereafter so the PL analysis falls over at that point. So we need to go back to the original objective

$$\Phi(K_0) = \sup E\left[\int_0^T e^{-\rho t} U(C_t, L_t)\,dt + e^{-\rho T}\varphi(K_T)\right]$$

for some concave increasing φ defined and finite on $\mathbb{R}$, and introduce η to solve $d\eta = \eta(a\,dW + b\,dt)$ for some η_0 and we get that for any feasible (K, C, L)

$$E \int_0^T \{e^{-\rho t} U(C_t, L_t) + \eta(Zf(K,L) - \delta K - C) + \eta K b\}\,dt + \varphi(K_T)e^{-\rho T} + K_0\eta_0 - K_T\eta_T$$

$$= E\left[\int_0^T e^{-\rho t} U(C_t, L_t)\,dt + e^{-\rho T}\varphi(K_T)\right]$$

If we now dig out the FOCs for $t \in (0,T)$ we have

$$\begin{cases} e^{-\rho t} U_C = \eta \\ e^{-\rho t} U_L + \eta Z f_L = 0 \\ Z f_K - \delta + b = 0 \end{cases}$$

so this allows us to use (Z, K, η) as system state, which we can now evolve forward to generate $K^*, C^*, L^*, \eta^*, b^*$, depending on K_0, η_0 and Z. The FOC at T will typically <u>not</u> be satisfied by this trajectory.

What we shall have, since (K^*, C^*, L^*) is feasible, is that

$$\Phi(K_0) \geq E\left[\int_0^T e^{-\rho t} U(C_t^*, L_t^*)\,dt + e^{-\rho T}\varphi(K_T^*)\right]$$

$$= E\Big[\int_0^T \big\{e^{-\rho t} U(C_t^*, L_t^*) + \eta_t^*\big(Z_t f(K_t^*, L_t^*) - \delta K_t^* - C_t^*\big) + \eta_t^* K_t^* b_t^*\big\}\,dt + e^{-\rho T}\varphi(K_T^*) - \eta_T^* K_T^* + \eta_0 K_0\Big]$$

If we now set

$$H(t, \eta_t, b_t, Z) \equiv \sup_{K,C,L}\left[e^{-\rho t}U(C,L) + \eta_t\big(Zf(K,L) - \delta K - C\big) + \eta_t b_t K\right]$$

then the value for any feasible K, C, L is bounded by

$$E\left[\int_0^T H(t, \eta_t^*, b_t^*, Z_t)\,dt + e^{-\rho T}\varphi(K_T) - \eta_T^* K_T + \eta_0 K_0\right]$$

$$\leq E\left[\int_0^T H(t, \eta_t^*, b_t^*, Z_t)\,dt + e^{-\rho T}\tilde{\varphi}(e^{\rho T}\eta_T^*) + \eta_0 K_0\right]$$

So we get a bracket on the value

$$\boxed{\begin{aligned} &E\left[\int_0^T H(t, \eta_t^*, b_t^*, Z_t)\,dt + e^{-\rho T}\varphi(K_T^*) - \eta_T^* K_T^* + \eta_0 K_0\right] \\ &\leq \Phi(K_0) \\ &\leq E\left[\int_0^T H(t, \eta_t^*, b_t^*, Z_t)\,dt + e^{-\rho T}\tilde{\varphi}(e^{\rho T}\eta_T^*) + \eta_0 K_0\right] \end{aligned}}$$

The gap is now $E\left[e^{-\rho T}\tilde{\varphi}(e^{\rho T}\eta_T^*) - e^{-\rho T}\varphi(K_T^*) + \eta_T^* K_T^*\right]$ which we'd like to make small.

Of course, we could vary η_0 <u>independently</u> on the two sides of the inequality to get things even closer!

Another simple stochastic control example (24/10/12)

Here's a little example that Pawel Z. came up with – it's quite cute.
We got an income stream δ_t which is log-Brownian

$$d\delta = \delta(\sigma dW + \mu dt)$$

and then this is used to buy a stock of consumption good ξ_t which is perishable and evolves as

$$\dot{\xi}_t = \delta_t - \lambda\xi_t - c_t$$

If the objective is to attain

$$V(\xi,\delta) = \sup E\left[\int_0^\infty e^{-\rho t} U(c_t)\, dt \,\middle|\, \xi_0 = \xi, \delta_0 = \delta\right]$$

for CRRA U, then the HJB would be

$$0 = \sup\left[-\rho V + U(c) + \mu\delta V_\delta + \tfrac{1}{2}\sigma^2\delta^2 V_{\delta\delta} + (\delta - \lambda\xi - c) V_\xi\right]$$

and from scaling we would have

$$V(\alpha\xi, \alpha\delta) = \alpha^{1-R} V(\xi,\delta) \Rightarrow V(\xi,\delta) = \xi^{1-R} v(\delta/\xi) \equiv \xi^{1-R} v(x).$$

Hence HJB is

$$0 = \sup_{q\geq 0} \xi^{1-R}\left[-\rho v + U(q) + \mu x v' + \tfrac{1}{2}\sigma^2 x^2 v'' + (x - \lambda - q)((1-R)v - xv')\right]$$

so when we do the optimization this becomes

$$\boxed{0 = -\rho v + \tilde{U}((1-R)v - xv') + \mu x v' + \tfrac{1}{2}\sigma^2 x^2 v'' + (x-\lambda)((1-R)v - xv')}$$

Which you would think should be a candidate for the $\mathcal{L}$-$\mathcal{N}$ method

For the record: if $\theta \in (0,\infty)$

$$\int_{-\infty}^{\infty} \frac{e^{i\theta x}}{x - z_0} \frac{dx}{2\pi i} = \begin{cases} 0 & \text{if } \operatorname{Im} z_0 < 0 \\ e^{i\theta z_0} & \text{if } \operatorname{Im} z_0 > 0 \end{cases}$$

More general winding-number stories (24/10/12)

(i) A variant of McKean's winding number example takes for some $\alpha > 0$, $k > 0$

$$v(x) = \begin{cases} |x|^\alpha & (x > 0) \\ -K|x|^\alpha & (x < 0) \end{cases}$$

and then sets $\varphi_t = \int_0^t v(X_s)\,ds$, where X is a standard BM, not necessarily started at zero.

We would like to be able to calculate for θ in some interval the function

$$f(x,y) \equiv E\left[X_\tau^\theta \;\middle|\; X_0 = x,\ \varphi_0 = y \right]$$

which must satisfy

$$\boxed{\tfrac{1}{2} f_{xx} + v(x) f_y = 0}$$

with $f(x,0) = x^\theta$ for all $x > 0$. Here, as usual, $\tau \equiv \inf\{u : \varphi_u > 0\}$.

(ii) If we start the BM X at x, then

$$\tilde{X}_t \equiv c X_{t/c^2}$$

is a BM started at cx. We have that

$$\tilde{\varphi}_t - \tilde{\varphi}_0 \equiv \int_0^t v(\tilde{X}_s)\,ds = \int_0^t v(cX_{s/c^2})\,ds = c^{2+\alpha}\left(\varphi_{t/c^2} - \varphi_0\right).$$

So if we started $X_0 = x$, $\varphi_0 = y < 0$, $\tilde{\tau} \equiv \inf\{u : \tilde{\varphi}_u > 0\}$ will be exactly τ/c^2 if we start with $\tilde{\varphi}_0 = c^{2+\alpha} y$. Therefore scaling gives us

$$\boxed{f(cx, c^{2+\alpha} y) = f(x,y).\, c^\theta}$$

Accordingly, if we choose $c = |y|^{-1/(2+\alpha)}$, we shall see that for $y < 0$

$$f(x,y) = |y|^{\theta/(2+\alpha)} f\left(x/|y|^{1/(2+\alpha)}, -1 \right)$$

$$= |y|^{\theta/(2+\alpha)} g(\xi), \text{ say}$$

where $\xi \equiv x/|y|^{1/(2+\alpha)}$. The PDE for f tells us therefore that

$$\boxed{0 = |y|^{(\theta-2)/(2+\alpha)} \left[\tfrac{1}{2} g''(\xi) - v(\xi) \left\{ \frac{\theta}{2+\alpha} g(\xi) - \frac{1}{2+\alpha} \xi g'(\xi) \right\} \right].}$$

This is consistent for $\alpha = 1$ with the story on p 70 of WN <u>XXXIII</u> if you bear in mind $|y|$ vs y renormalization.

Martingales in the Style of Azéma (1/11/12)

(i) The classical Azéma martingales for a continuous martingale X_t with $S_t \equiv \sup_{u\le t} X_u$, $I_t \equiv \inf_{u\le t} X_u$ are things of the form

$$F(S_t) - (S_t - X_t) F'(S_t)$$

$$F(I_t) - (I_t - X_t) F'(I_t)$$

and these can be thought of as lying behind the paper "The joint law of the max and terminal value of a martingale". Is there some analogue for (I, X, S)?

(ii) Suppose we want

$$F(S_t, I_t) - (S_t - X_t) F_S(S_t, I_t)$$

to be a martingale. This only happens if

$$F_I = (S-I) F_{SI}$$

from which

$$\boxed{F_I(S,I) = q(I)(S-I)}$$

for some function q. This would give us

$$\boxed{F(S,I) = g(S) - \int_I^S q(y)(S-y)\,dy}$$

for some $g(\cdot)$. The analogue for the inf would be

$$\boxed{G(S,I) = h(I) + \int_I^S \tilde{q}(y)(I-y)\,dy}$$

and $G(S,I) - (I-X) G_I(S,I)$ is a martingale.

Any good choices of $g, h, q, \tilde{q}$?

(iii) Let's look at the first with $g(S) = (S-b)^+$, $q(y) = I_{\{y \le -a\}}$ for some $a, b > 0$.

We have (writing $\varepsilon \equiv (-a-I)^+$)

$$F(S,I) = (S-b)^+ - \int_{-a-\varepsilon}^{-a} (S-y)\,dy = (S-b)^+ - \tfrac{1}{2}\varepsilon^2 - \varepsilon(S+a)$$

and so the martingale becomes

$$M_t = (S_t-b)^+ - \tfrac{1}{2}\varepsilon_t^2 - \varepsilon_t(S_t+a) - (S_t-\cancel{X_t})\left[I_{\{S_t>b\}} - (-a-I_t)^+\right]$$

Up until the time that X leaves $[-a,b]$ this martingale will be zero. If we let $F = \{H_b \wedge H_{-a} < \infty\}$, we therefore get

$$0 = E\left[(S-b)^+ - \tfrac{1}{2}((-a-I)^+)^2 - (-a-I)^+(S+a) - (S-X)(I_{\{S>b\}} - (-a-I)^+) : F\right]$$

$$= E\left[(X-b)I_{\{S>b\}} - \tfrac{1}{2}((-a-I)^+)^2 - (-a-I)^+(a+X) : F\right]$$

We could equally integrate over $\{H_b < \infty,\ H_b < H_{-a}\}$, or $\{H_{-a} < H_b\}$.

(iv) There doesn't appear to be anything really new here. If we considered very special forms of $q(y)\,dy$, such as $\delta_b(dy)$, we would just get

$$\cancel{F(S,I) = g(S)} - (S-b)^+$$

and this is the old Azéma martingale. Anything else will simply be mixtures of such things, so again of the old form.

(v) If we use the Azéma martingale $(X_t-b)I_{\{S_t>b\}}$ we get (if we just turn this on on the event $I(H_b) < -a$)

$$\cancel{0 = E\left[X-b : S>b,\ I(H_b) < -a\right]}$$

$$= E\left[X-b : S>b,\ I<-a\right] - E\left[X-b : I(H_b) > -a > I,\ H_b < \infty\right]$$

$$= E\left[X-b : S>b,\ I<-a\right] + (a+b)\ P\left[H_b < H_{-a} < \infty\right]$$

So we could deduce $P[H_b < H_{-a} < \infty]$ from the joint law it seems!

Evolution of firm sizes (6/11/12)

We took initially a density for (θ, σ) of the form

$$m(\theta,\sigma) = B_\lambda \exp\left(-A\sqrt{\theta^2+2\tilde{\varepsilon}}\right)\left(\theta^2+2\tilde{\varepsilon}\right)^{\lambda/2} \sigma^{-\nu}$$

with $\tilde{\varepsilon} \equiv \varepsilon/\sigma^2$, and $\lambda = 0$ or 1. However, it appears possible to make the distribution asymmetric in θ and still keep quite a bit of tractability: let's see what happens if we try

$$m(\theta,\sigma) = B_\lambda \exp\left(\xi\theta - A\sqrt{\theta^2+2\tilde{\varepsilon}}\right)\left(\theta^2+2\tilde{\varepsilon}\right)^{\lambda/2} \sigma^{-\nu}$$

for some $|\xi| < A$. Normalization integral?

Case $\lambda = 0$:

We have

$$1 = B_0 \int_0^\infty \sigma^{-\nu}\, d\sigma \int \exp\left\{\xi\theta - A\sqrt{\theta^2+2\tilde{\varepsilon}}\right\} d\theta$$

$$= B_0 \int_0^\infty \sigma^{-\nu}\; 2A\, K_1(\gamma\delta)\, \delta/\gamma \; d\sigma \qquad \left[\delta \equiv \sqrt{2\varepsilon}/\sigma,\; \gamma \equiv \sqrt{A^2-\xi^2}\right]$$

$$= \frac{2AB_0}{\gamma}\int_0^\infty \sigma^{-\nu}\,\frac{\sqrt{2\varepsilon}}{\sigma}\, K_1\left(\gamma\sqrt{2\varepsilon}/\sigma\right) d\sigma \qquad \left[b = \gamma\sqrt{2\varepsilon},\; \sigma = b/v\right]$$

$$= \frac{2AB_0\sqrt{2\varepsilon}}{\gamma}\, b^{-\nu}\int_0^\infty v^{\nu-1} K_1(v)\, dv$$

$$= \frac{2AB_0\sqrt{2\varepsilon}}{\gamma}\, b^{-\nu}\, 2^{\nu-2}\,\Gamma\left(\tfrac{\nu-1}{2}\right)\Gamma\left(\tfrac{\nu+1}{2}\right).$$

Case $\lambda = 1$:

We have

$$1 = B_1 \int_0^\infty \sigma^{-\nu}\, d\sigma \int \exp\left\{\xi\theta - A\sqrt{\theta^2+\delta^2}\right\}\sqrt{\theta^2+\delta^2}\; d\theta$$

$$= B_1 \int_0^\infty \sigma^{-\nu}\left[\left(2\delta\xi^2/\gamma^3\right) K_1(\delta\gamma) + \left(\delta A/\gamma\right)^2\left\{K_0(\delta\gamma) + K_2(\delta\gamma)\right\}\right] d\sigma$$

$$= B_1 \int_0^\infty \sigma^{-\nu}\left[\frac{2\xi^2\sqrt{2\varepsilon}}{\gamma^3}\,\frac{1}{\sigma}\, K_1(b/\sigma) + \frac{2A^2\varepsilon}{\gamma^2}\,\frac{1}{\sigma^2}\left\{K_0(b/\sigma) + K_2(b/\sigma)\right\}\right] d\sigma$$

$$= B_1\left[\frac{2\xi^2\sqrt{2\varepsilon}}{\gamma^3}\, b^{-\nu}\int_0^\infty v^{\nu-1} K_1(v)\, dv + \frac{2A^2\varepsilon}{\gamma^2}\, b^{-\nu-1}\int_0^\infty v^{\nu}\left(K_0(v) + K_2(v)\right) dv\right]$$

$$= B\left[\frac{2\xi^2\sqrt{2\varepsilon}}{\gamma^3}\, b^{-\nu}\, 2^{\nu-2}\,\Gamma\left(\tfrac{\nu-1}{2}\right)\Gamma\left(\tfrac{\nu+1}{2}\right) + \frac{2A^2\varepsilon}{\gamma^2}\, b^{-\nu-1}\, 2^{\nu-1}\left\{\Gamma\left(\tfrac{\nu+1}{2}\right)^2 + \Gamma\left(\tfrac{\nu-1}{2}\right)\Gamma\left(\tfrac{\nu+3}{2}\right)\right\}\right]$$

The relaxed investor: a comment (13/11/12)

(i) When we looked at the relaxed investor, we had wealth dynamics

$$dW_t = rW_t\,dt + \theta_t \cdot \sigma\,(dW_t + (\alpha - r\sigma^{-1}1)\,dt) - c_t\,dt$$

but the parameter α wasn't known. If we see $X_t = W_t + \alpha t$, then giving ourselves a prior density $f_0(\cdot)$ for α, the posterior for α will be

$$\propto f_0(\alpha) \exp\left[\alpha \cdot X_t - \tfrac{1}{2}|\alpha|^2 t\right]$$

Now the state-price density if α was known would be

$$\zeta_t = \exp\left(-\kappa\, W_t - \tfrac{1}{2}|\kappa|^2 t - rt\right) = \exp\left(-\kappa\,(X_t - \alpha t) - \tfrac{1}{2}|\kappa|^2 t - rt\right)$$

where $\kappa = \alpha - r\sigma^{-1}1 = \sigma^{-1}(\mu - r)$. Thus if we mix over the density of α, we get

$$q_t^{-1} \int f_0(\alpha) \exp\left[\alpha \cdot X_t - \tfrac{1}{2}|\alpha|^2 t - \kappa \cdot (X_t - \alpha t) - \tfrac{1}{2}|\kappa|^2 t - rt\right] d\alpha$$

where q_t is the normalization $\int f_0(\alpha) \exp\left[\alpha \cdot X_t - \tfrac{1}{2}|\alpha|^2 t\right]$. So the SPD will be

$$q_t^{-1} \int f_0(\alpha) \exp\left[r\,(\sigma^{-1}1) \cdot X_t - \tfrac{1}{2}|\alpha - \kappa|^2 t - rt\right] d\alpha$$

$$= q_t^{-1} \int f_0(\alpha) \exp\left[r\,(\sigma^{-1}1) \cdot X_t - \tfrac{1}{2} r^2 |\sigma^{-1}1|^2 t - rt\right] d\alpha$$

$$= q_t^{-1} \exp\left[r\,(\sigma^{-1}1) \cdot X_t - \tfrac{1}{2} r^2 |\sigma^{-1}1|^2 t - rt\right]$$

So the trick here is to simplify / evaluate $q_t = \int f_0(\alpha) \exp(\alpha X_t - \tfrac{1}{2}|\alpha|^2 t)\,d\alpha$

(ii) If we were using $f_0(\alpha) = \exp\left(-\tfrac{1}{2}(\alpha - \hat\alpha_0)\cdot\tau_0(\alpha - \hat\alpha_0)\right)(2\pi)^{-d/2}(\det \tau_0)^{\frac{1}{2}}$ we obtain

$$q_t = \sqrt{\frac{\det \tau_0}{\det \tau_t}}\, \exp\left[\tfrac{1}{2}\hat\alpha_t \cdot \tau_t \hat\alpha_t - \tfrac{1}{2}\hat\alpha_0 \cdot \tau_0 \hat\alpha_0\right]$$

Using this, with some calculation we arrive at the same expression as in the relaxed investor paper.

<u>Simulation methodology for liquidity problem (18/11/12)</u>

The simulation approach to FBH gives a way to deal with some stochastic control problems. A couple of examples.

(i) If we have a vector of assets to invest in,

$$dw_t = r w_t \, dt + \theta_t \cdot (\sigma \, dW_t + (\mu - r) \, dt)$$

and objective $\max E\, U(w_T)$, then introducing the Lagrangian process $d\eta = \eta(a \, dW + b \, dt)$ along with the P-L approach gives

$$\max E\Big[U(w_T) + \int_0^T \Big\{ \eta_t \big(\theta_t (\mu - r) + r w \big) + w \eta b + \eta \, a^T \sigma^T \theta \Big\} dt + \eta_0 w_0 - \eta_T w_T \Big]$$

so optimizing over w, θ would tell us that

$$\begin{cases} r + b = 0 \\ \sigma a + \mu - r = 0 \end{cases}$$

but we <u>can't discover θ from this</u>. Nevertheless, in a complete market with an η which is unique up to a scalar factor, we could still find the value.

(ii) If we look at the liquidity costs example,

$$dw_t = r w_t \, dt + H_t (dS_t - r S_t \, dt) - c_t \, dt - \varepsilon f(h_t) S_t \, dt, \quad h \equiv \dot{H}$$

and we try to $\max E\, U(w_T - S_T \varphi(H_T))$ (so there's a penalty for trying to liquidate stock at the end) we could do PL to get

$$\max E\Big[U(w_T - S_T \varphi(H_T)) + \int_0^T \Big\{ \eta_t \big(r w + HS(\mu - r) - \varepsilon S f(h) \big) + w \eta b + a H S \sigma \eta + y(\dot{H} - h) \Big\} dt + \eta_0 w_0 - \eta_T w_T \Big]$$

$$= \max E\Big[U(w_T - S_T \varphi(H_T)) + \int_0^T \Big\{ \eta \big(w(r+b) + HS(a\sigma + \mu - r) - \varepsilon S f(h) \big) - \dot{y} H - y h \Big\} dt + \eta_0 w_0 - \eta_T w_T + y_T H_T - y_0 H_0 \Big]$$

Now we get

$$\begin{cases} r + b = 0 \\ \eta S (a\sigma + \mu - r) - \dot{y} = 0 \\ \eta \varepsilon S f'(h) = -y \end{cases}$$

The state variable for the evolution would be (w, S, η, y, H), but the problem is that we seem to need to make a choice of a ... Not unexpected, as this is an incomplete market ...

(iii) A related problem would seem to be how we could solve the Brownian integral representation pathwise, because this is in effect what is required in problem (i); we know that we want to achieve terminal wealth $I(\lambda \zeta_T)$, but the issue is how we can do that. The classical approach of calculating $E[F(W_T) | \mathcal{F}_t]$ isn't going to work in higher dimensions, because storing an approximation to that function is too much.

So let's suppose that the terminal random variable can be represented as

$$Y = G\left((\xi_{j/N})_{j=1,\dots,N} \right)$$

where $\xi_{j/N} = B_{j/N} - B_{(j-1)/N}$ and G is fairly nice. Now let's make ourselves a population $\xi^{(k)}$, $k=1,\dots,K$, of paths which are to be our representative sample space. Each of the terms $\xi^{(k)}_j$ will be a vector in $\mathbb{R}^d$ where all but one of the components is zero, and the remaining component takes one of the values $\pm\varepsilon$, where $\varepsilon = \sqrt{d/N}$. We pick the component that's non-zero uniformly from $1,\dots,d$, and then we pick the sign independently equiprobably.

Now we simulate an actual path $(x_j)_{j=1}^N$ of RW increments just as we did the $\xi^{(k)}$ and we define a path $\xi^{(k,m)}$ as

$$\xi^{(k,m)}_j = \begin{cases} x_j & (j \le m) \\ \xi^{(k)}_j & (j > m) \end{cases}$$

When we are at time m, and have seen x up to time m, we approximate $Y_{m/N} = E(Y | \mathcal{F}_{m/N})$ by

$$\frac{1}{K} \sum_{k=1}^{K} G(\xi^{(k,m)})$$

Now at the next time step, the value x_{m+1} will be revealed, and $\xi^{(k,m)}$ will get changed to $\xi^{(k,m+1)}$, by altering <u>just</u> the time-$(m+1)$ component. For each of the $2d$ possible values of $\xi^{(k,m+1)}$ we can compute $G(\xi^{(k,m+1)})$ and the changes gives us a good idea what the hedge should be there.

We write $\xi^{(k,m)}(i,\pm)$ for the path $\xi^{(k,m)}$ where the $(m+1)^{\text{th}}$ entry is changed to $\pm\varepsilon$ in the i^{th} position. We would ideally hope to find

$$G(\xi^{(k,m)}) - \tfrac{1}{2}\left[G(\xi^{(k,m)}(i,+)) - G(\xi^{(k,m)}(i,-)) \right] = 0$$

This probably won't happen, but the extent to which it fails could be useful in assessing the hedging error ...?

Winding number problem again (24/11/12)

(i) We have X_t is a BM, $Y_t = \int_0^t v(X_s)\,ds$ where $x v(x) > 0$ for $x \neq 0$. The original approach to these problems was to look for separable solutions. But there are other things we could try. For example, if $G(x) = \int_0^x \int_0^y 2v(z)\,dz\,dy$, then

$$G(X_t) - Y_t \quad \text{~~is a martingale~~}$$

and G is positive convex in $(0,\infty)$, negative concave in $(-\infty,0)$ and increasing everywhere.

We could therefore regard $u(x,y) \equiv G(x) - y$ as a harmonic coordinate

(ii) More generally, if we consider the exponential martingales

$$e^{\lambda Y_t} f(X_t;\lambda) = e^{\lambda Y_t} \sum_{n\geq 0} \lambda^n f_n(X_t)$$

we get that $\frac{1}{2} f'' + \lambda v f = 0$, and we have

$$\begin{cases} f_0(x) \equiv 1 \\ \frac{1}{2} f_{n+1}'' + v f_n = 0 \end{cases}$$

giving us

$$e^{\lambda Y_t} f(X_t;\lambda) = \sum_{m\geq 0} \lambda^m \left\{ \sum_{j=0}^{m} \frac{Y_t^j}{j!} f_{m-j}(X_t) \right\}$$

and a sequence of martingales

$$M_t^n = \sum_{j=0}^{n} \frac{Y_t^j}{j!} \cdot f_{n-j}(X_t).$$

The first of these is the martingale identified in (i) above. Just scraps really.

(iii) Note Comparing the solutions for (a) $v(x) = I_{(x>0)} - I_{[-1,0]}(x)$ and (b) $v(x) = |x|^\alpha \operatorname{sgn}(x)$ we find the special function for case (a) has to be

$$f(x) = \begin{cases} \sinh \frac{1}{2}\pi x & (x>0) \\ \pm \sin \frac{1}{2}\pi x & (x<0) \end{cases}$$

which solves $\frac{1}{2} f'' - (\frac{1}{2}\pi)^2 . 2^{-1} . f . v(x) = 0$; but the special function in case (b) is

$$f(x) = \begin{cases} x^{2+\alpha} & (x>0) \\ \pm x^{2+\alpha} & (x<0) \end{cases}$$

which does not satisfy any corresponding equation $f'' + \lambda v f = 0$.

Agents with different lookbacks (28/11/12)

Let's return to this story which was introduced in WN XXXII, 3-4. The idea there was to a discrete-time model where we observe $Y_t = \mu_t + X_t$, where μ_t is a RW, and X_t are IID Gaussian noises. Could we tell a continuous-time story with similar elements?

(i) Let's suppose that we observe $(Y_t)_{t\geq 0}$ with dynamics

$$\begin{cases} dY_t = dW_t + \mu_t dt \\ d\mu_t = \sigma dW'_t \end{cases}$$

Now when we do steady-state Kalman filter on this, we get for some q

$$\begin{cases} dY_t = d\hat{W}_t + \hat{\mu}_t dt \\ d\hat{\mu}_t = q\, d\hat{W}_t \end{cases}$$

so that we shall see that Y solves

$$dY_t = d\hat{W}_t + (\hat{\mu}_0 + q\hat{W}_t)\, dt$$

in the observation filtration. By absorbing $\hat{\mu}_0$ into the initial condition for $\hat{W}_t$, we can suppose $\hat{\mu}_0 = 0$; then we get

$$\hat{W}_t = e^{-qt}\left(\hat{W}_0 + \int_0^t e^{qs}\, dY_s\right)$$

so that

$$dY_t = d\hat{W}_t + \bar{Y}_t(q)\, dt$$

where

$$\bar{Y}_t(q) \equiv q e^{-qt}\hat{W}_0 + \int_0^t q e^{q(s-t)}\, dY_s$$

is a EWMA of the increments of Y. With this model for Y, the law of Y has density

$$\Lambda_t \equiv \exp\left[\int_0^t \bar{Y}_s(q)\, dY_s - \frac{1}{2}\int_0^t \bar{Y}_s(q)^2\, ds\right]$$

w.r.t. Wiener measure. We also have

$$d\bar{Y}_t(q) = -q\bar{Y}_t(q)\, dt + q\, dY_t.$$

(ii) In an equilibrium story for agents with CARA preferences, we'd get

$$e^{-\rho t - \gamma_j c^j_t}\Lambda^j_t \propto \zeta_t$$

so

$$\log c^j_t = \frac{1}{\gamma_j}\left\{\log\Lambda^j_t - \rho t - \log\zeta_t\right\} + \text{const}$$

Supposing that Y is total output, $\Gamma^{-1} = \sum \gamma_j^{-1}$, we get that

$$Y_t = \frac{1}{\Gamma}\left\{ -\rho t - \log \zeta_t \right\} + \sum \tfrac{1}{\gamma_j} \log \Lambda^j_t \quad + \text{const.}$$

Hence

$$\boxed{\log \zeta_t = -\rho t - \Gamma Y_t + \sum p_j \log \Lambda^j_t \quad + \text{const}}$$

where $p_j = \Gamma/\gamma_j$. We lose no generality in taking the constant to be zero.

How about calculating the price of the asset? We'd need to calculate

$$\frac{1}{\zeta_t} E_t \left[\int_t^\infty \zeta_u Y_u \, du \right]$$

or perhaps more tractably

$$\frac{1}{\zeta_t} E_t \left[\int_t^\infty \zeta_u \exp(-\alpha Y_u) \, du \right]$$

which we could then differentiate. For this, it would be (in principle) enough to find

$$E\left[\zeta_T \exp(-\alpha Y_T) \mid \mathcal{F}_0 \right]$$

and due to the exponential-quadratic nature of Λ^j, ζ, we may expect that

$$E\left[\zeta_T \exp(-\alpha Y_T) \mid \mathcal{F}_0 \right] = \exp\left[-\alpha Y_0 - \tfrac{1}{2} \bar{Y}_0 \cdot A(T) \bar{Y}_0 - B(T) \cdot \bar{Y}_0 - C(T) \right]$$

where $\bar{Y}_t$ is the vector $(Y_t(q_1), \dots, Y_t(q_J))$, and the functions A, B, C must be identified. The key fact that gets us there is to use

$$\zeta_t e^{-\alpha Y_t} \exp\left[-\tfrac{1}{2} \bar{Y}_t \cdot A(T-t) \bar{Y}_t - B(T-t) \cdot \bar{Y}_t - C(T-t) \right] \text{ is a martingale}$$

Thus if we write (with T fixed for now)

$$Z_t = \log \zeta_t - \alpha Y_t - \tfrac{1}{2} \bar{Y}_t \cdot A(T-t) \bar{Y}_t - B(T-t) \cdot \bar{Y}_t - C(T-t)$$

then it's all about understanding Z. Abbreviate $\bar{Y}_t(q_i) \equiv \bar{Y}^i_t$, and notice that $d\bar{Y}^i = q_i \, dY - q_i \bar{Y}^i dt$. Hence

$$dZ = -\rho dt - \Gamma dY + \sum p_j \left(\bar{Y}^j dY - \tfrac{1}{2} (\bar{Y}^j)^2 dt \right) - \alpha dY - \sum \bar{Y}^i A_{ij} d\bar{Y}^j - \sum B_i d\bar{Y}^i$$
$$+ \left\{ \tfrac{1}{2} \bar{Y} \cdot \dot{A} \bar{Y} + \dot{B} \cdot \bar{Y} + \dot{C} \right\} dt - \tfrac{1}{4} \sum A_{ij} d\bar{Y}^i d\bar{Y}^j$$

$$= \left\{ -\Gamma - \alpha + \sum p_j \bar{Y}_j - \sum q_i A_{ij} \bar{Y}^j - \sum q_i B_i \right\} dY +$$

$$+\Big\{-\rho - \tfrac{1}{2}\sum p_j (\bar{y}^j)^2 + \sum q_i \bar{y}_i A_{ij} \bar{y}^j + \sum q_i B_i \bar{y}^i \quad - \tfrac{1}{4}\sum q_i q_j A_{ij}$$

$$+\left(\tfrac{1}{2}\bar{y}\cdot\dot{A}\bar{y} + \dot{B}\bar{y} + \dot{C}\right)\Big\}dt$$

Using the fact that $\exp(Z)$ is a martingale, we conclude that the drift has to be zero:

$$0 = -\rho - \tfrac{1}{2}\sum p_j (\bar{y}^j)^2 + \sum q_i \bar{y}_i A_{ij} \bar{y}_j + \sum q_i B_i \bar{y}_i - \tfrac{1}{4}\sum q_i q_j A_{ij}$$

$$+ \tfrac{1}{2}\bar{y}\dot{A}\bar{y} + \dot{B}\bar{y} + \dot{C}$$

$$+\tfrac{1}{2}\left(-\Gamma - \alpha + p\cdot\bar{y} - q\cdot A\bar{y} - q\cdot B\right)^2$$

Pull out the terms quadratic in $\bar{y}$. We find

$$\boxed{0 = q_i A_{ij} + \tfrac{1}{2}\dot{A}_{ij} + \tfrac{1}{2}(p - A^T q)_i (p - A^T q)_j}$$

Next the linear terms in $\bar{y}$:

$$\boxed{0 = q_i B_i + \dot{B}_i - (\Gamma + \alpha + q\cdot B)(p - A^T q)_i}$$

And finally the terms that don't depend on $\bar{y}$:

$$\boxed{0 = -\rho - \tfrac{1}{4} q\cdot Aq + \dot{C} + \tfrac{1}{2}(\alpha + \Gamma + q\cdot B)^2}$$

This is a Ricatti-style equation for A, linear for B, an integration for C.

This might be OK numerically, but any chance of an analytic solution ...?

If we check out the KF story, we find that H^{y} must be symmetric, and in fact $H^{y} = \Sigma_{\alpha}$, where Σ_{α} is the covariance of α.

Agents with different views: another try? (1/12/12)

Let's suppose that we have K assets,

$$dS^k_t = S^k_t \, \sigma_{kj} \, dY^j_t$$

where

$$dY_t = dW_t + \alpha_t \, dt.$$

Now we have different agents who observe the market; all see the same process Y, and agent y thinks that α is evolving as a Brownian motion with zero drift and some covariance. So when agent y does the filtering calculation, he forms estimate $\hat{\alpha}^y$ evolving like

$$\begin{cases} dY = d\hat{W}^y + \hat{\alpha}^y \, dt \\ d\hat{\alpha}^y = H^y \, d\hat{W}^y = H^y (dY - \hat{\alpha}^y dt) \end{cases}$$

Now let's suppose that he calculates his $\hat{\alpha}^y$ but for the purpose of investment pretends that $\hat{\alpha}^y$ is the true constant value of the drift. So the fractions of wealth he puts in the different assets are given by $(a \equiv \sigma\sigma^T)$

$$\pi^y = R^{-1} a^{-1} (\sigma \hat{\alpha}^y - r 1) = R^{-1} (\sigma^T)^{-1} (\hat{\alpha}^y - b)$$

where $b = r\sigma^{-1}1$. All agents share same R, let's suppose. For now, there is no consumption. Market clearing would require

$$\sum_y w^y \pi^y = S$$

if we suppose each asset is in unit net supply. Now we have for the evolution of w^y

$$dw^y = w^y \left[r\,dt + \pi^y \cdot \sigma (dY - b\,dt) \right]$$

$$= w^y \left[r\,dt + \frac{1}{R} (\hat{\alpha}^y - b) \cdot (dY - b\,dt) \right]$$

and

$$d\pi^y = R^{-1} (\sigma^T)^{-1} H^y (dY - \hat{\alpha}^y dt)$$

so we get

$$d(\pi^y w^y) = \pi^y w^y \left[r\,dt + \frac{1}{R} (\hat{\alpha}^y - b) (dY - b\,dt) \right] + w^y R^{-1} (\sigma^T)^{-1} H^y (dY - \hat{\alpha}^y dt)$$

$$+ \frac{dt}{R} (\sigma^T)^{-1}_{\cdot i} H^y_{ij} \frac{w^y}{R} (\hat{\alpha}^y_j - b_j)$$

$$= \left(\frac{\pi^y w^y}{R} (\hat{\alpha}^y - b) + w^y R^{-1} (\sigma^T)^{-1} H^y \right) dY$$

$$+ \left\{ \pi^y w^y \left(r - \frac{1}{R} (\hat{\alpha}^y - b) \cdot b \right) - w^y R^{-1} (\sigma^T)^{-1} H^y \hat{\alpha}^y \right.$$

$$\left. + \frac{1}{R^2} (\sigma^T)^{-1} H^y (\hat{\alpha}^y - b) w^y \right\} dt.$$

Let's first look at the coefficient of $\sigma \, dY$, which comes out as

$$w^y \pi^y (\pi^y)^T + w^y R^{-1} (\sigma^T)^{-1} H^y \sigma^{-1}$$

and when we sum this on y we should get $\mathrm{diag}(S)$: that is,

$$\boxed{\sum_y \left\{ w^y \pi^y (\pi^y)^T + w^y R^{-1} (\sigma^T)^{-1} H^y \sigma^{-1} \right\} = \mathrm{diag} \left(\sum w^y \pi^y \right)} \qquad (1)$$

Similarly, if we sum over y, the dt term should vanish. Let's simplify for now by assuming $\boxed{R = 1}$ so that the drift terms give

$$0 = \sum_y w^y \left\{ \pi^y \left(r - (\hat{\alpha}^y - b) \cdot b \right) - (\sigma^T)^{-1} H^y b \right\}$$

$$= \sum_y w^y \left\{ \pi^y \left(r - (\pi^y)^T \sigma b \right) - (\sigma^T)^{-1} H^y \sigma^{-1} \sigma b \right\}$$

$$= r S - \sum_y \left\{ w^y \pi^y (\pi^y)^T + w^y (\sigma^T)^{-1} H^y \sigma^{-1} \right\} \sigma b$$

$$= r S - \mathrm{diag}(S) \, \sigma b \qquad \text{using (1)}$$

$$= 0$$

so this is OK. So the key thing is to ensure that (1) holds (where $R = 1$):

$$\sum_y w^y \left\{ \pi^y (\pi^y)^T + (\sigma^T)^{-1} H^y \sigma^{-1} \right\} = \mathrm{diag} \left(\sum w^y \pi^y \right)$$

Two-sided exit problem again (17/12/12)

(1) One thing we don't yet seem to be using much is the information about the sets where the measures are concentrated. If we care about exit from $[-a, b]$, where $a, b > 0$, then we have

$$f(x) \equiv \int_0^\infty c e^{-ct} P(X_t \in dx)\Big/dx = \int_0^\infty c\, P(X_t \in dx, \tau > t)\Big/dx + \int_0^\infty e^{-c\tau} P(X_\tau \in dy)\, f(x-y)$$

$$= m(dx)/dx + \int \pi(dy)\, f(x-y)$$

if we assume wmlog that there is a density f for $X(T)$, with $T \sim \exp(c)$. Now the measure m lives in $[a,b]$, π lives in $[a,b]^c$ (here, of course, $\tau = \inf\{t : X_t \notin [a,b]\}$).
In terms of the Fourier transforms

$$\boxed{\hat f(\theta) = \hat m(\theta) + \hat\pi(\theta)\, \hat f(\theta)}$$

(2) The Fourier transform is a map $\mathcal{F} : L^2(\mathbb{R};\mathbb{C}) \to L^2(\mathbb{R};\mathbb{C})$ with the properties

$$\mathcal{F}(f * g) = (\mathcal{F}f).(\mathcal{F}g), \qquad (\mathcal{F}^2 f)(x) = 2\pi f(-x).$$

If we want to drop a measure down onto $[-a, b]$, and we set $h(x) = I_{[-a,b]}(x)$, we have

$$\hat h(\theta) = \left(e^{i\theta b} - e^{-i\theta a}\right)/i\theta.$$

Now if we have a measure with density g, we want the Fourier transform of gh. Since

$$g(x) = (2\pi)^{-1}(\mathcal{F}\hat g)(-x), \qquad h(x) = (2\pi)^{-1}(\mathcal{F}\hat h)(-x)$$

we get

$$\mathcal{F}(gh)(\theta) = (2\pi)^{-2}\, \mathcal{F}((\mathcal{F}\hat g)(\mathcal{F}\hat h))(-\theta)$$

$$= (2\pi)^{-2}\, \mathcal{F}(\mathcal{F}(\hat g * \hat h))(-\theta)$$

$$= (2\pi)^{-1}(\hat g * \hat h)(\theta)$$

So we take the FT of g and we form

$$\boxed{(P\hat g)(\theta) = \frac{1}{2\pi}\int \hat g(\theta - t)\, \hat h(t)\, dt = \int \frac{e^{ibt} - e^{-iat}}{it}\, \hat g(\theta - t)\, \frac{dt}{2\pi}.}$$

If we wanted to restrict g to the complement of $[a,b]$, then we would use (of course!)

$$(Q\hat g)(\theta) \equiv \hat g(\theta) - (P\hat g)(\theta).$$

One thing that may be worth remarking: for $\theta \in \mathbb{R}$,

$$1 - \hat{\pi}(\theta) = 1 - \int e^{i\theta x} E[e^{-c\tau} ; X_\tau \in dx]$$

has strictly positive real part, and is bounded. Since $\hat{f}(\theta) = c/(c - \psi(i\theta))$ also has strictly positive real part, and is bounded by 1 in modulus, we can make a continuous determination of $\arg \hat{m}(\theta) \in (-\pi, \pi)$, θ in $\mathbb{R}$.

No, what happens is this. If there are J agents, A assets, N states we have to choose $\theta_{ja}(x)$ (so $(J-1)AN$ things to choose once we respect market clearing) and then we define $S_a(x)$ by

$$S_a(x) = \frac{1}{f_j(x)} (\rho - \Omega)^{-1} (f_j \delta_a Y)(x)$$

for some particular choice j^* of j, which leaves $(J-1)$ other agents to check for.

A question for Ezequiel Antor (1/1/13)

1) Ezequiel has been looking at various questions about securitization of risks in an incomplete market, and it's not really yielding much. So far, he has been looking mainly at terminal wealth criteria in a Brownian setting.

To me, it seems that if there's no intermediate consumption, the notion of prices at intermediate times is rather ill-specified; and secondly that unless there is a Markovian structure to the model we won't ever be able to solve anything in closed form.

2) So maybe the thing to do is to work with infinite-horizon Markov chain stories instead! Let's suppose the chain is finite and ergodic, and let's suppose that agent j receives a flow $\xi_j(X_t)\,dt$ of consumption good, and has objective

$$E\left[\int_0^\infty e^{-\rho t}\, U_j(c_t)\, dt\right]$$

where ρ is for simplicity assumed to be the same for all j. This might not be a whole lot easier though. The two extreme situations are for a central planner, where all consumption good is optimally redistributed, and the situation where each agent consumes only his endowment.

3) Let's now imagine a situation where new assets (in zero net supply) are introduced to create a financial market. For simplicity, I want to consider only steady-state markets. So we want that asset a should pay a stream $\delta_a(X_t)\,dt$ of consumption good, and have market price $S_a(X_t)$ at time t. If agent j holds θ_{ja} of asset a, then his wealth evolves as

$$dw^j = \sum_a \theta_{ja}\,(dS_a + \delta_a\, dt) + (\xi_j - c^j)\, dt. \qquad (?)$$

Let's look for a solution where $w^j_t = w_j(X_t)$, $c^j_t = c_j(X_t)$, $\theta_{ja} = \theta_{ja}(X_t)$. This way, we'd be insisting that (with $f_j(x) \equiv U_j'(c_j(x))$)

$$\boxed{\begin{aligned}
c_j(x) &= \xi_j(x) + \sum_a \theta_{ja}(x)\,\delta_a(x)\\
w_j(y) - w_j(x) &= \sum_a \theta_{ja}(x)\{S_a(y) - S_a(x)\}\\
f_j(x)\,w_j(x) &= (\rho - Q)^{-1}(f_j\,\xi_j)(x)\\
f_j(x)\,S_a(x) &= (\rho - Q)^{-1}(f_j\,\delta_a)(x)\\
\sum_j \theta_{ja}(x) &= 0
\end{aligned}}$$

How will defaults affect firm capital? (3/1/13)

(i) Thinking back to the FBH story where now we propose to have $O(1)$ defaults rather than the infinitesimal defaults of the original story... how is this to be done?

One story which would be rather neat would be if there was some probability density $\rho(\cdot)$ of individual firm sizes, and that firms of size x get killed at rate $\lambda(x)$. If we require that at all times $t \geq 0$ the density of surviving firms should be a scaled version of the original density, then we would have the condition

$$\boxed{\rho(x)\, e^{-t\lambda(x)} = \varphi(t)\, \rho(x\theta(t))}$$

for some functions $\varphi(t)$, $\theta(t)$ to be discovered.

(ii) Writing $\rho(x) = \exp q(x)$, the boxed equation says $\quad g(t) = \log \varphi(t)$

$$(1) \qquad q(x) - t\lambda(x) = g(t) + q(x\theta_t)$$

$$\Rightarrow \left(\tfrac{\partial}{\partial t}\right) \qquad -\lambda(x) = g'(t) + x\theta'\, q'(x\theta)$$

$$(1) \Rightarrow \left(\tfrac{\partial}{\partial x}\right) \qquad -\lambda'(x) = \theta'\left[\, q'(x\theta) + x\theta\, q''(x\theta)\right]$$

$$\Rightarrow \left(\tfrac{\partial}{\partial t}\right) \qquad 0 = \theta''\left[q'(x\theta) + (x\theta)q''(x\theta)\right] + x\theta'^2\left[2q''(x\theta) + x\theta\, q'''(x\theta)\right]$$

and this last holds identically for $x > 0$, $t > 0$. By taking $x = 1/\theta(t)$ we learn that

$$0 = \theta''(t)\left[q'(1) + q''(1)\right] + \frac{(\theta')^2}{\theta}\left[2q''(1) + q'''(1)\right]$$

so we deduce that $\frac{d}{dt}\left(\theta_t^{\nu}\, \theta_t'\right) = 0$ for some $\nu = \left(2q''(1) + q'''(1)\right)/\left(q'(1) + q''(1)\right)$

This gives us

$$\theta_t^{\nu+1} = A + Bt$$

for some constants A, B; since $\theta_0 = 1$, we shall have $\theta(t) = (1+Bt)^{\gamma}$ for some exponent γ, and constant $B \geq 0$. Thus

$$\theta' = \gamma B (1+Bt)^{\gamma-1} = \gamma B\, \theta^{(\gamma-1)/\gamma}$$

so if we multiply both sides of (1) by $x^{(\gamma-1)/\gamma}$ we learn

$$-x^{(\gamma-1)/\gamma}\lambda'(x) = \gamma B\,(x\theta)^{(\gamma-1)/\gamma}\left[q'(x\theta) + (x\theta)q''(x\theta)\right]$$

so the RHS depends on θ, the LHS does not; therefore both are constants, and we deduce that $\boxed{\lambda(x) = a + b x^{1/\gamma}}$ for some constants a, b. It seems reasonable

$$\rho(x) = \alpha \left(\frac{b}{B}\right)^{(k-1)/\alpha} x^{-k} \exp\left\{-\frac{b}{B}x^{-\alpha}\right\} \Big/ \Gamma\left(\frac{k-1}{\alpha}\right)$$

if you want the constants

to suppose that λ is decreasing since smaller firms would be more likely to go bust than big ones, so we'd postulate $1/\gamma = -\alpha < 0$:

$$\lambda(x) = a + bx^{-\alpha}$$

for some $a, b \geq 0$, $\alpha \geq 0$.

Now we have $\theta(t) = (1+Bt)^{-1/\alpha} \Rightarrow t = (\theta^{-\alpha}-1)/B$, and so returning to (2) we see that

$$q(x) - \frac{\theta^{-\alpha}-1}{B}(a+bx^{-\alpha}) = g(t) + q(x\theta)$$

Differentiate with respect to x to learn that

$$q'(x) + \frac{\theta^{-\alpha}-1}{B}\alpha b x^{-\alpha-1} = \theta q'(x\theta)$$

so

$$x q'(x) - \frac{\alpha b}{B} x^{-\alpha} = \theta x q'(x\theta) - (\theta x)^{-\alpha}\frac{\alpha b}{B}$$

Since θ varies independently of x, the only possibility is that this expression is constant and

$$\boxed{q(x) = -\frac{b}{B}x^{-\alpha} - k\log x}$$

The only possible profiles ρ therefore are

$$\boxed{\rho(x) \propto x^{-k}\exp\left(-\frac{b}{B}x^{-\alpha}\right)}$$

where we must have $k > 1$ for integrability.

(iii) How would this profile get killed off? If you look at the total mass of the profile, this drops off initially at rate $\frac{k-1}{\alpha} + a$, and in fact we have

$$K(t) = (1+Bt)^{-(k-1)/\alpha} e^{-at} K(0)$$

If we suppose that $a = 0$, then

$$\boxed{\frac{\dot{K}}{K} = -B\frac{k-1}{\alpha}(1+Bt)^{-1} = -B\frac{k-1}{\alpha}\left(K/K_0\right)^{\alpha/(k-1)}}$$

which is quite neat. Probably more of a fiddle than we want to be dealing with just at this moment.

$$\sigma = I - \frac{\gamma\gamma^T}{1 + |\gamma|^2}$$

Pricing impact of differing views again (8/1/13)

(1) The story from p 20 is promising, but the assumption of constant covariation can't work with the structural form we seek. Ideally we would want to have

$$dS^i = S^i\left[\,\sigma_{ij}\,(dW^j + \kappa^j dt) \;+ r\,dt\right]$$

for some constant σ. By taking suitable portfolios of the assets, we could wlog take σ to be the identity, and we shall do so. Now the idealization would be

$$dS^i = S^i\left[\,\delta_{ij}\,(dW^j + \kappa^j dt) + r\,dt\right]$$

but we shall modify this to (re-using the symbol σ)

$$dS^i = S^i\left[\,\sigma_{ij}\,(dW^j + \kappa^j dt) + r\,dt\right]$$

where $\boxed{\sigma = I + \tilde{z}\tilde{z}^T}$ (*)

for some vector semimartingale $\tilde{z}$.

[Aside: the only equation we have for determining σ will be market clearing, which tells us that N semimartingales are identically zero, where N is the number of assets. We cannot therefore expect to identify a general $N\times N$ semimartingale σ. We could try to perturb σ away from I by taking some diagonal σ, but numerical exploration shows that this doesn't easily account for increased correlation. This approach (*) could be interpreted as what would happen if the drivers dW^j were each affected by a common independent factor]

Notice that a consequence of (*) is that

$$(\sigma^T)^{-1} = I + z z^T$$

for some z (which is proportional to $\tilde{z}$), and this is the form in which we will apply it. Let's have notation

$$\begin{cases} dY^j = dW^j + \kappa^j dt \\ dZ = \sigma\, dY \end{cases}$$

so that $dS/S = dZ + r\,dt$, and let's also suppose that

$$dz_k = \alpha_{kj}\, dZ^j + \beta_k\, dt$$

(2) As before, all agents observe the process Y, but have different beliefs about the evolution of κ. Telling a Kalman filtering story (which of course is not strictly

justified here) gives as before

$$\begin{cases} dY = d\hat{W}^y + \hat{\kappa}^y dt \\ d\hat{\kappa}^y = H^y d\hat{W}^y = H^y(dY - \hat{\kappa}^y dt) \end{cases}$$

for some matrix H^y which we'll assume has a constant steady-state value. Agent y will choose portfolio proportions (Merton rule)

$$\pi^y = R^{-1}(\sigma\sigma^T)^{-1}\hat{\kappa}^y = R^{-1}(I + \zeta\zeta^T)\hat{\kappa}^y$$

and wealth will evolve as

$$dw^y = w^y\left[r\,dt + \pi^y\cdot\sigma\,dY\right] = w^y\left[r\,dt + \pi^y\cdot dZ\right].$$

The market clearing condition is

$$\boxed{\sum_y w^y \pi^y = S}$$

so we now must understand the evolution of $w^y\pi^y$.

(3) To begin with, let's develop $d\pi^y$, where for now we drop the superscript y. We'll use the notation σ^{ij} for the (ij) entry of σ^{-1}. Then

$$R\,d\pi_i = (\delta_{ij} + \zeta_i\zeta_j)H_{j\ell}(dY^\ell - \hat{\kappa}_\ell dt) + \hat{\kappa}_j\{\zeta_i(\alpha_{jm}dZ^m + \beta_j dt) + \zeta_j(\alpha_{im}dZ^m + \beta_i dt)$$

$$+ \alpha_{jm}\alpha_{ip}dZ^m dZ^p\} + (\zeta_i\alpha_{jm} + \zeta_j\alpha_{im})H_{j\ell}dY^\ell dZ^m$$

$$= \{\sigma^{ij}H_{j\ell}\sigma^{\ell m} + \hat{\kappa}_j(\zeta_i\alpha_{jm} + \zeta_j\alpha_{im})\}\,dZ^m +$$

$$\Big[-\sigma^{ij}H_{j\ell}\hat{\kappa}_\ell + \hat{\kappa}_j\big(\zeta_i\beta_j + \zeta_j\beta_i + \alpha_{jm}\alpha_{ip}(\sigma\sigma^T)_{mp}\big)$$

$$+ (\zeta_i\alpha_{jm} + \zeta_j\alpha_{im})H_{jp}\sigma_{mp}\Big]\,dt$$

$$\equiv q^y_{im}\,dZ^m + \mu^y_i\,dt$$

for short. Then we shall have

$$R\,d(w\pi_i) = w\left[\pi_i(r\,dt + \pi_\ell dZ^\ell) + q_{im}dZ^m + \mu_i dt + \pi_j q_{im}dZ^j dZ^m\right]$$

$$= w\left[(\pi_i\pi_\ell + q_{i\ell})dZ^\ell + \{r\pi_i + \mu_i + \pi_j q_{im}(\sigma\sigma^T)_{jm}\}dt\right]$$

Now we invoke market clearing:

$$\sum_y' w^y \Big[(\pi^y_i \pi^y_l + q^y_{il}) dZ^l + \{ r\pi^y_i + \mu^y_i + \pi^y_j q^y_{im} (\sigma\sigma^T)_{jm} \} dt \Big]$$

$$= RS_i \, (dZ^i + r\,dt)$$

If we match the terms in dZ, we discover:

$$\boxed{RS_i \, \delta_{il} = \sum_y w^y \{ \pi^y_i \pi^y_l + \sigma^{ij} H^y_{jm} \sigma^{ml} + \hat{\kappa}^y_j (z_i \alpha_{jl} + z_j \alpha_{il}) \}}$$

From this we can work out α. Matching up the terms in dt leads to

$$\boxed{RrS_i = \sum_y w^y [r\pi^y_i + \mu^y_i + \pi^y_j q^y_{im} (\sigma\sigma^T)_{jm}]}$$

From this, we can deduce what β is.

If we simplify by assuming $H^y = \lambda^y I$ (this isn't really necessary though) and we set $Q = \sum_y w^y (\pi^y (\pi^y)^T + \lambda^y a^{-1})$, $\bar{\kappa} = \sum w^y \hat{\kappa}^y$, then we express α as

$$\boxed{\alpha = \frac{1}{z \cdot \bar{\kappa}} \left(I - \frac{1}{2} \frac{z \bar{\kappa}^T}{z \cdot \bar{\kappa}} \right) \left(R \,\mathrm{diag}(S) - Q \right)}$$

We can similarly find an expression for β; maybe nicer now to assume $\boxed{R = 1}$?
This way we get rid of the terms in r. If we write $\bar{\lambda} = \sum w^y \lambda^y$, $\bar{\kappa} = \sum w^y \hat{\kappa}^y$ we get for β in the end

$$\boxed{\begin{aligned} rRS = rS - \sigma^{-1} \sum_y \lambda^y w^y \hat{\kappa}^y + \beta (\bar{\kappa} \cdot z) + z (\bar{\kappa} \cdot \beta) + \alpha a \alpha^T \bar{\kappa} \\ + \bar{\lambda} \{ z \,\mathrm{tr}(a\sigma) + \alpha \sigma z \} + \sum_y w^y \lambda^y \pi^y \\ + \sum_y w^y \{ z \, (\hat{\kappa}^y)^T \alpha + (\hat{\kappa}^y \cdot z) \alpha \} \pi^y \end{aligned}}$$

FBH: doing defaults (10/1/13)

(i) Suppose we have some part of the firm's capital with face value F getting into difficulties. This (sub)-firm is put in the hands of an administrator, who sells the capital (or some part of it) for $a = a_S + a_D$, where a_S is the payment from households to buy up some capital, a_D is the new loan from the bank. Thus the value of retained deposits drops by $a_S + a_D$. The administrator passes $(\lambda F) \wedge (a_S + a_D)$ to the bank $(\lambda \equiv D/(S+D))$, and then returns the residue $(a_S + a_D - \lambda F)^+$ to the shareholders who deposit this cash. At this point, the value of deposits $(\Delta = x + D)$ is short by λF, so the bank's equity has to make this up. So we have a sequence of transfers:

$$\begin{array}{l} x \\ S \\ D \\ Q \\ \text{Admin} \end{array} \quad \begin{pmatrix} x \\ S-(1-\lambda)F \\ D-\lambda F \\ Q \\ F \end{pmatrix} \rightarrow \begin{pmatrix} x - a_S - a_D \\ S-(1-\lambda)F + a_S \\ D - \lambda F + a_D \\ Q \\ a_S + a_D \end{pmatrix} \rightarrow \begin{pmatrix} x - a_S - a_D + (a_S + a_D - \lambda F)^+ \\ S-(1-\lambda)F + a_S \\ D - \lambda F + a_D \\ Q + (\lambda F) \wedge (a_S + a_D) \\ 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} x + (\lambda F - a_S - a_D)^+ \\ S - (1-\lambda)F + a_S \\ D - \lambda F + a_D \\ Q - (\lambda F - a_S - a_D)^+ \\ 0 \end{pmatrix} \equiv \begin{pmatrix} x' \\ S' \\ D' \\ Q' \\ 0 \end{pmatrix} \quad \text{say}$$

(ii) We also need to care about the inequalities $D \leq b(D+S)$, $D \leq \alpha x$, $D \leq \kappa Q$, where $b \in (0,1)$, but α, κ are both > 1. Can we be sure that $a_D \leq \lambda F$? If we could satisfy the inequalities with some $a_D > \lambda F$, a little thought shows that we could reduce a_D to λF and the inequalities would still hold. So we <u>may and shall</u> assume that $a_D \leq \lambda F$. Notice that Q' is not able to go negative, so actually

$$Q' = \{ Q - (\lambda F - a_S - a_D)^+ \}^+, \quad x' = \left(x + (\lambda F - a_S - a_D)^+ \right) \wedge M.$$

We propose that the amount recovered should be a fraction β of the face value (or, perhaps, at most this). In order to give the best chance of satisfying the leverage inequality, we would want D' small, S' big, so take $a_D = 0$, $a_S = \beta F$. Then we need

$$(1-b) D' = (1-b)\{D - \lambda F\} \leq bS' = b\{S + \beta F - (1-\lambda)F\}$$

which holds iff

$$(b-\lambda-b\beta)F \leq bS-(1-b)D$$

The inequality $D' \leq \alpha x'$ will be OK with $a_s = 0$ if we already had $D \leq \alpha x$, so that's no problem with this. For the final inequality, $D' \leq \kappa Q'$, we'll be OK if

$$\{\kappa(\lambda-\beta)^+ - \lambda\}F \leq \kappa Q - D$$

so in particular if $\beta > \lambda$ we would be OK.

If these inequalities are satisfied, then there is a way to do the default with no further selling of capital. If we can do this, we realize $\beta F = a_s + a_D$, and we make a_D as high as we can, up to $\lambda\beta F$, which would maintain the predefault level of leverage.

(iii) If the inequalities cannot be satisfied, then there is too much debt, + the firm has to do some selling to reduce the debt. If the firm chooses to sell ε (cash value) of capital, it raises $\beta\varepsilon$ in cash, and has to scrap $(1-\beta)\varepsilon$ which didn't find a buyer. So we get

$$\begin{pmatrix} x' \\ S' \\ D' \\ Q' \end{pmatrix} \longmapsto \begin{pmatrix} x' \\ S' - (1-2\beta)\varepsilon \\ D' - \beta\varepsilon \\ Q' \end{pmatrix} = \begin{pmatrix} x'' \\ S'' \\ D'' \\ Q'' \end{pmatrix}$$

(Debt is reduced by the cash value $\beta\varepsilon$ of the sale, the overall value of capital falls by $(1-\beta)\varepsilon$, the quantity scrapped, and the total value of deposits $x'+D'$ falls by $\beta\varepsilon$, the amount of cash withdrawn to buy the capital) For there to be any chance of getting $(1-b)D'' \leq bS''$, which is the leverage constraint, we have to have

$$(1-b)\beta > b(1-2\beta)$$

which is the condition for the LHS of the leverage inequality to fall faster than the RHS. This is the same inequality as we got before, viz, $\gamma < 1/(1+b)$.

So to summarise: we first try to satisfy the inequalities with $a_D = 0$, $a_s = \beta F$. If this can be done, raise $a_D = \beta F - a_s$ from zero as far as possible up to a max of $\lambda\beta F$. If it can't, there has to be extra selling of stock, and we do the least possible to restore the bounds.

Leverage forced sales? Could happen continuously. If $b\,pK - D = bS - (1-b)D$ happens to hit 0 in a C^1 fashion, we need to sell off at rate $\dot{A}$, in which case

$$\dot{S} \mathrel{+}= -(1-2\beta)\dot{A}, \qquad \dot{D} \mathrel{+}= -\beta\dot{A}$$

So we find

$$(b\,pK - D)^{\circ} \mathrel{+}= \{\beta(1-b) - b(1-2\beta)\}\dot{A} = (\beta + b\beta - b)\dot{A}$$

and now $\dot{A}$ is chosen to keep the overall drift equal to zero

(iv) While this is a fine story as it stands, the problem in using it is that we will have a pretty good idea about the distribution of F, which would be the $O(1)$ hit to capital, but we would also to know how S, D will respond to this in a sufficiently simple form that we may work out martingales for the jumps of S, D etc.

So let's demand that if there is a fire sale of distressed capital, then a fraction $\beta = 1-\gamma$ of it <u>will</u> be bought, and the ratio of new equity to new debt <u>will</u> result in no change of leverage. Thus $a_S = (1-\lambda)\beta F$, $a_D = \lambda\beta F$, and we get

$$\begin{pmatrix} x' \\ S' \\ D' \\ Q' \end{pmatrix} = \begin{pmatrix} x + (\lambda-\beta)^+ F \\ S - (1-\lambda)\gamma F \\ D - \lambda\gamma F \\ Q - (\lambda-\beta)^+ F \end{pmatrix}$$

For our three inequalities, $D' \leq \alpha x'$ is always OK <u>except</u> if all bank equity is lost, that is, $F > Q/(\lambda-\beta)^+$? No, even then it's OK, because x is no bigger than x', and D' is smaller than D.

So if there is to be trouble, it will come from the other two. There's trouble with $D' \leq b(D'+S')$ iff

$$(1-\beta)(b-\lambda)F \leq bS - (1-b)D \quad \underline{\text{fails}}$$

There's trouble with $D' \leq \kappa Q'$ iff

$$\left[\kappa(\lambda-\beta)^+ - \lambda(1-\beta)\right] F \leq \kappa Q - D \quad \underline{\text{fails}}$$

(so this is never a problem if $\beta > \lambda$). Together, these two give a safe region $F \leq F^*$ where resolving the default will leave everything ok. Here we have

$$F^* = F^*(S, D, Q, \lambda)$$

If we get $F > F^*$, then we will do the initial resolution of the default, and sell off more capital to come within the constraints.

This way, when a hit of size F comes along, the effect on S, D and Q is to reduce these by some reasonably explicit piecewise linear functions of F, so for compensating the jumps in S, D, Q we actually stand a chance of working stuff out!

[The forced sales ε will be

$$\boxed{\varepsilon = \max\left\{ \frac{(D'-\kappa Q')^+}{\beta}, \frac{((1-b)D' - bS')^+}{\beta(1+b)-b} \right\}}$$

<u>needs</u> $\beta(1+b) > b$.]

Pricing impact of different views again (11/1/13)

(i) Lukas has followed through on the story we discussed where the agent calculates the proportions of wealth to invest using some estimate of σ instead of the actual value. He finds that things work nicely if the estimate used is some EWMA of the actual values, which is very neat.

However, this feels a bit impractical to me: How are we to estimate σ? All we could hope to know from the data is $a \equiv \sigma\sigma^T$. For this reason, it seems better to try to tell a story in terms of a as much as we can.

(ii) For the log prices, we have

$$d(\log S^i) = \sigma_{ij}\, dW^j + \mu^i dt - \tfrac{1}{2} a_{ii}\, dt.$$

So assuming we had a reasonable idea about a, we could suppose we observed

$$dY = \sigma\, dW + \mu\, dt,$$

where $d\mu = \tilde\sigma\, d\tilde W$ for some BM independent of W. The terms $\sigma, \tilde\sigma$ are not expected to be constant, but supposing for the moment that they were, we could do a conventional Kalman filtering

$$\begin{cases} dY = \sigma\, d\hat W + \hat\mu\, dt \\ d\hat\mu = H(dY - \hat\mu\, dt) \end{cases}$$

It's a tedious calculation, but in the end we find that

$$\boxed{H = v a^{-1}}$$

where v is the covariance matrix characterised by $\boxed{\tilde a \equiv (\tilde\sigma\tilde\sigma^T) = v a^{-1} v.}$

If we imagine that $\tilde\sigma = \sigma q$, so the drifts of various assets move faster/slower as the assets themselves do, we would be able to find explicitly

$$\boxed{v = \sigma q \sigma^T}$$

and hence $\boxed{H = \sigma q \sigma^{-1}.}$ If we make the further simplifying assumption $\boxed{q = \lambda I}$ we shall have $H = \lambda$, and we end up with

$$\boxed{d\hat\mu = \lambda(dY - \hat\mu\, dt)} \qquad (1)$$

not depending on the volatility σ. This is interesting because under these special assumptions, it suggests that we might reasonably try to estimate μ by the

EWMA recipe (1), and then do the Merton portfolio selection

$$\pi = R^{-1} a^{-1} (\hat{\mu} - r)$$

for a CRRA agent. Different agents would have different values λ^γ for λ, but would all use the SDE (1), driven by the common process Y, to construct their estimates $\hat{\mu}^\gamma$.

What do we do about the interest rate? I'd prefer to work with the assumption that the observed process is actually

$$dY = \sigma dW + (\mu - r) dt$$

so that we form an estimate of the excess return $\hat{\beta} = (\hat{\mu} - r)$ following

$$d\hat{\beta} = \lambda (dY - \hat{\beta} dt)$$

$$\text{so } \hat{\beta}_t = \int_{-\infty}^{t} \lambda e^{\lambda(s-t)} dY_s = \int_{-\infty}^{t} \lambda^2 e^{\lambda(u-t)} (Y_t - Y_u) du$$

$$= \lambda \left\{ Y_t - \int_{-\infty}^{t} \lambda e^{\lambda(u-t)} Y_u du \right\}$$

(iii) Unfortunately this gets stuck. The thing that worked for Lukas in the coefficient of dW in market clearing is that we had σ and $\hat{\sigma}$ appearing, so that it was possible to work out σ from that; but this time we don't ...

Firm size story again (16/1/13)

We saw that given firm log size x and vol σ, the value of $\theta = \mu/\sigma^2 - \frac{1}{2}$ has a $GH(\cdot \mid \lambda, \alpha, \beta, \delta, 0)$ distribution.

Suppose then that at time 0 we observe firm sizes x_i, $i = 1, \dots, N$, with corresponding volatilities σ_i, $i = 1, \dots, N$ and that the firms evolve independently (later we will want to understand the situation where there is some overall market fluctuation common to all). Now suppose the investor plans to follow a fixed-mix strategy, putting $\pi_i W_t$ of his wealth at time t into asset i; what would he actually do? Suppose the horizon is T, and U is CRRA, $U'(x) = x^{-R}$. Then

$$dW_t = W_t \{ r\,dt + \pi \cdot (\sigma\, dW + (\mu - r)\,dt) \}$$

$$\Rightarrow \quad W_t = \exp\{ \pi \cdot \sigma W_t + (r + \pi \cdot (\mu - r) - \tfrac{1}{2} |\sigma^T \pi|^2) t \}$$

$$= \exp\{ \pi \cdot \sigma W_t + rt + \pi \cdot (\sigma^2(\theta + \tfrac{1}{2}) - r) t - \tfrac{1}{2} |\sigma^T \pi|^2 t \}$$

So the objective $EU(W_T)$ is

$$\frac{1}{1-R} E \exp\left[(1-R)\, \pi \cdot \sigma W_T + (1-R)(rT + \pi \cdot (\mu - r) T - \tfrac{1}{2} |\sigma^T \pi|^2 T) \right]$$

$$= \frac{1}{1-R} \exp\left[T(1-R) \{ r + \pi \cdot (\mu - r) - \tfrac{1}{2} R |\sigma^T \pi|^2 \} \right]$$

$$= \frac{1}{1-R} \prod_{i=1}^{N} \exp\left[T(1-R) \{ \tfrac{r}{N} + \pi_i (\mu_i - r) - \tfrac{1}{2} R \sigma_i^2 \pi_i^2 \} \right]$$

$$= \frac{1}{1-R} \exp\{ r T(1-R) - \tfrac{1}{2} R T (1-R) \sum \pi_i^2 \sigma_i^2 \} \prod_{i=1}^{N} e^{T(1-R)\pi_i (\mu_i - r)}$$

Now given the firm sizes and the vols, we need to take the expectation of the product over the GH distribution of the μ_i. At present, these are all taken to be independent, so we just get a product of MGFs. We get

$$\frac{1}{1-R} \exp\left\{ T(1-R) \{ r(1 - \sum \pi_i) + \tfrac{1}{2} \sum \sigma_i^2 \pi_i (1 - R\pi_i) \} \right\} \prod_{i=1}^{N} \varphi_i (\pi_i \sigma_i^2 T(1-R))$$

where

$$\varphi_i(y) = \frac{\gamma_i^\lambda}{(\alpha_i^2 - (\beta_i + y)^2)^{\lambda/2}} \; \frac{K_\lambda (\delta_i \sqrt{\alpha_i^2 - (\beta_i + y)^2})}{K_\lambda (\delta_i \gamma_i)} \qquad \left[z_i \equiv T(1-R)\sigma_i^2 y \right]$$

and λ is either 0 or 1, $\alpha_i = A + |x_i|$, $\beta_i = x_i + \xi$, $\delta_i = \sqrt{2\varepsilon}/\sigma_i$.

Chucking away the factor $\exp(-(R-1)Tr)$, what we want to maximize is

$$\frac{1}{1-R}\exp\left\{-r\sum z_i/\sigma_i^2 + \frac{1}{2}\sum z_i\left(1-\frac{Rz_i}{T(1-R)\sigma_i^2}\right)\right\}\prod_{i=1}^{N}\varphi_i(z_i)$$

where we use $z_i \equiv \pi_i(1-R)T\sigma_i^2$. This should be OK numerically?

Scaling: if $X \sim GH(\lambda,\alpha,\beta,\delta,\mu)$ and $b>0$, then

$$bX \sim GH\left(\lambda, \frac{\alpha}{b}, \frac{\beta}{b}, b\delta, b\mu\right)$$

So if $\theta \sim GH(\lambda,\alpha,\beta,\delta,m)$ then $\mu = \sigma^2(\theta+\frac{1}{2})$ will have a

$$GH\left(\lambda, \frac{\alpha}{\sigma^2}, \frac{\beta}{\sigma^2}, \sigma^2\delta, \sigma^2(m+\tfrac{1}{2})\right)$$

The thing we want to maximize is $(z = \pi T(1-R))$

$$\frac{1}{1-R}\exp\left\{-\frac{\sigma^2 R}{2T(1-R)}z^2\right\} E\exp(z(\mu-r))$$

Now when $\lambda = 0$ or 1, we have the density of θ given log size X and vol σ is

$$\propto \exp\left\{(\xi+X)\theta - (|X|+A)\sqrt{\theta^2+2\tilde{\varepsilon}}\right\}(\theta^2+2\tilde{\varepsilon})^{(\lambda-1)/2} \qquad (\tilde{\varepsilon} = \varepsilon/\sigma^2)$$

$$\sim GH(\lambda, A+|X|, \xi+X, 2\tilde{\varepsilon}, 0)$$

so that $\mu - r \sim GH\left(\lambda, \frac{A+|X|}{\sigma^2}, \frac{\xi+X}{\sigma^2}, \sigma\sqrt{2\varepsilon}, \frac{\sigma^2}{2}-r\right)$

FBH : gathering equations (21/1/13)

(i) Using our existing notation in the jump default version of the problem, we have that all of the processes in the story should be FV, and they will jump only when the default jumps happen. At other times then, we shall write $\mathring{\Delta}$, $\mathring{S}$, ... for the derivatives. When a hit to the capital of size $\xi > 0$ occurs, we shall write $J_S(\xi)$, $J_Q(\xi)$, $J_D(\xi)$, $J_x(\xi)$ for the changes in value of S, Q, D, x respectively. These depend on current values of the variables of the problem in some rather complicated fashion as discussed on pp 30–32. Apart from the times of the jumps, things evolve in a nice differentiable way.

Suppose A_t is the cumulative quantity of capital disposed of in forced sales by time t. Some of this happens at the times of defaults, but some can happen when the bound $D_t \leq b\, p_t K_t$ is tight. A fraction γ (which may be non-constant) of the forced selling has to be scrapped, a fraction $\beta \equiv 1-\gamma$ survives. So we have

$$\mathring{K} = I - \delta K - \gamma \mathring{A}, \qquad \mathring{D} = \ell - \beta p \mathring{A}$$

for the evolution of capital between defaults.

(ii) We have quantity identities:

$$pK = S + D$$

$$f = C + I$$

and cashflow identities:

$$pC + \ell = a + wL + RD \qquad \text{(firm's cashflow)}$$

$$wL + a + \tilde{a} + r\Delta - pC = \Lambda, \qquad \text{(household's saving rate)}$$

$$\mathring{Q} = RD - r\Delta - \tilde{a} \qquad \text{(bank equity cashflow)}$$

$$\mathring{\Delta} = \Lambda - \beta p \mathring{A}$$

Then there are optimality equations:

$$f_L = \theta$$

$$0 = \theta U_c + U_L$$

leading to the state-price density process

$$\zeta_t = e^{-\rho t}\, U_c(C, L) / p_t$$

and the NPV pricing equations:

$$\begin{cases} \zeta_t S_t + \int_0^t \zeta_u (a_u du - p_u \dot{p}_u dA_u) & \text{is a martingale} \\ \zeta_t Q_t + \int_0^t \zeta_u \tilde{a}_u \, du & \text{is a martingale} \\ \zeta_t \Delta_t + \int_0^t r_u \Delta_u \zeta_u \, du & \text{is a martingale} \end{cases}$$

Then there is the firm's cost of capital calculation: borrowing results in cost of capital of

$$\Big\{ R_t D_t + \psi_t \int_0^1 J_Q(K_t z) \, F(dz) \Big\} / D_t$$

where ψ_t is the current rate at which defaults happen, $F(\cdot)$ is the CDF of the fraction of the total capital that gets into difficulties when defaults occur. Note that $J_Q(\xi) \leq 0$ is the loss to bank equity when defaults happen, and that has to be the capital writedown, which is a credit to the firm. Compare this with the cost of financing via firm equity

$$\Big\{ a_t + p_t \gamma_t \dot{A}_t + \psi_t \int_0^1 \tilde{J}_S(K_t z) \, F(dz) \Big\} / S_t$$

where $\tilde{J}_S(\xi)$ is the write-down of firm capital that happens when a default occurs (notice this is made up of default losses on the original default, along with losses on any LBS that may result – we aren't seeing the actual change in S at those times, because the change in S includes any purchases of distressed capital)

In the interior, these two rates have to be the same.

(iii) The shadow firm story we told earlier needs little modification. The capital now dies at rate

$$\delta + \gamma(q_t) \psi(\bar{q}_t) \int_0^1 z F(dz)$$

which is the only modification to the story (previously we had 1 in place of $\int z F(dz)$)

Comparing models for option price surfaces (30/1/13)

(i) Moritz is looking at a Bayesian model comparison for option pricing models, but is finding that the fitting of models to data is problematic; he is getting log-likelihoods which are way too negative. Can we find a way round this, and some sort of justification?

(ii) Suppose that $C^*(T,K)$ is the market price of a call option strike K, expiry T, which in practice we know only for $K_1, \dots, K_n$, $T_1, \dots, T_m$. A model produces model prices $C(T,K)$; how well does it fit the data?

Our first attempt here has been to look at a log-likelihood

$$-\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\left(I^*(T_j,K_i) - I(T_j,K_i)\right)^2 / 2\sigma_{ij}^2 + \tfrac{1}{2}\log \sigma_{ij}^2 \right)$$

where $I^*(T_j,K_i)$ is the log-implied-vol, $I(T_j,K_i)$ is the log-implied vol for the model.

This approach leads to very big numerical values if you choose the σ_{ij} to be of the order of magnitude implied by the bid-ask spread; what is going wrong?

I think the problem is that the log-likelihood form would be correct if we thought that the observed log-implied-vols I^* were expressible as model value + Gaussian noise; but we clearly don't believe this. We don't believe this because the observed call surface $C^*(T,K)$ has to have the no-arbitrage properties

$$\begin{cases} C^*(T,0) = S_0 \quad \forall T \\ K \mapsto C^*(T,K) \text{ is convex, decreasing to } 0 \\ C^*(T,K) \text{ increases to } S_0 \text{ as } T \uparrow \infty \end{cases}$$

- in other words, we will <u>never</u> expect to see C^* looking like C plus a load of independent Gaussian errors.

So maybe we ought to use the call prices to deduce the risk-neutral density of the underlying

$$C_{KK}(T,K) = e^{-rT} P(S_T \in dK)/dK$$

and do an L^2-comparison of those densities:

$$\boxed{\int_0^\infty e^{-\alpha T} \int \left(C^*_{KK}(T,K) - C_{KK}(T,K)\right)^2 dK \, dT}$$

for some $\alpha > 0$. This is a reasonable metric on option price surfaces: if it has value 0, the two call surfaces are the same, since they both have to satisfy $C(T,0) = S_0$, $C(T,\infty) = 0$.
(iii) How would we evaluate this metric when we only have discrete data? The question is really a question about evaluating

$$\int_0^\infty \left(C^*_{KK}(T,K) - C_{KK}(T,K) \right)^2 dK$$

for some fixed T_j; if we can do this for each T_j, then we'll just do a trapezium rule integration.

So let's suppose we just have call prices for a finite set of strikes, with T held fixed and omitted from the notation. If we assumed that the call function was just the convex hull of the prices at strikes K_i, then the implied distribution would be atomic, and we see easily that

$$\frac{C(K_{i+1}) - C(K_i)}{K_{i+1} - K_i} - \frac{C(K_i) - C(K_{i-1})}{K_i - K_{i-1}} = e^{-rT} P(S_T = K_i)$$

$$\approx \tfrac{1}{2} C''(K_i) (K_{i+1} - K_{i-1})$$

using Taylor expansion. Thus we can calculate the risk neutral probability at each strike K_i. Now we'd like to turn that into a density, which I'd propose most simply should be constant on each interval (K_{i-1}, K_i), (K_i, K_{i+1}) of lengths $\Delta_- \equiv K_i - K_{i-1}$, $\Delta_+ = K_{i+1} - K_i$. If we insist that the mean is at K_i, then the density on (K_{i-1}, K_i) is a_-, on (K_i, K_{i+1}) is a_+, where

$$a_\pm = \frac{p}{2\Delta_\pm}$$

and $p = P(S_T = K_i)$. We use this piecewise-linear density for evaluating the metric (for the highest strike, need to make some story - maybe just introduce a strike $K_{n+1} = 2K_n - K_{n-1}$ and spread the probability symmetrically? No... there is already an issue about what $P(S_T = K_n)$ should be).

The <u>other</u> issue is about an appropriate scaling of this quadratic form; and also choice of α...

The KF story says

$$V_{t+1} = V_t + V_\eta - (V_t + V_\eta)(V_t + V_\eta + V_\varepsilon)^{-1}(V_t + V_\eta)$$

$$= V_\varepsilon (V_t + V_\eta + V_\varepsilon)^{-1} (V_t + V_\eta)$$

so $V_{t+1}^{-1} = V_\varepsilon^{-1} + (V_t + V_\eta)^{-1}$ so if $\tilde{A} = V_\varepsilon^{-1/2} A V_\varepsilon^{-1/2}$, we see

$\tilde{V}_{t+1}^{-1} = I + (\tilde{V}_t + \tilde{V}_\eta)^{-1}$ so if $\tilde{V}_\eta = R \Lambda R^T$ is diagonal representation, the steady-state equation must be

$$\tilde{V}^{-1} = I + (\tilde{V} + R\Lambda R^T)^{-1} \Rightarrow R^T \tilde{V}^{-1} R = I + (R^T \tilde{V} R + \Lambda)^{-1}$$

and so $R^T \tilde{V} R$ has diagonal form.

Price impact story again (31/1/13)

1) The earlier story on pp 27–29 is rather a mess ... Lukas & I have been independently simulating it, and it's highly unstable. Maybe we can do better with a discrete time model. Let's try to tell a story where

(i) Agents are CARA, one-step-ahead optimizers

(ii) Agents observe market prices, but understand the dynamics differently

(iii) Market prices are market-clearing, but the total supply is variable.

Here are some comments on each of these modelling assumptions. We will suppose that there are d assets, $S_t = (S_t^1, \dots, S_t^d)'$ the vector of prices at day t, as well as cash, where the interest rate should also be set by market clearing.

2) Agents think

$$\left.\begin{aligned} S_{t+1} - S_t \equiv Y_{t+1} &= \mu_{t+1} + \varepsilon_{t+1} \\ \mu_{t+1} &= \mu_t + \eta_{t+1} \end{aligned}\right\}$$

where the noises (ε_t) and (η_t) are independent IID zero-mean gaussians. If the agent thinks $(\mu_t \mid \mathcal{Y}_t) \sim N(\hat{\mu}_t, V_t)$, then ($V_\eta, V_\varepsilon$ may differ for different agents!)

$$\begin{pmatrix} \mu_{t+1} \\ Y_{t+1} \end{pmatrix} \Big| \mathcal{Y}_t \sim N\left(\begin{pmatrix} \hat{\mu}_t \\ \hat{\mu}_t \end{pmatrix}, \begin{pmatrix} V_t + V_\eta & V_t + V_\eta \\ V_t + V_\eta & V_t + V_\eta + V_\varepsilon \end{pmatrix} \right)$$

so by the usual KF stuff

$$(\mu_{t+1} \mid \mathcal{Y}_{t+1}) \sim N(\hat{\mu}_t + K(Y_{t+1} - \hat{\mu}_t), V)$$

where K, V are steady-state values.

So each agent j is doing his own KF story with steady-state covariance V_j;

$$\boxed{\hat{\mu}^j_{t+1} = (I - K_j)\hat{\mu}^j_t + K_j Y_{t+1}}$$

3) What does the investment decision of agent j at time t look like? He thinks

$$S_{t+1} = S_t + \mu^j_{t+1} + \varepsilon_{t+1} + \eta_{t+1}$$

$$\sim N(S_t + \hat{\mu}^j_t, V_j + V_{\varepsilon,j} + V_{\eta,j}) \equiv N(S_t + \hat{\mu}^j_t, \Sigma_j).$$

But notice that $\hat{\mu}^j_t = (I - K_j)\hat{\mu}^j_{t-1} + K_j(S_t - S_{t-1})$ depends on S_t, so the demand of agent j at the end of day t to carry into day t+1 will be

$$\theta_t^j = (\gamma_j \Sigma_j)^{-1}(S_t + \hat{\mu}_t^j - (1+r)S_t)$$

$$= (\gamma_j \Sigma_j)^{-1}((K_j - r)S_t + (I - K_j)\hat{\mu}_{t-1}^j - K_j S_{t-1})$$

Hence the total demand vector of the agents will be

$$\boxed{\bar{\theta}_t \equiv \sum_j (\gamma_j \Sigma_j)^{-1}\{(K_j - r)S_t + (I - K_j)\hat{\mu}_{t-1}^j - K_j S_{t-1}\}} \quad (*)$$

which is equated to the total supply $\bar{\xi}_t$ to allow us to deduce the market-clearing price vector.

4) What story could we tell for $\bar{\xi}_t$? Here are some possibilities.

(i) $\bar{\xi}_t = 1$ for all t - assets are in unit net supply and remain so. In this story, the evolution involves no randomness!

(ii) $\bar{\xi}_t$ could be some IID sequence $N(1, V_\xi)$ for some small covariance, which introduces some randomness.

(iii) Here is an intriguing possibility. There is some large population of agents all watching the market, but only occasionally participating in it. Depending on what happens in the market on day t, each agent makes a (randomized) decision whether or not to enter the market on day $t+1$. The supply $\bar{\xi}_{t+1}$ is then the total holdings of the agents who chose to enter the market on day $t+1$.

One way to do this would be to look at what expected utility gain would result from entering the market, and act if this exceeded some probabilistic threshold

5) Numerics on the simple stories (i), (ii) above suggest that prices just keep gradually rising. I wonder whether there is a missing element of the story here, about r, or money supply...

6) Notice that we could use $(*)$ to discover a 'steady state' value $S_t = S_{t-1}$ given the values $\bar{\theta}$, $\hat{\mu}^j$, just by setting $S_t = S_{t-1}$ and solving for S_t. However, although this would lead to $Y_t \equiv S_t - S_{t-1} = 0$, the $\hat{\mu}^j$ wouldn't stay the same, so all we can hope for is that we would at least be ball park correct...

(7) (11/2/13) If we assume a constant supply vector b, then the evolution of beliefs and prices is a deterministic linear system! In more detail, if

$$A = \sum (\gamma_j \Sigma_j)^{-1} (K_j - \Lambda), \quad B = \sum (\gamma_j \Sigma_j)^{-1} K_j$$

then market clearing (*) gives

$$b = A S_t + \sum_j (\gamma_j \Sigma_j)^{-1} (I - K_j) \hat{\mu}^j_{t-1} - B S_{t-1}$$

So if we write $S_t = x_t + \alpha$, where $b = (A - B)\alpha$, we can reduce this to

$$x_t = A^{-1} B x_{t-1} - \sum_{j=1}^{J} A^{-1} (\gamma_j \Sigma_j)^{-1} (I - K_j) \hat{\mu}^j_{t-1}$$

$$= M_0 x_{t-1} + \sum_{j=1}^{J} M_j \hat{\mu}^j_{t-1}$$

where $M_0 = A^{-1} B$, $M_j = A^{-1} (\gamma_j \Sigma_j)^{-1}$. Likewise, we have

$$\hat{\mu}^j_t = (I - K_j) \hat{\mu}^j_t + K_j (x_t - x_{t-1}).$$

So if we stack $Z_t = [x_t ; \hat{\mu}^1_t ; \ldots ; \hat{\mu}^J_t]$ as a single vector, we get

$$Z_t = \begin{pmatrix} I \\ K_1 \\ \vdots \\ K_J \end{pmatrix} (M_0, M_1, \ldots, M_J) Z_{t-1} + \begin{pmatrix} 0 & 0 & 0 & \cdots \\ -K_1 & I - K_1 & 0 & \cdots \\ -K_2 & 0 & I - K_2 & \cdots \\ \vdots & \vdots & \ddots & \end{pmatrix} Z_{t-1}$$

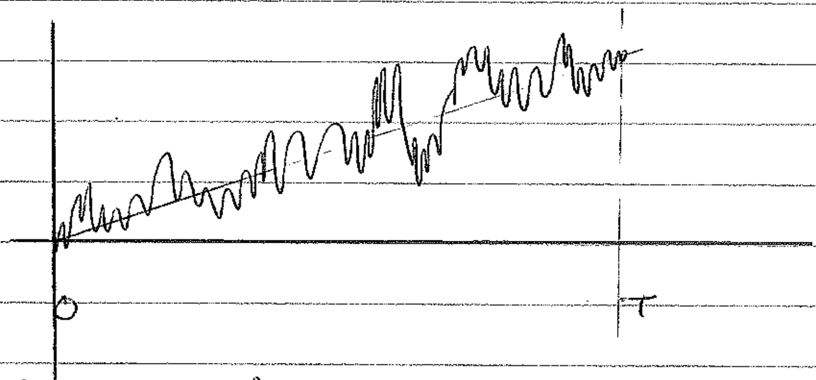
A natural question is "When (if ever) is this stable?" So if $Z_t = C Z_{t-1}$ is the recursion, we want all eigenvalues of C to be ≤ 1 in modulus.

(8) Actually, we would want eigenvalues of C to be $\leq 1 + r$ in modulus to stop prices going up faster than the bank account. Simulated examples show that this rarely happens, so unstable examples are the rule.

One thing we could do is to put into the market some traders who never change their views about μ. This can have a stabilizing effect.

An alternative performance measure? (18/2/13)

If we want to determine how good some trading strategy might be, it's a common criterion to take the Sharpe ratio of the PnL. However, you might see something that looks like



which would have a poor Sharpe ratio because there's a lot of quadratic variation (So, if we were seeing a drift plus a high vol OU process with strong mean reversion, the Sharpe ratio criterion would say this was not very good, though from the point of view of performance it should be reckoned good.)

Something you could do is to change the estimate of vol, which is most simply the sample variance of returns. But if we were seeing a BM with a drift, then $X_t - \frac{t}{T} X_T$ would be a Brownian bridge, and if we use

$$V \equiv \int_0^T \left(X_t - \frac{t}{T} X_T\right)^2 dt$$

then

$$E V = \sigma^2 \int_0^T \left(t - \frac{t^2}{T}\right) dt = \sigma^2 T^2 / 6$$

So we could form a different estimator

$$\boxed{\hat{\sigma} = \frac{\sqrt{6}}{T} \sqrt{V}}$$

for the denominator.

[We could also try things like the Rogers-Satchell estimator if we wanted, but that's a bit too vulnerable to $I = X'$ problems ...]

Of course, we could combine several estimators of σ^2 if we wished to.

Another version of the drawdown tale (21/2/13)

(i) Maybe we want to be a bit less absolute about drawdown, and maybe allow ourselves to relax a bit after some time has passed from a drop. Thus if we define for the wealth process w the auxiliary process

$$\bar{w}_t = \int_{-\infty}^t \lambda e^{\lambda(s-t)} w_s \, ds$$

then $d\bar{w} = \lambda(w-\bar{w})dt$, and we could choose the objective

$$\max E\left[\int_0^\infty e^{-\rho t} u(c_t) \left(w_t/\bar{w}_t\right)^{-\alpha} dt \,\Big|\, w_0 = w, \bar{w}_0 = \bar{w}\right] \equiv V(w,\bar{w})$$

where $u'(x) = x^{-R}$ for some $R > 1$, and $\alpha > 0$. Then we would have the scaling relationship for any $q > 0$ that

$$V(qw, q\bar{w}) = q^{1-R}\, V(w,\bar{w})$$

so we may write

$$V(w,\bar{w}) = \bar{w}^{1-R}\, \tilde{v}(w/\bar{w}) \equiv u(\bar{w})\, v(w/\bar{w}),$$

The HJB for this problem now will be

$$0 = \sup\left[-\rho V + u(c)\left(\tfrac{w}{\bar{w}}\right)^{-\alpha} + (rw + \theta(\mu-r) - c)V_w + \tfrac{1}{2}\sigma^2\theta^2 V_{ww} + \lambda(w-\bar{w})V_{\bar{w}}\right]$$

and if we write $x = w/\bar{w}$, $y = \theta/\bar{w}$, $z = c/\bar{w}$, this becomes

$$0 = \sup u(\bar{w})\Big[-\rho v(x) + z^{1-R}x^{-\alpha} + \{rx + y(\mu-r) - z\}v' + \tfrac{1}{2}\sigma^2 y^2 v''$$
$$+ \lambda(x-1)\{(1-R)v - xv'(x)\}\Big]$$

Optimising over z gives

$$(1-R)z^{-R}x^{-\alpha} = v' \qquad \Rightarrow z = \left((-x^\alpha v')/(R-1)\right)^{-1/R}$$

and over y gives

$$y = -\frac{(\mu-r)}{\sigma^2}\frac{v'}{v''}$$

so altogether the HJB appears as

$$\boxed{0 = -\rho v + R\left(\frac{-x^\alpha v'}{R-1}\right)^{1-1/R} x^{-\alpha} + rxv' - \frac{1}{2}\kappa^2\frac{(v')^2}{v''} + \lambda(x-1)\{(1-R)v - xv'\}}$$

Notice that in this formulation v must be decreasing convex.

(ii) From the point of view of a fund manager, the notion of running consumption is not really correct. You might try an additive story for w, but suppose that AUM is of the form $f(w-\bar{w})$. For example,

$$V(w,\bar{w}) = \sup E\left[\int_0^\infty e^{-\rho t} f(w_t - \bar{w}_t)\,(\varepsilon dt + \alpha dw_t) \,\Big|\, w_0 = w, \bar{w}_0 = \bar{w}\right]$$

Now clearly this has an additive structure if $r=0$: $V(w,\bar{w}) = v(w-\bar{w})$

So HJB is

$$0 = \sup\left[-\rho V + f(w-\bar{w})\{\varepsilon + \alpha\theta(\mu-r)\} + \theta(\mu-r)V_w + \tfrac{1}{2}\sigma^2\theta^2 V_{ww} + \lambda(w-\bar{w})V_{\bar{w}}\right]$$

but this is indeterminate for θ..? No, it's OK; we have

$$0 = \sup\left[-\rho v + f\{\varepsilon + \alpha\mu\theta\} + \theta\mu v' + \tfrac{1}{2}\sigma^2\theta^2 v'' - \lambda x v'\right]$$

where $x \equiv w - \bar{w}$ is the independent variable. So the optimality condition is

$$\boxed{\sigma^2\theta v'' + \alpha\mu f + \mu v' = 0}$$

and HJB becomes finally

$$\boxed{0 = -\rho v(x) + \varepsilon f(x) - \frac{(\alpha\mu f(x) + \mu v'(x))^2}{2\sigma^2 v''(x)} - \lambda x v'(x)}$$

Models for option price surfaces again (21/2/13)

Maybe the story we need to tell is that the difference between the model and market prices is indeed a zero-mean Gaussian process, but one which has much better sample paths?

(i) If we take a one-dimensional version of this firstly, we could suppose that $(X_t)_{t\in\mathbb{R}}$ was a stationary OU process with covariance

$$E(X_s X_t) = \sigma^2 \exp(-\lambda|s-t|)$$

If we saw $(X_{t_1}, \dots, X_{t_n})$ for some $t_1 < \dots < t_n$, what does the inverse covariance matrix look like?

What we have is that $E X_s X_t \propto G(s,t)$ where G is the Green function for a BM killed at rate $\frac{1}{2}\lambda^2$, so $G = (\frac{1}{2}\lambda^2 - \frac{1}{2}D^2)^{-1}$ so the inverse covariance matrix will be (formally)

$$\tfrac{1}{2}\lambda^2 - \tfrac{1}{2}D^2$$

and numerics bear this out. This is good, because it will give a sensible log likelihood.

(ii) What if we try some two dimensional analogue of this? We could try to define a random process

$$X(s,t) = \int_{-\infty}^{s}\int_{-\infty}^{t} e^{\alpha(u-s)+\beta(v-t)}\, dW(u,v)$$

for a Brownian sheet, and the covariance would look like

$$E\left[X(s,t)\,X(s',t')\right] = \frac{1}{4\alpha\beta} e^{-\alpha(s+s')-\beta(t+t')}\; e^{2\alpha(s\wedge s')+2\beta(t\wedge t')}$$

$$= \frac{1}{4\alpha\beta}\exp\left[-\alpha|s-s'| - \beta|t-t'|\right]$$

Numerics show this also is well behaved, but it's not so clear if there is a Markov process this time with this Green function

(iii) Go back to RW I.25 where the covariances of stationary Gaussian processes are represented as

$$\rho(x) = \int e^{i\theta\cdot x}\, F(d\theta)$$

for some $F \geq 0$. If $\int \left(\prod_1^n \theta_j^{2\alpha_j}\right)|\theta|^{\varepsilon} F(d\theta) < \infty$, then $D^\alpha X$ is a cts Gaussian process

– SUGGESTS WE USE A NICE F (eg Gaussian) to generate the covariance of the noise

Price impact story: determining the riskless rate? (26/2/13)

In the price impact story on pp 41–43, the riskless rate is assumed fixed and given, but could we tell some equilibrium story to account for its evolution?

(i) Agent j enters period t with φ^j_t bonds, θ^j_t of the stocks, which were chosen at period $t-1$. The value of this portfolio as we enter period t is therefore

$$\varphi^j_t(1+r_t) + \theta^j_t \cdot S_t$$

where r_t is the interest rate which prevails from period $t-1$ to period t. The new portfolio θ^j_{t+1} chosen by agent j leaves cash φ^j_{t+1} satisfying

$$\varphi^j_{t+1} + \theta^j_{t+1} \cdot S_t = \varphi^j_t(1+r_t) + \theta^j_t \cdot S_t$$

and so if we sum on j we get

$$\text{cash held at time } t = \sum \varphi^j_{t+1} \equiv M_t = \sum \varphi^j_t(1+r_t) = M_{t-1}(1+r_t)$$

since the total holdings of stocks are not altered by the portfolio rebalancing. This is rather unsatisfactory, because it seems that the riskless rate is driven by (exogenous?) money supply values...

(ii) Here is another way we could model this situation. Suppose that there is a (representative) agent who never buys stock, but just invests in the bank account, and consumes. He sees a riskless rate r which he imagines will prevail for all time and then does a simple dynamic-programming optimization

$$V(w) = \sup_c \left[U(c) + \beta V((1+r)(w-c)) \right]$$

To solve this, we can pass to the convex dual

$$\tilde V(\lambda) = \sup \{ V(x) - \lambda x \}$$

$$= \tilde U(\lambda) + \beta \tilde V\left(\frac{\lambda}{\beta(1+r)} \right)$$

$$\equiv \tilde U(\lambda) + \beta \tilde V(\alpha\lambda) \qquad \alpha \equiv 1/\beta(1+r)$$

so by repeated substitution,

$$\boxed{\tilde V(\lambda) = \sum_{n\ge 0} \beta^n \tilde U(\alpha^n \lambda)}$$

Example. If $U(x) = x^{1-R}/1-R$, we get $\tilde U(\lambda) = -\dfrac{\lambda^{1-1/R}}{1-1/R}$, so

$$\tilde V(\lambda) = \sum_{n\ge 0} \beta^n \alpha^{n(1-1/R)} \tilde U(\lambda) = \tilde U(\lambda) / \left(1 - \beta\alpha^{1-1/R}\right).$$

Perhaps a more realistic story is to suppose that agents receive an income $\varepsilon \geq 0$ per unit time, and then the DP equation is

$$V(w) = \sup \Big[U(c) + \beta V\big((1+r)(w-c) + \varepsilon \big) \Big]$$

and

$$\tilde{V}(y) = \tilde{U}(y) + \sup_x \Big[\beta V((1+r)x + \varepsilon) - xy \Big]$$

$$= \tilde{U}(y) + \beta \sup \Big(V((1+r)x + \varepsilon) - \frac{y}{\beta(1+r)} \{ (1+r)x + \varepsilon \} + \frac{\varepsilon y}{\beta(1+r)} \Big)$$

$$= \tilde{U}(y) + \beta \tilde{V}\Big(\frac{y}{\beta(1+r)} \Big) + \frac{\varepsilon y}{1+r}$$

$$\equiv \tilde{U}_0(y) + \beta \tilde{V}(\alpha y)$$

where $\alpha \equiv 1/\beta(1+r)$ and $\tilde{U}_0(y) = \tilde{U}(y) + \varepsilon y/(1+r)$. Again we have the expression

$$\boxed{\tilde{V}(y) = \sum_{n \geq 0} \beta^n \tilde{U}_0(\alpha^n y)}$$

Example: For CRRA utility U, we get

$$\tilde{V}(y) = \frac{\tilde{U}(y)}{1 - \beta\alpha^{1-1/R}} + \frac{\varepsilon}{1+r} \frac{y}{1-\alpha\beta} = \frac{\tilde{U}(y)}{1 - \beta\alpha^{1-1/R}} + \frac{\varepsilon}{r} y$$

$$\equiv A \tilde{U}(y) + By \qquad \boxed{A = \{1 - \beta\alpha^{1-1/R}\}^{-1}, \; B = \varepsilon/r}$$

so if we invert we get

$$\boxed{V(x) = A U\Big(\frac{B+x}{A} \Big)}$$

So how does this non-speculative investor choose to consume/save? Some calculations give

$$c\big(1 + (1+r)(\alpha^{1/R} - \alpha\beta) \big) \equiv c\big((1+r)\alpha^{1/R} \big)$$

$$= \big((1+r)\alpha^{1/R} - 1 \big) \frac{rw + \varepsilon}{r}$$

so

$$\boxed{c = \frac{(1+r)\alpha^{1/R} - 1}{(1+r)\alpha^{1/R}} \; \frac{rw + \varepsilon}{r}}$$

This gives that the wealth saved will be

$$(w-c) = \frac{w}{(1+r)\,\alpha^{1/R}} - \frac{(1+r)\alpha^{1/R}-1}{(1+r)\alpha^{1/R}} \cdot \frac{\varepsilon}{r}$$

Thus the disposable wealth that the agent will have next period will be

$$(1+r)(w-c)+\varepsilon = \frac{w}{\alpha^{1/R}} + \frac{(1-\alpha^{1/R})\,\varepsilon}{r\,\alpha^{1/R}}$$

Notice that this is equal to his original wealth w iff

$$(rw+\varepsilon)(1-\alpha^{1/R}) = 0$$

which in effect would require $\beta(1+r)=1$ (that is, $\alpha = 1$).

Example: If we have $U(x) = -\frac{1}{\gamma}e^{-\gamma x}$, then $\tilde{U}(y) = \frac{y}{\gamma}(\log y - 1)$, and some calculations lead to

$$\tilde{V}(y) = A\,\tilde{U}(y) + By \qquad \begin{cases} A = \frac{1+r}{r} \\ B = \frac{\alpha\beta}{(1-\alpha\beta)^2}\,\frac{\log\alpha}{\gamma} + \frac{\varepsilon}{r} \\ \quad = \frac{1+r}{r^2}\,\frac{\log\alpha}{\gamma} + \frac{\varepsilon}{r} \end{cases}$$

Hence $V(x) = A\,U\left(\frac{B+x}{A}\right)$

Optimizing $U(c) + \beta V((1+r)(w-c)+\varepsilon)$ over c leads to optimal choice

$$c^* = \frac{rw+\varepsilon}{1+r} + \frac{\log(\alpha)}{\gamma r}$$

Available wealth for saving

$$= w - c^* = \frac{w-\varepsilon}{1+r} - \frac{\log\alpha}{\gamma r} = \frac{w-\varepsilon}{1+r} + \frac{\log(\beta(1+r))}{\gamma r}$$

Cash available next period

$$= (1+r)(w-c^*) + \varepsilon = w + \frac{(1+r)\log(\beta(1+r))}{\gamma r}$$

Firm size: beyond independence (28/2/13)

The firm size story pretty much requires independence of the evolutions of the different firms, but we may be able to put a little dependence in by supposing the evolution X_t of an individual log size gets changed to

$$\tilde{X}_t = X_t + mt + \sqrt{v_*}\,\tilde{W}_t$$

for some common but independent BM $\tilde{W}$. If we now try a fixed-mix investment to please the log investor, we find objective

$$\pi \cdot E(\mu + m - r) - \tfrac{1}{2}\pi \cdot E(\sigma\sigma^T + v_* 11^T)\pi$$

Now the original $E\,\sigma\sigma^T$ will be diagonal, $\mathrm{diag}(v(x_i))$, say, so we have optimality equation

$$(V + v_* 11^T)\pi = m - r + E\mu,$$

that is

$$(I + zz^T)V^{\frac{1}{2}}\pi = V^{-\frac{1}{2}}(m - r + E\mu)$$

where $z = \sqrt{v_*}\,V^{-\frac{1}{2}}1$. This gives us

$$V^{\frac{1}{2}}\pi = \left(I - \frac{zz^T}{1+|z|^2}\right)V^{-\frac{1}{2}}(m - r + E\mu)$$

$$= V^{-\frac{1}{2}}(m - r + E\mu) - \frac{zz^T V^{-\frac{1}{2}}(m - r + E\mu)}{1+|z|^2}$$

$$\therefore \quad \pi = V^{-1}(m - r + E\mu) - V^{-\frac{1}{2}}z \cdot \frac{z^T V^{-\frac{1}{2}}(m - r + E\mu)}{1+|z|^2}$$

$$= V^{-1}(m - r + E\mu) - v_* V^{-1}1 \; \frac{1^T V^{-1}(m - r + E\mu)}{1 + v_*\, 1 \cdot V^{-1}1}$$

Fitting options data again (1/3/13)

1) Moritz was finding unpleasantly large log-likelihoods in the Gaussian random fields structure I was proposing ... can we do better?

I was wondering about defining a random field

$$X(t_1,t_2) = \int_0^{t_2} W(t_1,s_2)\,ds_2 - t_2\int_0^1 W(t_1,s_2)\,ds_2$$

$$- t_1\left\{\int_0^{t_2} W(1,s_2)\,ds_2 - t_2\int_0^1 W(1,s_2)\,ds_2\right\}$$

where W is a standard Brownian sheet. This has the property that X vanishes along the edges of the unit square, and at least in the t_2 direction the loglikelihood looks like $-\frac{1}{2}\int_0^1 \left|\frac{\partial X}{\partial t_2}\right|^2 dt_2$.

The key thing is to calculate the covariance structure. After some calculations I came to the conclusion that

$$E\left[X(t_1,t_2)X(u_1,u_2)\right]$$

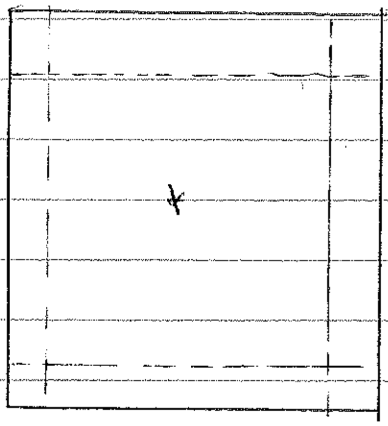
$$= (t_1\wedge u_1 - t_1u_1)\left[\frac{(u_2\wedge t_2)^3}{3} + |t_2-u_2|\frac{(u_2\wedge t_2)^2}{2}\right.$$

$$\left. + \frac{1}{6}u_2t_2(u_2^2+t_2^2) - \frac{u_2t_2(u_2+t_2)}{2} + u_2t_2/3\right]$$

Maybe this works better

2) How should covariance get scaled? I'm not sure how to do this yet, and to some extent it is an arbitrary choice reflecting the relative importance of getting the transitions good, and getting the derivative prices good ...

It may be possible to do a linear scaling of log moneyness to put lowest value at $\frac{1}{4}$, highest at $\frac{3}{4}$, and likewise for log expiry to put lowest value at $\frac{1}{4}$, highest at $3/4$, and then just think of this as a Gaussian random field on the unit square, maybe with inverse covariance given by $\alpha|\nabla f|^2$.



Now to scale this let's take a function which is 0 at the edges of the square, 1 spread at the centre, piecewise linear in between, and calculate the log-likelihood for this function — now pick α so that this log-likelihood scales to the value $\frac{1}{2}$ ...?

Another example for Optimal Investment (8/3/13)

1) I set the following as an example on one of the sheets. You may invest in a stock

$$dS = \sigma S \, d\tilde{W} + \mu S dt$$

and receive income stream

$$d\varepsilon = \varepsilon \, (\sigma_\varepsilon dW + \alpha dt) \qquad d\tilde{W} dW = \rho dt$$

and your objective is

$$V(w,\varepsilon) = \sup E\left[\int_0^\infty e^{-\beta t} U(c_t) \, dt \;\Big|\; W_0 = w, \; \varepsilon_0 = \varepsilon \right]$$

where as usual $U'(x) = x^{-R}$. We have modified wealth equation

$$dW = rW dt + \theta \, (\sigma d\tilde{W} + (\mu - r) dt) - c dt + \varepsilon dt$$

and it's not hard to see that V is concave, and scales:

$$V(w,\varepsilon) = \varepsilon^{1-R} f(w/\varepsilon) \equiv \varepsilon^{1-R} f(x).$$

This question was a lot harder than was really fair for an example sheet. If we let Maple thrash through the calculations, and let it convert into the dual variables, what results is the ODE

$$\tfrac{1}{2}(\kappa - \rho R \sigma_\varepsilon)^2 z^2 J'' + \left(\beta - r + \alpha R - \tfrac{1}{2}\sigma_\varepsilon^2 R(1+R) - \rho \sigma_\varepsilon \kappa + \rho^2 \sigma_\varepsilon^2 R\right) z J'$$

$$- \left(\beta + \tfrac{1}{2}\sigma_\varepsilon^2 R(1-R) + \alpha R - \alpha\right) J + z + \tilde{U}(z)$$

$$- \tfrac{1}{2}(1-\rho^2)\sigma_\varepsilon^2 \, (J')^2 / J''$$

and the final term makes this non-linear. Still, it ought not to be too bad.

If we had a constant wage rate a, then you could convert that into initial wealth a/r in the complete market calculation, and the value would then be

$$V(w) = V_M \, (w + a/r)$$

with dual function $J(z) = J_M(z) + az/r$. This could be used as a first guess for a numerical procedure...

2) There is however a smarter way, and that is to consider the dynamics of $x_t \equiv W_t/\varepsilon_t$. We have wealth equation and evolution of $1/\varepsilon$ given by

$$\begin{cases} dw = rw\,dt + \theta\left(\sigma d\tilde{W} + (\mu - r)dt\right) - c\,dt + \varepsilon\,dt \\ d(1/\varepsilon) = 1/\varepsilon\left(-\sigma_\varepsilon dW - (\alpha + \sigma_\varepsilon^2)dt\right) \end{cases}$$

therefore

$$dx \equiv d(w/\varepsilon) = rx\,dt + \tilde{\theta}\left(\sigma d\tilde{W} + (\mu - r)dt\right) - \tilde{c}\,dt + dt + x\left(-\sigma_\varepsilon dW - (\alpha + \sigma_\varepsilon^2)dt\right)$$
$$- \rho\sigma\tilde{\theta}\sigma_\varepsilon\,dt$$

$$= (r - \alpha - \sigma_\varepsilon^2)x\,dt + \tilde{\theta}\left[\sigma d\tilde{W} + (\mu - r - \rho\sigma\sigma_\varepsilon)dt\right] - \tilde{c}\,dt + dt - \sigma_\varepsilon x\,dW.$$

The objective of the agent is now

$$\sup E\left[\int_0^\infty e^{-\beta t}\varepsilon_t^{1-R} U(\tilde{c}_t)dt \,\middle|\, w_0 = w, \varepsilon_0 = \varepsilon\right] = \varepsilon^{1-R} f(x)$$

and if we write

$$\varepsilon_t^{1-R} = Z_t e^{qt}$$

where

$$\varepsilon_t^{1-R} = \exp\Big[(1-R)\sigma_\varepsilon W_t - \tfrac{1}{2}(1-R)^2\sigma_\varepsilon^2 dt + \underbrace{(1-R)\left(\alpha - \tfrac{1}{2}R\sigma_\varepsilon^2\right)}_{q} t\Big]$$

the expectation is with respect to measure P^* with LR martingale Z:

$$f(x) = E^*\left[\int_0^\infty e^{-\beta t + qt} U(\tilde{c}_t)dt \,\middle|\, x_0 = x\right]$$

and under P^*, $dW = dW^* - (R-1)\sigma_\varepsilon dt$, where W^* is a P^*-BM. Now we write $d\tilde{W} = \rho\,dW + \rho' dW'$, $\rho' \equiv \sqrt{1-\rho^2}$, W' an independent BM, and we get for the evolution of x:

$$dx = (r - \alpha - \sigma_\varepsilon^2)x\,dt + \tilde{\theta}\left[\sigma\left(\rho\,dW^* - \rho(R-1)\sigma_\varepsilon dt + \rho' dW'\right) + (\mu - r - \rho\sigma\sigma_\varepsilon)dt\right]$$
$$+ dt - \tilde{c}\,dt - \sigma_\varepsilon x\left(dW^* - (R-1)\sigma_\varepsilon dt\right)$$

$$= \left(r - \alpha - \sigma_\varepsilon^2 + \sigma_\varepsilon^2(R-1)\right)x\,dt$$
$$+ \tilde{\theta}\left[\sigma\left(\rho\,dW^* + \rho' dW'\right) + (\mu - r - \rho R\sigma\sigma_\varepsilon)dt\right] - \tilde{c}\,dt + dt$$
$$- \sigma_\varepsilon x\,dW^*$$

and, were it not for the very last term here, this would be just like the case of constant income; looks rather intractable else.

Price impact again 13/3/13

It seems that if we want to determine the riskless rate process we need a further element of the modelling story.

Agent j starts day t with a cash balance $(1+r_t)\varphi_t^j$ and portfolio θ_t^j, both known at the end of day $t-1$. So he has wealth

$$w_t = (1+r_t)\varphi_t + \theta_t \cdot S_t$$

where we omit the index j from the notation for the calculation. This agent now chooses consumption c and portfolio θ_{t+1}. So he has $\varphi_{t+1} = w_t - c - \theta_{t+1}\cdot S_t$ in the bank account. His aim is to achieve

$$\max_{c,\theta_t} E\Big[U(c) + aU\big((1+r_{t+1})\varphi_{t+1} + \theta_{t+1}\cdot S_{t+1}\big)\Big]$$

where U is CARA, $a>0$ (but not necessarily <1), and $S_{t+1} \sim N(\hat{\mu}_t + S_t, \Sigma)$. The optimal choice for θ is

$$\boxed{\theta_{t+1}^j = (\gamma_j \Sigma^j)^{-1}\{\hat{\mu}_t^j - r_{t+1} S_t\}}$$

exactly as before. The optimization over c requires us to maximise

$$-e^{-\gamma c} - a\exp\Big[-\gamma(1+r)(w-c) - \tfrac{1}{2}(\hat{\mu} - rS_t)\cdot\Sigma^{-1}(\hat{\mu}-rS_t)\Big]$$

so with calculus we get

$$-\gamma c = \log(a(1+r)) - \gamma(1+r)(w-c) - \tfrac{1}{2}(\hat{\mu}-rS_t)\cdot\Sigma^{-1}(\hat{\mu}-rS_t)$$

so

$$\boxed{(2+r)c = -\frac{1}{\gamma}\log(a+ar) + (1+r)w + \frac{1}{2\gamma}(\hat{\mu}-rS_t)\cdot\Sigma^{-1}(\hat{\mu}-rS_t)}$$

The market-clearing condition for the θ's will be exactly the same as before. As for the cash, the change in the agent's cash holdings will be

$$\varphi_{t+1} - (1+r_t)\varphi_t = w_t - c - \theta_{t+1}\cdot S_t - (1+r_t)\varphi_t$$

$$= \theta_t\cdot S_t - c - \theta_{t+1}\cdot S_t$$

and in aggregate this has to be covered by the lending from the non-speculative agent

Firm sizes with different killing rates (28/3/13)

(1) Lukas has raised a number of quite searching questions about the firm size story if we allowed the killing rate to depend on σ. Let's see if we can deal with these by supposing that there can be only finitely many types of firm, and firms of type i are created at rate ρ_i and killed at rate ε_i, $i=1,\dots,I$.

(2) Temporarily dropping the label i, let's just deal with a $M/M/\infty$ queue with arrival rate ρ, departure rate ε. In steady state, the distribution of the number in the queue is $\mathcal{P}(\rho/\varepsilon)$. If we see at $t=0$ that there are K people in the queue, what is the joint distribution of the times $-T_j$, $j=1,\dots,K$ at which these people joined the queue?

The answer is that the T_j are IID $\exp(\varepsilon)$; the neat way to see this is that the steady-state queue is reversible, and in reversed time the T_j become the times at which people leave the queue!

(3) Now let $r_i(x) \equiv \int_0^\infty \varepsilon_i \exp(-\varepsilon_i t)\, p_t^i(x)\, dt$ be the density of where the log-price for a type-i firm has got to after an $\exp(\varepsilon_i)$ time has elapsed. Set $\lambda_i \equiv \rho_i/\varepsilon_i$, and $\lambda = \sum \lambda_i$, $\pi_i \equiv \lambda_i/\lambda$. Now we look for

$$P(N \text{ firms at time } 0;\ \text{of types } i_1,\dots,i_N;\ \text{with sizes } x_1,\dots,x_N)$$

$$= \frac{\lambda^N e^{-\lambda}}{N!} \cdot \prod_{\ell=1}^N \pi_{i_\ell}\, r_{i_\ell}(x_\ell)$$

$$= \frac{e^{-\lambda}}{N!} \prod_{\ell=1}^N \lambda_{i_\ell}\, r_{i_\ell}(x_\ell)$$

(4) Suppose at time zero we see N firms with sizes $x_1,\dots,x_N$; what is the distribution of the types given this information?

$$P(\text{firm } \ell \text{ is of type } i_\ell \mid x_1,\dots,x_N) = \prod_{\ell=1}^N \pi_{i_\ell} r_{i_\ell}(x_\ell) \Big/ \sum_{j_1,\dots,j_N} \prod_{\ell=1}^N \pi_{j_\ell} r_{j_\ell}(x_\ell)$$

$$= \prod_{\ell=1}^N \pi_{i_\ell} r_{i_\ell}(x_\ell) \Big/ \prod_{\ell=1}^N \Big(\sum_{j=1}^I \pi_j r_j(x_\ell) \Big)$$

– So given the firm sizes $x_1,\dots,x_N$, the firm types are <u>independent</u> and the probability of firm of size x being of type j is $\propto \pi_j r_j(x) \propto \rho_j \int_0^\infty \exp(-\varepsilon_j t)\, p_t^j(x)\, dt$.

Questions relating to purchasing a block of stock (16/4/13)

Here are some natural and interesting questions around the following theme: You have the task of buying A units of stock by time T. The market price evolves as

$$dX_t = \sigma dW_t + \mu dt$$

(we could make the evolution log-Brownian but it probably won't make much difference qualitatively).

Your aim is to max $E\, U(-Z_T)$, where Z_t is total cost of acquiring the assets by time t. We could consider various forms:

(1) All the assets have to be bought in one go; or

(2) You may buy the assets gradually;

(3) You may know σ, μ; or

(4) You may know σ, but doing Bayesian inference on μ;

(5) You may be trading continuously, but have to pay $X_t - \eta \dot{A}_t$, where A_t is the amount still to buy at time t, and η is a positive constant;

(6) The price may experience permanent price impact as you buy

(7) You may place some limit bid orders, which get taken out at some rate in local time ...

Price impact model again (26/4/13)

(i) Let's return to the modelling story presented on pp 41 – 43. There were some things to notice there, concerning the stability of the linear system. With a constant interest rate r, we derived ((*) on p 42) the updating recursions

$$\begin{cases} b = \sum (\gamma_j \Sigma_j)^{-1} \{ (K_j - r) S_t + (I - K_j) \hat{\mu}^j_{t-1} - K_j S_{t-1} \} \\ \hat{\mu}^j_{t+1} = (1-K_j)\hat{\mu}^j_t + K_j Y_{t+1} = (1-K_j)\hat{\mu}^j_t + K_j (S_{t+1} - S_t) \end{cases}$$

where we assume all agents are in the market everyday, and that b is the fixed total supply vector of the assets.

Introduce $P = \sum (\gamma_j \Sigma_j)^{-1} K_j$, $Q = \sum (\gamma_j \Sigma_j)^{-1}$ so that the updating recursions read

$$\begin{cases} b = P(S_t - S_{t-1}) - r Q S_t + \sum (\gamma_j \Sigma_j)^{-1} (1 - K_j) \hat{\mu}^j_{t-1} \\ \hat{\mu}^j_t = (1-K_j)\hat{\mu}^j_{t-1} + K_j (S_t - S_{t-1}) \end{cases}$$

Let's assume $\Sigma_j = V_j + V_{\varepsilon,j} + V_{\eta,j}$, $V_{\varepsilon j}$ and $V_{\eta,j}$ are all strictly positive-definite. Writing $Z_t = [S_t ; \hat{\mu}^1_t ; \dots ; \hat{\mu}^J_t]$, we have the linear recursion

$$Z_t = A Z_{t-1} + f$$

for some fixed vector f, as we assume that Q is invertible. For this system to be stable, we would require all eigenvalues of A to be in the unit circle.

(ii) Suppose λ is an eigenvalue of A, with eigenvector $[z_0; z_1; \dots; z_J]$. Then the equations

$$\begin{cases} 0 = \lambda (P - rQ) z_0 - P z_0 + \sum (\gamma_j \Sigma_j)^{-1} (1 - K_j) z_j \\ \lambda z_j = (1 - K_j) z_j + K_j (\lambda - 1) z_0 \qquad (j = 1, \dots, J) \end{cases}$$

must hold. From the second, $(\lambda - 1 + K_j) z_j = (\lambda - 1) K_j z_0$, so $z_j = (1-\lambda)(1-\lambda-K_j)^{-1} K_j z_0$. Substitute this into the first equation to learn

$$\boxed{0 = ((\lambda-1)P - \lambda r Q) z_0 + \sum (\gamma_j \Sigma_j)^{-1} (1-\lambda)(1-\lambda-K_j)^{-1}(1-K_j) K_j z_0}$$

Dividing by $(1-\lambda)$, using the definition of P and doing some rearranging gives us the equivalent statement

$$\boxed{0 = \sum (\gamma_j \Sigma_j)^{-1} (1-\lambda-K_j)^{-1} K_j z_0 - \frac{r Q z_0}{1-\lambda}} \qquad (**) \qquad (\lambda \neq 0)$$

which has to have a non-zero solution z_0.

(iii) Now let's think what we know about Σ_j, K_j. Since K_j is the steady-state Kalman gain matrix, we see (look at p 41 again) that

$$K_j = (V_j + V_{\eta,j})(V_j + V_{\eta,j} + V_{\varepsilon,j})^{-1} \equiv (V_j + V_{\eta,j})\Sigma_j^{-1}$$

Now define the symmetric matrix

$$\tilde{K}_j = \Sigma_j^{-1/2} K_j \Sigma_j^{1/2}$$

which clearly has the same eigenvalues as K_j, which must therefore be real and positive. It is a simple exercise to show that all eigenvalues of K_j are in $(0,1)$. Hence the eigenvalue equation $(**)$, re-expressed in terms of $\tilde{K}_j$, reads

$$\boxed{0 = \sum \gamma_j^{-1} \Sigma_j^{-1/2} (1-\lambda-\tilde{K}_j)^{-1} \tilde{K}_j \Sigma_j^{-1/2} z_0 - \frac{r Q z_0}{1-\lambda}} \qquad (***)$$

The nice thing here is that the matrix applied to z_0 is symmetric

(iv) Special case: $r=0$

In this situation, the equation for the eigenvalue λ becomes (if $\lambda \neq 0$)

$$\boxed{0 = \sum \gamma_j^{-1} \Sigma_j^{-1/2} \tilde{K}_j^{1/2} (1-\lambda-\tilde{K}_j)^{-1} \tilde{K}_j^{1/2} \Sigma_j^{-1/2} z_0} \qquad (\dagger)$$

Thus since all e-values of $1-K_j$ are in $(0,1)$ for all j, it follows that if $\lambda \notin (0,1)$ then λ cannot be an eigenvalue, and the system is stable.

Note: if $\lambda = 0$, then certainly λ is an eigenvalue, with eigenvector $[w; 0; 0; \ldots; 0]$ for any non-zero w.

Note: There can be no complex solutions to $(\dagger)$ ... go figure!

(v) General r. This time, we use the definition of Q to rework $(***)$

$$0 = \sum \gamma_j^{-1} \Sigma_j^{-1/2} \left\{ (1-\lambda-\tilde{K}_j)^{-1} \tilde{K}_j - \frac{r}{1-\lambda} \right\} \Sigma_j^{-1/2} z_0$$

$$= \sum \gamma_j^{-1} \Sigma_j^{-1/2} \left\{ \tilde{K}_j - \frac{r}{1-\lambda}(1-\lambda-\tilde{K}_j) \right\} (1-\lambda-\tilde{K}_j)^{-1} \Sigma_j^{-1/2} z_0$$

$$= \frac{1-\lambda+r}{1-\lambda} \sum \gamma_j^{T} \Sigma_j^{-\frac{1}{2}} \left\{ \tilde{K}_j - \frac{r(1-\lambda)}{1-\lambda+r} \right\} (1-\lambda-\tilde{K}_j)^{-1} \Sigma_j^{-\frac{1}{2}} z_0$$

When can we say that this matrix is positive-definite or negative-definite? We know all eigenvalues x of $\tilde{K}_j$ are in $(0,1)$, so what we would look for (a sufficient condition) is that λ has the property

$$\left(x - \frac{r(1-\lambda)}{1-\lambda+r} \right) (1-\lambda-x)^{-1}$$

has the same sign for all $x \in (0,1)$. Now if $\lambda > 1$, or if $\lambda < 0$, the second factor $(1-\lambda-x)^{-1}$ does not change sign as x varies through $(0,1)$. So we must ask of $\lambda \notin (0,1)$ that the first factor doesn't change sign in $(0,1)$.

Assuming that $r \in (0,1)$, we have the condition

$$\frac{-r(1-\lambda)}{1-\lambda+r} \cdot \left(1 - \frac{r(1-\lambda)}{1-\lambda+r} \right) > 0$$

equivalently,

$$-r(1-\lambda)(1-\lambda+r\lambda) > 0$$

If we have $\lambda < 0$, then $(1-\lambda) > 0$, so we must have $1-\lambda(1-r) < 0$, which is impossible. If we have $\lambda > 1$, then we shall have to have $1-\lambda(1-r) > 0$ so that $\lambda < 1/(1-r)$. All in all, this is not very encouraging; we have learned that

if $r \in (0,1)$ then no $\lambda \in \left(1, \frac{1}{1-r}\right)$ can be an eigenvalue.

Alternative price impact story (15/5/13)

(i) The story on pp 41–43 seems to have some quite unrealistic features. What we see if $r=0$ is that the prices S_t seem to grow linearly with t, and this would appear to be a consequence of the modelling assumption that the trend in price is a RW; what seems to be happening is that this trend vector seems to be settling down to a more-or-less constant value.

(ii) As an alternative, suppose we postulate

$$\begin{cases} S_t = \mu_t + \varepsilon_t \\ \mu_t = B\mu_{t-1} + \eta_t \end{cases}$$

Where B has spectral radius ≤ 1 (probably take diagonal B to be getting started). Now we can again do the KF story: if $(\mu_t|\mathcal{Y}_t) \sim N(\hat{\mu}_t, V_t)$ then given $\mathcal{Y}_t$

$$\begin{pmatrix}\mu_{t+1}\\ S_{t+1}\end{pmatrix} \sim N\left(\begin{pmatrix} B\hat{\mu}_t \\ B\hat{\mu}_t\end{pmatrix}, \begin{pmatrix} BV_tB^T + V_\eta & BV_tB^T + V_\eta \\ BV_tB^T + V_\eta & BV_tB^T + V_\eta + V_\varepsilon \end{pmatrix}\right)$$

so

$$\boxed{\begin{aligned} \hat{\mu}_{t+1} &= B\hat{\mu}_t + K\left(S_{t+1} - B\hat{\mu}_t\right) \\ K &= \left(BVB^T + V_\eta\right)\left(BVB^T + V_\eta + V_\varepsilon\right)^{-1} \\ V &= BVB^T + V_\eta - \left(BVB^T + V_\eta\right)\left(BVB^T + V_\eta + V_\varepsilon\right)^{-1}\left(BVB^T + V_\eta\right) \\ &= V_\varepsilon\left(BVB^T + V_\eta + V_\varepsilon\right)^{-1}\left(BVB^T + V_\eta\right) \end{aligned}}$$

in steady state. Each agent will do his own Kalman filtering, as before.

(iii) The demand of agent j on day t will now be $\left(\Sigma_j \equiv B_jV_jB_j^T + V_{\varepsilon,j} + V_{\eta,j}\right)$

$$(\gamma_j\Sigma_j)^{-1}\left\{ B_j\hat{\mu}_t^j - (1+r)S_t\right\}$$

So total demand is

$$\boxed{\sum_j (\gamma_j\Sigma_j)^{-1}\left\{ B_j\left(K_jS_t + (I-K_j)B_j\hat{\mu}_{t-1}^j\right) - (1+r)S_t\right\}}$$

We can take this a bit further. If we write $\bar{\theta}_t$ for the total supply in period t, and def $M_0 \equiv \sum (\gamma_j \Sigma_j)^{-1}\{1+r-B_j K_j\}$, $M_j \equiv (\gamma_j \Sigma_j)^{-1} B_j (1-K_j) B_j$, then the market clearing condition gives us

$$\bar{\theta}_t = -M_0 S_t + \sum_1^J M_j \hat{\mu}^j_{t-1}$$

so

$$\boxed{\begin{aligned} S_t &= M_0^{-1}\left(\sum_1^J M_j \hat{\mu}^j_{t-1} - \bar{\theta}_t\right) \\ \hat{\mu}^j_t &= (1-K_j)B_j \hat{\mu}^j_{t-1} + K_j S_t \end{aligned}}$$

The conclusion is that we evolve the $\hat{\mu}^j$, and deduce the S_t as we go:

$$\hat{\mu}^j_t = (1-K_j)B_j \hat{\mu}^j_{t-1} + K_j M_0^{-1} \sum_{\ell=1}^J M_\ell \hat{\mu}^\ell_{t-1} - K_j M_0^{-1} \bar{\theta}_t .$$

Hunger as objective (19/6/13)

(1) Suppose we think that an agent invests in a conventional market

$$dw_t = r w_t \, dt + \theta_t \{ \sigma dW_t + (\mu - r) dt \} - dc_t$$

where the non-decreasing consumption process is used to assuage hunger, which evolves as

$$dh_t = \varphi(h_t) \, dt - dc_t$$

and the objective of the agent is to attain

$$V(w, h) = \inf E\left[\int_0^\infty e^{-\rho t} f(h_t) \, dt \;\middle|\; w_0 = w, \; h_0 = h \right]$$

with $w_t \geq 0$ for all t. Here we shall suppose that f is non-negative, convex, increasing. Running the usual MPOC story leads to

$$\left.\begin{aligned} &0 \leq \inf \left[-\rho V + (rw + \theta(\mu - r)) V_w + \tfrac{1}{2} \sigma^2 \theta^2 V_{ww} + \varphi V_h + f \right] \\ &0 \leq -(V_w + V_h) \end{aligned}\right\}$$

with one of these holding with equality at each (w, h).

(2) Now in general this won't be easy to work with, but if we suppose

$$\varphi(h) = ah, \qquad f(h) = h^\varepsilon$$

for some $a > 0$, $\varepsilon > 1$ constant, then we have scaling of the evolution equation, so we shall expect

$$V(\lambda w, \lambda h) = \lambda^\varepsilon V(w, h) \quad \Rightarrow \quad V(w, h) = h^\varepsilon v(w/h)$$

and the HJB equations give us

$$0 \leq \inf \left[-\rho v + (r + z(\mu - r)) x v' + \tfrac{x^2}{2} \sigma^2 z^2 v'' + a\{ \varepsilon v - x v'(x) \} + 1 \right]$$

$$0 \leq -\{ v'(x) + \varepsilon v(x) - x v'(x) \}$$

with $x \equiv w/h$. Doing the optimization in the first leads to

$$\begin{cases} 0 \leq -(\rho - a\varepsilon) v + (r - a) x v' - \dfrac{\kappa^2}{2} \dfrac{(v')^2}{v''} + 1 \\ 0 \leq (x - 1) v' - \varepsilon v \end{cases}$$

We also need the non-infinitesimal form of the second, that $V(w - c, h - c) \geq V(w, h)$ for all $c \geq 0$. Looks like the control will be to do some little consumptions so as to keep $w/h \leq x_*$?

(3) How about we try dual variables? But maybe first it makes sense to simplify.

What we could do is to introduce $\tilde{h}_t = e^{-at} h_t$, $\tilde{\theta}_t = e^{-at}\theta_t$, $\tilde{w}_t = e^{-at} dt$, $d\tilde{c}_t = e^{-at} dc_t$ so that the evolution of the wealth $\tilde{w}$ and hunger $\tilde{h}$ becomes

$$\begin{cases} d\tilde{w}_t = (r-a)\tilde{w}_t\, dt + \tilde{\theta}_t \{\sigma dW + (\mu - r)\, dt\} - d\tilde{c}_t \\ d\tilde{h}_t = - d\tilde{c}_t \end{cases}$$

with objective $E\left[\int_0^\infty e^{-(\rho - a\varepsilon)t} \tilde{h}_t^{\varepsilon}\, dt\right]$ so looks like we must insist that

$$\rho > a\varepsilon$$

for well posed problem, and at that point we could just suppose that $a = 0$ in the original formulation. The HJB then simplifies to

$$\begin{cases} 0 \leq -\rho v + rxv' - \frac{\kappa^2}{2}\frac{(v')^2}{v''} + 1 \\ 0 \leq (x-1)v' - \varepsilon v \end{cases}$$

Now it's clear that $V(w,h)$ decreases with w, increases with h, and $J(z) \equiv \inf_{x>0} \{V(x) - zx\}$ is only finite valued if $z < 0$; and J has to be non-negative, decreasing, concave in $(-\infty, 0)$. The dual equations give us

$$\begin{cases} 0 \leq -\rho J - (r-\rho) z J' + \frac{1}{2}\kappa^2 z^2 J'' + 1 \\ 0 \leq -z - \varepsilon J + (\varepsilon - 1) z J' \end{cases}$$

and we expect the first to hold if $z \leq z_* = v'(x_*)$, the second to hold at $z \geq z_*$. Now the first equation has a solution of the form

$$J(z) = \frac{1}{\rho} + A|z|^{-\alpha} + B|z|^{\beta}$$

where α, β are roots of

$$\tfrac{1}{2}\kappa^2 t(t-1) - (r-\rho)t - \rho = 0$$

so one root $-\alpha$ is negative, the other root $\beta > 1$; so since J has to be concave, positive decreasing in $(-\infty, z_*)$ we see that $B = 0$, and

$$\boxed{J(z) = \frac{1}{\rho} + A|z|^{-\alpha} \qquad (z \leq z_*)}$$

and for concavity and $J\downarrow$ we will have to have $A < 0$.

Solving the other equation in $(z_*, 0)$ gives us a solution

$$J(z) = -z + \lambda |z|^{\varepsilon/(\varepsilon-1)}$$

for some λ, and as before we would need $\lambda < 0$ in order to preserve the concavity of J.

Writing the solution as

$$J(z) = \begin{cases} \frac{1}{\rho} - b \left| z/z_* \right|^{-\alpha} & (z \le z_*) \\ -z - k \left| z/z_* \right|^{\varepsilon/(\varepsilon-1)} & (z_* \le z < 0) \end{cases}$$

we get the conditions for a C^1 join:

$$\frac{1}{\rho} - b = -z_* - k$$

$$-\frac{\alpha b}{|z_*|} = -1 + \frac{\varepsilon k}{(\varepsilon-1)|z_*|}$$

and for a C^2 join it's going to require

$$\alpha(\alpha+1) b = \frac{\varepsilon}{(\varepsilon-1)^2} k$$

which should allow us to solve out completely. This leads to

$$\boxed{z_* = -\frac{\alpha\varepsilon}{\rho(1+\alpha)}, \quad b = \frac{\varepsilon}{\rho(1+\alpha)(\alpha\varepsilon - \alpha + \varepsilon)}, \quad k = \frac{(\varepsilon-1)^2 \alpha}{\rho(\alpha\varepsilon - \alpha + \varepsilon)}}$$

Financing a firm (24/6/13)

Zhiguo He gave a talk in Chengdu about a firm financing itself by bond issues which got rolled over. The bondholders could renew the debt at its face value, or could decline to do so, and if they declined, there was a chance the firm failed. I wasn't sure I liked all of the assumptions, but it looked to me like we might try to take the ingredients and mix in with the firm's policy along the lines of Leland + Toft. Let's try this.

At time t, there is debt of face value F_t issued, on which coupons $cF_t\,dt$ are being paid. Issued bonds have $\exp(\delta)$ lifetimes, and new bonds are issued at rate dA_t. The firm pays out dividends at rate e_t to its shareholders. The market value at time t of debt with face value 1 is denoted D_t. The evolution of the firm's wealth w_t is given as

$$dw_t = w_t(\sigma dW_t + r\,dt) - e_t\,dt - (c+\delta)F_t\,dt + D_t\,dA_t$$

and of the face value of debt by

$$dF_t = -\delta F_t\,dt + dA_t$$

Here, W is a BM in the pricing measure. The firm's objective is to achieve

$$V(w,F) = \sup E\left[\int_0^\tau e^{-\rho t}\Lambda_t\, U(e_t)\,dt \;\Big|\; w_0 = w,\ F_0 = F\right]$$

where $d\Lambda_t = \Lambda_t\,\kappa\,dW_t$, $\kappa = (\mu - r)/\sigma$, is the change-of-measure to the real-world probability, and τ is the first time wealth hits zero. The value of debt is

$$D(w,F) = E\left[\int_0^{\tau\wedge T} c\,e^{-rt}\,dt + e^{-r(T\wedge\tau)}\, I_{\{T<\tau\}} \;\Big|\; w_0 = w,\ F_0 = F\right]$$

where $T \sim \exp(\delta)$. Simple calculations give

$$D(w,F) = \frac{c+\delta}{r+\delta}\, E\left[1 - e^{-(r+\delta)\tau} \;\Big|\; w_0 = w,\ F_0 = F\right].$$

The HJB story is now

$$0 = \sup\left[-\rho V + (\mu w - (c+\delta)F - e)V_w + \tfrac{1}{2}\sigma^2 w^2 V_{ww} - \delta F V_F + U(e)\right]$$

with the condition $D V_w + V_F \le 0$. To have some chance of solving, let's suppose $U(x) = x^{1-R}$ for some $R \in (0,1)$, which would imply the scaling

$$V(w,F) = F^{1-R}\, v(x) \equiv F^{1-R}\, v(w/F)$$

In terms of the scaled value function v, the HJB now reads

$$0 = \sup_z \Big[-\rho v + U(z) + (\mu x - z - c - \delta) v' + \tfrac{1}{2}\sigma^2 x^2 v'' - \delta\{(1-R)v - xv'\}\Big]$$

$$0 \geq Dv' + (1-R)v - xv'$$

or again doing the optimization over z

$$\boxed{\begin{aligned} 0 &= -\rho v + \tilde{U}(v') + \big((\mu+\delta)x - c - \delta\big)v' + \tfrac{1}{2}\sigma^2 x^2 v'' - \delta(1-R)v \\ 0 &\geq (D - x)v' + (1-R)v \end{aligned}}$$

If we've found the optimal v, then we chose $z = I(v'(x))$, and the dynamics of $x \equiv W/F$ are obtained from Itô:

$$dx = x(\sigma dW + r dt) - z_t dt - (c+\delta)dt + (D - x)\frac{dA}{F} + \delta x\, dt$$

$$= \sigma x\, dW + \{(r+\delta)x - (c+\delta) - I(v'(x))\}dt + (D-x)\, dA/F$$

We can find D from the ODE

$$\begin{cases} 0 = \tfrac{1}{2}\sigma^2 x^2 D'' + \{(r+\delta)x - c - \delta - I(v'(x))\}D' - (r+\delta)D + c + \delta \\ D(0) = 0 = D'(x_*) \end{cases}$$

Merton problem with liquidity costs (18/7/13)

(i) A long time ago, Sunb Singh, Johnathan Evans & I were looking at the Merton problem with liquidity costs. The dynamics studied was

$$\begin{cases} dW_t = H_t dS_t - h_t S_t f(\varepsilon h_t)\,dt - c_t\,dt + r(W_t - H_t S_t)\,dt \\ dH_t = h_t\,dt \end{cases}$$

where $dS = S(\sigma dW + \mu dt)$ is a standard log-Brownian asset, and $f : \mathbb{R} \to [-1,\infty)$ is convex increasing, $f(0)=0$, representing the effects of trading rapidly through the limit order book. The agent has objective

$$V(w,H,S) = \sup E\left[\int_0^\infty e^{-\rho t} U(c_t)\,dt \;\Big|\; W_0 = w,\ H_0 = H,\ S_0 = S\right]$$

where $U'(x) = x^{-R}$. It is not hard to see that we must have $H_t \geq 0$, $W_t - H_t S_t \geq 0$. Exploiting scaling, and writing $Y_t \equiv W_t/S_t - H_t$, the value of cash in terms of the stock, we got to the HJB equation

$$\boxed{0 = \sup_{c,h}\left[U(c) - \tilde{\rho} F + \tfrac{1}{2}\sigma^2 Y^2 F_{YY} - \{h + h f(\varepsilon h) + c + \alpha Y\} F_Y + h F_H\right]}$$

where $\tilde{\rho} = \rho + (R-1)(\mu - \sigma^2 R/2)$, $\alpha = \mu - r - \sigma^2 R$, where

$$V(w,H,S) = S^{1-R}\, F(Y,H).$$

How could this be solved?

(ii) Notice that the PDE has a parabolic form, which suggests we might try to solve it numerically by starting at $H=0$ and working out to $H = \Delta H, 2\Delta H, 3\Delta H, \ldots$

But how would we start? One thing we might guess is that if H is very small, it should be possible to exchange cash for stock at low cost, so we might guess $F_H \approx F_Y$?

Set down some grid $Y_1 < \ldots < Y_N$ of Y-values, and write

$$F_k^n = F(Y_k, n\Delta H)$$

Then if we knew the values of $F^n_\cdot$ and if L_0 denotes the matrix approximation to $\frac{1}{2}\sigma^2 Y^2 D_Y^2 - \alpha D_Y - \tilde{\rho}$, the Crank-Nicolson recipe would give us

$$0 = \sup_{c,h}\left[U(c) + L_0\left(\tfrac{1}{2}F^{n+1} + \tfrac{1}{2}F^n\right) - \tfrac{1}{2}\left(h + h f(\varepsilon h) + c\right) D_Y\left(F^{n+1} + F^n\right) + h\,\frac{F^{n+1} - F^n}{\Delta H}\right]$$

as a way of finding F^{n+1} from F^n, and this could perhaps be done iteratively.

[We could also play it back to get F^n from F^{n+1}??]

(iii) This looks quite a mess numerically: one thing might be to just fix the consumption at $\gamma_M w = \gamma_M (Y+H)S$, so that we are only optimizing over R. Although it's a parabolic-style PDE, the direction of the first-order component changes with the sign of h, so likely we won't get a stable method going in just one direction...

(iv) Another thought would be to set a boundary condition at some H_* large, namely, that when H reaches H_* all the frictions are turned off, so we just get the Merton solution:

$$F(Y, H_*) = \gamma_M^{-R}\, U(H_* + Y)$$

There still remains the question about this will get stepped back into $H < H_*$ but it is at least a good start.

~~Suppose we have $F(\cdot, H)$ and we want to get $F(\cdot, H-\Delta H) = F(\cdot, H) - \eta(\cdot)\Delta H$.~~
~~We will have (references to F are evaluated at H)~~

This approach to the numerics seems to be very unsuccessful.

An interesting dynamic contracting problem (1/8/13)

(i) Suppose there's a single risky asset whose value at time t, S_t, evolves as

$$dS_t = S_t\,(\sigma\,dW_t + (\mu + a_t)\,dt)$$

where $a_t \geq 0$ is the agent's level of effort at time t. The principal will pay the agent wages at rate y_t, and the agent's objective will be

$$v(w) = \sup E\left[\int_0^\tau e^{-\gamma s}\{U_A(y_s) - f(a_s)\}\,ds \;\Big|\; w_0 = w\right]$$

where U_A is concave increasing, f is convex increasing, and τ is the time when the agent walks away from the contract, when $v(w)$ falls to his reservation value $\underline{v}$. Assume $U_A \geq 0$. The wealth dynamics of the principal are

$$dw_t = r w_t + \theta_t\{\sigma\,dW_t + (\mu + a_t - r)\,dt\} - (c_t + y_t)\,dt$$

and he seeks the objective

$$V(w) = \sup E\left[\int_0^\infty e^{-\rho s}\,U_p(c_s)\,ds \;\Big|\; w_0 = w\right].$$

Notice that the principal could decide to pay the agent nothing at all, the agent would walk away immediately, so V must be at least the value of the Merton problem!

How to solve this?

(ii) Even a single-agent version of the problem looks interesting:

$$\sup E\left[\int_0^\infty e^{-\rho t}\,(U(c_t) - f(a_t))\,dt \;\Big|\; w_0 = w\right] \equiv V(w)$$

where

$$dw_t = r w_t\,dt + \theta_t\{\sigma\,dW_t + (\mu + a_t - r)\,dt\} - c_t\,dt$$

The HJB is

$$0 = \sup_{\theta, c, a}\left[-\rho V + (rw + \theta(\mu + a - r) - c)V' + \tfrac{1}{2}\sigma^2\theta^2 V'' + U(c) - f(a)\right]$$

Not immediately obvious that the HJB is concave in (θ, c, a) ...

Merton liquidity problem again (7/8/13)

(i) Looks like we really need to tell a better probabilistic story to solve this problem. The co-ordinate system $(Y,H) \equiv (w/s - H, H)$ seems a sensible choice. Seems like we need to be more explicit about boundary conditions when we solve in some box $[0,Y^*] \times [0,H^*]$. So let's suppose the following:

(a) when $Y > Y^*$, all trading of the stock is forbidden until $Y \le Y^*$

(b) when $Y = 0$, we are compelled to do an emergency sale of one unit of stock

(c) once H reaches H^*, all the liquidity costs are turned off and the value is just the Merton value:

$$F(Y,H) = \gamma_M^{-R}\, U(Y+H) \qquad \text{for } H \ge H^*.$$

For the notion of emergency sale of stock to be meaningful, we need to suppose that there is some $\beta \in (0,1)$ such that

$$\lim_{t \to -\infty} f(t) = -\beta.$$

Assuming this, $\Phi(a) \equiv \sup\{at - tf(t)\}$ is infinite for $a < -\beta$, is convex, non-negative, and $\Phi(0) = 0$. What I'd propose is to define

$$\Phi(a) = \frac{\lambda}{a+\beta} - \frac{\lambda}{\beta} + \frac{\lambda a}{\beta^2} + \frac{\varepsilon}{2} a^2$$

for some $\lambda > 0$. Recovering f would require solution of some cubic, but it's not essential to have f in closed form.

We may as well think of H as being quantized in multiples of some unit ΔH. If we try to purchase at rate h, then the mean time Δt until we achieve the purchase will satisfy $\Delta H = h\,\Delta t$, and in this time we suppose the effect will be to have pushed Y down by $\lambda\,\Delta t$, where we want

$$\frac{\lambda\,\Delta t}{\Delta H} = \frac{h + \tilde{f}(h)}{h} \quad \text{so} \quad \boxed{\lambda = h + \tilde{f}(h)} \qquad [\tilde{f}(h) = hf(h)]$$

(ii) We can cast the problem in the form

$$\max E\left[\int_0^\infty e^{-\rho t}\, U(c_t)\,dt \;\Big|\; Y_0 = Y,\ H_0 = H\right] \equiv F(Y,H)$$

where

$$\begin{cases} dY_t = Y(\sigma dW - \alpha dt) - (c_t + h_t + h_t f(h_t))\,dt \\ dH_t = h_t\,dt \end{cases}$$

(iii) How does it look to the right of Y^*? Maybe what we'll do here to spare ourselves the trouble of doing a load of optimization is to assume that we shall consume as we would for the Merton problem, so take $c = \gamma_M (Y+H)$; this would give

$$dY = Y\left(\sigma\, dW - (\alpha + \gamma_M)\, dt\right) - \gamma_M H\, dt$$

for the dynamics of Y and to discover for test function q

$$\varphi(y) = E^y \left[\int_0^{\tau} e^{-\tilde{\rho} s} q_1(Y_s)\, ds \right]$$

we'd need to solve

$$0 = -\tilde{\rho}\varphi + \tfrac{1}{2}\sigma^2 y^2 \varphi'' - (\alpha y + \gamma_M (y+H))\varphi' + q. \qquad (*)$$

We're interested in the two cases $q(y) = U(\gamma_M (y+H))$ and $q(y) = 1$ in order to identify the value V^* at (Y^*, H).

In more detail, if φ_1 solves the ODE $(*)$ in $[Y^*, \bar{Y}]$ with boundary conditions

$$\varphi_1(Y^*) = 0, \quad \varphi_1'(\bar{Y}) = (1-R)\varphi_1(\bar{Y})/\bar{Y}, \quad q(y) = U(\gamma_M (y+H))$$

and φ_2 solves $(*)$ in $[Y^*, \bar{Y}]$ with conditions $q(y) \equiv 1$,

$$0 = \varphi_2(Y^*), \quad \varphi_2(\bar{Y}) = 1/\tilde{\rho}$$

then we would have

$$V(Y^* + \Delta Y) = \varphi_1(Y^* + \Delta Y) + \{1 - \tilde{\rho}\varphi_2(Y^* + \Delta Y)\} V(Y^*).$$

This allows us to deduce

$$\boxed{V'(Y^*) \approx \frac{V(Y^* + \Delta Y) - V(Y^*)}{\Delta Y} = \frac{\varphi_1(Y^* + \Delta Y) - \tilde{\rho}\varphi_2(Y^* + \Delta Y) V(Y^*)}{\Delta Y}}$$

This is a linear relation between V and its derivative at Y^*, or between the values of V at Y^* and at $Y^* + \Delta Y$.

(iv) At $Y = 0$, we must jump from $(0, H)$ to $((1-\beta)\Delta H,\ H - \Delta H)$, so the smart thing to do would be to put the first positive Y grid point at $(1-\beta)\Delta H = Y_1$, say. Then the boundary condition would say

$$\boxed{F(0, H) = F(Y_1, H - \Delta H) \approx 2F(Y_1, H) - F(Y_1, H + \Delta H)}$$

again a linear relation, but we probably need to account for concavity also.

(v) It may in fact be better to the right of Y^* to suppose that consumption happens at rate $c = \gamma_M Y$. Thus if we solve for

$$g(y) = E^y\left[\int_0^\tau e^{-\tilde{\rho} t}\, U(\gamma_M Y_t)\,dt + e^{-\tilde{\rho}\tau} K\right] \qquad \left[\tilde{\alpha} \equiv \alpha + \gamma_M\right]$$

we get $0 = \frac{1}{2}\sigma^2 y^2 g'' - \tilde{\alpha} y g' - \tilde{\rho} g + U(\gamma_M y)$, $g(Y^*) = K$. If we look at the auxiliary quadratic $\frac{1}{2}\sigma^2 z(z-1) - \tilde{\alpha} z - \tilde{\rho}$ this has roots $-a < 0 < b$, so the ODE has homogeneous solutions y^b, y^{-a}. There is a particular solution

$$g(y) = A y^{1-R}$$

where

$$0 = A\left[-R(1-R)\frac{\sigma^2}{2} - \tilde{\alpha}(1-R) - \tilde{\rho}\right] + U(\gamma_M).$$

After some calculations

$$0 = A\left[-\rho - (R-1)(r-\gamma_M)\right] + U(\gamma_M)$$

which determines A, and the solution to the right of Y^* has to look like

$$g(y) = A y^{1-R} + B\left(y/Y^*\right)^{-a}$$

where we choose B to match $\boxed{g(Y^*) = K:}$ $\quad B = K - A(Y^*)^{1-R}$, so we conclude

$$g(Y^* + \Delta Y) = A(Y^* + \Delta Y)^{1-R} + B\left(\frac{Y^* + \Delta Y}{Y^*}\right)^{-a}$$

$$= A(Y^* + \Delta Y)^{1-R} + \left\{K - A(Y^*)^{1-R}\right\}\left(\frac{Y^* + \Delta Y}{Y^*}\right)^{-a}$$

which provides a linear relation between g at Y^* and g at $Y^* + \Delta Y$:

$$g(Y^* + \Delta Y) = \left(\frac{Y^* + \Delta Y}{Y^*}\right)^{-a} g(Y^*) + A(Y^* + \Delta Y)^{1-R}\left\{1 - \left(\frac{Y^* + \Delta Y}{Y^*}\right)^{-a+R-1}\right\}$$

Alternatively, we can have

$$g'(Y^*) = (1-R)A(Y^*)^{-R} - a\,B/Y^*$$

$$= (1-R)A(Y^*)^{-R} - \frac{a}{Y^*}\left\{K - A(Y^*)^{1-R}\right\}$$

$$\boxed{g'(Y^*) = A(Y^*)^{-R}\left\{1 - R + a\right\} - aK/Y^*}$$

expressing a linear dependence ($K \equiv g(Y^*)$)